

1. INTEGRAL MANIFOLDS FOR TRIVIAL LINE BUNDLES

Let M be a C^∞ manifold (without corners) and let $E \subseteq TM$ be a subbundle of the tangent bundle. In class we discussed the notion of integral manifolds for E in M (as well as maximal ones), essentially as a generalization of the theory of integral curves for vector fields. Roughly speaking, in the special case that E is a *trivial* line bundle we are in the setup of integral curves for (non-vanishing) vector fields but with a fundamental difference: we do *not* specify the trivialization. What is the impact of this?

To motivate what is to follow, we shall now undertake a close study of the effect of changing the trivialization. Say \vec{v} and \vec{w} are two trivializations for a line subbundle L in TM , which is to say that these are non-vanishing smooth vector fields which are pointwise proportional, so we have $\vec{w} = f\vec{v}$ for a necessarily non-vanishing smooth function f on M . We shall prove that the associated maximal integral curves are “the same” up to a unique reparameterization in time that fixes $t = 0$:

Theorem 1.1. *The respective maximal integral curves $c : I \rightarrow M$ and $\tilde{c} : J \rightarrow M$ for \vec{v} and \vec{w} through m_0 at time 0 satisfy $\tilde{c} = c \circ F$ for a unique C^∞ isomorphism $F : J \simeq I$ preserving 0. In particular, $c(I) = \tilde{c}(J)$ as subsets of M .*

If you visualize the meaning of this theorem in terms of motion of two particles (parameterized by time), the assertion becomes “physically obvious”. The rigorous proof thereby illustrates the difference between physics and mathematics. Keep the visualization in mind when reading the proof, since it both motivates the entire strategy of proof and makings it easy to understand.

Proof. By definition, $c'(t) = \vec{v}(c(t))$ for all $t \in I$ with $c(0) = m_0$, and I is the unique maximal open interval in \mathbf{R} with this property (it contains all others). Since $f \circ c$ is a non-vanishing continuous function on the open interval I , it has constant sign. If f is negative then we can replace \vec{v} with $-\vec{v}$, I with $-I$, and $t \mapsto c(t)$ on I with $t \mapsto c(-t)$ on $-I$ without changing the image of c but bringing us to the case $f > 0$. Hence, we now suppose $f > 0$. Consider the I -valued initial-value problem $F' = (f \circ c) \circ F$ with initial condition $F(0) = 0 \in \text{int}(I)$ for smooth maps $F : J' \rightarrow I$ on intervals J' around 0. This is a non-linear ODE, and the local existence theorem ensures that there exists such a solution on some interval J' around 0. Since f is positive, so is F' , and hence F is a strictly increasing function. Thus, F is a C^∞ order-preserving isomorphism of J' onto $F(J')$.

The old arguments via uniqueness of solutions to ODE’s provide a maximal open interval J' on which there is a solution (satisfying $F(J') \subseteq I$), and it is unique. Taking J' to be maximal, we *must* have $F(J') = I$: if not then the strictly increasing F is bounded away from endpoints of I as t approaches some endpoint of J' yet the smooth $f \circ c$ persists across all of I and so by Corollary 2.5 in the handout on ODE’s (!) the interval J' would not be maximal after all. The smooth map $c \circ F : J' \rightarrow M$ carries 0 to m_0 and satisfies

$$(c \circ F)'(t) = F'(t) \cdot c'(F(t)) = (f \circ c)(F(t)) \cdot \vec{v}(c(F(t))) = f((c \circ F)(t)) \cdot \vec{v}((c \circ F)(t)) = \vec{w}((c \circ F)(t)).$$

This says that $c \circ F : J' \rightarrow M$ is an integral curve for \vec{w} through m_0 at time 0. In particular, J' is contained in the open interval of definition J for the maximal integral curve $\tilde{c} : J \rightarrow M$ for \vec{w} passing through m_0 at time 0.

Note that $\vec{w} = (1/f) \cdot \vec{v}$. By the formula for the derivative of an inverse function in 1-variable calculus, the smooth strictly increasing map $F^{-1} : I \simeq J' \subseteq J$ sending 0 to 0 solves the ODE $y' = ((1/f) \circ \tilde{c}) \circ y$ for J -valued y on I because $\tilde{c}|_{J'} = c \circ F$. By the argument given above, if we consider the solution $H : \tilde{I} \rightarrow J$ to this initial-value problem on a maximal open interval of definition around 0 (so $I \subseteq \tilde{I}$ and $H|_{\tilde{I}} = F^{-1}$) then $H(\tilde{I}) = J$ and $\tilde{c} \circ H : \tilde{I} \rightarrow M$ is an integral curve for \vec{v}

through m_0 at time 0. This forces $\tilde{I} \subseteq I$, so in fact $\tilde{I} = I$ and $J = H(\tilde{I}) = H(I) = F^{-1}(I) = J'$. We have therefore proved that $J' = J$, so F is a strictly increasing C^∞ isomorphism between the maximal intervals of definition for the integral curves of \vec{v} and $\vec{w} = f\vec{v}$ through m_0 at time 0, and composition with F carries the maximal integral curve for \vec{v} to the maximal integral curve for \vec{w} .

It remains to check that F is the unique solution to our problem: if $H : J \simeq I$ is a C^∞ isomorphism fixing the origin such that $\tilde{c} = c \circ H$, then $H = F$. By differentiating and using the “integral curve” properties of c and \tilde{c} ,

$$f(\tilde{c}(t)) \cdot \vec{v}(\tilde{c}(t)) = \vec{w}(\tilde{c}(t)) = H'(t) \cdot \vec{v}(c(H(t))) = H'(t) \cdot \vec{v}(\tilde{c}(t)).$$

Since $\vec{v}(\tilde{c}(t)) \neq 0$ for all t , we get $H'(t) = f(\tilde{c}(t)) = (f \circ c)(H(t))$. Hence $H : J \rightarrow I$ is a solution to the same initial-value problem as F (i.e., $y'(t) = (f \circ c)(y(t))$ for I -valued y on an open interval around 0, with $y(0) = 0$). This forces $H = F$. ■

We have just shown that if we replace a non-vanishing vector field \vec{v} on M with the line subbundle $L \subseteq TM$ that it generates (this is just the C^∞ subbundle inclusion $M \times \mathbf{R} \rightarrow TM$ given by $(m, a) \mapsto a\vec{v}(m)$), then since L only “knows” \vec{v} up to multiplication by a non-vanishing smooth function it only “knows” the maximal integral curve $c_{\vec{v}, m_0} : I_{\vec{v}, m_0} \rightarrow M$ up to (possibly order-reversing) composition with some C^∞ isomorphism $F : J \simeq I_{\vec{v}, m_0}$ for an open subinterval J of \mathbf{R} around 0. In particular, the *image* subset $c_{\vec{v}, m_0}(I_{\vec{v}, m_0})$ and the property of whether or not $c_{\vec{v}, m_0}$ is injective depend only on m_0 and *not* on the choice of trivialization \vec{v} for L . Let us write N_{m_0} for this image subset. In the non-injective case, we know from Example 5.7 in the handout on integral curves that N_{m_0} is a smoothly embedded circle in M . As we have seen long ago for general embedded submanifolds of a manifold, this is the unique possible C^∞ submanifold structure on the subset N_{m_0} in M in such cases. In the injective case, the non-vanishing of the vector field implies that $c_{\vec{v}, m_0} : I_{\vec{v}, m_0} \rightarrow M$ is an injective *immersion*, though this may *not* be an embedding (think of the line densely wrapping the torus). We thereby get a (possibly non-embedded) submanifold structure on the subset $N_{m_0} \subseteq M$ by using the bijection $c_{\vec{v}, m_0}$ and the C^∞ manifold structure on $I_{\vec{v}, m_0}$ to put both the topology *and* differentiable structure on N_{m_0} . If we change \vec{v} to some other \vec{w} trivializing L then the preceding arguments show that there is a (unique) C^∞ isomorphism $F : I_{\vec{w}, m_0} \simeq I_{\vec{v}, m_0}$ such that $c_{\vec{w}, m_0} = c_{\vec{v}, m_0} \circ F$, and so in particular the C^∞ submanifold structure put on N_{m_0} is *independent* of the choice of trivializing section of $L \subseteq TM$.

We have done the hard part in the proof of:

Theorem 1.2. *Let M be a smooth manifold and let L be a trivial line subbundle of the tangent bundle. Let $m_0 \in M$ be a point. There exists a unique maximal integral submanifold N_{m_0} for L through m_0 . Upon choosing a trivialization L we get a C^∞ -isomorphism of N_{m_0} with an embedded circle when N_{m_0} is compact and with an open interval in \mathbf{R} otherwise.*

Proof. The uniqueness goes by the usual kind of argument: if $i : N \hookrightarrow M$ and $i' : N' \rightarrow M$ are two maximal integral submanifolds for L and both pass through m_0 then by the property of maximal integral submanifolds each of i and i' must smoothly factor through the other. It follows for set-theoretic reasons (due to i and i' being injective) that these two factorizations must be inverse C^∞ isomorphisms between N and N' , so both (N, i) and (N', i') “coincide” as submanifolds of M . (That is, they put the *same* topology and differentiable structure on the *same* subset of M .)

Now we turn to existence. That is, we have to prove that the submanifold N_{m_0} as constructed above really is a maximal integral submanifold for L . It is built as the image of an integral curve mapping for a vector field trivializing L , so it is certainly an integral submanifold (i.e., it is connected and its tangent space at each point is the fiber-line of L inside the tangent space to M at that point). Is it maximal? Let $i : N \hookrightarrow M$ be a connected submanifold that is an integral curve for L ,

so N is 1-dimensional. We assume $i(N)$ meets N_{m_0} and we seek to prove that i factors (necessarily uniquely) through a C^∞ map from N to N_{m_0} .

The preimage $i^{-1}(N_{m_0})$ is closed and non-empty in the connected manifold N , so to prove it equals N it suffices to prove openness. This is a local problem near each point of $i^{-1}(N_{m_0})$. Also, once this problem is settled, the problem of proving that the unique set-theoretic map $N \rightarrow N_{m_0}$ through which i factors is a C^∞ map is a local problem on N . Hence, for our purposes the entire problem is local near points of $i^{-1}(N_{m_0})$. Pick a point $n_0 \in i^{-1}(N_{m_0})$. Since the submanifold N_{m_0} is “unaffected” by replacing m_0 with any other point in N_{m_0} (as this corresponds to just making a time translation in the setup with integral curves for vector fields), we may rename $i(n_0)$ as m_0 . Thus, we can assume $m_0 \in i(N)$ and that N is an arbitrarily short embedded open interval in M through m_0 , and we just have to prove that it lies in N_{m_0} set-theoretically and in fact as a submanifold of M . In particular, we may assume that N is an open interval in \mathbf{R} around 0 with $i(0) = m_0$, and the integral manifold condition implies that the velocity vector field $i'(t)$ is a non-vanishing smooth multiple of $\vec{v} \circ i$ (with \vec{v} a trivialization of L). Negating time if necessary, we can assume that the non-vanishing multiplier function is positive.

Smooth functions along an embedded submanifold locally lift to smooth functions on the ambient manifold (by the immersion theorem), and the property of positivity is inherited locally by such liftings, so with the help of a smooth function on M that equals 1 away from a small neighborhood of m_0 and equals the lift of this smooth positive multiplier function near m_0 we may modify the choice of \vec{v} so that $\vec{v} \circ i = i'$. Hence, $i : N \rightarrow M$ is an integral curve for \vec{v} passing through m_0 at time 0. By the existence theorem for maximal integral curves of vector fields, the interval N equipped with its mapping i to M is a subinterval of the interval $I_{\vec{v}, m_0}$ equipped with its canonical mapping to M . Thus, the embedding $i : N \rightarrow M$ factors smoothly through N_{m_0} . ■

2. EXAMPLES WITH CIRCLES AND THE MÖBIUS STRIP

Theory of integral curves is a “coordinatized” version of the theory of integral manifolds for trivial line subbundles of the tangent bundle. Upon *choosing* a trivialization (a non-vanishing vector field) we get the maximal integral curves that are precisely maximal integral manifolds with a “preferred” (possibly non-injective) parameterization dictated by the choice of vector field. Changing the choice changes the parameterization but leaves the *image submanifold* (image subset endowed with suitable topology and differentiable structure) unaffected. This is the content of Theorem 1.2.

Example 2.1. Consider the counterclockwise circular vector field (say with constant speed a) in $\mathbf{R}^2 - \{0\}$. The integral curves for this vector field are the mappings $c : \mathbf{R} \rightarrow \mathbf{R}^2 - \{0\}$ given by

$$c(t) = (r \cos((at/r) + b), r \sin((at/r) + b))$$

with r and b depending on the initial position. In contrast, the integral manifolds for the associated line subbundle L in $T(\mathbf{R}^2 - \{0\})$ are just the “bare” circles of radius r centered at the origin (without any parameterization data) viewed as embedded submanifolds of the punctured plane. In particular, when taking the viewpoint of integral manifolds for L the “wrapping around” aspect from the trigonometric mapping c is *forgotten* because the time parameterization has been eliminated from consideration by the abandonment of velocity vector data.

Observe likewise that whereas in the theory of integral curves it was meaningful to ask if the curve “returns to itself” in the sense of the parameterization map c being non-injective (and in Example 5.7 in the handout on integral curves we saw that there was a very satisfying description of non-injective situations), there is no such meaningful question in the context of integral manifolds for subbundles of the tangent bundles: you either have an integral manifold or you don’t (and it

may or may not be maximal), but it is just “there” and is not parameterized by anything so it is not meaningful to ask if it “returns to itself”. (Of course, you can imagine moving along some trajectories within the submanifold and asking how that parameterized trajectory behaves as time evolves, but this is a different issue.) Perhaps a convenient way to summarize the dichotomy is to say that the theory of integral curves for vector fields is manifestly *dynamic* but the theory of integral manifolds for subbundles of the tangent bundle is essentially *static* (the proof of existence of such integral manifolds in the general case has certain dynamic aspects much as does the proof of Theorem 1.2, but once the proof is over, that’s it for the dynamics).

For a nontrivial line subbundle of the tangent bundle, the “integral curve” viewpoint only makes good sense on opens over which we can find trivializations. If we pick such trivializations locally then by (Example 5.7 from the integral curves handout and) uniqueness aspects in the theory of integral curves these do patch up as mere submanifolds (without parameterization!) when we move from one trivializing domain to another but we cannot expect the actual parameterizations to patch up in a manner that is well-behaved with respect to changing the initial position (as otherwise we’d be able to define globally consistent nonzero velocity vectors at all points and so we’d get a non-vanishing global section to our nontrivial line subbundle, a contradiction).

Example 2.2. We now illustrate how the framework of integral manifolds for line bundles allows us to consider *global* problems that simply cannot be formulated in terms of the language of integral curves for vector fields. Consider trying to make a non-vanishing vector field on the Möbius strip M such that the vectors lie in the vertical direction at all points (i.e., lie on the tangent lines perpendicular to the right/left motion relative to the usual visualization of M). I claim no such vector field exists. Viewing M as a quotient of $(-a, a) \times S^1$, such a vector field must descend a vector field \vec{v} of the form $h\partial_t$ on $(-a, a) \times S^1$ for some $h \in C^\infty((-a, a) \times S^1)$. But we know from Exercise 2(iv) in Homework 5 that \vec{v} must be invariant by the action of the order-2 group that defines the quotient M , and so by Exercise 2(ii) in Homework 5 (replace \mathbf{R} there with $(-a, a)$) it follows that h has to vanish somewhere (even somewhere along the central circle). This is a contradiction.

Now suppose we abandon the desire to specify the non-vanishing vector field and we only care about the (vertical) *line* it spans in the tangent plane at each point on M . This is much better, because the negation problem from the group action that forced the vanishing of h at some point (by Intermediate Value Theorem) is eliminated: lines are stable under negation (whereas nonzero vectors are not). If you stare at a picture of the Möbius strip, you can see that there is a globally consistent sense of verticalness even though there is no globally consistent sense of up or down. The resulting family of lines based at each point is a line subbundle L_0 of TM that has no non-vanishing global section due to the preceding paragraph.

[Let us justify the last sentence rigorously. On the product $(-a, a) \times S^1$ the tangent bundle is the direct sum of two line bundles, namely the pullbacks of the tangent bundles of the factors, and the pullback L of the tangent bundle of $(-a, a)$ (tangent lines along the $(-a, a)$ -factor) is *stable* under the group action on the tangent planes to $(-a, a) \times S^1$ (check!). By Exercise 3 in Homework 5, this line bundle must therefore descend to a line bundle L_0 on M . In fact, by the naturality of the construction in that exercise, since TM descends $T((-a, a) \times S^1)$ the bundle inclusion $L \rightarrow T((-a, a) \times S^1)$ descends to a bundle mapping $L_0 \rightarrow TM$ over M that is injective on fibers and hence is a subbundle. On fibers, it is exactly the line we want. This completes the verification.]

By inspection, integral manifolds for this line bundle L_0 are the vertical lines along the Möbius strip M . Since L_0 is *non-trivial* there is no global vector field for which these are the (images of)

integral curves, as such a vector field would be a non-vanishing global section of L_0 . If we remove a single vertical segment from M then we can trivialize L_0 and so we return to the setting of integral curves. But if we wish to work across the entire manifold at once then we're sunk if we try to use the framework of integral curves. The intervention of non-trivial vector bundles is a fundamental dichotomy between local and global aspects of geometry and topology on manifolds.

3. SOME DEFINITIONS

In some introductory books on differential geometry, a lot of definitions are too global. (The reason is that the existence of bump functions causes the theory of sheaves to not play a significant role in differential geometry, as it does in algebraic and complex-analytic geometry.) Here is the definition that one often finds for integrability of a subbundle of the tangent bundle, but we modify the terminology to avoid conflict with our terminology:

Definition 3.1. A subbundle E in TM is *globally integrable* if $[X, Y] \in E(M)$ for all $X, Y \in E(M)$.

Remark 3.2. The phrase of “subbundle of TM ” is avoided in many books; the classical terminology for a rank- r subbundle of TM is *C^∞ distribution of r -planes over M* . This latter concept is classically defined to be a choice of r -dimensional subspace $W_m \subseteq T_m(M)$ for each $m \in M$ such that there exist local collections of r independent smooth vector fields spanning the W_m 's on fibers. This is exactly one of our criteria for defining a C^∞ subbundle of a C^∞ vector bundle, applied with vector bundle TM .

The difference between Definition 3.1 and our definition of integrability is that we require $[X, Y]_U \in E(U)$ for all $X, Y \in E(U)$ for *all* open $U \subseteq M$. We shall now use the crutch of bump functions to prove that the much weaker condition of being globally integrable implies the condition that we have defined as integrability (and it is our definition that must be used in the real-analytic and complex-analytic cases).

Theorem 3.3. *If a subbundle E in TM is globally integrable, then it is integrable.*

Proof. We pick an open set $U \subseteq M$ and $X, Y \in E(U)$, and we want to prove $[X, Y]_U \in E(U)$. That is, for each $u \in U$ we want $[X, Y](u) \in E(u)$. Choose $u_0 \in U$. Let $\phi \in C^\infty(M)$ be a function equal to 1 near u_0 and supported inside of a compact subset $K \subseteq U$. Thus, $\phi X, \phi Y \in E(U)$ are compactly supported inside of U and so “extend by zero” to elements of $E(M)$ that vanish on the open set $M - K$. We write X' and Y' denote these elements of $E(M)$. By the hypothesis of global integrability, $[X', Y'] \in E(M)$, so $[X', Y'](u_0) \in E(u_0)$. But the formation of the Lie bracket on vector fields is compatible with shrinking the open domain, so in particular $[X', Y']|_U \in E(U)$ is equal to $[X'|_U, Y'|_U]_U = [\phi X, \phi Y]_U$. The function ϕ is equal to 1 on some open subset $U_0 \subseteq U$ around u_0 , so shrinking to U_0 gives $[X|_{U_0}, Y|_{U_0}]_{U_0} = [X', Y']|_{U_0} \in E(U_0)$. Passing to u_0 -fibers therefore gives $[X, Y](u_0) \in E(u_0)$. ■

The reason for the terminology of “integrability” for the stability of a subbundle under the bracket operation is that it is closely related to the existence of lots of integral submanifolds to the subbundle:

Theorem 3.4. *Let E be a subbundle of TM and assume that for all $m \in M$ there exists an integral submanifold $i : N \hookrightarrow M$ to E with $m \in i(N)$. The subbundle E is integrable: for all open $U \subseteq M$ and vector fields $X, Y \in E(U) \subseteq (TM)(U) = \text{Vec}_M(U)$, the bracket vector field $[X, Y]_U \in \text{Vec}_M(U)$ lies in $E(U)$.*

Proof. Without loss of generality we may assume $U = M$, so X and Y are global smooth vector fields on M . Fix $m_0 \in M$, and we wish to prove that the tangent vector $[X, Y](m_0) \in T_{m_0}(M)$ lies in the subspace $E(m_0)$. Let $i : N \rightarrow M$ be an integral submanifold to E with $m_0 = i(n_0)$ for a (necessarily unique) point $n_0 \in N$. In particular, the subspace $E(m_0) \subseteq T_{m_0}(M)$ is the image of the injective linear map $di(n_0) : T_{n_0}(N) \rightarrow T_{m_0}(M)$. By shrinking N around n_0 we can assume that the injective immersion i is an *embedding* (immersion theorem!). Hence, by Exercise 3 in Homework 6 the total tangent mapping $di : TN \rightarrow TM$ over $i : N \rightarrow M$ is also an embedding of smooth manifolds. For all $n \in N$ we have that $E(i(n)) = di(n)(T_n(N))$ by the integral submanifold property of N with respect to E , so the vector $X(i(n)) \in E(i(n))$ has the form $X(i(n)) = di(n)(\tilde{X}(n))$ for a unique tangent vector $\tilde{X}(n) \in T_n(N)$. Likewise we have $Y(i(n)) = di(n)(\tilde{Y}(n))$ for a unique $\tilde{Y}(n) \in T_n(N)$ for all $n \in N$. The first key point is that the set-theoretic vector fields $\tilde{X} : n \mapsto \tilde{X}(n)$ and $\tilde{Y} : n \mapsto \tilde{Y}(n)$ on N are actually *smooth* vector fields on N ; that is, the two maps $\tilde{X}, \tilde{Y} : N \rightarrow TN$ (sections to the natural projection $TN \rightarrow N$) are smooth maps of smooth manifolds. Indeed, by their very definition the composite maps $di \circ \tilde{X}, di \circ \tilde{Y} : N \rightarrow TM$ are smooth because these are respectively equal to the maps $X \circ i$ and $Y \circ i$ that are visibly smooth (since $i : N \rightarrow M$ is smooth and both sections $X, Y : M \rightarrow TM$ are smooth), and so since di is a C^∞ -embedding it follows that \tilde{X} and \tilde{Y} are smooth. (Here we use the fact that if $j : M_1 \rightarrow M_2$ is a C^p -embedding of C^p premanifolds and $f : M' \rightarrow M_1$ is a set-theoretic map from a third C^p premanifold such that $j \circ f$ is a C^p map then f is a C^p map: by our long-ago discussion of mapping properties submanifolds the C^p property of f only requires checking f is continuous, given that j is an injective immersion, and this purely topological property of f is immediate from the hypothesis that j is even an embedding.)

By the very construction of the smooth vector fields \tilde{X} and \tilde{Y} on N , we have that $X \circ i = di \circ \tilde{X}$ and $Y \circ i = di \circ \tilde{Y}$. In other words, X is i -related to \tilde{X} and Y is i -related to \tilde{Y} . Hence, by the good behavior of the bracket of vector fields with respect to the property of “ φ -relatedness” for smooth maps φ between smooth manifolds (as discussed already in class), the bracket vector fields $[\tilde{X}, \tilde{Y}]$ on N and $[X, Y]$ on M are i -related. That is, for each $n \in N$ we have

$$[X, Y](i(n)) = di(n)([\tilde{X}, \tilde{Y}](n)) \in di(n)(T_n(N)) = E(i(n))$$

for all $n \in N$. Taking $n = n_0$ (with $i(n_0) = m_0$) gives the desired result: $[X, Y](m_0) \in E(m_0)$ with $m_0 \in M$ our initial arbitrary choice of point. ■

On the theme of comparing our definitions with the “too global” definitions in various books, the rest of this handout is devoted to showing how to use bump functions to verify some more equivalences among competing definitions. The arguments all go pretty much the same way, though it is sometimes a bit tedious to write out the mechanical details; such reading is best left for a rainy day, since the details are not especially interesting (but the *technique* is definitely a very important one for making global constructions in differential geometry, so it is worthwhile to read a couple such proofs to see how the procedure goes). None of what follows will ever be used in this course, but for ease of communication with other people later in life you may wish to read some or all of it.

Theorem 3.1 in the old handout on construction of vector fields gave the equivalence between our definition of smooth vector field on a smooth manifold with corners and the “global” definition found in many books. A related method works for comparing definitions of differential forms and tensor fields of type (r, s) . We begin with the case of type $(0, 1)$, which is to say global differential 1-forms.

Theorem 3.5. *Let M be a smooth manifold with corners. Any $C^\infty(M)$ -linear mapping $T : \text{Vec}_M(M) \rightarrow C^\infty(M)$ has the form $\vec{v} \mapsto \omega(\vec{v})$ for a unique $\omega \in \Omega_M^1(M)$, where $\omega(\vec{v})$ is the smooth function $m \mapsto \omega(m)(\vec{v}(m))$.*

In many books, such global mappings T are the *definition* of $\Omega_M^1(M)$! Naturally this makes the passage from $\Omega_M^1(M)$ to $\Omega_M^1(U) = \Omega_U^1(U)$ an arduous task, since there is no natural map from $\text{Vec}_M(U) = \text{Vec}_U(U)$ to $\text{Vec}_M(M)$. In effect, one has to create a recipe with lots of bump functions, and this is tantamount to running through the proof of the present theorem without actually saying so (since our natural and easily localizable notion of the \mathcal{O} -module Ω_M^1 is not introduced *a priori* in such treatments, but is only built *a posteriori* through the muck of bump functions and a bad global definition).

Proof. This theorem has nothing at all do with vector fields and differential forms: more generally, for any C^∞ vector bundles E and E' on M and $C^\infty(M)$ -linear map $T : E(M) \rightarrow E'(M)$, we claim that there exists $\ell \in \text{Hom}_M(E, E')$ such that T is the effect of ℓ on global sections. (Taking $E = \text{Vec}_M(M)$ and $E' = M \times \mathbf{R}$ then gives the lemma as a special case.)

We first show uniqueness, which is to say if $\ell : E \rightarrow E'$ is a bundle mapping such that $\ell(s) = 0$ for all $s \in E(M)$ then $\ell = 0$ (i.e., ℓ is the 0-map on fibers). Choose $m \in M$, so we want to show $\ell(m) \in \text{Hom}(E(m), E'(m))$ vanishes. Clearly any $s_0 \in E(m)$ is the m -fiber of a smooth section $s_U \in E(U)$ for some open U around m . Multiplying s_U by some smooth bump function equal to 1 near m and compactly supported in U allows us to then “extend by zero” to build $s \in E(M)$ that equals s_U near m and so $s(m) = s_0$. Thus, $(\ell(m))(s_0) = (\ell(s))(m) = 0$ in $E'(m)$. This settles uniqueness.

With uniqueness proved, we turn to existence. The first step is to “localize” the problem. Let $\{\phi_i\}$ be a smooth partition of unity with each ϕ_i compactly supported inside of an open subset domain U_i in M , with $\{U_i\}$ a locally finite collection of open sets in M that are so small that E and E' are trivial over each U_i . Let $T_i = \phi_i T$, so $T = \sum T_i$; this is a locally finite sum and hence is well-posed. Suppose we can find ℓ_i solving the problem for T_i . The preceding argument with bump functions shows that since elements in the image of T_i restrict to 0 outside of the compact support of ϕ_i , ℓ_i vanishes on fibers away from this support. Thus, the sum $\sum \ell_i$ is locally finite and so makes sense. Taking this to be ℓ then gives a solution to our problem for T .

In this way, it suffices to solve the problem for each of the T_i 's, and so we may assume elements in the image of T vanish outside of a compact set K contained in an open set U over which E and E' have trivial restriction. Let $U_0 \subseteq U$ be an open neighborhood of K with compact closure. I next claim that for any $s \in E(M)$, $T(s)$ only depends on $s|_{U_0}$: if $s_1, s_2 \in E(M)$ satisfy $s_1(m) = s_2(m)$ for all $m \in U_0$ then $T(s_1) = T(s_2)$. Passing to $s_1 - s_2$, we have to show that if $s(m) = 0$ for all $m \in U_0$ then $T(s) = 0$. By hypothesis on T we know $T(s) \in E'(M)$ vanishes on all fibers outside of $K \subseteq U_0$, so we just have to show vanishing on fibers over K . Let ϕ be a smooth function equal to 1 on K and compactly supported inside of U_0 , so $\phi \cdot T = T$ since elements in the image of T in $E'(M)$ vanish on fibers away from the locus K over which ϕ is equal to 1. Hence, $T(s) = \phi T(s) = T(\phi \cdot s)$ since T is $C^\infty(M)$ -linear. But s vanishes on U_0 and ϕ vanishes outside of a compact subset of $V \subseteq U_0$, so for all $m \in M$ either $s(m) \in E(m)$ vanishes or $\phi(m) \in \mathbf{R}$ vanishes. Hence, $\phi \cdot s \in E(M)$ vanishes on all fibers and so is 0. This forces $T(\phi \cdot s) = 0$, so $T(s) = 0$ as desired. This completes the proof that $T(s)$ only depends on $s|_{U_0}$ for an open subset $U_0 \subseteq U$ containing K and having compact closure.

Now we use the triviality of $E|_U$ and $E'|_U$. Choose $s_1, \dots, s_n \in E(U)$ trivializing $E|_U$ and $s'_1, \dots, s'_{n'} \in E'(U)$ trivializing $E'|_U$. Fix a choice of U_0 as above, and choose $\phi \in C^\infty(M)$ equal to 1 on U_0 and compactly supported inside of U . Thus, for all $s \in E(U)$ the section $\phi s \in E(U)$

is compactly supported inside of U and so “extends by zero” to give a global section $\tilde{s} \in E(M)$ that is equal to s on U_0 . In particular, we get elements $\tilde{s}_1, \dots, \tilde{s}_n \in E(M)$ with U_0 -restrictions in $E(U_0)$ agreeing with $s_1|_{U_0}, \dots, s_n|_{U_0}$. We therefore get elements $T(\tilde{s}_j) \in E'(M)$ and by hypothesis on T these elements are supported inside of K . We may uniquely write $T(\tilde{s}_j)|_U \in E'(U)$ as $\sum a_{ij}s'_i$. Since the s'_i 's are a trivializing frame and $T(s)$ vanishes on fibers away from K for all $s \in E(M)$, it follows that all a_{ij} 's are supported inside of K . Motivated by this, we define $\ell_U \in \text{Hom}(E, E')(U) = \text{Hom}_U(E|_U, E'|_U)$ by the condition $s_j \mapsto \sum a_{ij}s'_i$. (That is, over an open subset $U' \subseteq U$, we send $\sum c_j s_j|_{U'}$ to $\sum_i (\sum_j a_{ij}|_{U'} c_j) s'_i|_{U'}$.)

By definition, ℓ_U vanishes on fibers away from K . Thus, we may “extend by zero” to $\ell \in \text{Hom}(E, E')$ that vanishes on fiberwise outside of K . In particular, for $s \in E(M)$ vanishing over K , $\ell(s) = 0$. Hence, $\ell(s)$ only depends on $s|_K$. We claim $\ell(s) = T(s)$ in $E'(M)$ for all $s \in E(M)$ (thereby solving our problem). We know that both sides vanish on fibers outside of K , so problem is to compare values on fibers of E' over $K \subseteq U_0$. We also know that $\ell(s)$ and $T(s)$ only depend on $s|_{U_0}$, so it suffices to check the equality after replacing s with some other element of $E(M)$ having the same U_0 -restriction. But U_0 is contained in U and $E|_U$ is trivialized by s_1, \dots, s_n , so $s|_U$ is a $C^\infty(U)$ -linear combination of the s_j 's. Multiplying the coefficient functions by the bump function ϕ that is compactly supported in U and equal to 1 on U_0 , we see that ϕs is a linear combination of the \tilde{s}_j 's. Since s and ϕs agree over U_0 , we are therefore reduced to the case when s is a $C^\infty(M)$ -linear combination of the \tilde{s}_j 's. By $C^\infty(M)$ -linearity of T , we are thereby reduced to the special case $s = \tilde{s}_j$ for some j . In this case $T(\tilde{s}_j)|_K = \ell(\tilde{s}_j)|_K$ due to how ℓ was defined. This gives equality on fibers over K , which is all we needed to show. ■

Now we can handle arbitrary tensor fields, differential forms, etc.

Theorem 3.6. *Let M be a smooth manifold with corners. Any $C^\infty(M)$ -multilinear mapping*

$$T : \text{Vec}_M(M)^{\times s} \rightarrow C^\infty(M)$$

has the form

$$(\vec{v}_1, \dots, \vec{v}_s) \mapsto t(\vec{v}_1 \otimes \dots \otimes \vec{v}_s)$$

*for a unique global section t of $(T^*M)^{\otimes s} \simeq ((TM)^{\otimes s})^\vee$.*

In some books, the collection of T 's as in this theorem is taken as the *definition* of the $C^\infty(M)$ -module of global tensor fields of type $(0, s)$ on M . This leads to the same unpleasantness as was discussed following the statement of the previous theorem.

Proof. Once again, this has nothing at all do with vector fields. Rather generally, we claim that if E_1, \dots, E_n, E' are vector bundles on M and

$$T : E_1(M) \times \dots \times E_n(M) \rightarrow E'(M)$$

is a $C^\infty(M)$ -multilinear map then there exists a unique $t \in \text{Hom}_M(E_1 \otimes \dots \otimes E_n, E')$ such that

$$T(s_1, \dots, s_n) = t(s_1 \otimes \dots \otimes s_n) \in E'(M)$$

for all $s_j \in E_j(M)$. The proof goes *exactly* like the proof of the preceding theorem (which was the case $n = 1$) except that we work with multilinear mappings and have to trivialize all E_i 's and E' over the open sets U . We leave it to the reader to check that the method does carry over with essentially no changes. ■

Theorem 3.7. *Let M be a smooth manifold with corners. Any alternating $C^\infty(M)$ -multilinear mapping*

$$T : \text{Vec}_M(M)^{\times k} \rightarrow C^\infty(M)$$

has the form $(\vec{v}_1, \dots, \vec{v}_k) \mapsto \omega(\vec{v}_1 \wedge \dots \wedge \vec{v}_k)$ for a unique $\omega \in \Omega_M^k(M)$, where $\omega(\vec{v}_1 \wedge \dots \wedge \vec{v}_k)$ is the function whose value at $m \in M$ is the evaluation of the alternating functional $\omega(m) \in \wedge^k(T_m^*(M)) \simeq (\wedge^k(T_m(M)))^\vee$ on the vector $(\vec{v}_1 \wedge \dots \wedge \vec{v}_k)(m) = \vec{v}_1(m) \wedge \dots \wedge \vec{v}_k(m)$ in $\wedge^k(T_m(M))$.

In some books the $C^\infty(M)$ -module of T 's as in this theorem is given as the *definition* of the module of smooth global differential k -forms on M . The variant for symmetric tensor fields goes similarly.

Proof. The proof is identical to the preceding proof, except we track the alternating property throughout. ■

We conclude with a comment on the case of Riemannian manifolds. Later in the course when we study Riemannian geometry, there will be an extra structure put on the manifold (in effect, a smoothly varying inner product on the fibers) that provides a preferred identification of the tangent and cotangent bundles. Once we permit ourselves to thereby identify TM with T^*M , we can identify $(TM)^{\otimes r} \otimes (T^*M)^{\otimes s}$ with $(T^*M)^{\otimes(r+s)}$. By Theorem 3.6, global sections of the latter may be identified with $C^\infty(M)$ -multilinear maps $\text{Vec}_M(M)^{\times(r+s)} \rightarrow C^\infty(M)$. Provided we keep track of the naturalities in the first r and last s factors (using tangent mappings on the first r and adjoints of tangent mappings on the last s), this latter monstrosity is sometimes presented as the *definition* of a tensor field of type (r, s) on a Riemannian manifold. In some other treatments of more classical nature, the definition of such global tensor fields is given locally in terms of swarms of indices (upper and lower, according to some convention that I can never keep straight).