Math 396. Local structure theorem for C^p maps

Let $(X', \mathscr{O}')$ and $(X, \mathscr{O})$ be C^p premanifolds with corners, $1 \le p \le \infty$. Let $F : X' \to X$ be a C^p map. An important technique in local studies in differential geometry is the intelligent selection of local C^p coordinates that are well-adapted to the problem at hand. In the case that X' is a premanifold (i.e., $\partial X' = \emptyset$) and the tangent maps $\mathrm{d}F(\xi') : \mathrm{T}_{\xi'}(X') \to \mathrm{T}_{\xi}(X)$ (with $\xi = F(\xi')$) have constant rank $r \ge 0$ independent of $\xi' \in X'$ (a rather typical situation in practice), it turns out that F takes on an especially simple form in well-chosen local C^p coordinates around both ξ' and ξ. As we shall see, the essential content in the proof is the inverse function theorem.

1. Statement of structure theorem

We now require X' to be a premanifold (i.e., $\partial X'$ is empty).

Theorem 1.1. *Let (φ, U) be a local C^p chart at $\xi_0 = F(\xi_0') \in X$, with $\varphi(U)$ open in a sector in some $\mathbf{R}^n$ and $\{x_1, \dots, x_n\}$ the component functions of φ. Fix a non-negative integer r.*

(1) *We have $\mathrm{rank}(\mathrm{d}F(\xi_0')) \ge r$ if and only if there exists a local C^p chart (ψ, U') around ξ_0' in X' with a small open $U' \subseteq F^{-1}(U)$ and $\psi(U')$ open in some $\mathbf{R}^{n'}$ such that, after perhaps rearranging the x_i's, we have*

$$(\varphi \circ F \circ \psi^{-1})(a_1, \dots, a_{n'}) = (a_1, \dots, a_r, h_{r+1}(a_1, \dots, a_{n'}), \dots, h_n(a_1, \dots, a_{n'}))$$

for C^p functions $h_{r+1}, \dots, h_n$ near $\psi(\xi_0') \in \mathbf{R}^{n'}$.

In other words, $\mathrm{rank}(\mathrm{d}F(\xi_0')) \ge r$ if and only if some r among $x_1 \circ F, \dots, x_n \circ F$ form part of a local C^p coordinate system on X' near ξ_0'.

(2) *Assume $\mathrm{rank}(\mathrm{d}F(\xi_0')) \ge r$, and choose (ψ, U') as above. We have $\mathrm{rank}(\mathrm{d}F(\xi_0')) = r$ if and only if $\partial_j h_{r+1}, \dots, \partial_j h_n$ all vanish at $\psi(\xi_0') \in \mathbf{R}^{n'}$ for all $j > r$.*

(3) *Continuing with the assumption that the tangent map has rank at least r at ξ_0', the rank of $\mathrm{d}F(\xi')$ is equal to r for all ξ' near ξ_0' if and only if $h_{r+1}, \dots, h_n$ near $\psi(\xi_0') \in \mathbf{R}^{n'}$ are C^p functions of the first r coordinates on $\mathbf{R}^{n'}$. In such cases, we may replace x_j with $x_j - h_j(x_1, \dots, x_r)$ for $r < j \le n'$ in the definition of ψ near ξ_0' to get to the case $h_j = 0$ for $j > r$:*

$$(\varphi \circ F \circ \psi^{-1})(a_1, \dots, a_{n'}) = (a_1, \dots, a_r, 0, \dots, 0) \tag{1.1}$$

near $\psi(\xi_0')$.

The most important part is the final one: with respect to good choices of local C^p coordinates on the source and target, maps of constant rank locally look just like linear maps of constant rank in "good" linear coordinates. It is important that we can essentially use any initial choice of local C^p coordinate system that we wish around ξ_0 (up to perhaps rearranging the coordinate functions to get the 0's to appear "at the end" as in (1.1)). The converse to the last part of (3) is immediate: if locally near ξ_0' and ξ_0 we can find C^p coordinate systems (ψ, U') and (φ, U) such that (1.1) holds then the tangent mappings $\mathrm{d}F(\xi')$ have constant rank r for all ξ' near ξ_0'.

We should also note that it seems to be rather problematic to state a reasonable yet general version of the structure theorem when X' is allowed to have points of positive index, even if we require the hypothesis $F(\partial X') \subseteq \partial X$ that intervenes in the general version of the inverse function theorem on premanifolds with corners. For example, if $X' = [0, 1)$ and X is the closed unit disc $\{\|v\| \le 1\}$ in the plane then we can map X' to X so that 0 lands in the boundary and the curve is tangential to the boundary but otherwise nicely maps into the interior of the disc. In such a case the mapping is an injective immersion (so of constant rank 1) but the tangential condition rules out the possibility of finding local coordinates of the sort described in the final part of the theorem.

2. Proof of structure theorem

The implication "$\Leftarrow$" in (1) is clear, and by calculation of the Jacobian matrix for the proposed coordinatized formula in (1) we see that (2) is obvious. Also, since a C^p function on an open set is locally constant if and only if its first-order partials all vanish, once the remaining implication in (1) is proved then the "if and only if" assertion in (3) follows from the obvious fact if h is a C^p function near a point $\xi'_0 = (a_1, \dots, a_{n'}) \in \mathbf{R}^{n'}$ such that the last $n' - r$ partials of h all vanish then for the C^p function

$$g(t_1, \dots, t_r) = h(t_1, \dots, t_r, a_{r+1}, \dots, a_{n'})$$

at points $(t_1, \dots, t_r)$ near $(a_1, \dots, a_r)$ we have

$$h(t_1, \dots, t_{n'}) = g(t_1, \dots, t_r)$$

for $(t_1, \dots, t_{n'})$ near ξ'_0. Continuing to assume (1) is settled, we may also verify (1.1) by checking that the mapping

$$(t_1, \dots, t_n) \mapsto (t_1, \dots, t_r, t_{r+1} - h_{r+1}(t_1, \dots, t_r), \dots, t_n - h_n(t_1, \dots, t_r))$$

on an open neighborhood of $\varphi(\xi_0)$ in $\mathbf{R}^n$ is a C^p isomorphism onto an open set in $\mathbf{R}^n$ (as then the n-tuple of functions

$$\{x_1, \dots, x_r, x_{r+1} - h_{r+1}(x_1, \dots, x_r), \dots, x_n - h_n(x_1, \dots, x_r)\}$$

is a local C^p coordinate system near $\xi_0 \in X$, as required for (1.1)). To verify this local isomorphism claim, we use the inverse function theorem to reduce to checking that this mapping has invertible derivative at $\varphi(\xi_0)$, and such invertibility is clear because with respect to standard coordinates this derivative map has block matrix form

$$\begin{pmatrix} 1_r & * \\ 0 & 1_{n-r} \end{pmatrix}$$

where 1_m is the $m \times m$ indentity matrix.

Thus, the problem is now just to prove the implication "$\Rightarrow$" in (1). Pick *any* local C^p chart (ψ, U') around ξ' with $U' \subseteq F^{-1}(U)$ and $y'_1, \dots, y'_{n'}$ the C^p component functions of ψ on U'. Letting $F_i = x_i \circ F|_{U'}$ we see that the matrix $((\partial F_u / \partial y'_j)(\xi'_0))$ computes $\mathrm{d}F(\xi'_0)$. Thus, this matrix has rank $\geq r$, so there exist r linearly independent columns. In the submatrix of such independent columns there must exist r linearly independent rows (since row rank equals column rank for this submatrix). Hence, there is an invertible $r \times r$ submatrix, and by rearranging the ordering among the x_i's and among the y'_j's we can assume that the "upper left" submatrix $((\partial_{y'_j} F_i)(\xi'_0))_{1 \leq i,j \leq r}$ is invertible.

Now let $\psi : U' \to \mathbf{R}^{n'}$ be the C^p mapping given by

$$\psi'(u') = (F_1(u'), \dots, F_r(u'), y'_{r+1}(u'), \dots, y'_{n'}(u'))$$

with $F_i = x_i \circ F|_{U'}$. If this map is a C^p isomorphism near ξ'_0 then we can take is to be (ψ, U') to complete the proof of (1) (and hence of the structure theorem). By the inverse function theorem on premanifolds (applied to the above C^p mapping $U' \to \mathbf{R}^{n'}$ on an open set in X') it is equivalent to check that the tangent mapping $\mathrm{d}\psi(\xi'_0)$ is a linear isomorphism. Using the ψ'-coordinates on U' and the standard coordinates on $\mathbf{R}^{n'}$ gives this tangent mapping the associated matrix

$$\begin{pmatrix} (((\partial_{y'_j} F_i)(\xi'_0))_{1 \leq i,j \leq r} & 0 \\ * & 1_{n'-r} \end{pmatrix}$$

that is invertible because of the rearranging among the x_i's and y'_j's that we made above.