MATH 396. METRIC TENSOR ON HYPERSURFACES

1. MOTIVATION

Let $U \subseteq \mathbf{R}^n$ be a non-empty open subset and $f : U \to \mathbf{R}$ a C^∞-function. Let $\Gamma \subseteq U \times \mathbf{R}$ be the graph of f. The closed subset Γ in $U \times \mathbf{R}$ projects homeomorphically onto U with inverse

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, f(x_1, \ldots, x_n))$$

that is a smooth mapping from U to $U \times \mathbf{R}$. In fact, this latter mapping is trivially an immersion, so in this way its image Γ acquires a (necessarily unique) structure of closed smooth submanifold of $U \times \mathbf{R}$. In particular, the coordinate functions $x_1, \ldots, x_n$ on $\mathbf{R}^{n+1}$ restrict to a global C^∞ coordinate system on Γ. In what follows, we let $x'_j = x_j|_\Gamma$ for $1 \le j \le n$.

Example 1.1. The most classical example is $n = 2$, which are the old "$z = f(x, y)$" parametric surfaces in $\mathbf{R}^3$ over open parts of the xy-plane. Whenever meeting a new concept in differential geometry, it is always a good idea to figure out how it works in the case of such surfaces in $\mathbf{R}^3$, or more generally for hypersurface graphs Γ in $\mathbf{R}^{n+1}$ as above (as the immersion theorem tells us that all hypersurfaces in a manifold locally have this form).

Using the standard Riemannian metric on $U \times \mathbf{R} \subseteq \mathbf{R}^{n+1}$, we wish to compute the "induced metric" on Γ, which is to say the induced inner product on each $\mathrm{T}_p(\Gamma) \subseteq \mathrm{T}_p(\mathbf{R}^{n+1})$ for $p \in \Gamma$. We want to express this metric tensor in terms of the coordinate system $\{x'_1, \ldots, x'_n\}$ on Γ. The metric tensor at $p \in \Gamma$ is

$$\sum_{i,j} \langle \partial_{x'_i}|_p, \partial_{x'_j}|_p \rangle_p \cdot \mathrm{d}x'_i(p) \otimes \mathrm{d}x'_j(p),$$

with the ij-coefficient computed as an inner product in $\mathrm{T}_p(\mathbf{R}^{n+1})$ (since the induced metric on Γ is given pointwise by the inclusion of $\mathrm{T}_p(\Gamma)$ into $\mathrm{T}_p(\mathbf{R}^{n+1})$ for each $p \in \Gamma$). Since the $\partial_{x_i}|_p$'s are an orthonormal basis of $\mathrm{T}_p(\mathbf{R}^{n+1})$, the problem is therefore largely that of figuring out the image of each $\partial_{x'_i}|_p \in \mathrm{T}_p(\Gamma)$ in $\mathrm{T}_p(\mathbf{R}^{n+1})$ as a linear combination in $\{\partial_{x_1}|_p, \ldots, \partial_{x_{n+1}}|_p\}$.

An easy "beginner's mistake" is to think that since $x'_i = x_i|_\Gamma$, $\partial_{x'_i}|_p \in \mathrm{T}_p(\Gamma)$ viewed in $\mathrm{T}_p(\mathbf{R}^{n+1})$ should equal $\partial_{x_i}|_p$ for each $i \le n$. That this is nonsense is easy to recognize geometrically as follows: the "x_i-coordinate line" through p in $\mathbf{R}^{n+1}$ is usually *not* tangent to Γ at p and so its tangent line at p is generally not contained in the tangent hyperplane $\mathrm{T}_p(\Gamma)$ (so the nonzero tangent vector $\partial_{x_i}|_p$ in this line cannot belong to the hyperplane $\mathrm{T}_p(\Gamma) \subseteq \mathrm{T}_p(\mathbf{R}^{n+1})$). To better appreciate what is happening, consider a simple example:

Example 1.2. Let $p = (a, b, c)$ be a point on the open upper unit hemisphere

$$H = \{z = \sqrt{1 - x^2 - y^2}\}$$

in $\mathbf{R}^3$ (with (x, y) inside the open unit disc Δ centered at the origin in $\mathbf{R}^2$). Geometric intuition suggests that away from the intersection of H with the yz-plane, the vector $\partial_x|_p \in \mathrm{T}_p(\mathbf{R}^3)$ is not tangent to any curve in the sphere at p and so it certainly cannot be a velocity vector along a coordinate line for a local coordinate system on H near p. In particular, it generally cannot equal $\partial_{x'}|_p$ when the latter is viewed in $\mathrm{T}_p(\mathbf{R}^3)$.

What is the image of $\partial_{x'}|_p \in \mathrm{T}_p(H)$ in $\mathrm{T}_p(\mathbf{R}^3)$ as a linear combination of $\partial_x|_p, \partial_y|_p, \partial_z|_p$? There is a simple geometric procedure to figure this out. On any manifold, the partial derivative operator with respect to a coordinate parameter is just the velocity vector field for the coordinate line with time given by the chosen coordinate. Thus, we have to take the x-coordinate line $y = b$ in Δ passing through (a, b), consider it as parameterized via x, and map this parametric curve into H using the $\{x, y\}$ parameterization of H. That is, we take the embedded parametric curve $\sigma(t) = (t, b, \sqrt{1-t^2-b^2})$ in H for t near a, and its velocity vector in $\mathrm{T}_{\sigma(t)}(H) \subseteq \mathrm{T}_{\sigma(t)}(\mathbf{R}^3)$ at time t is

$$\sigma'(t) = \partial_x|_{\sigma(t)} - 2t(1-t^2-b^2)^{-1/2}\partial_z|_{\sigma(t)}.$$

At $t = a$, since $c = \sqrt{c^2}$ (as $c > 0$) we get $\partial_{x'}|_p = \partial_x|_p - 2(a/c)\partial_z|_p$ in $\mathrm{T}_p(\mathbf{R}^3)$. Note that this is equal to $\partial_x|_p$ if and only if $a = 0$, which is to say along the yz-plane (exactly as we see physically). At all other points of H, the vectors $\partial_{x'}|_p$ and $\partial_x|_p$ in $\mathrm{T}_p(\mathbf{R}^3)$ are linearly independent.

2. Some tangent vectors and normal vectors

The preceding example with the sphere suggests a general method for computing the image of $\partial_{x'_i}|_p \in \mathrm{T}_p(\Gamma)$ in $\mathrm{T}_p(\mathbf{R}^{n+1})$ in terms of the basis of $\partial_{x_j}|_p$'s. But before we do that, we want to address another possible beginner's mistake, which is to think that maybe $\partial_{x'_i}|_p$ should be the orthogonal projection of $\partial_{x_i}|_p$ onto $\mathrm{T}_p(\Gamma)$. A moment of geometric thought shows that this is generally false, for two reasons: (i) the problem of computing the image of $\partial_{x'_i}|_p$ in $\mathrm{T}_p(\mathbf{R}^{n+1})$ has absolutely nothing to do with Riemannian metrics, and so the answer cannot possibly involve the crutch of a metric (as is implicit with notions such as "orthogonal projection"), and (ii) visualizing the case of the sphere shows that the angle between $\partial_{x'_i}|_p$ and $\partial_{x_i}|_p$ in $\mathrm{T}_p(\mathbf{R}^{n+1})$ can be pretty much anything in $(0, \pi/2)$, so again orthogonal projection should have nothing to do with the answer. We shall sort out the exact relationship with orthogonal projection from $\mathrm{T}_p(\mathbf{R}^{n+1})$ onto $\mathrm{T}_p(\Gamma)$, as an application of solving our first problem (to compute the image of each $\partial_{x'_i}|_p$ in $\mathrm{T}_p(\mathbf{R}^{n+1})$ in terms of the basis of $\partial_{x_j}|_p$'s).

Let us now give the formula for $\partial_{x'_j}|_p$ in $\mathrm{T}_p(\mathbf{R}^{n+1})$:

$$\partial_{x'_j}|_p = \partial_{x_j}|_p + \partial_{x_j} f(p) \cdot \partial_{x_{n+1}}|_p.$$

To justify this, consider the parametric path $\sigma_j(t)$ in Γ which (in the x'-coordinate system) has ith coordinate fixed at p_i for $i \neq j$, and has jth coordinate $p_j + t$. In $\mathbf{R}^{n+1}$ this is the path with the same first n coordinates, and with last coordinate $t \mapsto f(p_1, \dots, p_j + t, \dots, p_n)$. Thus, we readily compute $\sigma'_j(t)$ much as we did for the hemisphere example above, and for $t = 0$ this gives the asserted formula. (In the case of the upper unit hemisphere in $\mathbf{R}^3$, we take $f(x, y) = \sqrt{1 - x^2 - y^2}$ and recover the formula found in Example 1.2.)

Theorem 2.1. *The nonzero vector*

$$N_p = \sum_{i \leq n} \partial_{x_i} f(p) \cdot \partial_{x_i}|_p - \partial_{x_{n+1}}|_p \in \mathrm{T}_p(U \times \mathbf{R}) = \mathrm{T}_p(\mathbf{R}^{n+1})$$

(i.e., $(\dots, \partial_{x_i} f(p), \dots, \partial_{x_n} f(p), -1)$ *if we identify* $\mathrm{T}_p(U \times \mathbf{R})$ *with* $\mathbf{R}^{n+1}$ *in the usual way) is a non-zero normal vector to* $\mathrm{T}_p(\Gamma)$*, and the vectors* $N_p, \partial_{x_1}|_p, \dots, \partial_{x_n}|_p \in \mathrm{T}_p(U \times \mathbf{R})$ *are a basis.*

Proof. A normal vector to $\mathrm{T}_p(\Gamma)$ in $\mathrm{T}_p(\mathbf{R}^{n+1})$ is one that is normal to each of the n vectors $\partial_{x'_i}|_p$ (as these span $\mathrm{T}_p(\Gamma)$). The images of these n vectors in $\mathrm{T}_p(\mathbf{R}^{n+1})$ are given above, and N_p as defined in the theorem is clearly perpendicular to all of them. Since N_p has nonzero coefficient for $\partial_{x_{n+1}}|_p$, the final part of the theorem is clear. ■

Since orthogonal projection to $\mathrm{T}_p(\Gamma)$ kills N_p, it follows from Theorem 2.1 that the orthogonal projections $\{v_j(p)\}$ of the $\partial_{x_j}|_p$'s ($j \leq n$) are a basis of $\mathrm{T}_p(\Gamma)$ (so in particular, they are all nonzero!).

Explicitly, the orthogonal projection $v_j(p)$ is given by

$$v_j(p) = \partial_{x_j}|_p - \frac{\langle \partial_{x_j}|_p, N_p\rangle_p}{\langle N_p, N_p\rangle_p} N_p$$

inside of $\mathrm{T}_p(\mathbf{R}^{n+1})$. What is this in terms of the basis of $\partial_{x'_i}|_p$'s for $\mathrm{T}_p(\Gamma)$? We know that $v_j(p)$ must be a unique linear combination of the $\partial_{x'_i}|_p$'s, and in $\mathrm{T}_p(\mathbf{R}^{n+1})$ the expansion of $\partial_{x'_i}|_p$ involves $\partial_{x_i}|_p$ with a coefficient of 1 and no $\partial_{x_k}|_p$'s for $k \leq n$ with $k \neq i$. Hence, by considering coefficients of $\partial_{x_i}|_p$'s for $i \leq n$ the only possibility is

$$v_j(p) = \sum_{i \neq j} \frac{\partial_{x_i} f(p) \partial_{x_j} f(p)}{1 + \sum_{r \leq n} (\partial_{x_r} f(p))^2} \cdot \partial_{x'_i}|_p + \frac{1 + \sum_{r \neq j, r \leq n} (\partial_{x_r} f(p))^2}{1 + \sum_{r \leq n} (\partial_{x_r} f(p))^2} \cdot \partial_{x'_j}|_p.$$

(The reader may verify as a safety check that the coefficients for $\partial_{x_{n+1}}|_p$ implicit on both sides are indeed equal, as they must be.) This is quite different from $\partial_{x'_j}|_p$ in general!

Using the determination of the image of $\partial_{x'_j}|_p \in \mathrm{T}_p(\Gamma)$ in $\mathrm{T}_p(\mathbf{R}^{n+1})$ as a linear combination of the $\partial_{x_i}|_p$'s, we also see immediately that

$$\langle \partial_{x'_j}|_p, \partial_{x'_i}|_p \rangle_p = \delta_{ij} + \partial_{x_i} f(p) \partial_{x_j} f(p),$$

where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. Hence, we arrive at the desired formula for the metric tensor of Γ in the x'-coordinates:

$$\sum_{i=1}^{n} (1 + (\partial_{x_i} f(p))^2) \cdot \mathrm{d}x'_i \otimes \mathrm{d}x'_i + \sum_{i \neq j} (\partial_{x_i} f(p) \partial_{x_j} f(p)) \cdot \mathrm{d}x'_i \otimes \mathrm{d}x'_j.$$

Example 2.2. Consider the classical case of a surface $z = f(x, y)$ in $\mathbf{R}^3$. In this case, the metric tensor on the surface in $\{x, y\}$ coordinates is

$$(1 + f_x^2) \cdot \mathrm{d}x^{\otimes 2} + (1 + f_y^2) \cdot \mathrm{d}y^{\otimes 2} + f_x f_y \cdot \mathrm{d}x \otimes \mathrm{d}y + f_x f_y \cdot \mathrm{d}y \otimes \mathrm{d}x,$$

where we indulge in the standard shorthand $f_x = \partial_x f$ and $f_y = \partial_y f$.

3. Surfaces of revolution

In addition to the graph surfaces "$z = f(x, y)$", another interesting class of surfaces in $\mathbf{R}^3$ is the surfaces of revolution. We shall focus on a special subclass. Let $I \subseteq (0, \infty)$ be a nontrivial interval and let $f : I \to \mathbf{R}$ be a positive smooth function such that f' is nowhere zero. Let $S \subseteq I \times \mathbf{R}^2 \subseteq \mathbf{R}^3$ be the surface of revolution obtained from revolving the graph of f (in $I \times \mathbf{R} \subseteq \mathbf{R}^2$, the xy-plane) about the x-axis. This surface does not touch the x-axis (since f is positive). By Exercise 3 in Homework 4 (which used an interval (a, b), but works the same with (a, b) replaced by an arbitrary non-trivial in $\mathbf{R}$), S is a smooth submanifold of $\mathbf{R}^3$ if I is open, and if I has endpoints then S is a smooth submanifold with boundary in $\mathbf{R}^3$. Since f' has constant sign (as it is non-vanishing and continuous over the interval I) the growth of f is strictly monotone. Hence, the surface either uniformly "approaches" the central axis or "diverges" from it; it does not "wiggle". Hence, the surface S naturally parameterized by polar coordinates in the yz-plane.

More rigorously, by the inverse function theorem from calculus (and a moment of thought for the endpoints) f must be a C^∞ isomorphism onto a nontrivial interval $J = f(I)$ in $(0, \infty)$ and so it has an inverse function $g : J \to I \subseteq \mathbf{R}$. (Classically, g is "x as a function of y".) We then have a smooth map $J \times S^1 \to \mathbf{R}^3$ given by

$$h : (r, \theta) \mapsto (g(r), r \cos\theta, r \sin\theta)$$

that is a bijection onto S. It must therefore be a C^∞ isomorphism. This can also be verified "by hand", as follows. The tangent mappings for h are injective (look at the y and z parts), so as maps into the 2-dimensional tangent spaces of the embedded submanifold S the tangent mappings of the C^∞ map $J \times S^1 \to S$ are injective. Thus, these latter tangent maps are isomorphisms for dimension reasons, so $J \times S^1 \to S$ is a bijective local C^∞ isomorphism. It is therefore a C^∞ isomorphism.

What is the metric tensor on S in terms of its $\{r, \theta\}$ parameterization? The simplest way to solve this is to recognize that the parameterization of S is better adapted to cylindrical coordinates (with x-axis as the axis of symmetry) rather than rectangular coordinates. Thus, we will find our task easier if we pull back the standard metric tensor from $\mathbf{R}^3$ when it is expressed in cylindrical coordinates rather than in rectangular coordinates. Let us therefore digress to compute the tensor on $\mathbf{R}^3$ in such coordinates, adapted to the x-axis as the axis of symmetry ("traditionally" it is the z-axis that is taken as the axis of symmetry, but the distinction is a triviality):

Lemma 3.1. *The C^∞ mapping $\mathbf{R} \times (0,\infty) \times S^1 \to \mathbf{R}^3$ defined by $(x, r, \theta) \mapsto (x, r\cos\theta, r\sin\theta)$ is a C^∞ isomorphism onto the complement U of the x-axis in $\mathbf{R}^3$, and on U the standard metric tensor of $\mathbf{R}^3$ has restriction $\mathrm{d}x^{\otimes 2} + \mathrm{d}r^{\otimes 2} + r^2 \mathrm{d}\theta^{\otimes 2}$.*

The absence of "cross-terms" in this formula reflects the mutual orthogonality (in tangent spaces to $\mathbf{R}^3$) of the tangent lines along each of the cylindrical coordinate directions through any points of $\mathbf{R}^3$ away from the axis of symmetry.

Proof. Polar coordinates provide the C^∞ isomorphism $(0,\infty) \times S^1 \simeq \mathbf{R}^2 - \{(0,0)\}$, so taking the product against $\mathbf{R}$ on both sides gives the first part of the lemma (as $U = \mathbf{R} \times (\mathbf{R}^2 - \{(0,0)\})$). On a product of Riemannian manifolds, endowed with the product Riemannian metric (via the identification of the tangent bundle to a product manifold with the direct sum of the pullbacks of the tangent bundles of the factors), the metric tensor is built fiberwise as the orthogonal sum of the inner products on the tangent spaces to the factor manifolds; the same goes for manifolds with corners. Since $\mathbf{R}^3$ equipped with its standard metric tensor is the product of the Riemannian manifolds $\mathbf{R}$ (the x-axis) and $\mathbf{R}^2$ (the yz-plane) equipped with their standard metric tensors, we get the asserted formula for the metric tensor in cylindrical coordinates by using the result from the theory of polar coordinates that on $\mathbf{R}^2 - \{(0,0)\}$ the restriction of the standard metric tensor on $\mathbf{R}^2$ is $\mathrm{d}r^{\otimes 2} + r^2\mathrm{d}\theta^{\otimes 2}$. ∎

Returning to the situation with our surface of revolution the induced metric tensor is the pullback of $\mathrm{d}x^{\otimes 2} + \mathrm{d}r^{\otimes 2} + r^2\mathrm{d}\theta^{\otimes 2}$ under the mapping $(r,\theta) \mapsto (g(r), r, \theta)$ in terms of the cylindrical coordinate system on $\mathbf{R} \times (\mathbf{R}^2 - \{(0,0)\})$. In other words, the metric tensor on S (identified with $J \times S^1$) is

$$\mathrm{d}s^2 = \mathrm{d}(g(r))^{\otimes 2} + \mathrm{d}r^{\otimes 2} + r^2\mathrm{d}\theta^{\otimes 2} = (g'(r)\mathrm{d}r)^{\otimes 2} + \mathrm{d}r^{\otimes 2} + r^2\mathrm{d}\theta^{\otimes 2} = (1 + g'(r)^2)\mathrm{d}r^{\otimes 2} + r^2\mathrm{d}\theta^{\otimes 2}.$$

Recall that $g : J \to I \subseteq \mathbf{R}$ is the inverse function to $f : I \simeq J \subseteq (0,\infty)$ (so classically one would write $(1 + x'(r)^2)$ for the coefficient of $\mathrm{d}r^{\otimes 2}$ in this formula). This is the most general formula for the induced metric tensor on a surface of revolution about the x-axis when the surface admits a parameterization by polar coordinates in the yz-plane (i.e., the graph being rotated is strictly monotone).

Example 3.2. A very famous example of such a surface of revolution is the Beltrami surface that is given explicitly in "inverse function" form by

$$x(y) = -\sqrt{a^2 - y^2} + \frac{a}{2} \cdot \log\left(\frac{a + \sqrt{a^2 - y^2}}{a - \sqrt{a^2 - y^2}}\right)$$

for $0 < y < a$. This is an antiderivative to $-\sqrt{a^2 - y^2}/y < 0$ and it approaches ∞ as $y \to 0^+$ and approach 0 as $y \to a^-$, so the graph in the first quadrant for the inverse y as a function of x is strictly decreasing from the point $(0, a)$ on the positive y-axis asymptotically down toward the x-axis. In particular, it satisfies the above requirements. This surface is interesting because (in terms of concepts to be introduced later) it has constant negative curvature (equal to $-1/a^2$).

Let us compute the metric tensor on the Beltrami surface in the polar coordinate system from the yz-plane. Since $x'(r) = -\sqrt{a^2 - r^2}/r$, clearly $1 + x'(r)^2 = a^2/r^2$. Thus, by the general formula given above, the metric tensor is $(a^2/r^2)\mathrm{d}r^{\otimes 2} + r^2 \mathrm{d}\theta^{\otimes 2}$.

4. The torus

We conclude by studying a very classical example, the donought (or torus) $T \subseteq \mathbf{R}^3$ with inner radius $r - |a|$ and outer radius $r + |a|$, $0 < |a| < r$. Explicitly, T is the image of the smooth closed embedding $S^1 \times S^1 \to \mathbf{R}^3$ defined by

$$(\theta, \psi) \mapsto ((a + r\cos\theta)\cos\psi, (a + r\cos\theta)\sin\psi, r\sin\theta).$$

Geometrically, θ is the angle measure for the circles of radius r that go "through the hole" and ψ is the angle measure for the other family of circles that have radii varying from $r - |a|$ to $r + |a|$.

There is a global trivialization of the tangent bundle specified by the ordered pair of vector fields $\{\partial_\theta, \partial_\psi\}$ (that are globally well-defined, even though θ and ψ are not), and under the embedding i of $S^1 \times S^1$ into $\mathbf{R}^3$ and hence of $T(S^1 \times S^1)$ into $i^*(T(\mathbf{R}^3)) = (S^1 \times S^1) \times \mathbf{R}^3$ these go over to the vector fields

$$\partial_\theta = (-r\sin\theta\cos\psi, -r\sin\theta\sin\psi, r\cos\theta), \ \ \partial_\psi = (-(a + r\cos\theta)\sin\psi, (a + r\cos\theta)\cos\psi, 0).$$

That is, for a point $\xi \in S^1 \times S^1$ with angle coordinates (θ_0, ψ_0), the injection of $\mathrm{T}_\xi(S^1 \times S^1)$ into $\mathrm{T}_{i(\xi)}(\mathbf{R}^3)$ satisfies

$$\partial_\theta|_\xi = -r\sin\theta_0\cos\psi_0\partial_x|_{i(\xi)} - r\sin\theta_0\sin\psi_0\partial_y|_{i(\xi)} + r\cos\theta_0\partial_z|_{i(\xi)}$$

and

$$\partial_\psi|_\xi = -(a + r\cos\theta_0)\sin\psi_0\partial_x|_{i(\xi)}|(a + r\cos\theta_0)\cos\psi_0\partial_y|_{i(\xi)}.$$

The lack of a ∂_z-term in this final expression is suggested by the picture: the coordinate lines for ψ are parallel to the xy-plane.

As a picture suggests and a calculation with the orthonormal frame $\{\partial_x, \partial_y, \partial_z\}$ at $i(\xi)$ confirms, the vector fields ∂_θ and ∂_ψ are pairwise orthogonal with respect to the induced metric. By direct calculation, the self inner-products are r^2 for $\partial_\theta|_\xi$ and $(a + r\cos\theta_0)^2$ for $\partial_\psi|_\xi$. Hence, the induced metric tensor is

$$r^2\mathrm{d}\theta^{\otimes 2} + (a + r\cos\theta)^2\mathrm{d}\psi^{\otimes 2}.$$

The constancy of the first coefficient and the varying nature of the second coefficient reflect some basic geometric properties of the surface of the torus *as it sits in* $\mathbf{R}^3$. That is, this geometry is very sensitive to the chosen embedding, insofar as this is what determined the metric tensor.

The inherent asymmetry in the roles of θ and ψ is not apparent when considering the "bare" manifold $S^1 \times S^1$, but the chosen embedding into $\mathbf{R}^3$ singles out different roles for these two factors and so leads to the asymmetric nature of their appearance in the metric tensor. Can you see a geometric "explanation" (within $\mathbf{R}^3$) for the varying length of ∂_ψ as we vary θ, but the constant length of ∂_θ as we wander across the surface?