Math 396. Gauss map and scalar curvature

1. Motivation

In Gauss' fundamental work on the geometry of surfaces in $\mathbf{R}^3$, he introduced an important object of study, the *Gauss map*, that encodes how a surface stretches inside of $\mathbf{R}^3$. More specifically, if $i : S \hookrightarrow \mathbf{R}^3$ is a smooth embedded oriented surface in $\mathbf{R}^3$, then the Gauss map $G : S \to S^2$ is defined as follows. Let $\mathbf{n}$ be the unit normal field along S determined by the orientation, so this trivializes the orthogonal bundle $(TS)^{\perp}$ to the plane bundle TS inside of $i^*(T(\mathbf{R}^3)) \simeq S \times \mathbf{R}^3$; here, on each tangent space $\mathrm{T}_x(\mathbf{R}^3)$ for $x \in \mathbf{R}^3$ we put the "standard" inner product arising from the natural isomorphism $j_x : \mathrm{T}_x(\mathbf{R}^3) \simeq \mathbf{R}^3$. Gauss defined $G : S \to \mathbf{R}^3$ to be the map that associates to each point of the surface its unit normal vector arising from the orientation. That is, $G : x \mapsto j_x(\mathbf{n}(x)) \in \mathbf{R}^3$; traditionally one abuses notation and writes $\mathbf{n}(x)$ rather than $j_x(\mathbf{n}(x))$.

If you think about it for a minute, the map G does encode information about how S streches in $\mathbf{R}^3$. (For example, if S is a translated linear subspace of $\mathbf{R}^3$ then G is a constant map that depends up to sign on the orientation. Try to visualize G in the case of a cylinder by working out the image under G of various curves on the cylinder; do the same for a torus.) We will see in §2 that G is a C^∞ map and that for any $x \in S$ the tangent mapping $\mathrm{d}G(x) : \mathrm{T}_x(S) \to \mathrm{T}_{G(x)}(\mathbf{R}^3) \simeq \mathbf{R}^3$ has image contained in $\mathrm{T}_x(S)$ when this plane is viewed as a subspace of $\mathrm{T}_x(\mathbf{R}^3) \simeq \mathbf{R}^3$. Hence, $\mathrm{d}G(x)$ may be viewed as a linear endomorphism of the 2-dimensional space $\mathrm{T}_x(S)$. As such, Gauss showed that it is *self-adjoint* for the inner product on the plane $\mathrm{T}_x(S)$ in $\mathbf{R}^3$, so it has real eigenvalues by the spectral theorem, and (building on earlier results of Euler) he realized that (i) the two eigenvalues (perhaps equal) have a geometric interpretation in terms of extremal values of the "curvature at x" for slices of S by planes in $\mathbf{R}^3$ containing the normal line to S at x, and (ii) the determinant $k(x) = \det(\mathrm{d}G(x))$ of this self-map of $\mathrm{T}_x(S)$ is *intrinsic* to S equipped with the Riemannian metric it inherits from $\mathbf{R}^3$. That is, if we view S as a Riemannian 2-manifold equipped with an *isometric* embedding into $\mathbf{R}^3$, then the smooth function $k : S \to \mathbf{R}$ depends only on the Riemannian structure on S and does not depend on the choice of isometric embedding into $\mathbf{R}^3$ nor on the orientation.

As a very simple example of rather different isometric embeddings of the same Riemannian 2-manifold into $\mathbf{R}^3$, an open square with the standard flat metric can be put in the xy-plane as usual or can be put on the surface of a cylinder with one pair of opposite sides along the direction of the axis of symmetry. Since you can do this with a piece of paper, it really does not introduce stretching. (To be rigorous, one computes the metric tensor on the radius-R cylinder with z-axis as the axis of symmetry: in cylindrical coordinates (z, θ) it is the tensor $\mathrm{d}z^{\otimes 2} + (1/R^2)\mathrm{d}\theta^{\otimes 2} = \mathrm{d}z^{\otimes 2} + \mathrm{d}(\theta/R)^{\otimes 2}$ that becomes the standard flat metric tensor in the $\{z, \theta/R\}$ coordinate system; note R is a constant.)

This function k was called the *scalar curvature* by Gauss, and it is also called the *Gaussian curvature*. The intrinsic nature of k (i.e., its dependence on only the Riemannian structure and not on the chosen isometry into $\mathbf{R}^3$) was such a striking discovery that Gauss called it his *Theorem Egregium* (Remarkable Theorem). Gauss did not use the language of modern coordinate-free differential geometry when stating his results, but he certainly had a clear picture of what was going on. Our aim in this handout is to develop a bit of the theory of the Gauss map for oriented embedded smooth hypersurfaces in any finite-dimensional inner product space (not just $\mathbf{R}^3$), and to study the historically important cases of curves (in $\mathbf{R}^2$) and surfaces (in $\mathbf{R}^3$) in order to see how Gauss' work built on earlier ideas of Euler. Along the way, we shall investigate some interesting examples.

2. Hypersurfaces

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space, and $S = \{v \in V \mid \langle v, v\rangle = 1\}$ the unit sphere in V. We assume $n = \dim V > 1$, and we fix an orientation on V. For example, upon choosing an oriented orthonormal basis of V we may suppose $V = \mathbf{R}^n$ with the standard inner product and standard orientation; this is the classical formulation.

Let $i : H \hookrightarrow V$ be an oriented embedded hypersurface. The natural isomorphism $j_x : \mathrm{T}_x(V) \simeq V$ puts an inner product $\langle \cdot, \cdot \rangle_x$ on $\mathrm{T}_x(V)$ for all $x \in V$; this is a Riemannian metric on the vector bundle $TV \to V$. The orientation trivializes the orthogonal line bundle $(TH)^{\perp} \subseteq i^*(TV)$. More specifically, since the vector bundle $i^*(TV) = (TH)^{\perp} \oplus TH$ and its hyperplane subbundle TH are oriented, there is a unique orientation on the line bundle $(TH)^{\perp}$ such that the natural isomorphism

$$(TH)^{\perp} \otimes \det(TH) \overset{\wedge}{\simeq} \det(i^*TV) \tag{2.1}$$

is orientation-preserving on fibers. More concretely, for each $h \in H$ we take $\mathbf{n}(h) \in \mathrm{T}_h(H)^{\perp}$ to be the unique unit vector such that for any positive basis $\{v_1, \dots, v_{n-1}\}$ of $\mathrm{T}_h(H)$ the ordered basis $\{\mathbf{n}(h), v_1, \dots, v_{n-1}\}$ of $\mathrm{T}_{i(h)}(V) \simeq V$ is positive. For example, in the case of an oriented surface S in $\mathbf{R}^3$, the condition on the normal vector $\mathbf{n}(x) \in \mathrm{T}_x(S)^{\perp}$ is that for a positive ordered basis $\{v, w\}$ of $\mathrm{T}_x(S)$ the 3×3 coefficient matrix for the ordered basis $\{\mathbf{n}(x), v, w\}$ of $\mathrm{T}_x(\mathbf{R}^3) \simeq \mathbf{R}^3$ has positive determinant. Eqiuvalently, using the vector cross-product on the oriented inner product space $\mathbf{R}^3$ (*cf.* the end of the old handout on the Hodge star operator),

$$\mathbf{n}(x) = \frac{v \times w}{\|v \times w\|}.$$

In general, by the universal nature of Gramm-Schmidt and the smoothness of the square-root function on $(0, \infty)$, $h \mapsto \mathbf{n}(h)$ is a smooth section $\mathbf{n} \in (TH)^{\perp}(H)$ that is the unit normal field determined pointwise by the orientations on H and V. Using the natural isomorphism $j_x : \mathrm{T}_x(V) \simeq V$ that carries $\langle \cdot, \cdot \rangle_x$ to $\langle \cdot, \cdot \rangle$, for $h \in H$ the vector $j_{i(h)}(\mathbf{n}(h)) \in V$ is a unit vector and so lies in the unit sphere S in V.

Definition 2.1. The *Gauss map* of the oriented smooth embedded hypersurface $i : H \hookrightarrow V$ is the map $G : H \to S$ to the unit sphere in V given by $G(h) = j_{i(h)}(\mathbf{n}(h)) \in S$.

It is traditional to abuse notation and write $\mathbf{n}(h)$ for $G(h)$. We will hold off on this abuse of notation for a short while. Since the definition of G is pointwise, we should first check:

Lemma 2.2. *The map G is C^{∞}.*

Proof. The pointwise isomorphisms $j_x : \mathrm{T}_x(V) \simeq V$ for $x \in V$ fit together to define a C^{∞} vector bundle isomorphism $TV \simeq V \times V$ over V (using $\mathrm{pr}_1 : V \times V \to V$) recovering j_x on fibers over each $x \in V$. Pulling back along $i : H \hookrightarrow V$ gives a C^{∞} vector bundle isomorphism $i^*(TV) \simeq H \times V$ over H. The unit normal field $\mathbf{n}$ is a C^{∞} section of the subbundle $(TH)^{\perp}$ over H, so it is also a C^{∞} section of the ambient bundle $i^*(TV) \simeq H \times V$. Hence, we get a C^{∞} composite map

$$H \overset{\mathbf{n}}{\to} (TH)^{\perp} \hookrightarrow i^*(TV) \simeq H \times V \overset{\mathrm{pr}_2}{\to} V$$

that is exactly the composite of $G : H \to S$ with the inclusion $j : S \hookrightarrow V$. Since j is a C^{∞}-embedding and $j \circ G$ is C^{∞}, it follows that G is also C^{∞}. ∎

For $h \in H$, consider the tangent mapping $\mathrm{d}G(h) : \mathrm{T}_h(H) \to \mathrm{T}_{G(h)}(S)$ between two hyperplanes in V, namely $\mathrm{T}_h(H) \subseteq \mathrm{T}_{i(h)}(V) \simeq V$ and $\mathrm{T}_{G(h)}(S) \subseteq \mathrm{T}_{G(h)}(V) \simeq V$. I claim that the hyperplanes $\mathrm{T}_h(H) \subseteq \mathrm{T}_{i(h)}(V)$ and $\mathrm{T}_{G(h)}(S) \subseteq \mathrm{T}_{G(h)}(V)$ are "parallel", or in other words that as hyperplanes in V they coincide:

Lemma 2.3. *The hyperplanes* $\mathrm{T}_h(H)$ *and* $\mathrm{T}_{G(h)}(S)$ *in* V *are equal.*

Proof. To show equality of subspaces in an inner product space, it is equivalent to show equality of the orthogonal subspaces. Hence, we shall prove equality of the normal lines to these hyperplanes. For any $v \in S$, the line $\mathbf{R}v$ is the normal line to the hyperplane $\mathrm{T}_v(S) \subseteq \mathrm{T}_v(V) \simeq V$ (proof: pass to orthonormal coordinates and do a direct calculation of the tangent hyperplane to the standard sphere S^{n-1} in $\mathbf{R}^n$ at the point $(1, 0, \dots, 0)$). Hence, taking $v = G(h)$, we need to prove that $G(h) = j_{i(h)}(\mathbf{n}(h))$ spans the normal line to $\mathrm{T}_h(H)$ in $\mathrm{T}_{i(h)}(V) \overset{j_{i(h)}}{\simeq} V$. But by definition $\mathbf{n}(h)$ spans the normal line $\mathrm{T}_h(H)^{\perp}$ in $\mathrm{T}_{i(h)}(V)$, and so applying the isometry $j_{i(h)}$ implies that $j_{i(h)}(\mathbf{n}(h)) = G(h)$ spans the normal line to $j_{i(h)}(\mathrm{T}_h(H))$ in V. ■

By the lemma, we may (and do) identify the tangent mapping $\mathrm{d}G(h)$ with a linear *endomorphism* of $\mathrm{T}_h(H)$. In particular, it makes sense to speak of eigenvalues for $\mathrm{d}G(h)$ (a concept that only makes sense for endomorphisms of a vector space, not general linear maps between a pair of vector spaces).

Example 2.4. Choose $a > 0$. Let $S \subseteq (0, a) \times (0, \infty) \times \mathbf{R} \subseteq \mathbf{R}^3$ be the Beltrami surface B_a from Example 3.2 in the handout on metric tensors for hypersurfaces. This is the surface of revoluion of the parametric curve $(x(y), y)$ in the first quadrant given by $x'(y) = -\sqrt{a^2 - y^2}/y$ (with $0 < y < a$) and $x(y) \to 0$ as $y \to \infty$. That is, B_a is the image of the embedding $\iota : (0, a) \times S^1 \hookrightarrow \mathbf{R}^3$ given by $(t, \theta) \mapsto (x(t), t\cos\theta, t\sin\theta)$.

Let us compute the Gauss map $G : B_a \to S^2$ explicitly and then compute its induced tangent mappings so as to see Lemma 2.3 worked out in this example. We fix an orientation of B_a by trivializing the tangent bundle using the standard trivialization $\{\partial_t, \partial_\theta\}$ on $(0, a) \times S^1$, and we will use a cross-product normal field within $T(\mathbf{R}^3)|_{B_a}$ to define the orientation. Explicitly, under ι these tangent fields go over to the sections of the subbundle $T(B_a) \subseteq T(\mathbf{R}^3)|_{B_a}$ over B_a given by

$$(\mathrm{d}\iota)(\partial_t) = x'(t) \cdot \partial_x|_{B_a} + \cos\theta \cdot \partial_y|_{B_a} + \sin\theta \cdot \partial_z|_{B_a}, \quad (\mathrm{d}\iota)(\partial_\theta) = -t\sin\theta \cdot \partial_y|_{B_a} + t\sin\theta \cdot \partial_z|_{B_a}.$$

In particular, for $\xi_0 = \iota(t_0, \theta_0) \in B_a$, under the isomorphism $\mathrm{T}_{\xi_0}(\mathbf{R}^3) \simeq \mathbf{R}^3$ the tangent plane $\mathrm{T}_{\xi_0}(B_a)$ goes over to the plane in $\mathbf{R}^3$ spanned by the two vectors

$$(x'(t_0), \cos\theta_0, \sin\theta_0), \quad (0, -t_0\sin\theta_0, t_0\cos\theta_0) \in \mathbf{R}^3. \tag{2.2}$$

The vector cross product in $\mathrm{T}_{\xi_0}(\mathbf{R}^3)$ is

$$(\mathrm{d}\iota)(\partial_t)_{\xi_0} \times (\mathrm{d}\iota)(\partial_\theta)_{\xi_0} = \partial_x|_{\xi_0} - x'(t_0)\cos\theta_0 \cdot \partial_y|_{\xi_0} - x'(t_0)\sin\theta_0 \cdot \partial_z|_{\xi_0}.$$

Since $x'(t) = -\sqrt{a^2 - t^2}/a$, dividing by the length and using the usual isomorphism $\mathrm{T}_{\xi_0}(\mathbf{R}^3) \simeq \mathbf{R}^3$ gives the unit vector

$$G(t_0, \theta_0) = \left(-\frac{t_0}{a}, -\frac{\sqrt{a^2 - t_0^2}}{a} \cdot \cos\theta_0, -\frac{\sqrt{a^2 - t_0^2}}{a} \cdot \sin\theta_0 \right) \in S^2.$$

This is the Gauss map. To which orientation of the surface B_a does it correspond? Since B_a is connected, there are two orientations and these correspond to the two kind of unit normal fields: one pointing inward toward the x-axis and the one points outward. To figure out the one we've got, we just have to check at one point. Working at a point $\iota(t_0, 0) = (x(t_0), t_0, 0)$ in the first quadrant of the xy-plane, the normal line points roughly in the northeast (outward) and southwest (inward) directions as distinguished by the xy-components of the normal vector being positive and negative respectively. Since $G(t_0, 0)$ has negative coordinates, we conclude that we have chosen the inward normal orientation.

The tangent mapping $\mathrm{d}G(t_0,\theta_0) : \mathrm{T}_{(t_0,\theta_0)}(B_a) \to \mathrm{T}_{G(t_0,\theta_0)}(S^2) \subseteq \mathrm{T}_{G(t_0,\theta_0)}(\mathbf{R}^3)$ is given by the 3×2 matrix

$$\begin{pmatrix} -1/a & 0 \\ \frac{t_0\cos\theta_0}{a\sqrt{a^2-t_0^2}} & \frac{\sqrt{a^2-t_0^2}\sin\theta_0}{a} \\ \frac{t_0\sin\theta_0}{a\sqrt{a^2-t_0^2}} & -\frac{\sqrt{a^2-t_0^2}\cos\theta_0}{a} \end{pmatrix}$$

with respect to the ordered bases

$$\{\partial_t|_{(t_0,\theta_0)}, \partial_\theta|_{(t_0,\theta_0)}\},\ \ \{\partial_x|_{G(t_0,\theta_0)}, \partial_y|_{G(t_0,\theta_0)}, \partial_z|_{G(t_0,\theta_0)}\}$$

on the source and target respectively. We have seen that when the source $\mathrm{T}_{(t_0,\theta_0)}(B_a)$ is viewed as a plane in $\mathrm{T}_{(t_0,\theta_0)}(\mathbf{R}^3) \simeq \mathbf{R}^3$, it is spanned by (2.2). The image in $\mathrm{T}_{G(t_0,\theta_0)}(S^2) \subseteq \mathrm{T}_{G(t_0,\theta_0)}(\mathbf{R}^3) \simeq \mathbf{R}^3$ is the span of the columns of the above 3×2 matrix, and those columns are respectively $t_0/a\sqrt{a^2-t_0^2}$ times the first vector in (2.2) and $-\sqrt{a^2-t_0^2}/at_0$ times the second vector in (2.2). Hence, we recover the "equality" of planes as predicted by Lemma 2.3.

Upon identifying the source and target planes $\mathrm{T}_{(t_0,\theta_0)}(B_a)$ and $\mathrm{T}_{G(t_0,\theta_0)}(S^2)$ for $\mathrm{d}G(t_0,\theta_0)$, the above calculation shows that the resulting self-map of this plane has matrix

$$\begin{pmatrix} t_0/a\sqrt{a^2-t_0^2} & 0 \\ 0 & -\sqrt{a^2-t_0^2}/at_0 \end{pmatrix}$$

with respect to the ordered basis $\{\partial_t|_{(t_0,\theta_0)}, \partial_\theta|_{(t_0,\theta_0)}\}$ of $\mathrm{T}_{(t_0,\theta_0)}(B_a)$. For example, in this case we see that for any $\xi_0 \in B_a$ corresponding to $(t_0,\theta_0) \in (0,a)\times S^1$, when $\mathrm{d}G(\xi_0)$ is viewed as an endomorphism of $\mathrm{T}_{\xi_0}(B_a)$ then it has eigenvalues $t_0/a\sqrt{a^2-t_0^2}$ and $-\sqrt{a^2-t_0^2}/at_0$ that only depend on t_0 (not on the angle) and it has determinant $-1/a^2$ that is *constant*: independent of ξ_0. This concludes the present example.

In the definition of G we had to use an orientation on H in order to define the unit normal field $\mathbf{n}$ along H. If we pass to the opposite orientation (the only other one when H is connected) then to keep (2.1) an orientation-preserving isomorphism we must use the unit normal field $-\mathbf{n}$ and hence the associated Gauss map is $-G$. Likewise, if we negate the orientation of V but do not change the one on H then we must use the unit normal field $-\mathbf{n}$. If *both* orientations are negated, then neither G nor the unit normal field $\mathbf{n}$ are changed (since $-(-1)=1$). We leave it to the reader to check that the outer part of the diagram

(2.3)
$$\begin{array}{ccccccc} \mathrm{T}_h(H) & \xrightarrow{\mathrm{d}G(h)} & \mathrm{T}_{G(h)}(S) & \longrightarrow & \mathrm{T}_{G(h)}(V) & \xrightarrow{\simeq} & V \\ & {\scriptstyle \mathrm{d}(-G)(h)}\searrow & \downarrow{\scriptstyle \mathrm{d}(-\mathrm{id}_S)} & & \downarrow{\scriptstyle \mathrm{d}(-\mathrm{id}_V)} & & \downarrow{\scriptstyle -1} \\ & & \mathrm{T}_{-G(h)}(S) & \longrightarrow & \mathrm{T}_{-G(h)}(V) & \xrightarrow[\simeq]{} & V \end{array}$$

commutes (check the smaller subdiagrams commute, using the Chain Rule). Hence, the self-mapping $\mathrm{d}G(h) \in \mathrm{End}(\mathrm{T}_h(H))$ changes by a sign when we pass to the opposite orientation on exactly one of H and V, and it does *not* change when both are negated. In particular, when H is connected (so there are exactly two orientations on H, given that there is at least one), $\mathrm{d}G(h) : \mathrm{T}_h(H) \simeq \mathrm{T}_h(H)$ depends on the orientations of H and V only up to a sign.

The assignment $h \mapsto \mathrm{d}G(h) \in \mathrm{Hom}(\mathrm{T}_h(H), \mathrm{T}_h(H))$ is a set-theoretic section of the C^∞ vector bundle $\mathrm{Hom}(TH, TH) \to H$. As with any natural construction, it is better than just set-theoretic:

Lemma 2.5. *The set-theoretic section $h \mapsto \mathrm{d}G(h)$ of the vector bundle* $\mathrm{Hom}(TH, TH)$ *over H is a C^∞ section.*

Proof. Since TH is a C^∞ subbundle of $i^*(TV) \simeq H \times V$, $\mathrm{Hom}(TH, TH)$ is a C^∞ subbundle of $\mathrm{Hom}(TH, i^*(TV)) = \mathrm{Hom}(TH, H \times V)$. A C^∞ subbundle inclusion is a C^∞ embedding, so to check that our set-theoretic section $H \to \mathrm{Hom}(TH, TH)$ is a smooth mapping it is equivalent to check that the composite mapping $H \to \mathrm{Hom}(TH, H \times V)$ is smooth. This problem is local on H, and so we may work over open subsets U on which there are C^∞ coordinates $\{x_1, \dots, x_n\}$. In this case $TH|_U$ is trivialized by the sections $\partial_{x_1}, \dots, \partial_{x_n}$, so we just have to show that $h \mapsto \mathrm{d}G(h)(\partial_{x_i}) \in \mathrm{T}_{G(h)}(V) \simeq V$ is a smooth map from H to V for each i. Upon picking linear coordinates $t_1, \dots, t_n$ of V, and letting $G_j = t_j \circ G$ be the component functions of $G : H \to S \subseteq V$, this mapping $H \to V$ has component functions $\partial_{x_i}(G_j)$. These are smooth, by Lemma 2.2. ■

Definition 2.6. The *extrinsic scalar curvature* of the smooth embedded oriented hypersurface $i : H \hookrightarrow V$ is the map $k : H \to \mathbf{R}$ defined by $k(h) = \det(-\mathrm{d}G(h) : \mathrm{T}_h(H) \to \mathrm{T}_h(H)) = (-1)^{\dim H} \det(\mathrm{d}G(h))$.

For example, the calculations in Example 2.4 show that the Beltrami surface $B_a \subseteq \mathbf{R}^3$ has constant negative extrinsic scalar curvature $-1/a^2$. In the special case $a = 1$ we get constant negative curvature -1. For this reason, B_1 is often called the *pseudosphere* (as a sphere of radius 1 in $\mathbf{R}^3$ has constant extrinsic scalar curvature 1, to be proved in Example 4.5). The pseudosphere provides a "model" for Lobachevsky's non-Euclidean geometry (where a "line" is a certain kind of length-minimizing curve on the surface).

Note that $k : H \to \mathbf{R}$ is a smooth function. Indeed, up to the constant sign $(-1)^{\dim H}$ it is the composite of three maps

$$H \overset{\mathrm{d}G}{\to} \mathrm{Hom}(TH, TH) \overset{\det}{\to} H \times \mathbf{R} \overset{\mathrm{pr}_2}{\to} \mathbf{R}$$

where the middle step is the fiberwise determinant on fibers over H (*not* a map of vector bundles over H) and the first step is smooth by Lemma 2.5. The smoothness of the third step is clear, and the smoothness of the middle step is seen by direct calculation with local trivializing frames. (More generally, replacing $TH \to H$ with an arbitrary C^p vector bundle $E \to X$ over a C^p premanifold with corners, the fiberwise determinant mapping $\det_E : \mathrm{Hom}(E, E) \to X \times \mathbf{R}$ over X (*not* a map of vector bundles!) is C^p because over small opens $U \subseteq X$ on which there is a trivialization $E|_U \simeq U \times \mathbf{R}^n$ we get a C^p isomorphism $\mathrm{Hom}(E, E) \simeq X \times \mathrm{Mat}_{n \times n}(\mathbf{R})$ over X whose inverse composes with $\det_E$ to give the visibly C^p mapping $X \times \mathrm{Mat}_{n \times n}(\mathbf{R}) \to X \times \mathbf{R}$ defined by $(x, (a_{ij})) \mapsto (x, \det(a_{ij}))$.)

Remark 2.7. Negating the orientation on both H and V simultaneously does not change G and so does not change k. Negating exactly one of these orientations does negate G and so changes k by a sign of $(-1)^{\dim H}$.

Remark 2.8. The reason for the sign in Definition 2.6 is due to compatibility with more general constructions in the theory of curvature on Riemannian manifolds. In the case of surfaces (or hypersurfaces in an odd-dimensional V) one does not perceive the sign. In particular, in Gauss' study of surfaces he did not need to pick an orientation for his surfaces. (This is fortunate, since the phenomenon of non-orientability was not discovered until Möbius came along some time after Gauss' work.)

The reason for the word "extrinsic" in Definition 2.6 is because this definition uses the embedding of H into V (and not just, say, the Riemannian metric and orientation on H). The reason for calling this a "curvature" will be explained by some later examples with curves and surfaces. The embedding $H \hookrightarrow V$ puts a Riemannian structure on the oriented manifold H. It was Gauss who first had the idea to consider the Riemannian structure on H as *fixed* and to contemplate different isometric embeddings of H into V (i.e., embeddings whose induced metric on H is the one fixed at the outset). Does k depend on the actual embedding into V (say, up to translation on V or orthogonal orientation-preserving linear transformations of V), or is it intrinsic to the oriented Riemannian manifold H in the sense that it depends only on the Riemannian structure of the oriented manifold H? It was a well-known (elementary) result of the classical (i.e., pre-Gauss) geometers that an oriented curve in the plane is determined up to translation and orthogonal orientation-preserving linear transformation of the plane by the specification of its curvature function, and so for $\dim V = 2$ the affirmative answer is not particularly deep. Gauss' Theorem Egregium gives the affirmative answer in the case $\dim V = 3$. In the case $\dim V > 3$, the situation becomes more complicated: there is a higher-dimensional version of Gauss' Theorem Egregium, but it involves another notion, that of *sectional curvature*, which coincides with k for surfaces but is otherwise of rather different nature. In particular, for $\dim V > 3$ the extrinsic scalar curvature k depends very much on how H is isometrically embedded into V. That is, the naive higher-dimensional version of the Theorem Egregium is *false*.

Remark 2.9. To compute the extrinsic scalar curvature for surfaces exhibiting some symmetry, we can considerably simplify computational efforts by recording the behavior of k with respect to certain transformations of the ambient vector space. Let $L : \mathbf{R}^n \simeq \mathbf{R}^n$ be an orthogonal linear change of coordinates, so L respects the inner product. We endow $L(H)$ with the orientation "induced" by H via $L : H \simeq L(H)$. By inspecting the definitions, it follows that the extrinsic scalar curvatures k_H and $k_{L(H)}$ are related by the identity $k_H = \varepsilon^{\dim H} k_{L(H)} \circ L$ with $\varepsilon = \det(L) = \pm 1$ the sign that encodes whether or not L is orientation-preserving. In particular, as we would expect, an orientation-preserving orthogonal linear transformation on V does not affect the curvature. An additive translation is similarly "harmless". (If $\dim V$ is odd, so $\dim H$ is even, then even orientation-reversing orthogonal L are "harmless".) Of special interest is the case of such an L for which $L(H) = H$ and $\dim H$ is even. In this case we may say $k_H(x) = k_H(L(x))$ for all $x \in H$. This is especially useful for the case of surfaces in $\mathbf{R}^3$.

By Remark 2.9, when trying to compute the extrinsic scalar curvature in a specific example we may always apply a translation or an orthogonal change of coordinates (provided we keep track of whether the coordinate change has negative determinant, at least when $\dim H$ is odd). For example, let us return to Example 2.4. We noted by calculation that the self-maps $\mathrm{d}G(\xi_0)$ were "independent" of the angular parameter θ_0, and in particular the eigenvalues were independent of θ_0. But this can now be seen by geometry without any calculation: rotation about the x-axis is an orthogonal transformation of $\mathbf{R}^3$ that carries the surface of revolution B_a back to itself, and so it commutes with formation of the Gauss map and hence with formation of its tangential self-maps. Since all points on a common circular slice (for fixed t_0) can be carried to each other by a suitable such rotation, it follows that the intrinsic concept of "eigenvalues of $\mathrm{d}G(\xi_0)$ as a self-map of $\mathrm{T}_{\xi_0}(B_a)$" is independent of the angle and must depend only on t_0. What geometrical reasoning does not predict is that the determinant, which is to say the extrinsic scalar curvature, is in fact constant across the surface. The reason that this cannot be seen by geometry alone is that it is not a general feature of surfaces of revolution. One has to make use of the specific definition of B_a to verify such constancy.

3. Curves in $\mathbf{R}^2$

Let us now study the most classical case of all, $\dim V = 2$. These are smooth embedded curves in $\mathbf{R}^2$ with its standard inner product. (The topic of geometry for curves in $\mathbf{R}^3$ is another rich and important one in the historical development of differential geometry, and it preceded Gauss' work on surfaces, but we will not dwell on it here.) We now write C instead of H, since it is a curve in the 2-dimensional V. We fix a unit vector $\mathbf{e} \in V$ and let $\mathbf{e}'$ be the unique unit vector perpendicular to $\mathbf{e}$ such that the ordered basis $\{\mathbf{e}, \mathbf{e}'\}$ is positive with respect to the given orientation on V. Thus, the unit sphere S (circle) in V is parameterized by $\mathbf{R}/2\pi\mathbf{Z}$ via trigonometry:

$$t \mapsto \cos(t)\mathbf{e} + \sin(t)\mathbf{e}'.$$

This defines a C^∞ isomorphism $\mathbf{R}/2\pi\mathbf{Z} \simeq S$. For any unit vector $v \in V$, we may therefore uniquely write $v = \cos(\theta)\mathbf{e} + \sin(\theta)\mathbf{e}'$ with $\theta \in \mathbf{R}/2\pi\mathbf{Z}$, and we call $\theta = \theta(v)$ the *angle* of v with respect to $\mathbf{e}$ (given the orientation on V). We usually choose a lift of $\theta(v)$ in $\mathbf{R}$ and we call this real number an "angle" of v with respect to $\mathbf{e}$ (understanding that it is only well-defined up modulo $2\pi\mathbf{Z}$).

Let $\theta_0 \in \mathbf{R}$ be an angle with respect to $\mathbf{e}$ for a unit vector $v_0 \in V$. There is a unique continuous (and even smooth) way to extend this to an angle function at all points of any open arc in S that contains v_0 and omits at least one point of S. This is geometrically obvious, and is rigorously proved as follows. Pick $v_1 \neq v_0$ in S. The claim is that there is a unique continuous function $S - \{v_1\} \to \mathbf{R}$ given by angle representatives and taking v_0 to θ_0, and that this function is C^∞. Uniqueness is clear because any two such have difference that is locally constant and hence constant on the connected arc $S - \{v_1\}$. For existence, compose the inverse isomorphism $S \simeq \mathbf{R}/2\pi\mathbf{Z}$ with a suitable C^∞-section to $\mathbf{R} \to \mathbf{R}/2\pi\mathbf{Z}$ over the complement of a point to arrange that v_0 is carried to $\theta_0 \in \mathbf{R}$.

Let $i : C \to V$ be the inclusion of the oriented curve C into the plane V. Choose a point $c \in C$, and let $\sigma : (-\varepsilon, \varepsilon) \to C$ be a C^∞ parameterization by arc length near c with $\sigma(0) = c$. There are two such parameterizations (depending on the direction of motion along C), and we fix one via the orientation of C: we choose the one such that for all t the unit vector $\sigma'(t)$ in the line $\mathrm{T}_{\sigma(t)}(C)$ lies in the "positive" half-line as determined by the orientation. It is geometrically obvious that this can be done in a unique way, and a rigorous proof goes as follows. A choice of local parameterization by arc length σ is an open immersion, and hence the non-vanishing velocity vector field σ' defines a trivialization of the tangent bundle on C over the connected open neighborhood $U = \mathrm{image}(\sigma)$ of c in C. (Since σ is C^∞, by calculation in coordinates we see that $t \mapsto \sigma'(t) \in V$ is a C^∞ map from $(-\varepsilon, \varepsilon)$ to V.) Hence, this is an orientation form for $TC|_U$, and by connectivity of U either this trivialization or its negative is compatible with the chosen orientation on C. If there is not compatibility, then we switch to the parameterization $t \mapsto \sigma(-t)$ (whose velocity vector field is negative that of σ; check!). Hence, we may and do uniquely parameterize C by arc length near c such that the velocity vector is in the positive half-line of the tangent line at all points.

Because the parameterization is by arc length, $\sigma'(t) \in \mathrm{T}_{\sigma(t)}(C) \subseteq \mathrm{T}_{\sigma(t)}(V) \simeq V$ is a unit vector for all t. The smooth map $t \mapsto \sigma'(t) \in V$ from $(-\varepsilon, \varepsilon)$ to V has image contained in the unit circle S (an embedded smooth submanifold of V) because the velocity vectors are unit vectors, so we get a smooth map $\sigma' : (-\varepsilon, \varepsilon) \to S$.

We have seen above that "angle" is a well-defined smooth function (up to an integral multiple of 2π) on the complement of a single point in S. By continuity of σ', if we shrink ε then we can arrange that for all $t \in (-\varepsilon, \varepsilon)$ the point $\sigma'(t) \in S$ is near $\sigma'(0)$. In particular, the function $\theta : (-\varepsilon, \varepsilon) \mapsto \mathbf{R}$ given by the angle of $\sigma'(t)$ with respect to $\mathbf{e}$ is *smooth* in t and is well-defined up to adding an integral multiple of 2π (and the natural choice is to require that the element $\theta(0) \in 2\pi\mathbf{Z}$ vanishes).

Hence, the old-fashioned calculus derivative $\theta' : (-\varepsilon, \varepsilon) \to \mathbf{R}$ is well-defined (the additive constant ambiguity is eliminated by the differentiation).

Changing $\mathbf{e}$ changes θ by an additive constant, so θ' is also independent of this choice too. In other words, the "rate of angle change" $t \mapsto \theta'(t)$ as we traverse the curve C near c at unit speed only depends on three things: the curve C in V, the orientation of V, and the orientation of C (i.e., the direction of motion in which we traverse C). If we negate the orientation of exactly one of C or V then our notion of angle gets negated, but if we negate both orientations then our notion of angle is unaffected.

Example 3.1. Let us take C to be the circle of radius $r > 0$ centered at the origin in V, and give it the "clockwise" orientation (corresponding to the trivialization by the vector field $-\partial_\theta$ with respect to the standard angular parameterization using an oriented orthonormal basis of V). In view of the specified orientation, the parameterization at unit speed is

$$\sigma(t) = r\cos(-t/r)\mathbf{e} + r\sin(-t/r)\mathbf{e}'$$

with $\{\mathbf{e}, \mathbf{e}'\}$ a positive orthonormal basis of V, since

$$\sigma'(t) = r\sin(-t/r)\mathbf{e} - r\cos(-t/r)\mathbf{e}' = r(-\partial_\theta|_{\sigma(t)})$$

with $r > 0$. In this case locally $\theta(t) = -t/r$ (up to adding a constant integral multiple of 2π), so $\theta'(t) = -1/r$.

Theorem 3.2. *With notation as above,* $k(c) = -\theta'(0)$.

By Example 3.1, for a circle of radius r with "clockwise" orientation in V, the extrinsic scalar curvature is the constant $1/r > 0$ at all points. Since a circle with radius r has curvature $1/r$ (up to sign, depending on the orientation), the circles with high curvature (in absolute value) are those with very small radius. Anyone who has driven a car around a tight bend in the road with small turning radius will recognize that this merits being considered highly curved (in contrast with the case of large radius: as the radius tends to infinity, the *local* geometry becomes nearly linear for motion *at unit speed*).

Proof. By definition of the extrinsic scalar curvature, we want $-\mathrm{d}G(c)$ to be multiplication by $-\theta'(0)$, or in other words that $\mathrm{d}G(c)$ is multiplication by $\theta'(0)$ as a self-map on $\mathrm{T}_c(C)$. The line $\mathrm{T}_c(C) = \mathrm{T}_{\sigma(0)}(C)$ is spanned by the unit vector $\sigma'(0)$, so it is equivalent to prove

$$\mathrm{d}G(\sigma(0))(\sigma'(0)) = \theta'(\sigma(0)) \cdot \sigma'(0).$$

More generally, we claim that for all $t \in (-\varepsilon, \varepsilon)$,

$$\mathrm{d}G(\sigma(t))(\sigma'(t)) = \theta'(\sigma(t)) \cdot \sigma'(t).$$

Since $\sigma'(t) = \mathrm{d}\sigma(t)(\partial|_t)$ for the standard vector field ∂ on $(-\varepsilon, \varepsilon)$, by the Chain Rule

$$\mathrm{d}G(\sigma(t))(\sigma'(t)) = \mathrm{d}(G \circ \sigma)(t)(\partial|_t)$$

with $G \circ \sigma : (-\varepsilon, \varepsilon) \to S \subseteq V$ the Gauss map in terms of the arc-length parameter.

We have $\sigma(t) = x(t)\mathbf{e} + y(t)\mathbf{e}'$ for some smooth functions x and y on $(-\varepsilon, \varepsilon)$. The isomorphism $j_{\sigma(t)} : \mathrm{T}_{\sigma(t)}(V) \simeq V$ carries the unit velocity vector $\sigma'(t)$ to the unit vector $x'(t)\mathbf{e} + y'(t)\mathbf{e}'$ in V (why?). By the definition of the Gauss map, $G(\sigma(t)) \in V$ is the unique unit vector such that the ordered basis

$$\{G(\sigma(t)), x'(t)\mathbf{e} + y'(t)\mathbf{e}'\}$$

is a positive orthonormal basis of V. The ordered basis

$$\{y'(t)\mathbf{e} - x'(t)\mathbf{e}', x'(t)\mathbf{e} + y'(t)\mathbf{e}'\}$$

is orthonormal and positive (since its change of basis matrix with respect to the positive ordered basis $\{\mathbf{e}, \mathbf{e}'\}$ is $y'^2 + x'^2 = 1 > 0$). Hence,

$$G(\sigma(t)) = y'(t)\mathbf{e} - x'(t)\mathbf{e}',$$

and since $\{G(\sigma(t)), j_{\sigma(t)}(\sigma'(t))\}$ is a positive orthonormal basis of V it follows that the angles these make against a fixed unit vector $\mathbf{e}$ differ by a constant amount (of the form $\pm\pi/2 + 2\pi\mathbf{Z}$).

Since we are interested in θ' and not θ, changing θ by a constant amount is harmless. Hence, we may work with the angle $\Theta(t)$ of $G(\sigma(t))$ against $\mathbf{e}$ rather than with the angle $\theta(t)$ of $j_{\sigma(t)}(\sigma'(t))$ against $\mathbf{e}$. For t near 0 we have

$$\cos\Theta(t)\mathbf{e} + \sin\Theta(t)\mathbf{e}' = G(\sigma(t)) = y'(t)\mathbf{e} - x'(t)\mathbf{e},$$

so $y' = \cos\Theta$ and $x' = -\sin\Theta$. Since $\mathrm{d}(G\circ\sigma)(t)(\partial|_t) \in \mathrm{T}_{G(\sigma(t))}(V)$ is the "vector-valued" derivative $(G\circ\sigma)'(t) \in V$ of the mapping $G\circ\sigma : t \mapsto \cos\Theta(t)\mathbf{e}+\sin\Theta(t)\mathbf{e}'$ in the sense of classical multivariable calculus (componentwise differentiation), it equals

$$\Theta'(t)\cdot(-\sin\Theta(t)\mathbf{e} + \cos\Theta(t)\mathbf{e}') = \Theta'(t)\cdot(x'(t)\mathbf{e} + y'(t)\mathbf{e}') = \Theta'(t)\cdot j_{\sigma(t)}(\sigma'(t)).$$

■

Remark 3.3. It follows from the proof that $k(c) \neq 0$ if and only if the acceleration of the unit-speed parameterization is nonzero at c.

As a consequence of the theorem, $k(c) > 0$ precisely when traversing C at unit speed in the "positive" direction (as determined by the orientation) gives rise to a unit tangent field (of velocity vectors) whose angle against a fixed direction is decreasing at c. To say the angle is decreasing is exactly to say that the local trajectory through C near c is "clockwise" in the oriented plane V. More precisely, we get the following geometric interpretation of the extrinsic scalar curvature for oriented curves in an oriented plane at points c of the curve at which k is nonzero:

$$k(c) = \lim_{s\to 0^+}\frac{-\ell(s)}{s} \tag{3.1}$$

where $\ell(s)$ is the signed length of the (nontrivial!) path $\sigma'([0,s])$ in the unit circle S, with σ the oriented parameterization if C near c by arc length (and $\sigma(0) = c$); the sign of the length is that of the small (nonzero!) angle of $\sigma'(s)$ against the unit vector $\sigma'(0)$ for small positive s. (The definition of angle uses the orientation of V.) The formula is an immediate consequence of Theorem 3.2 and the fact that *signed* arc length in the unit circle is the same as angle difference (in "radians"!). for small arcs on the unit circle S. A moment's reflection shows that (3.1) "justifies" the name "curvature" in the case of oriented curves in the plane (with the sign interpreted in the manner just explained).

4. Surfaces in $\mathbf{R}^3$

We now turn to the case of smooth embedded surfaces H in a 3-dimensional inner product space V. We seek a geometric interpretation of the extrinsic scalar curvature in the spirit of (3.1), but with the arc-length replaced by area. Also, we seek an interpretation of the sign of this scalar curvature at points where it is not zero. The sign has a simple interpretation, as follows. Give the unit sphere $S \subseteq V$ the "outward normal" orientation, which is to say that for $x \in S$ an ordered basis $\{v, w\}$ of $\mathrm{T}_x(S)$ is positive when the ordered basis $\{\mathbf{n}(x), v, w\}$ for $\mathrm{T}_x(V) \simeq V$ is positive, where the unit normal $\mathbf{n}(x) \in \mathrm{T}_x(V) \simeq V$ is the point on the unit sphere given by x. (Draw a picture!) Since $\dim H = 2$ is even, so $k(h) = \det(-\mathrm{d}G(h)) = \det(\mathrm{d}G(h))$ for all $h \in H$, we see that $k(h)$ is the determinant of the tangent map $\mathrm{d}G(h) : \mathrm{T}_h(H) \to \mathrm{T}_{G(h)}(S)$ when the source and target are naturally identified with the same plane in V (via Lemma 2.3). In particular, by the inverse function theorem, k is nonzero at precisely those points where $G : H \to S$ is a local C^∞ isomorphism. Near such points, we get an isomorphism of oriented vector bundles $TH \simeq G^*(TS)$, and so passing to second exterior powers we see (upon choosing orientation forms over small opens) that if $\mathrm{d}G(h) : \mathrm{T}_h(H) \to \mathrm{T}_{G(h)}(S)$ is an orientation-preserving (resp. orientation-reversing) isomorphism then the same holds upon slightly moving h. Moreover, this condition is exactly the property that G pulls a local oriented coordinate chart on S back to a local oriented coordinate chart on H (resp. a local coordinate chart on H whose orientation is opposite the one given on H). The sign of k detects how G interacts with the chosen orientation on H and the "outward normal" orientation on S:

Theorem 4.1. *The extrinsic scalar curvature k on the surface H is positive at exactly those points near which G is an orientation-preserving local C^∞ isomorphism, using the outward-normal orientation on the unit sphere $S \subseteq V$.*

If we change the orientation on H then G is negated, and negation on S is orientation-reversing. Hence, the assertion in the theorem is independent of the choice of orientation on H. This is as it must be, since we know *a priori* that for even-dimensional H the extrinsic scalar curvature is independent of the choice of orientation on H (though up to sign the Gauss map certainly depends on this choice).

Proof. The problem is a local one, and we at least know that k is nonzero precisely where G is a local C^∞ isomorphism. The problem is therefore one of comparing signs *at* a point $h_0 \in H$ where k is nonzero (and so $\mathrm{d}G(h_0) : \mathrm{T}_{h_0}(H) \to \mathrm{T}_{G(h_0)}(S)$ is a linear isomorphism). One can proceed by busting out lots of coordinates and Jacobian matrices, but an alternative to avoid such muck is to argue as follows.

The orientation-preserving property of G near h_0 is exactly the condition that the linear isomorphism $\mathrm{d}G(h_0)$ respects the given orientation on $\mathrm{T}_{h_0}(H)$ and the "outward normal" orientation on $\mathrm{T}_{G(h_0)}(S)$. Let $W \subseteq V$ be the plane that "is" the common subspace as in Lemma 2.3 for h_0. By *definition* of $G(h_0) \in S$, an ordered basis $\{w, w'\}$ of W is positive for the orientation from $\mathrm{T}_{h_0}(H)$ precisely when the ordered basis $\{G(h_0), w, w'\}$ of V is positive. By the *definition* of the outward-normal orientation on S, when the oriented $\mathrm{T}_{G(h_0)}(S)$ is identified with W its positive bases are characterized by the exact same condition (since for $x \in S$ the "outward normal" in $\mathrm{T}_x(S)^\perp \subseteq \mathrm{T}_x(V)$ goes over to the unit vector in $S \subseteq V$ that *is* x when $\mathrm{T}_x(V)$ is identified with V in the usual manner). Hence, when the *oriented* tangent spaces $\mathrm{T}_{h_0}(H)$ and $\mathrm{T}_{G(h_0)}(S)$ (using the given orientation on H and the "outward normal" orientation on $S \subseteq V$) are identified with the common plane $W \subseteq V$, both identifications put *the same* orientation on W. It follows that when $\mathrm{d}G(h_0) : \mathrm{T}_{h_0}(H) \simeq \mathrm{T}_{G(h_0)}(S)$ is identified with a self-map of W (or equivalently, of $\mathrm{T}_{h_0}(H)$, as in the *definition* of the extrinsic scalar curvature) its determinant is positive if and only if it respects this common orientation. That is, $k(h_0) > 0$ if and only if $\mathrm{d}G(h_0)$ is orientation-preserving as a self-map of $\mathrm{T}_{h_0}(H)$. ■

The unit sphere $S \subseteq V$ inherits a Riemannian metric from V, as does the embedded surface H. For points $h \in H$ at which k is non-vanishing, the inverse function theorem implies that the Gauss map $G : H \to S$ is a local C^∞ isomorphism. Hence, for small rectifiable open subsets B in H around h the image $G(B)$ is a small rectifiable open subset of S around $G(h)$ (say with smallness measured by discs in the Riemannian metric, or just naively in a coordinate patch on the surfaces H and S), and so it makes sense to consider the positive areas $\mathrm{Area}(B)$ and $\mathrm{Area}(G(B))$. When we consider a limit as "$B \to h$" what we mean is that the expression gets as close as we please to the proposed limiting value by taking B inside of a sufficiently small open around h in H.

Theorem 4.2. *If* $\dim V = 3$ *and* $k(h_0) \neq 0$ *for some* $h_0 \in H$*, then*

$$|k(h_0)| = \lim_{B \to h_0} \frac{\mathrm{Area}(G(B))}{\mathrm{Area}(B)}.$$

In particular, this limit exists for such h_0*.*

If we assign $G(B)$ a "signed area" by inserting a sign when G is orientation-reversing with respect to the chosen orientation on H and the outward-normal orientation on S then Theorem 4.1 ensures that the above limit formula can be promoted to a formula for $k(h_0)$ and not just $|k(h_0)|$ (when $k(h_0) \neq 0$). This is analogous to the "signed length" condition that gives (3.1) without the intervention of absolute values.

Proof. Let $\{u, v\}$ be an *oriented* C^∞ coordinate system on a connected open set U around h_0 in H. In order to compute the area of a small region B contained in U, we use the area form associated to the Riemannian metric on H induced by the standard Riemannian metric on $\mathbf{R}^3$. For each $h \in H$, the area form in $\Omega^2_H(H)$ has h-fiber in $\wedge^2(T_h(H)^\vee)$ given by $\sqrt{\det(\langle v_i, v_j \rangle_h)}v_1^* \wedge v_2^*$ where $\{v_1, v_2\}$ is any positive basis of the oriented plane $\mathrm{T}_h(H)$ and $\langle \cdot, \cdot \rangle_h$ is the inner product on $\mathrm{T}_h(H)$ induced by the standard one on $\mathrm{T}_h(\mathbf{R}^3) = \mathbf{R}^3$. A frame of positive bases for $TH|_U$ is given by the ordered pair of vector fields $\{\partial_u, \partial_v\}$, so the area form over U is

$$\sqrt{\begin{vmatrix} \langle \partial_u, \partial_u \rangle & \langle \partial_u, \partial_v \rangle \\ \langle \partial_u, \partial_v \rangle & \langle \partial_v, \partial_v \rangle \end{vmatrix}} \mathrm{d}u \wedge \mathrm{d}v.$$

We let $A(u, v)$ denote this coefficient, so the area form on U is $A\mathrm{d}u \wedge \mathrm{d}v$ in $\{u, v\}$ coordinates.

Recall that if W is a d-dimensional oriented inner product space, then each exterior power $\wedge^i(W)$ is endowed with an induced inner product and on the line $\wedge^d(W)$ the length of $w_1 \wedge \cdots \wedge w_d$ is $\sqrt{\det(\langle w_i, w_j \rangle)}$ for any $w_1, \ldots, w_d \in W$. This is related to the old fact from linear algebra that the "volume form" on an oriented inner product space is the unique positive unit vector in the top exterior power (with its induced orientation and inner product). The relevance of this to the present circumstances is that when we view $\mathrm{T}_h(H)$ and $\mathrm{T}_{G(h)}(S)$ as subspaces of $\mathbf{R}^3$ they coincide (Lemma 2.3), and hence the induced inner products on these spaces *coincide*. Use this identification of planes to view $\wedge^2(\mathrm{d}G(h))$ as a self-map of the 1-dimensional oriented inner product space $\wedge^2(\mathrm{T}_h(H))$, so this map is multiplication by the scalar $k(h)$ (by *definition* of the extrinsic scalar curvature as a determinant), so it follows that the map $\wedge^2(\mathrm{d}G(h)) : \wedge^2(\mathrm{T}_h(H)) \to \wedge^2(\mathrm{T}_{G(h)}(S))$ distorts the length by a factor of $|k(h)|$: it sends unit vectors to vectors of length $|k(h)|$.

Shrink U around h_0 so that G is a C^∞ isomorphism of U onto an open $G(U)$ around $G(h_0)$ in S. Thus, we may use $u' = u \circ G^{-1}$ and $v' = v \circ G^{-1}$ as C^∞ coordinates on $G(U)$. Since $G(U)$ is connected, either $\{u', v'\}$ or $\{v', u'\}$ is an oriented coordinate system on S (with its outward normal orientation). Change the orientation on H if necessary (harmless for our purposes) – which is to say swap u and v if necessary – so that $\{u', v'\}$ is an oriented coordinate system on $G(U)$. The preceding discussion implies that

$$\langle \partial_{u'}, \partial_{u'} \rangle = |k \circ G^{-1}| \cdot \langle \partial_u, \partial_u \rangle \circ G^{-1}, \;\; \langle \partial_{u'}, \partial_{v'} \rangle = |k \circ G^{-1}| \cdot \langle \partial_u, \partial_v \rangle \circ G^{-1},$$
$$\langle \partial_{v'}, \partial_{v'} \rangle = |k \circ G^{-1}| \cdot \langle \partial_v, \partial_v \rangle \circ G^{-1},$$

so the area form on $G(U) \subseteq S$ is $|k \circ G^{-1}| A \circ G^{-1} \cdot \mathrm{d}u' \wedge \mathrm{d}v'$. Let $Q \subseteq \mathbf{R}^2$ be the open subset that is the common image of the coordinate systems $\{u, v\}$ on U and $\{u', v'\}$ on $G(U)$. Let $\phi : U \simeq Q$ be the C^∞ isomorphism defined by $\{u, v\}$, and let $f = A \circ \phi^{-1}$ and $K = |k| \circ \phi^{-1}$. Note that K

and f are smooth functions on Q (as k is non-vanishing on U) and f is everywhere positive on Q. By *definition* of integration of differential forms we conclude that for any rectifiable open subset $B \subseteq U$, $\mathrm{Area}(B) = \int_{\phi^{-1}(B)} f$ and $\mathrm{Area}(G(B)) = \int_{\phi^{-1}(B)} Kf$.

Letting $q_0 = \phi(h_0) \in Q$, as we shrink B in U around h_0 the open set $\phi^{-1}(B)$ shrinks in Q around q_0. Hence, our problem is to prove

$$\lim_{Q' \to q_0} \frac{\int_{Q'} Kf}{\int_{Q'} f} = K(q_0)$$

as Q' ranges over rectifiable opens in $Q \subseteq \mathbf{R}^2$ that shrink down to q_0. (That is, for any $\varepsilon > 0$ there exists an open $Q'_0 \subseteq Q$ around q_0 such that if $Q' \subseteq Q'_0$ then $|(\int_{Q'} Kf / \int_{Q'} f) - K(q_0)| < \varepsilon$.) Since $K(q_0) = (Kf)(q_0)/f(q_0)$, it therefore suffices to prove more generally that for any smooth function g near q_0, $\int_{Q'} g \sim g(q_0) \int_{Q'} 1 = g(q_0)\mathrm{Area}(Q')$ as $Q' \to q_0$. Writing $g = g(q_0) + (g - g(q_0))$ we may assume $g(q_0) = 0$, so we want $\int_{Q'} g = o(\mathrm{Area}(Q'))$ as $Q' \to q_0$ if $g(q_0) = 0$.

We may translate coordinates in $\mathbf{R}^2$ (doesn't affect integrals!) so that $q_0 = (0,0)$. By the 2-variable Taylor formula for the smooth g at $q_0 = (0,0)$, since $g(0,0) = 0$ we have $g = uf_1 + vf_2$ for smooth f_1 and f_2 near the origin. Hence, if $|f_1|$ and $|f_2|$ are bounded by respective constants C_1 and C_2 near the origin then for small Q' around the origin we have $|\int_{Q'} g| \leq C_1 \int_{Q'} |u| + C_2 \int_{Q'} |v|$. It suffices to show $\int_{[-\varepsilon,\varepsilon]^2} |u| = o(\varepsilon^2)$ as $\varepsilon \to 0^+$, and this is trivial (the integral is ε^3). ∎

Example 4.3. Consider the case $V = \mathbf{R}^3$ and $H = \{f(x,y,z) = c\}$ a level set without critical points for a smooth function f on $\mathbf{R}^3$. In this case, for each $h \in H$ we have a non-vanishing gradient vector $(\nabla f)(h) \in \mathrm{T}_h(\mathbf{R}^3) \simeq \mathbf{R}^3$ defined by the condition that $\langle (\nabla f)(h), \cdot \rangle = \mathrm{d}f(h)$. By the handout on universal bundles and normal bundles, ∇f trivializes the orthogonal line bundle $(TH)^\perp$ and so defines an orientation on H. With this orientation the Gauss map $G : H \to S^2 \subseteq \mathbf{R}^3$ is

$$G(h) = \frac{(\nabla f)(h)}{\|(\nabla f)(h)\|} = \frac{((\partial_x f)(h), (\partial_y f)(h), (\partial_z f)(h))}{\sqrt{(\partial_x f)^2(h) + (\partial_y f)^2(h) + (\partial_z f)^2(h)}}.$$

The "explicit calculation" of $\mathrm{d}G(h)$ requires a choice of oriented coordinates on H near h. By the implicit function theorem, any smooth embedded hypersurface in $\mathbf{R}^3$ locally has a smooth parameterization by two of the three standard coordinates. (That is, near any point one of $\{x,y\}$, $\{x,z\}$, or $\{y,z\}$ is a local C^∞ coordinate chart, depending on which partials of f are non-vanishing on H near the point of interest, and we may need to swap the order of coordinates to make the coordinate system be oriented with respect to the orientation that we have chosen on H by means of the choice of f.) Using such coordinates permits a clean local formula for the scalar curvature, as we work out in the next example.

Example 4.4. Consider surfaces of the form $H = \{z = g(x,y)\} \subseteq U \times \mathbf{R}$ for a smooth function g on an open set $U \subseteq \mathbf{R}^2$. This is the zero locus of $f(x,y,z) = z - g(x,y)$. As we saw in our initial study of submanifolds, on the submanifold H there is a global C^∞ coordinate system provided by $\{x,y\}$. A normal vector field that corresponds to the orientation determined by the $\{x,y\}$-coordinate system is given by the vector cross-product $\partial_x \times \partial_y$ in $i^*(T(\mathbf{R}^3))$, with $i : H \hookrightarrow \mathbf{R}^3$ the embedding. Strictly speaking, we really compute the cross product

$$\mathrm{d}i(\xi)(\partial_x|_\xi) \times \mathrm{d}i(\xi)(\partial_y|_\xi) \in \mathrm{T}_{i(\xi)}(\mathbf{R}^3) = \mathbf{R}^3$$

with $i(x,y) = (x,y,g(x,y))$. The matrix for $\mathrm{d}i(\xi)$ gives

$$\mathrm{d}i(x_0,y_0)(\partial_x|_{(x_0,y_0)}) = (1,0,(\partial_x g)(x_0,y_0)), \;\; \mathrm{d}i(x_0,y_0)(\partial_y|_{(x_0,y_0)}) = (0,1,(\partial_y g)(x_0,y_0)) \tag{4.1}$$

in $\mathbf{R}^3 = \mathrm{T}_{i(x_0,y_0)}(\mathbf{R}^3)$ The inner products among these two vectors are easy to compute via the standard inner product on $\mathbf{R}^3$, and so we see that the metric tensor on H is given as follows in terms of $\{x, y\}$-coordinates:

$$\mathrm{d}s^2 = (1 + (\partial_x g)^2)\mathrm{d}x^{\otimes 2} + \partial_x g \partial_y g \cdot \mathrm{d}x \otimes \mathrm{d}y + \partial_x g \partial_y g \cdot \mathrm{d}y \otimes \mathrm{d}x + (1 + (\partial_y g)^2)\mathrm{d}y^{\otimes 2}.$$

From the explicit description (4.1) of the vector fields ∂_x and ∂_y on H in terms of the standard trivialization of $T(\mathbf{R}^3)$, their cross product is the vector field $(-\partial_x g, -\partial_y g, 1)$ in $i^*(T(\mathbf{R}^3))$. That is, the orientation of H defined by the $\{x, y\}$-coordinate system is associated to the unit normal field

$$\mathbf{n} = \frac{(-\partial_x g, -\partial_y g, 1)}{\sqrt{1 + (\partial_x g)^2 + (\partial_y g)^2}}$$

with values in $\mathbf{R}^3$. For $f = z - g(x, y)$, this is exactly the unit vector field along H obtained by dividing the associated gradient vector field by its length, so the $\{x, y\}$ coordinate system is oriented with respect to the orientation determined by using f as the smooth function with level-set H. (If we had used $-f$ then the orientation on H would flip and $\{y, x\}$ would be an oriented coordinate system.)

We conclude that the Gauss map associated to the oriented surface $H = \{z = g(x, y)\}$ with $\{x, y\}$-orientation is

$$G(x, y) = \frac{(-\partial_x g, -\partial_y g, 1)}{\sqrt{1 + (\partial_x g)^2 + (\partial_y g)^2}}. \tag{4.2}$$

By Lemma 2.3, the tangent plane $\mathrm{T}_{G(h)}(S^2)$ to the unit sphere and the tangent plane $\mathrm{T}_h(H)$ coincide when they are put inside of $\mathbf{R}^3$ (via $\mathrm{T}_h(\mathbf{R}^3) \simeq \mathbf{R}^3$ and $\mathrm{T}_{G(h)}(\mathbf{R}^3) \simeq \mathbf{R}^3$). When this is used to identify $\mathrm{d}G(h)$ with a self-map of the vector space $\mathrm{T}_h(H)$ with ordered basis $\{\partial_x|_h, \partial_y|_h\}$ then by computing partials of (4.2) and using (4.1) one finds

$$\mathrm{d}G(x, y) = \frac{1}{(1 + g_x^2 + g_y^2)^{3/2}} \cdot \begin{pmatrix} g_{xy}g_x g_y - g_{xx}(1 + g_y^2) & g_{yy}g_x g_y - g_{xy}(1 + g_y^2) \\ g_{xx}g_x g_y - g_{xy}(1 + g_x^2) & g_{xy}g_x g_y - g_{yy}(1 + g_x^2) \end{pmatrix}$$

(with $g_x = \partial_x g$, $g_{xy} = \partial_x \partial_y g$, and so on). Hence, the determinant is

$$k = \frac{g_{xx}g_{yy} - g_{xy}^2}{(1 + g_x^2 + g_y^2)^{3/2}}; \tag{4.3}$$

this is the extrinsic scalar curvature on $\{z = g(x, y)\}$ in terms of $\{x, y\}$-coordinates. Recall that the function k is independent of the choice of orientation on the surface, and this accords with the fact that (4.3) is unaffected by the roles of x and y (thanks to the equality $g_{xy} = g_{yx}$).

Example 4.5. Let us use (4.3) to compute the scalar curvatures for a couple of basic surfaces in $\mathbf{R}^3$: spheres and cylinders (say with infinite length). Let us first argue by pure thought that these surfaces must have constant curvature. By Remark 2.9, this scalar curvature is unaffected by translation and orientation-preserving orthogonal linear transformations on $\mathbf{R}^3$ that carry the surface *back to itself*. Hence, to do the calculation we lose no generality in supposing our sphere is centered at the origin and that our cylinder has the x-axis as its axis of symmetry. For any two points ξ_1 and ξ_2 on such a sphere, we can certainly find an orthogonal linear transformation of $\mathbf{R}^3$ carrying ξ_1 to ξ_2 (why?), so indeed the scalar curvatures at the two points must be the same (so it is a constant function on the sphere). Likewise, by using suitable additive translation we can bring any point of the cylinder into the yz-plane, and then we can use a rotation in that plane to bring it to any point of the unit circle in that plane. In this way we see that suitable translations and orientation-preserving orthgonal linear transformations on $\mathbf{R}^3$ that preserve the cylinder can be used to carry any point on the cylinder to any other point on the cylinder. Thus, again the scalar curvature must be constant.

Let $H \subseteq \mathbf{R}^3$ be a sphere of radius $R > 0$ centered at the origin. In this case I claim $k = 1/R^2$. By constancy it suffices to work with the oriented chart $\{z = \sqrt{R^2 - x^2 - y^2}\}$ (for $x^2 + y^2 < R^2$). By (4.3) $g(x, y) = \pm\sqrt{R^2 - x^2 - y^2}$, a direct calculation gives $k(x, y) = 1/R^2$. We conclude that on the sphere $k = 1/R^2 > 0$. In particular, the spheres with small radius have "large curvature" (as they should!). This is analogous to the earlier conclusion (Example 3.1 and Theorem 3.2) that the extrinsic scalar curvature of a circle of radius R with *clockwise* orientation in $\mathbf{R}^2$ is $1/R$.

Next, consider a cylinder in $\mathbf{R}^3$ with radius R and infinite length, having the x-axis as the axis of symmetry. I claim $k = 0$. (The vanishing of the "curvature" in the cylinder is due to that fact that a piece of paper may be smoothly rolled up to make a cylinder without distorting local distance on the surface – intuitively, we do not need to stretch or rip the paper.) It suffices to work in the chart $z = \sqrt{R^2 - y^2}$ (with $|y| < R$), and by direct calculation we get $k = 0$. The example of the cylinder drives home the fact that k measures "stretching" and not "bending".

Example 4.6. We now study a surface in $\mathbf{R}^3$ whose extrinsic scalar curvature is non-constant, and the variation reflects geometric properties of the surface. The surface we wish to consider is the donought (or torus) $T \subseteq \mathbf{R}^3$ with inner radius $r - |a|$ and outer radius $r + |a|$, $0 < |a| < r$. Explicitly, T is the image of the smooth closed embedding $S^1 \times S^1 \to \mathbf{R}^3$ defined by

$$(\theta, \psi) \mapsto ((a + r\cos\theta)\cos\psi, (a + r\cos\theta)\sin\psi, r\sin\theta).$$

Geometrically, θ is the angle measure for the circles of radius r that go "through the hole" and ψ is the angle measure for the other family of circles that have radii varying from $r - |a|$ to $r + |a|$.

There is a global trivialization of the tangent bundle (hence orientation of T) specified by the ordered pair of vector fields $\{\partial_\theta, \partial_\psi\}$ (that are globally well-defined, even though θ and ψ are not), and under the embedding i of $S^1 \times S^1$ into $\mathbf{R}^3$ and hence of $T(S^1 \times S^1)$ into $i^*(T(\mathbf{R}^3)) = (S^1 \times S^1) \times \mathbf{R}^3$ these go over to the vector fields

$$\partial_\theta = (-r\sin\theta\cos\psi, -r\sin\theta\sin\psi, r\cos\theta), \quad \partial_\psi = (-(a + r\cos\theta)\sin\psi, (a + r\cos\theta)\cos\psi, 0).$$

Forming their cross product and dividing by the length (via the Riemannian metric induced from $\mathbf{R}^3$) gives a formula for the Gauss map:

$$G(\theta, \psi) = (\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta) \in S^2 \subseteq \mathbf{R}^3.$$

By direct computation,

$$\partial_\theta G = (-\sin\theta\cos\psi, -\sin\theta, \sin\psi, \cos\theta) = r^{-1}\partial_\theta,$$

$$\partial_\psi G = (-\cos\theta\sin\psi, \cos\theta\cos\psi, 0) = \frac{\cos\theta}{a + r\cos\theta}\partial_\psi,$$

so we compute the extrinsic scalar curvature is the determinant

$$k(\theta, \psi) = \begin{vmatrix} 1/r & 0 \\ 0 & \frac{\cos\theta}{a + r\cos\theta} \end{vmatrix} = \frac{\cos\theta}{r(a + r\cos\theta)}. \tag{4.4}$$

Note that this is independent of ψ, as it should be because by rotating about the axis of symmetry we may move ψ arbitrarily without changing θ. (Recall that the scalar curvature is invariant under additive translation and orthogonal orientation-preserving linear transformation on the ambient oriented inner product space.) In contrast, the circles $\psi = c$ that pass "through the hole" encounter varying geometry on the surface as θ varies. On the "outer half" with $-\pi/2 < \theta < \pi/2$ we have $k > 0$ by (4.4), and at such points the surface lies on one side of the tangent plane (cutting the surface at an isolated point). On the "inner half" with $\pi/2 < \theta < 3\pi/2$ we have $k < 0$ by (4.4), and at such points the surface cuts through its tangent plane at an isolated point but the surface straddles both sides of the tangent plane. The case $k = 0$ happens exactly along the top and bottom circles $\theta = \pi/2, 3\pi/2$ at which the surface is "flat" (and the tangent plane meets the surface along a curve rather than at an isolated point).

5. Curves in $\mathbf{R}^2$ on surfaces in $\mathbf{R}^3$

We conclude by proving a wonderful theorem of Euler that relates the curvature of a surface (in 3-space) to the geometry of curves on the surface. This was apparently a big source of motivation for Gauss when he undertook his monumental study of the geometry of surfaces in 3-space, and the technique of studying a surface by understanding the geometry of curves on the surface is a powerful principle in algebraic geometry. Euler's theorem gives a very satisfying intepretation for the extrinsic scalar curvature of a surface in $\mathbf{R}^3$ in terms of the curvature of curves given by oriented planar slices of the surface. That is, it links the work in the preceding two sections.

Let $i : H \subseteq \mathbf{R}^3$ be a smooth embedded oriented surface, and let $\mathbf{n}$ be the associated unit normal field in $(TH)^\perp \subseteq i^*(T(\mathbf{R}^3))$. We fix a point $x \in H$ and we wish to study the geometry of $H \cap P$ near x for an affine plane P in $\mathbf{R}^3$ passing through x and containing the (affine) normal line to H at x (by "affine plane" we mean a translate of a 2-dimensional linear subspace; similarly for "affine line"). In particular, the smooth embedded surfaces P and H in $\mathbf{R}^3$ satisfy $\mathrm{T}_x(P) \neq \mathrm{T}_x(H)$ in $\mathrm{T}_x(\mathbf{R}^3) = \mathbf{R}^3$. A pair of distinct hyperplanes in a vector space is always transverse, so P and H have *transverse* intersection at x. It follows from our earlier work with transverse intersection of submanifolds that $P \cap H$ is an embedded smooth curve (in H, P, and $\mathbf{R}^3$) near x. Let C be a connected open around x in this smooth curve. We fix an orientation for P, so together with the orientation on H we get (via our earlier work with orientations and normal bundles) a preferred orientation on the transverse intersection C.

Concretely, there are exactly two smooth unit tangent fields $\mathbf{t}$ along the connected curve C, corresponding to the two orientations of C, and each is determined by its value at one point of C. Working at the point $x \in C$, the orientation on C is associated to the unique $\mathbf{t}$ such that $\{\mathbf{n}(x), \mathbf{t}(x)\}$ is a positive basis of the oriented plane P. Conversely, if we do not orient P but we fix a choice of orientation of C dictated by a choice of unit tangent field $\mathbf{t}$ on C then we orient P by requiring the ordered basis $\{\mathbf{n}(x), \mathbf{t}(x)\}$ of P to be positive.

Since C is an embedded smooth curve in the 2-dimensional inner product space P, and a choice of unit tangent field $\mathbf{t}$ picks out preferred orientations on C and P as explained above, for a given choice of $\mathbf{t}$ we can speak of the extrinsic scalar curvature $k_{\mathbf{t}} : C \to \mathbf{R}$. Now we use geometric input: since $P \subseteq \mathrm{T}_x(\mathbf{R}^3)$ is the span of the independent (in fact, orthogonal!) unit vectors $\mathbf{n}(x)$ and $\mathbf{t}(x)$, we can pass back and forth between C and P: as we let $\mathbf{t}_0 \in \mathrm{T}_x(H)$ vary over *all* unit vectors the oriented planes $P = \mathrm{span}(\mathbf{t}_0, \mathbf{n}(x))$ in $\mathrm{T}_x(\mathbf{R}^3)$ range over all *oriented* affine planes in $\mathbf{R}^3$ that contain the (affine) normal line to H at x. Thus, the oriented curve C near x is determined by the choice of unit tangent vector $\mathbf{t}_0 \in \mathrm{T}_x(H)$, and so we may write $C_{\mathbf{t}_0}$ and $P_{\mathbf{t}_0}$ for the resulting oriented curve and oriented plane through x. Note that $P_{-\mathbf{t}_0} = P_{\mathbf{t}_0}$ as affine planes in $\mathbf{R}^3$, but they have opposite orientations. The same goes for the curves $C_{-\mathbf{t}_0}$ and $C_{\mathbf{t}_0}$ near x.

Consider the extrinsic scalar curvature $k_{\mathbf{t}_0}(x)$ of $C_{\mathbf{t}_0}$ at the point x. As we let $\mathbf{t}_0$ vary through the unit circle in $\mathrm{T}_x(H)$, how does $k_{\mathbf{t}_0}(x)$ vary? In more geometric terms, as we let P vary through the set of *oriented* affine planes in $\mathbf{R}^3$ that contain the normal line to H at x, what can we say about the variation in the curvature at x for the resulting oriented curve $P \cap H$ (near x) in the oriented 2-dimensional inner product space P? It was this problem that Euler solved, and what he proved (in modern language) is that there is a quadratic form q on the plane $\mathrm{T}_x(H)$ such that

$q(\mathbf{t}_0) = k_{\mathbf{t}_0}(x)$ for all unit vectors $\mathbf{t}_0 \in \mathrm{T}_x(H)$. This quadratic form is a special case of a general construction ("second fundamental form") that appears in the higher-dimensional case.

Remark 5.1. Since $q(w) = q(-w)$ for any quadratic form q on a vector space W, if Euler's result is to be correct then it must be that $k_{-\mathbf{t}_0}(x) = k_{\mathbf{t}_0}(x)$. We can see such equality *a priori* as follows. By negating $\mathbf{t}_0$ we do not change the physical plane $P_{\mathbf{t}_0}$ nor the curve $C_{\mathbf{t}_0}$ in this plane near x, but the orientations on *both* are negated. But in general we know (Remark 2.7) that negating both the orientation on the ambient inner product space and on the hypersurface does not change the Gauss map and so does not change the associated scalar curvature function. Thus, indeed $k_{-\mathbf{t}_0}(x) = k_{\mathbf{t}_0}(x)$.

The basic problem we need to solve is to relate the curvature $k_{\mathbf{t}_0}(x)$ attached to the oriented curve $C_{\mathbf{t}_0}$ near x in the oriented plane $P_{\mathbf{t}_0}$ and the self-map $\mathrm{d}G(x) : \mathrm{T}_x(H) \to \mathrm{T}_x(H)$ (upon identifying $\mathrm{T}_x(H)$ and $\mathrm{T}_{G(x)}(S^2)$ in $\mathbf{R}^3$, via Lemma 2.3). The starting point is the fact that when the derivative of the Gauss map at x is viewed as a self-map of the inner product space $\mathrm{T}_x(H)$, it has a very nice interaction with the inner product structure:

Lemma 5.2. *The self-map* $\mathrm{d}G(x)$ *of* $\mathrm{T}_x(H)$ *is self-adjoint for the natural inner product on* $\mathrm{T}_x(H) \subseteq \mathrm{T}_x(\mathbf{R}^3) = \mathbf{R}^3$.

In the higher-dimensional case there is also such a self-adjointness result. Here, we give a proof in the case of surfaces by a local coordinate calculation. (In the general case, one can avoid passing to local coordinates in the proof.)

Proof. After suitable relabelling of the standard coordinates on $\mathbf{R}^3$ (say a cyclic permutation so as to not affect the orienation) and working locally near x in H, we may assume that H is given by $z = g(u, v)$. In this case, the tangent space $\mathrm{T}_x(H)$ has as basis $\partial_u|_x$ and $\partial_v|_x$ with $\{u, v\}$ a C^∞ coordinate system on H at x, so the symmetry condition

$$\langle \mathrm{d}G(x)(w), w' \rangle_x = \langle w, \mathrm{d}G(x)(w') \rangle_x$$

for $w, w' \in \mathrm{T}_x(H)$ is reduced to the special case $w = \partial_u|_x$ and $w' = \partial_v|_x$.

The vector fields ∂_u and ∂_v on the coordinate chart around x in H are velocity vectors along the parameteric "coordinate lines" (one coordinate fixed, the other varying). If the parameterization near x that is inverse to the $\{u, v\}$-coordinate system is denoted

$$(u, v) \mapsto (f_1(u, v), f_2(u, v), f_3(u, v)) \in H \subseteq \mathbf{R}^3$$

then in the tangent spaces to $\mathbf{R}^3$ we have

$$\partial_u = (\partial_u f_1)\partial_{t_1} + (\partial_u f_2)\partial_{t_2} + (\partial_u f_3)\partial_{t_3}$$

and likewise for ∂_v. By the definition of the Gauss map as a normal field to the surface, we have the pointwise identity $0 = \langle G(u, v), \partial_v \rangle$ at all points near x in H, so by applying ∂_v to this identity we get

$$0 = \langle \mathrm{d}G \circ \partial_v, \partial_u \rangle + \langle G(u, v), \sum_j (\partial^2 f_j / \partial v \partial u) \partial_{t_j} \rangle.$$

If we instead begin with the vanishing of $\langle G(u, v), \partial_u \rangle$ then we get a similar identity, except the partials with respect to u and v occur everywhere in swapped positions. By equality of mixed partials (for the f_j's), we get formulas for $\langle \mathrm{d}G \circ \partial_v, \partial_u \rangle$ and $\langle \mathrm{d}G \circ \partial_u, \partial_v \rangle$ that coincide. Evaluating at x, we are done. ∎

In view of the self-adjointness, $\langle -\mathrm{d}G(x)(w), w'\rangle_x$ is a *symmetric* bilinear form on the plane $\mathrm{T}_x(H)$, and we let q be the associated quadratic form. Thus, $q(w) = \langle -\mathrm{d}G(x)(w), w\rangle_x$ for $w \in \mathrm{T}_x(H)$, and by the spectral theorem the self-ajoint self-map $-\mathrm{d}G(x)$ of $\mathrm{T}_x(H)$ can be diagonalized with some eigenvalues $\lambda_1 \leq \lambda_2$ (with perpendicular eigenlines if $\lambda_1 \neq \lambda_2$). In particular, in suitable orthonormal linear coordinates $\{u_1, u_2\}$ on $\mathrm{T}_x(H)$ we have $q(u_1, u_2) = \lambda_1 u_1^2 + \lambda_2 u_2$, so for *unit vectors* $\mathbf{t}_0 \in \mathrm{T}_x(H)$ we have $\lambda_1 \leq q(\mathbf{t}_0) \leq \lambda_2$, and if $\lambda_1 \neq \lambda_2$ then the extremal values are attained along unique lines in $\mathrm{T}_x(H)$ that are moreover perpendicular. Classically, these lines are the "principal directions" at x on the surface H, and the eigenvalues λ_1 and λ_2 along these tangential lines are called the "principal curvatures" due to:

Theorem 5.3 (Euler)**.** *For each unit vector $\mathbf{t}_0 \in \mathrm{T}_x(H)$, $k_{\mathbf{t}_0}(x) = q(\mathbf{t}_0)$. In particular, as P varies through the oriented affine planes that contain the normal line to H at x, the oriented curve $P \cap H$ near x has extrinsic scalar curvature at x that ranges between the principal curvatures. If the extremal values λ_1 and λ_2 are distinct, each is attained only for the associated curve $C_{\pm\mathbf{t}_0}$ whose tangent line in $\mathrm{T}_x(H)$ is the corresponding principal direction.*

Moreover, the extrinsic scalar curvature $k(x)$ of H at x is the product $\lambda_1\lambda_2$ of the principal curvatures at x.

Proof. The final part follows from the definition of $k(x)$ as the determinant of $-\mathrm{d}G(x)$ when this is viewed as a self-map of $\mathrm{T}_x(H)$. Since $\mathbf{t}_0$ is a *unit* vector in $\mathrm{T}_x(H)$, the inner product $q(\mathbf{t}_0)$ is the $\mathbf{t}_0$-coefficient of the projection of $\mathrm{d}G(x)(\mathbf{t}_0) \in \mathrm{T}_x(H)$ on the line $\mathbf{R}\mathbf{t}_0$. Letting $C_{\mathbf{t}_0}$ denote the corresponding oriented curve through x (i.e., the oriented slice $P_{\mathbf{t}_0} \cap H$ near x with $P_{\mathbf{t}_0}$ the oriented span of $\mathbf{n}(x)$ and $\mathbf{t}_0$), the oriented line $\mathbf{R}\mathbf{t}_0$ in $\mathrm{T}_x(H)$ is the oriented tangent line $\mathrm{T}_x(C_{\mathbf{t}_0})$. Hence, it suffices to prove that near x the restriction $G|_{C_{\mathbf{t}_0}}$ of the Gauss map of H is the Gauss map of $C_{\mathbf{t}_0}$ in the oriented plane $P_{\mathbf{t}_0}$ (with values in the unit circle of $P_{\mathbf{t}_0}$, put inside of the unit circle of $\mathbf{R}^3$ via parallel translation in the evident manner). But this compatibility of Gauss maps follows from how $P_{\mathbf{t}_0}$ and $C_{\mathbf{t}_0} = P_{\mathbf{t}_0} \cap H$ are oriented (near x) using the data of the normal field $\mathbf{n}$ along H near x. More specifically, the crux is that for each point x' in $C_{\mathbf{t}_0}$, the unit normal vector $\mathbf{n}(x')$ to H at x' lies in the oriented inner product space $P_{\mathbf{t}_0}$ as a unit normal vector to the oriented curve $C_{\mathbf{t}_0}$ such that $\{\mathbf{n}(x'), \mathbf{t}(x')\}$ is a positive basis of $\mathrm{T}_{x'}(P_{\mathbf{t}_0}) \simeq P_{\mathbf{t}_0}$ for the positive unit vector $\mathbf{t}(x')$ in $\mathrm{T}_{x'}(C_{\mathbf{t}_0})$ (as we infer by connectivity of $C_{\mathbf{t}_0}$ and the special case $x' = x$. ■

Example 5.4. In the cases of a sphere of radius R (say given outward normal orientation) the oriented planar slices as in Theorem 5.3 are (great) circles of radius R with counterclockwise orientation in an oriented plane, so these all have scalar curvature $-1/R$. Taking the product of two gives $(-1/R)^2 = 1/R^2$.

For a cylinder of radius R (with a fixed choice of orientation), there is a unique planar slice that is a straight line (with curvature 0) and all others having "varying" velocity vector and so have nonzero curvature. Consideration the orientation, it is clear that these curvatures all have the same sign as we vary the plane, so the vanishing curvature is one of the two extremal possibilities (depending on how we orient the sphere). The other eigenline in the tangent space is perpendicular to this one, so the other principal direction must correspond to the circular slice of radius R whose tangential direction is perpendicular to the line. This has curvature $\pm 1/R$, the sign depending on how we orient the cylinder. The product of these two principal curvatures 0 and $\pm 1/R$ is 0.

Finally, consider the torus $T = S^1 \times S^1$ embedded in $\mathbf{R}^3$ as in Example 4.6, with orientation by $\{\theta, \psi\}$. In this case we diagonalized the differential of the Gauss map in the proof of (4.3), so the principal directions at any point are the tangent lines to integral curves for ∂_θ and ∂_ψ and the respective principal curvatures are r^{-1} and $(\cos\theta)/(a + r\cos\theta)$. The product is $(\cos\theta)/r(a + r\cos\theta)$ as it should be.

Euler's theorem explains the geometry of the surface H near x in terms of the sign of $k(x)$ (when it is nonzero): if $k(x) > 0$ then the principal curvatures have the same sign and so all (naturally oriented) planar slices C by oriented planes P containing the normal line at x have scalar curvature (in the oriented plane P) with a common sign. That is, near x all such naturally oriented curves through x are clockwise, or all are counterclockwise, in their natural oriented plane. In contrast, when $k(x) < 0$ there are some (naturally oriented) slices that are positively curved at x and some that are negatively curved. Since the curvature of the slice curve at x encodes the distinction between clockwise and counterclockwise motion through x (in an oriented plane), this explains the geometric dichotomy one sees in curve slices for Example 4.6 at points on the "outer" half (where $k > 0$) and on the "inner" half (where $k < 0$).

Example 5.5. Let us see what Euler's theorem tells us concerning the Beltrami surface B_a as in Example 2.4. In that example, we worked out the explicit eigenvalues and we even saw that the eigenlines in each tangent plane are precisely the perpendicular tangent lines along the polar coordinate directions through each point (i.e., parallel to the x-axis and parallel to the yz-plane). Let us now show how to see by pure geometry that the eigenlines must be these lines. In fact, the argument we give applies to an arbitrary surface of revolution. (Note, however, that the determination of the eigenvalues along these lines does require some calculation; it can be done for any surface of revolution without requiring an explicit computation of the Gauss map, but to do so requires some more geometrical input because the normal lines to the surface of revolution are usually not perpendicular to the axis of symmetry.)

Pick a point ξ on our surface of revolution. If the differential of the Gauss mapping has equal eigenvalues (which it actually does not in the case of B_a) then there is nothing to do because all lines in the tangent plane at ξ are eigenlines, so we may assume the eigenvalues are distinct. By self-adjointness of the differential of the Gauss map, it has exactly two eigenlines on each tangent plane and they are perpendicular in the tangent plane. We have to show that these lines are the ones along the polar coordinate directions with respect to the axis of symmetry. Suppose not. Consider reflection through the (affine) plane H_ξ containing the axis of symmetry and the point ξ. This is an orthogonal transformation of $\mathbf{R}^3$ (up to an additive translation before and after), so it respects the formation of the Gauss map. Since it fixes ξ and induces reflection through the r-coordinate line in the tangent plane to the surface at ξ, it follows that if L is an eigenline then so is its reflection L' through the r-coordinate line through ξ, and moreover the line L' must have the *same* eigenvalue as the line L. But the two distinct eigenlines have distinct eigenvalues, so $L = L'$. This forces the eigenlines L to satisfy $L = L'$, so these two lines must be the polar coordinate lines through ξ.

Remark 5.6. Say we fix the Riemannian structure on H and we consider only those embeddings $H \hookrightarrow \mathbf{R}^3$ that are isometric embeddings (i.e., induced upon H the chosen Riemannian structure). The geometry of the "planar slices" $P \cap H$ near x as in Theorem 5.3 (with P containing the normal line to H at x) depends very much on how H is embedded in $\mathbf{R}^3$. (Consider embedding an open square with flat metric into either a plane in $\mathbf{R}^3$ or onto the surface of a cylinder with one pair of sides parallel to the axis of symmetry.) The principal curvatures as an unordered pair of numbers are *not* intrinsic to H near x considered as an oriented Riemannian 2-manifold. Gauss' Theorem Egregium is the remarkable assertion that the *product* $k(x)$ of these principle curvatures at x is determined solely by the Riemannian structure on H near x. (On the cylinder, the straight line through any point has associated principal curvature equal to 0 for both orientations.) As we have noted earlier, in the case of higher-dimensional hypersurfaces, the product of the eigenvalues (i.e., the determinant of $\mathrm{d}G(x)$ as a self-map of $\mathrm{T}_x(H)$) is generally not determined just by the induced metric tensor on H.