Math 396. Premanifolds with corners and a theorem of Whitney

1. Topological preliminaries

Let W be an m-dimensional $\mathbf{R}$-vector space, $m \geq 1$. For $1 \leq k \leq m$ a *k-sector* in W is a non-empty subset of the form

$$\Sigma = \{w \in W \mid \ell_1(w) \geq c_1, \ldots, \ell_k(w) \geq c_k\}$$

with $c_1, \ldots, c_k \in \mathbf{R}$ and linearly independent $\ell_1, \ldots, \ell_k \in W^\vee$. A *0-sector* is $\Sigma = W$. A *sector* $\Sigma \subseteq W$ is a k-sector for some $0 \leq k \leq m$. If $w \in W$ is a point, then the translation $w + \Sigma$ is also a k-sector: we use the same ℓ_i's but replace c_i with $c_i + \ell_i(w)$.

Lemma 1.1. *Let Σ be a k-sector as above, so Σ is a closed set in W. Let $\partial_W \Sigma$ and $\mathrm{int}_W(\Sigma) = \Sigma - \partial_W \Sigma$ denote the topological boundary and interior of Σ in W.*

There are exactly k translated hyperplanes H in W such that $H \cap \partial_W \Sigma$ contains a non-empty open set in H. These H's are $H_i = \{\ell_i = c_i\}$. In particular, the subset $\Sigma \subseteq W$ uniquely determines k and the pairs (ℓ_i, c_i) up to positive scaling.

Proof. We may make an additive translation and choose linear coordinates on W so that $\Sigma = [0, \infty)^k \times \mathbf{R}^{m-k}$ in $W = \mathbf{R}^m$. In this case, it is easy to check that $\partial_W \Sigma$ is the union of the k sets $\Sigma \cap \{x_i = 0\}$ for $1 \leq i \leq k$, each of which contains a non-empty open in the hyperplane $H_i = \{x_i = 0\}$ (namely, it contains the locus in $\{x_i = 0\}$ where $x_j > 0$ for all $1 \leq j \leq k$ with $j \neq i$). Suppose $H \subseteq W$ is some other translated hyperplane such that $H \cap \partial_W \Sigma$ contains a non-empty open set in H. Since $H \neq H_i$, the intersection $H \cap H_i$ is a proper (translated) subspace of H for all i. Hence, $H \cap \partial_W \Sigma$ is contained in the union of the $H \cap H_i$'s, so to rule out H it suffices to show that no finite union of proper (translated) subspaces of H can contain a non-empty open set in H. This is a simple exercise left to the reader. Since the subset $\{\ell_i = c_i\}$ in W determines the pair (ℓ_i, c_i) up to a nonzero scaling factor (why?), it remains to prove that if we switch the order of any of the initial defining inequalities then the sector changes. But using linear coordinates extending the ℓ_i's makes this obvious. ∎

By the lemma, the *only* description of Σ by a system of finitely many linear inequalities of the form $\lambda_i \geq b_i$ with $b_i \in \mathbf{R}$ and linearly independent $\lambda_i \in W^\vee$ are precisely ones obtained from the system of k inequalities $\ell_i \geq c_i$ by positive scaling of these conditions. Thus, the subset $\Sigma \subseteq W$ uniquely determines the translated subspaces $H_i = \ell_i^{-1}(c_i)$ and in terms of the subset $\Sigma \subseteq V$ it is well-posed to say that a point $x \in \Sigma$ has *index* r if $\ell_i(x) = c_i$ for exactly r indices i (with $0 \leq r \leq k$). We define Σ_r to be the set of points $x \in \Sigma$ with index r, or equivalently $x \in H_j$ for exactly r values of j.

The following result summarizes some nice topological relations (easily visualized by picturing the non-negative orthant $\Sigma = [0, \infty)^3 \subseteq \mathbf{R}^3 = W$ and the 2-sector $[0, \infty)^2 \times \mathbf{R}$ in $\mathbf{R}^3$):

Theorem 1.2. *Let Σ be a k-sector in W, $1 \leq k \leq m = \dim W$. Let Σ_r be the set of points $x \in \Sigma$ with index r, and let $H_1, \ldots, H_k$ be the k translated hyperplanes uniquely determined by the subset $\Sigma \subseteq W$.*

- *The topological interior $\mathrm{int}_W(\Sigma)$ of Σ relative to W is Σ_0, and Σ is the topological closure of Σ_0 relative to W. In particular, Σ_0 is dense in Σ.*
- *For $1 \leq r \leq k$, $\Sigma_r \neq \emptyset$ and the connected components of Σ_r are open in Σ_r and are given by the intersections of Σ_r with $H_{i_1} \cap \cdots \cap H_{i_r}$ for each $1 \leq i_1 < \cdots < i_r \leq k$, with this intersection also open in $H_{i_1} \cap \cdots \cap H_{i_r}$. (This is also true for $r = 0$ by silly logic reasons: an intersection indexed by the empty set is the entire space.)*

- *For $0 \leq r \leq k$, the closure of Σ_r in Σ (or in V) is the union of the $\Sigma_{r'}$'s for $r' \geq r$.*
- *For $r \geq 1$, Σ_r is the set of $x \in \Sigma$ that lie in the closure (in Σ, or equivalently in V) of exactly r connected components of Σ_1. (This is also true for $r = 0$.)*

Remark 1.3. In particular, using just Σ and Σ_1 we can *locally* topologically encode the property of having index $r \geq 0$: $x \in \Sigma$ has index r if and only if x admits arbitrarily small open neighborhoods U in Σ that meet the closures of exactly r connected components of $U_1 = U \cap \Sigma_1$. This is tremendously important for globalization to manifolds with corners.

Proof. All assertions are unaffected by additive translation and linear isomorphism of vector spaces, so by using a translation and a choice of linear coordinates adapted to the ℓ_i's (i.e., a basis of $W^\vee$ extending the collection of ℓ_i's) we may suppose $W = \mathbf{R}^m$ and

$$\Sigma = \{x_1, \ldots, x_k \geq 0\} = [0, \infty)^k \times \mathbf{R}^{m-k}.$$

Thus, $\Sigma_0 = (0, \infty)^k \times \mathbf{R}^{m-k}$, and this is clearly the interior of Σ in W with closure in W equal to Σ. For $1 \leq r \leq k$, the locus Σ_r is exactly the disjoint union

$$\coprod_{1 \leq i_1 < \cdots < i_r \leq k} ((0, \infty)^{k-r} \times \mathbf{R}^{m-k}) \cap \{x_{i_1} = \cdots = x_{i_r} = 0\}),$$

and these overlaps are exactly the $\Sigma \cap (H_{i_1} \cap \cdots \cap H_{i_r})$'s in the statement of the theorem (up to our initial translation and choice of linear coordinates). Thus, for the second part of the theorem we must show that each such overlap is a connected open subset of Σ_r that is also open in $H_{i_1} \cap \cdots \cap H_{i_r}$. Relabelling the coordinates, we have to prove that

$$\{0\}^r \times (0, \infty)^{k-r} \times \mathbf{R}^{m-k}$$

is a connected open subset in Σ_r and in

$$H_1 \cap \cdots \cap H_r = \{0\}^r \times [0, \infty)^{k-r} \times \mathbf{R}^{m-k}.$$

Openness in the latter is obvious since $(0, \infty)$ is open in $[0, \infty)$, and for openness in Σ_r we note that it is the intersection of Σ_r with the open conditions $x_{r+1} > 0, \ldots, x_k > 0$ in $\mathbf{R}^m$.

Next, we have to prove that the closure of Σ_r is equal to the union of the sets Σ_j for $j \geq r$. This union is the locus of points with index $\geq r$, so its complement is the locus of points with index $< r$. Let us first prove that this union is closed. To say $x \in \Sigma$ has index $< r$ is to say that at most $r - 1$ of the inequalities $\ell_j(x) \geq c_j$ are equalities, or in other words at least $k - r + 1$ of the strict inequalities $\ell_j(x) > c_j$ hold. Hence, if we consider all $1 \leq j_1 < \cdots < j_{k-r+1} \leq k$ then the open simultaneous conditions $\ell_{j_1} > c_{j_1}, \ldots, \ell_{j_{k-r+1}} > c_{j_{k-r+1}}$ on $\mathbf{R}^m$ gives a collection of open subsets whose open union meets Σ in the complement of $\cup_{j \geq r} \Sigma_j$. Hence, this latter union is closed in Σ and it certainly contains Σ_r. To prove that it is the closure of Σ_r, we just have to prove that each Σ_j is in the closure of Σ_r for all $j \geq r$. We may suppose $j > r$, and by induction on $j - r$ (along with the fact that the closure of a closure is itself) it suffices to consider the case $j = r + 1$ for *all* $0 \leq r < k$. That is, we want Σ_{r+1} to be in the closure of Σ_r. It suffices to treat each of the connected components of Σ_{r+1} separately, so by choosing suitable linear coordinates (and a translation) we can focus attention on

$$\{0\}^{r+1} \times [0, \infty)^{k-r-1} \times \mathbf{R}^{m-k}.$$

Any point in here is a limit of points of the form

$$(0, \ldots, 0, 1/n, 0, \ldots, 0) \in \Sigma_r$$

with $1/n$ in the $(r + 1)$th slot. This settles the analysis of closures.

Finally, we have to check that the points of Σ_r are exactly those points $x \in \Sigma$ such that x is in the closure of exactly r connected components of Σ_1. Picking coordinates as above so that $\Sigma = [0, \infty)^k \times \mathbf{R}^{m-k}$, the connected components of Σ_1 are the loci

$$I_1 \times \cdots \times I_k \times \mathbf{R}^{m-k}$$

with $I_j = \{0\}$ for exactly one j and $I_j = (0, \infty)$ otherwise. Thus closure of each is given by replacing $(0, \infty)$ with $[0, \infty)$ in the factors, so a point lies in the closure of exactly r of these precisely when exactly r of its first k coordinates is zero. Such points are precisely those of index r. ∎

2. Calculus on sectors

Let V and V' be two finite-dimensional vector spaces over $\mathbf{R}$, and let $\Sigma \subseteq V$ and $\Sigma' \subseteq V'$ be two sectors. Fix $1 \leq p \leq \infty$. Suppose we are given non-empty open sets $U \subseteq \Sigma$ and $U' \subseteq \Sigma$. In class we defined the notion of a C^p-morphism $f : U \to U'$. For such f, since $p \geq 1$ we see that at each point $x \in U$ there is a derivative $Df(x)$ that is a linear map $V \to V'$, so by the Chain Rule if f is a C^p isomorphism then $Df(x)$ is a linear isomorphism and hence $\dim V = \dim V'$. In general, if f is a C^p map then it is impossible to say anything about the index of $f(x) \in U' \subseteq \Sigma'$ in terms of the index of $x \in U \subseteq \Sigma$. (For example, the index could go up or down; consider putting $[0, 1)$ into $\mathbf{R}$ or along the edge of a square in the plane.) However, to get the theory of C^p-premanifolds with corners off of the ground we just need to build a consistent theory of local C^p-charts, and so rather than studying general C^p maps what we need to study are C^p-isomorphisms. That is, we need to prove:

Theorem 2.1. *If $f : U \to U'$ is a C^p-isomorphism then $f(x)$ has the same index in Σ' as x has in Σ for all $x \in U$. That is, $f(U \cap \Sigma_r) = U' \cap \Sigma'_r$.*

To prove this theorem, let $g : U' \to U$ be the C^p-inverse of f. Since U and U' are non-empty, the Chain Rule ensures $\dim V = \dim V'$; let n be this common dimension. Let $U_r = U \cap \Sigma_r$ and $U'_r = U' \cap \Sigma'_r$. We first show that f must carry U_0 into U'_0 and g must carry U'_0 into U_0, so $U'_0 = f(U_0)$. By symmetry, we consider f. Since $U_0 = U \cap \Sigma_0 = U \cap \mathrm{int}_V(\Sigma)$ is an *open set* in the set $\mathrm{int}_V(\Sigma)$ (as U is open in Σ) which in turn is open in V, U_0 is open in V. The map $\widetilde{f} : U_0 \to V'$ defined by restriction of f is therefore a C^p mapping in the usual sense, with $D\widetilde{f}(u_0) = Df(u_0)$ as linear maps from V to V'. Since $Df(u_0)$ is a linear isomorphism (by the Chain Rule for f and g), the mapping $\widetilde{f} : U_0 \to V'$ between open sets in vector spaces satisfies the hypotheses for the usual inverse function theorem at u_0 (i.e., its total derivative map at u_0 is a linear isomorphism). Thus, by the usual inverse function theorem $\widetilde{f}$ gives a C^p isomorphism between small opens around u_0 and $f(u_0)$ in U_0 and V' respectively. In particular, $f(U_0) \subseteq U' \subseteq V'$ contains an *open* set around $f(u_0)$ in V'. Hence, $f(u_0) \in U' \cap \mathrm{int}_{V'}(\Sigma') = U' \cap \Sigma'_0 = U'_0$, as desired.

By using Remark 1.3 U_r for $r > 1$ is topologically determined in U by U_1 and U_0. More precisely, U_r is the set of points $x \in U - U_0$ admitting arbitrarily small open neighborhoods meeting the closures of exactly r connected components of U_1. The same holds for U'_r in terms of U'_0 and U'_1, so since f and g are inverse homeomorphisms and we have already proved that they identify U_0 and U'_0 we are reduced to the case of index 1. If $x \in U_1$ then $f(x) \notin f(U_0) = U'_0$, so $f(x)$ has index at least 1 in $U' \subseteq \Sigma'$. The problem is to prove that $f(x)$ has index exactly 1. Once this is settled, it makes sense to define the notion of a C^p-premanifold with corners as in class (in the sense of being a structured $\mathbf{R}$-space locally isomorphic to an open in a sector in a vector space equipped with its natural $\mathbf{R}$-space structure given by C^p-functions on its open subsets), but we will need to show more, namely that the locally closed set of points with a given index has a natural structure of C^p-premanifold. We take up these issues and more in what follows.

3. Points of index 1

Using the notation as in the preceding discussion, we have $x \in U_1$ and we seek a contradiction if $f(x) \in U'_r$ with $r \geq 2$, which is to say (after relabelling) that we seek a contradiction if $f(x) \in H'_1 \cap H'_2$ for two of the translated hyperplanes that give "faces" of Σ'. (This possibility can only occur if $n \geq 2$, so we now assume this to be the case.) By translation, we may and do assume (for simplicity of language) that x and $f(x)$ are the origin in their respective vector spaces. In particular, any translated hyperplane through these points is a genuine hyperplane.

We claim that in fact if $f(x) \in H'$ for a hyperplane H' that gives a "face" of Σ' then the map $Df(x) : V \to V'$ carries H into H', where H is the unique hyperplane in V that is a "face" of Σ and contains x (here we use that x has index 1, so $x \in \Sigma_1$). Granting this, it follows that $Df(x)$ sends H into $H'_1 \cap H'_2$, but this is impossible for dimension reasons because $Df(x) : V \to V'$ is an isomorphism and $H'_1 \cap H'_2$ has codimension 2 in V'. This contradiction settles the problem for points with index 1, granting the above claim that must now be proved.

By suitable choice of linear coordinates on V and V', we can assume $V = \mathbf{R}^n$, $\{t_n = 0\}$ is the unique hyperplane H in V through the origin x giving a face of Σ , and that associated to this hyperplane the inequality "$t_n \geq 0$" (rather than "$-t_n \geq 0$") arises in the definition of the sector Σ. We can likewise suppose $V' = \mathbf{R}^n$ with $H' = \{t'_n = 0\}$, and that "$t'_n \geq 0$" is the corresponding inequality that arises in the definition of Σ'. Since x is a point of index 1, near x an open set in Σ is open in the half-space $\{t_n \geq 0\}$. Thus, since our problems are local near x, we may replace Σ with $\mathbf{H} = \{t_n \geq 0\}$ and Σ' with $\mathbf{H}' = \{t'_n \geq 0\}$ to reduce to the setup in the following result:

Theorem 3.1. *Let V and V' be finite-dimensional nonzero vector spaces over $\mathbf{R}$, and let $\mathbf{H} = \{\ell \geq 0\}$ and $\mathbf{H}' = \{\ell' \geq 0\}$ be closed half-spaces defined by nonzero linear functionals $\ell \in V^\vee$ and $\ell' \in V'^\vee$. Let $U \subseteq \mathbf{H}$ be an open subset around a point $x \in \partial\mathbf{H} = \{\ell = 0\}$ and let $f : U \to \mathbf{H}'$ be a C^1-map such that $f(x) \in \partial\mathbf{H}' = \{\ell' = 0\}$. The map $Df(x) : V \to V'$ sends the hyperplane $\partial\mathbf{H}$ into the hyperplane $\partial\mathbf{H}'$.*

Note that this theorem allows $\dim V \neq \dim V'$, and in particular there is no local C^p-isomorphism assumption on f near x (nor a linear isomorphism hypothesis on $Df(x)$). Before giving the proof, we describe what the theorem says in concrete terms. Say we fix linear coordinates $t_1, \dots, t_n$ on V and $t'_1, \dots, t'_{n'}$ on V' so that $\mathbf{H} = \{t_n \geq 0\}$ and $\mathbf{H}' = \{t'_{n'} \geq 0\}$. The theorem says that $Df(x)$ sends the first $n-1$ basis vectors of V into the span of the first $n'-1$ basis vectors of V'.

Proof. Choose linear coordinates $t_1, \dots, t_n$ on V and $t'_1, \dots, t'_{n'}$ on V' such that $\mathbf{H} = \{t_n \geq 0\}$ and $\mathbf{H}' = \{t'_{n'} \geq 0\}$. Thus, we get $V = \partial\mathbf{H} \times \mathbf{R}$ and $V' = \partial\mathbf{H}' \times \mathbf{R}$. We write

$$f : U \to \mathbf{H}' \subseteq V' = \partial\mathbf{H}' \times \mathbf{R}$$

as $f = (\psi, f_{n'})$ where $\psi : U \to \partial\mathbf{H}'$ and $f_{n'} : U \to \mathbf{R}$ are C^1-maps. Since f lands inside of $\mathbf{H}' = \{t'_n \geq 0\}$, if we write $x = (a_1, \dots, a_{n-1}, 0)$ then $f_n(t_1, \dots, t_{n-1}, 0) \geq 0$ for $(t_1, \dots, t_{n-1})$ near $(a_1, \dots, a_{n-1})$ in the open set $\partial U \stackrel{\text{def}}{=} U \cap \partial\mathbf{H}$ in the hyperplane $\partial\mathbf{H}$, and hence $f_{n'}|_{\partial U}$ has a *local minimum* at $(a_1, \dots, a_{n-1})$. This is a local minimum for a C^1-function $f_{n'}|_{\partial U}$ on an *open* domain ∂U in a vector space $\partial\mathbf{H}$ (of dimension $n-1$), so for $1 \leq j \leq n-1$ we deduce the *vanishing* of

$$\frac{\partial(f_{n'}|_{\partial U})}{\partial t_j}(a_1, \dots, a_{n-1}).$$

But this partial derivative is the *same* as $(\partial f_{n'}/\partial t_j)(a_1, \dots, a_{n-1}, 0)$, due to how differentiation for functions on sectors in vector spaces (such as $f_{n'}$ on $\mathbf{H}'$) is defined. Thus, we conclude that $(\partial f_{n'}/\partial t_j)(x) = 0$ for all $j \leq n-1$.

For any $1 \le j \le n$ we have

$$Df(x)(e_j) = \sum_{i=1}^{n'} \frac{\partial f_i}{\partial t_j}(x) e'_i$$

where $x' = f(x)$, the functions $f_i = t'_i \circ f$ are the component functions of f around x in U, and $\{e_j\}$ and $\{e'_i\}$ are the chosen bases of V and V' (and $\partial f_i/\partial t_n$ on $U \subseteq \mathbf{H} = \{t_n \ge 0\}$ is computed as a limit with $h \to 0^+$). The calculation of the preceding paragraph shows that when $j < n$ the coefficient of $e'_{n'}$ in $Df(x)(e_j)$ vanishes. In other words, under the map $Df(x)$ each of the vectors $e_j \in V$ for $j < n$ gets sent into the span of the vectors $e'_i \in V'$ for $i < n'$. But the linear coordinates were rigged so that the e_j's for $j < n$ span $\partial\mathbf{H}$ and the vectors e'_i for $i < n'$ span $\partial\mathbf{H}'$. Thus, $Df(x) : V \to V'$ carries spanning vectors for the hyperplane $\partial\mathbf{H} \subseteq V$ over into the subspace $\partial\mathbf{H}' \subseteq V'$. This proves the desired result. ∎

This completes the proof of Theorem 2.1, and we get a refinement on each stratum:

Theorem 3.2. *Let f and g be as in Theorem 2.1. Considering the connected components of $U \cap \Sigma_r$ and $U' \cap \Sigma'_r$ as open sets in translated codimension-r subspaces of V and V', if C is a connected component of $U \cap \Sigma_r$ and it is carried to the connected component C' in $U' \cap \Sigma'_r$ then by viewing C and C' as opens in vector spaces (translated codimension-r linear subspaces of V and V') the induced homeomorphism between C and C' is a C^p isomorphism in the traditional sense.*

The content of the theorem is that the restricted inverse maps $C \to C'$ and $C' \to C$ are C^p mappings in the traditional sense (when C and C' are viewed as opens in vector spaces, as explained in the statement of the theorem; note that they are open because of the fact that the connected components of Σ_r and Σ'_r are open subsets of Σ_r and Σ'_r respectively). We may choose linear coordinates and a translation so that $\Sigma = [0,\infty)^k \times \mathbf{R}^{n-k}$ and $\Sigma' = [0,\infty)^{k'} \times \mathbf{R}^{n'-k'}$ in $V = \mathbf{R}^n$ and $V' = \mathbf{R}^{n'}$, and C and C' are respectively open in $\{0\}^r \times (0,\infty)^{k-r} \times \mathbf{R}^{n-k}$ and $\{0\}^r \times (0,\infty)^{k'-r} \times \mathbf{R}^{n'-k'}$. Let $W = \{x_1 = \dots = x_r\}$ and $W' = \{x'_1 = \dots = x'_r\}$ in $\mathbf{R}^n$ and $\mathbf{R}^{n'}$ respectively, so C and C' are identified with open sets in W and W' respectively. We want the restriction $f : C \to C'$ to be C^p in the traditional sense, so since C' is open in W' is is equivalent to say $f : C \to W'$ is C^p in the traditional sense. Since C is open in W and V' contains W' as a linear subspace, it is equivalent (via the classical "component function" criterion to be C^p) that the map $f : C \to V'$ is a C^p map in the traditional sense. The inclusion i of C into U is trivially C^p, as is the inclusion j of U' into V', and the map $f : C \to V'$ is really $j \circ f \circ i$ where $f : U \to U'$ is our initial C^p mapping. Thus, by stability of the C^p property under composition (for maps between opens in sectors in vector spaces), we are done.

4. C^p-structure on singular strata

We begin with a definition that has been discussed in class:

Definition 4.1. For $0 \le p \le \infty$, a C^p *premanifold with corners* is a structured $\mathbf{R}$-space $(X, \mathscr{O})$ that is locally isomorphic (in the sense of structured $\mathbf{R}$-spaces) to an open subset of a sector in a finite-dimensional vector space (equipped with its natural $\mathbf{R}$-space structure given by C^p-functions on open subsets of itself). If the underlying topological space is Hausdorff and second-countable then we call it a *C^p-manifold with corners*. We usually write X rather than $(X, \mathscr{O})$.

For $1 \le p \le \infty$, let X be a C^p premanifold with corners. In view of the local results on sectors, we may use any local C^p-chart to determine the property of $x \in X$ having index $r \ge 0$, and the subset $X_r \subseteq X$ of points with index r is locally closed in X. The subsets $X_{\ge r} = \cup_{i \ge r} X_i$ are closed in X, and $X_{\ge 1}$ is called the *boundary* of X and is denoted ∂X; this intrinsic notion (that makes no reference to an ambient topological space containing X) must not be confused with the (extrinsic) notion of topological boundary for a subset of a topological space.

We wish to give X_r a natural structure of C^p-premanifold. The idea is quite simple. Let (ϕ, U) be a C^p-chart on X, with

$$\phi : U \to \phi(U) \subseteq \Sigma \subseteq V$$

a C^p-isomorphism onto an open domain in a sector in a finite-dimensional vector space V, say with $n = \dim V \geq 1$ (as the case $n = 0$ offers nothing to be done). The set U_r of points with index r goes over to the open set $\phi(U) \cap \Sigma_r$ in Σ_r. Since Σ_r is a disjoint union of open subsets (of itself) that are each *open* in a *unique* translated codimension-r subspace of V, by topologically viewing U_r as a corresponding union of disjoint open subsets of itself we obtain (via translation in V) a homeomorphism of open sets in U_r onto open subsets of $(n - r)$-dimensional vector spaces, and these opens cover X_r as we vary (ϕ, U). But as we vary (ϕ, U) we may have overlaps among the open connected components of the varying U_r's and so to assert that we have built a C^p-atlas on X_r we have to verify that the transition maps for the overlaps are C^p isomorphisms.

The problem is to check that on non-empty overlaps of these charts on opens in X_r, the resulting transition maps between opens in Euclidean spaces are C^p-isomorphisms in the usual sense. Once this is done, we will have a C^p-atlas on X_r, and this defines the desired C^p-premanifold structure. Recall that we began with C^p-charts (ϕ, U) for the "C^p premanifold with corners" structure on X, for which the transition maps on overlaps are C^p in the sense of C^p-maps between opens in sectors in vector spaces. Hence, our problem is a local one: prove that a C^p-isomorphism f between non-empty open sets U and U' in sectors Σ and Σ' in finite-dimensional vector spaces V and V' induces a local C^p-isomorphism (in the traditional sense) between the connected components of the index-r loci U_r and U'_r considered as open sets in translated linear subspaces of V and V'. The work in the preceding sections above, coupled with the discussion in class, provides the crucial fact that f does restrict to a bijection (even homeomorphism) between the index-r loci of U and U'. Provided that in general f is proved to be a C^p-map (in the traditional sense) between these loci, we may apply the same conclusions to the inverse map to get the desired local C^p-isomorphism property (in the traditional sense) for f as a map between U_r and U'_r.

Working locally on U_r, we may assume that it is open in a translated codimension-r subspace in V. The inclusion of this translated subspace into V is C^p, so by the stability of the C^p-property under composition we may replace V with this subspace to get to the case when $\Sigma = V$, U is open in V, and $f(U)$ lies in a translated codimension-r subspace of V'. In particular, by the very definition of C^p-maps in the local theory with sectors, the map $f : U \to V'$ is C^p in the usual sense. Composing with a translation, we may assume that $f(U)$ lies in a codimension-r linear subspace $W' \subseteq V'$. Hence, $f : U \to V'$ is a C^p-map in the ordinary sense such that $f(U) \subseteq W'$, and the problem is to show that the map $f : U \to W'$ is C^p in the usual sense. We may choose linear coordinates on V' extending a system of linear coordinates on W', and the C^p-property of $f : U \to V'$ implies that all component functions of f are C^p-functions on U. Restricting to the collection of such component functions corresponding to the basis vectors in W' verifies the component-function criterion for $f : U \to W'$ to be a C^p-map in the usual sense.

Definition 4.2. If X is a C^p-premanifold with corners for $1 \leq p \leq \infty$ then we say it is a C^p-*premanifold with boundary* if $X_2 = \emptyset$ (i.e., all points have index ≤ 1). In this case, the closed subset X_1 is called the *boundaary.* If X is Hausdorff and second-countable then it is called a C^p-*manifold with boundary.* In case $\partial X = \emptyset$ we say X is a C^p-*premanifold* (resp. C^p-*manifold*) if it is also Hausdorff and second-countable.

Note that a C^p-premanifold with boundary is locally isomorphic to an open in a half-space in a finite-dimensional vector space (equipped with its usual C^p-structure).

Example 4.3. Suppose that X is a C^p-premanifold with boundary, $1 \leq p \leq \infty$. For $x \in X_1 = \partial X$, how does the pair (X_1, X) look near x in local C^p-coordinates? A typical chart (ϕ, U) carries a neighborhood of x in X to an open set $\phi(U)$ near the origin in some closed half-space $\mathbf{H} = \{(t_1, \dots, t_n) \in \mathbf{R}^n \mid t_n \geq 0\}$, with $X_1 \cap U$ going over to the open set $\phi(U) \cap \{t_n = 0\}$ in $\mathbf{R}^{n-1}$ and $(X - X_1) \cap U$ going over to the open set $\phi(U) \cap \{t_n > 0\}$ in $\mathbf{R}^{n-1} \times \mathbf{R}_{>0}$.

Example 4.4. For a C^p premanifold with corners and any $r \geq 0$, the inclusion map $i : X_r \to X$ is C^p. Indeed, upon working locally this is the assertion that if Σ is a sector in a finite-dimensional vector space V then the inclusion $\Sigma_r \to \Sigma$ is C^p. By definition of a C^p map, this is the same as saying that the map $\Sigma_r \to V$ is C^p with respect to the C^p premanifold structures on Σ_r and on V. The components of Σ_r are given a C^p-premanifold structure via how they sit as open sets in translated codimension-r linear subspaces $w_i + W_i$ of V, and so by the problem is to check that for any open $U_i \subseteq w_i + W_i$ the inclusion map $U_i \to V$ between open domains in vector spaces (identifying U_i with $-w_i + U_i \subseteq W_i$) is C^p. This is a restriction of a linear map, so it is obviously C^p.

Remark 4.5. Let X be a C^p-premanifold with corners. For any r, let $X_{\geq r}$ be the closed subset of points with index $\geq r$. In general this does not have a natural structure of C^p-manifold with corners (unless X_{r+1} is empty, in which case $X_{\geq r} = X_r$ and so $X_{\geq r}$ even has a natural structure of C^p-premanifold). The problem is best illustrated with an example. Let $X = [0, \infty)^2 \subseteq \mathbf{R}^2$ and $p = \infty$. In this case, $X_{\geq 1}$ is the union of the non-negative coordinate axes in the plane, and so at the origin the geometry (from the differentiable viewpoint) does not look like the local model space $[0, \varepsilon)$ for a 1-dimensional premanifold with corners. (To be rigorous, one should prove that it is impossible to give $X_{\geq 1}$ a structure of C^∞ premanifold with corners such that $X_{\geq 1} \to X$ is an immersion and that the C^∞-structure induced on the open set $X_1 \subseteq X_{\geq 1}$ is its usual C^∞ manifold structure as constructed on the locus X_r for any C^p premanifold with corners.) Naturally this suggests that any attempt to seriously investigate the geometry of manifolds with corners will require the introduction of a more general class of singular spaces to allow for crossing lines and so forth. The theory of C^p-premanifolds with corners provides a category of geometric objects that contains C^p-premanifolds with boundary and admits products. This suffices for our purposes, and so we will not investigate the further development of the theory of singular spaces in differential geometry (except to say that the more general theory of "manifolds with Lipschtiz boundary" gives a satisfactory framework for geometric investigation of singularities).

The next theorem provides a mapping property for the C^p-premanifold structure on the strata X_r.

Theorem 4.6. *Let $f : X \to X'$ be a C^p-map between C^p premanifolds with corners. Assume $f(X) \subseteq X'_{r'}$ set-theoretically for some $r' \geq 0$. The induced map of sets $\overline{f} : X \to X'_{r'}$ is C^p with respect to the C^p-premanifold structure put on $X'_{r'}$ above.*

For example, if $f : X \to X'$ is a C^p-map between C^p-premanifolds with boundary and $f(\partial X) \subseteq \partial X'$, then since the inclusion $i : \partial X \to X$ is C^p (by Example 4.4 with $r = 1$) it follows that $f \circ i$ is C^p. Thus, by the theorem, the set-theoretic map $\partial f : \partial X \to \partial X'$ induced by $f \circ i$ (or f) is therefore a C^p-map between C^p premanifolds.

Proof. The locus $X'_{\leq r}$ of points in X' with index $\leq r'$ is an *open* subset of X', so it has a natural structure of C^p-premanifold with corners (using C^p-charts on X' that are supported in this open subset), and it contains the image of f. By its openness, the restricted map $f : X \to X'_{\leq r'}$ is certainly C^p. Hence, we can rename $X'_{\leq r'}$ as X' to get to the case when $X'_{r'+1}$ is empty, so $X'_{r'}$ is a closed subset of X.

Our problem is local on X and X', so we may assume that X is an open set U in a sector Σ in some finite-dimensional vector space V and (since $X'_{r'+1} = \emptyset$) that

$$X' = [0, \varepsilon)^{r'} \times (-\varepsilon, \varepsilon)^{n'-r'} \subseteq \mathbf{R}^{n'}.$$

Thus, by definition of the notion of C^p map between C^p premanifolds with corners (a condition that may be checked using any local C^p-charts), the composite map

$$f = (f_1, \ldots, f_{n'}) : U \to X' \subseteq \mathbf{R}^{n'}$$

is a C^p map in the usual sense (so all $f_i : U \to \mathbf{R}$ are C^p functions in the usual sense on the open domain U in the sector $\Sigma \subseteq V$) and by hypothesis the image of f lies in the locus $X'_{r'}$ that is identified with $\{(0, \ldots, 0)\} \times (-\varepsilon, \varepsilon)^{n'-r'} \subseteq \mathbf{R}^{n'}$. That is, $f_1 = \cdots = f_{r'} = 0$ and the map $\overline{f} : X \to X'_{r'}$ is identified with the map

$$(f_{r'+1}, \ldots, f_{n'}) : U \to (-\varepsilon, \varepsilon)^{n'-r'} \subseteq \mathbf{R}^{n'-r'}$$

that is visibly C^p since its component functions are C^p. ■

5. Inverse and implicit function theorems on sectors

Having cleared up the remaining issues in the definition of C^p-premanifolds with corners, we want to provide some other essential tools in the local theory (if only to convince the reader that it *is* possible to work with such spaces): we want to prove the inverse and implicit function theorems "with corners". The proofs of these theorems rest on Whitney's extension theorem, which we prove in §6 in a special case that suffices for our needs. This special case of Whitney's theorem says that if $U \subseteq \Sigma$ is an open set in a sector in a finite-dimensional vector space over $\mathbf{R}$, and if $f : U \to W$ is a C^p map to another finite-dimensional $\mathbf{R}$-vector space (where the C^p property here is taken in the sense defined in class, internal to the sector), then for each $x \in U$ there is an open set $U'_x \subseteq V$ around x and a C^p mapping $f_x : U'_x \to W$ (in the traditional sense) such that $f_x|_{U'_x \cap U} = f|_{U'_x \cap U}$. In other words, C^p maps on U locally extend to C^p maps on open domains in the vector space V. As the reader will see, we only require this result in the C^1 case. Most introductory books take the existence of such local C^p extensions as the *definition* of a C^p mapping on an open set in a sector; such an *ad hoc* definition is entirely adequate to get the theory of C^p premanifolds with corners off of the ground, but it seems much more natural to use our "intrinsic" definition of C^p maps on open sets in sectors (not requiring any reference to functions on domains outside of the sector) and to deduce *a posteriori* via Whitney's theorem that our definition is equivalent to the apparently stronger-looking traditional definition.

The inverse and implicit function theorems in the presence of corners require extra hypotheses that are not detected (or rather, are vacuous) in their traditional form on open domains in finite-dimensional vector spaces. For example, the inclusion i of $[0, 1)$ into $[-1, 1)$ shows that a mere isomorphism condition on a derivative is insufficient for a map to be a local C^p-isomorphism (as $Di(0) : \mathbf{R} \to \mathbf{R}$ is an isomorphism, even the identity, yet i is not a local isomorphism at the index-1 point $0 \in [0, 1)$). At the very least, if $f : U \to U'$ is a C^p-map between open sets in sectors $\Sigma \subseteq V$ and $\Sigma' \subseteq V'$, and $x \in U$ is a point of positive index in Σ, then for an inverse function theorem we see that in addition to requiring $Df(x) : V \to V'$ to be an isomorphism we have to impose the (necessary) assumption that $f(x)$ has positive index (in Σ'). But even this is not enough.

For example, if $V = V' = \mathbf{R}^2$ and $\Sigma = \Sigma'$ are the sectors given by the upper half-plane, then for $U = U' = \Sigma'$ the map $f(u, v) = (u, v + u^2)$ is a C^∞-map preserving the origin but sending all other index-1 points $(u, 0)$ to index-0 points in Σ'. There is a local C^∞-inverse at the origin when working with open sets in $\mathbf{R}^2$, but not when working with open sets in the closed upper half-spaces Σ and Σ'. The derivative map $Df(0, 0)$ is an isomorphism that even preserves the tangent line along the loci of index-1 points, so a derivative condition at a point cannot ensure that the map locally respects the loci of positive-index points.

The topological boundaries $\partial\Sigma$ and $\partial\Sigma'$ in V and V' are also the loci of positive-index points, so we shall use the suggestive notation $\partial U \stackrel{\text{def}}{=} U \cap \partial\Sigma$ and $\partial U' \stackrel{\text{def}}{=} U' \cap \partial\Sigma$ to denote the sets $U - U_0$ and $U' - U'_0$ of points with positive index; these are generally not the topological boundaries of U or U' (in either the sectors or the ambient vector spaces)! We can summarize the preceding paragraph by saying that the topological condition $f(\partial U) \subseteq \partial U'$ should be assumed if we wish to state an inverse function theorem at arbitrary points of U; of course, if U is open in V then ∂U (in the sense just defined!) is empty and so there is no topological condition being imposed.

Theorem 5.1 (Inverse function theorem with corners)**.** *Let $f : U \to U'$ be a C^p map between open domains $U \subseteq \Sigma$ and $U' \subseteq \Sigma'$ in sectors in nonzero finite-dimensional vector spaces V and V', with $1 \le p \le \infty$. Let ∂U and $\partial U'$ denote the sets $U - U_0 = U \cap \partial\Sigma$ and $U' - U'_0 = U' \cap \partial\Sigma'$ of points with positive index. Assume $f(\partial U) \subseteq \partial U'$.*

For any $x_0 \in U$, if $Df(x_0) : V \to V'$ is an isomorphism then f induces a C^p-isomorphism between open neighborhoods of x_0 in U and $f(x_0)$ in U'. In particular, for x near x_0 in U the index of x is the same as that of $f(x)$.

Proof. Since $Df(x_0)$ is an isomorphism, V and V' have the same dimension, say n. If x_0 has index 0 then we can shrink U to be open in V and hence $f : U \to V'$ satisfies the hypotheses of the usual inverse function theorem at x_0. In particular, $f(U)$ is a neighborhood of $f(x_0)$ in V', yet $f(U) \subseteq \Sigma'$ and hence Σ' is a neighborhood of $f(x_0)$ in V'. This forces $f(x_0)$ to be a point of index 0 in Σ', and the conclusion of the usual inverse function theorem implies that f is a local C^p-isomorphism between small opens around x_0 and $f(x_0)$ in V and V' that we may suppose are contained in Σ and Σ'. This settles the case when $x_0 \notin \partial U$, so we now turn to the more interesting case when $x_0 \in \partial U$. Let $r > 0$ be the index of x_0 (so $n > 0$). Since our problem is local near x_0 we can enlarge Σ to be an r-sector with

$$\Sigma = \{(t_1, \dots, t_n) \in \mathbf{R}^n \,|\, t_1 \ge 0, \dots, t_r \ge 0\}$$

in $V = \mathbf{R}^n$ using suitable linear coordinates to also make $x_0 = 0$ (after a suitable translation).

Step 1. We first treat the case when x_0 has index 1. We choose coordinates so that $V = \mathbf{R}^n$ and $x_0 = 0$, with Σ the upper half-space $\{t_1 \ge 0\}$. Since our problem is local near x_0 and $f(x_0)$, it is harmless to drop any of the defining inequalities for the sector Σ' for which $f(x_0)$ is not in the corresponding hyperplane. By hypothesis $f(x_0) \in \partial\Sigma'$ since $x_0 \in \partial\Sigma$, so the index r' of $f(x_0)$ is positive and we can suppose (by composing with a translation on V' and using suitable linear coordinates) that $V' = \mathbf{R}^n$ with $f(x_0) = 0$ and Σ' defined by inequalities $t'_1 \ge 0, \dots, t'_{r'} \ge 0$. Let us check that necessarily $r' = 1$. For each $1 \le i \le r'$, composing f with the C^∞ inclusion of $U' \subseteq \Sigma'$ into the closed half-space $\mathbf{H}'_i = \{t'_i \ge 0\} \subseteq \mathbf{R}^n$ gives a C^p-map $f_i : U \to \mathbf{H}'_i$ such that $f(x_0) \in \partial\mathbf{H}'_i$ with $x_0 \in \partial\Sigma$. Clearly $Df_i(u) : V \to V'$ is equal to $Df(u)$ for all $u \in U$, so by Theorem 3.1 (taking $\mathbf{H} = \Sigma$), the map $Df(x_0)$ must carry the hyperplane $\partial\Sigma = \{t_1 = 0\} \subseteq V$ into the hyperplane $\partial\mathbf{H}'_i = \{t'_i = 0\}$. Hence, if $r' \ge 2$ (so $n \ge 2$) then the linear isomorphism $Df(x_0) : V \simeq V'$ carries the hyperplane $\partial\Sigma$ into an intersection of distinct hyperplanes, an impossibility for dimension reasons. This proves that $r' = 1$. Thus, $\Sigma' = \{t'_1 \ge 0\} \subseteq \mathbf{R}^n = V'$.

By Whitney's extension theorem (see §6) we can shrink U around x_0 so that f extends to a C^1-map $\widetilde{f}:\widetilde{U}\to V'$ for an open set $\widetilde{U}\subseteq V$ containing U. Clearly $D\widetilde{f}(0):V\to V'$ must coincide with $Df(0)$ (as a basis of V is given by tangent vectors to line segments lying in the sector Σ on which $\widetilde{f}$ recovers f), and so $D\widetilde{f}(0)$ is an isomorphism. Hence, we can apply the usual inverse function theorem to deduce that $\widetilde{f}$ is a local C^1-isomorphism between open neighborhoods of the origins in V and V'. How does the local C^1-inverse behave with respect to the closed half-spaces Σ and Σ'? More specifically, we have to show that near $f(x_0)$, all points of Σ' are hit by points of Σ under $\widetilde{f}$ (and so $\widetilde{f}^{-1}$ near $f(x_0)$ does restrict to a map between open neighborhoods of $f(x_0)$ and x_0 in the original sectors Σ and Σ', and not merely between opens in the ambient vector spaces).

By Theorem 3.1, $f(U\cap\partial\Sigma)\subseteq\partial\Sigma'$. Hence, the restriction $f_0:U\cap\partial\Sigma\to\partial\Sigma'$ of f makes sense and is a C^p map from an open domain in $\partial\Sigma$ into a vector space $\partial\Sigma'$ of the same dimension, and $Df_0(x_0):\partial\Sigma\to\partial\Sigma'$ is easily seen to be the restriction of the linear isomorphism $Df(x_0):V\simeq V'$. Thus, $Df_0(x_0)$ is injective and hence an isomorphism for dimension reasons. By the usual inverse function theorem applied to the C^p map f_0, it follows that f_0 near x_0 is a local C^p-isomorphism and so by shrinking U and $\widetilde{U}$ around x_0 we may arrange that

- $\widetilde{f}(\widetilde{U})\subseteq V'$ is open with $\widetilde{f}:\widetilde{U}\to\widetilde{f}(\widetilde{U})$ a C^1-isomorphism,
- $f_0(U\cap\partial\Sigma)\subseteq\partial\Sigma'$ is open with $f_0:U\cap\partial\Sigma\to f_0(U\cap\partial\Sigma)$ a C^p-isomorphism,
- $U=\widetilde{U}\cap\Sigma$ and $\widetilde{U}$ $U\cap\partial\Sigma=\widetilde{U}\cap\partial\Sigma$.

In particular,

$$\widetilde{f}(\widetilde{U}\cap\partial\Sigma)=f(U\cap\partial\Sigma)=f_0(U\cap\partial\Sigma)=f_0(U)\cap\partial\Sigma'=f(U)\cap\partial\Sigma'$$

is open in $\widetilde{f}(\widetilde{U})\cap\partial\Sigma'$, so by shrinking $\widetilde{U}$ (and hence $U=\widetilde{U}\cap\Sigma$) we can ensure $\widetilde{f}(\widetilde{U}\cap\partial\Sigma)=\widetilde{f}(\widetilde{U})\cap\partial\Sigma'$.

We may also assume $\widetilde{U}$ is an open ball centered at the origin x_0. In paricular, $\widetilde{U}^0\overset{\text{def}}{=}\widetilde{U}\cap\{t_n\neq 0\}$ has exactly two connected components and $\widetilde{f}$ carries this over to the part of $\widetilde{f}(\widetilde{U})$ not in $\partial\Sigma'$. Hence, the part of $\widetilde{f}(\widetilde{U})$ not in $\partial\Sigma'$ has exactly two connected components. Since $f(U)\subseteq\Sigma'=\{t_1'>0\}$, $\widetilde{f}$ must therefore carry $\widetilde{U}\cap\{t_1>0\}$ to $\widetilde{f}(\widetilde{U})\cap\{t_1'>0\}$ and $\widetilde{f}^{-1}$ must do likewise in the other direction. This shows that $f(U)=\widetilde{f}(U)$ is equal to $\widetilde{f}(\widetilde{U})\cap\Sigma'$, so $f(U)$ is *open* in Σ' and the C^1-map $\widetilde{f}^{-1}$ carries $f(U)$ back to U.

To summarize, we have constructed a local C^1-inverse to f as maps between open neighborhoods of x_0 and $f(x_0)$ in the respective sectors Σ and Σ'. Since f is actually a C^p-map (with $1\leq p\leq\infty$), we need to show that its local C^1-inverse is also C^p. This is proved by induction on p *exactly* as in the proof of the usual inverse function theorem (using the Chain Rule to express the derivative of the local C^1-inverse of f in terms of the derivative of f and the inversion operation on matrices). This concludes the case when x_0 has index 1.

Step 2. Now assume x_0 has index $r\geq 2$ in Σ. We shall first show that $f(x_0)$ has index at least r, and then we will reduce to the case when its index is exactly r. By working locally near x_0 we may ignore the components of Σ_1 whose closures do not contain x_0, and likewise for Σ' near $f(x_0)$, so if we let $r'\geq 1$ denote the index of $f(x_0)$ then after suitable translation and choice of linear coordinates on V we may assume $x_0=f(x_0)=0$, $V=V'=\mathbf{R}^n$,

$$\Sigma=\{(t_1,\dots,t_n)\in\mathbf{R}^n\,|\,t_1\geq 0,\dots,t_r\geq 0\},\ \ \Sigma'=\{(t_1',\dots,t_n')\in\mathbf{R}^n\,|\,t_1'\geq 0,\dots,t_{r'}'\geq 0\},$$

$$U=\{(t_1,\dots,t_n)\in\mathbf{R}^n\,|\,0\leq t_1<\varepsilon,\dots,0\leq t_r<\varepsilon,|t_{r+1}|<\varepsilon,\dots,|t_n|<\varepsilon\},\ \ U'=\Sigma'.$$

Since $u\mapsto\det Df(u)$ is continuous and it is non-vanishing at x_0, we can assume $\det Df(u)$ is nonzero for all $u\in U$ by shrinking the open $U\subseteq\Sigma$ around x_0. For every index-1 point $u\in U_1$ we may therefore use Step 1 at u to conclude that $f(u) \in \Sigma'_1$. Thus, f carries the locus $U_1 = U \cap \Sigma_1$ of index-1 points into the locus U'_1 of index-1 points of Σ'. The connected components of U_1 are

$$U_{1,i} = \{(t_1, \dots, t_n) \in U \mid t_i = 0\}$$

for $1 \le i \le r$, and the connected components $\Sigma'_{1,i'}$ of Σ'_1 correspond to the hyperplane "faces" $\Sigma' \cap \{t'_{i'} = 0\}$ for $1 \le i' \le r'$, so by continuity the map $f : U_1 \to \Sigma'_1$ must carry each $U_{1,i}$ into some $\Sigma'_{1,\phi(i)}$ for a unique $1 \le \phi(i) \le r'$.

Our next task is to get to the geometrically pleasing setup with $\phi(i) = i$ for all $1 \le i \le r$ (so in particular $r' \ge r$). It is equivalent to check that ϕ is injective, as then $r' \ge r$ and a renumbering of the first r' linear coordinates chosen on V' will bring us to the case $\phi(i) = i$ for $1 \le i \le r$. Since $r \ge 2$, we have $n \ge 2$. Let $H_i = \{t_i = 0\} \subseteq V$ and $H'_{i'} = \{t'_i = 0\} \subseteq V'$ be the unique hyperplanes containing $U_{1,i}$ and $\Sigma'_{1,i'}$ respectively for $1 \le i \le r$ and $1 \le i' \le r'$. These hyperplanes are nonzero since $n \ge 2$. It is easy to see that there are line segments in $\overline{U}_{1,i} \subseteq \Sigma$ through 0 whose velocity vectors at 0 give a basis of H_i. Hence, the linear isomorphism $Df(0) : V \to V'$ must carry H_i into $H'_{\phi(i)}$, and so for dimension reasons $Df(0)$ carries H_i isomorphically onto $H'_{\phi(i)}$. Since $Df(0)$ is an isomorphism and $H_{i_1} \ne H_{i_2}$ in V for $i_1 \ne i_2$, this forces $\phi(i_1) \ne \phi(i_2)$ for $i_1 \ne i_2$, as desired. This gives the desired injectivity of ϕ (and in particular shows $r' \ge r$).

Step 3. Now we have (after the relabelling of coordinates as indicated above) $f(U_{1,i}) \subseteq U'_{1,i}$ for $1 \le i \le r$, so f carries the closure of $U_{1,i}$ in U over into the closure of $U'_{1,i}$ in Σ' (or equivalently, in V'). Let $\Sigma'_{\le r}$ be the r-sector $\{t'_1 \ge 0, \dots, t'_r \ge 0\}$ in $\mathbf{R}^n$ that contains Σ', so we may view f as a C^p-map $f_{\le r} : U \to \Sigma'_{\le r}$ without losing the hypotheses on f. Suppose that can settle the problem for $f_{\le r}$. This forces $f(U) = f_{\le r}(U)$ to be a neighborhood of 0 in $\Sigma'_{\le r}$, yet it is contained in the r'-sector Σ' and so the subset $\Sigma' \subseteq \Sigma'_{\le r}$ would have to be a neighborhood of 0 in $\Sigma'_{\le r}$. This is obviously impossible if $r' > r$, and so would force $r' = r$, $\Sigma' = \Sigma'_{\le r}$, and $f = f_{\le r}$, thereby giving the desired result for f. We may therefore replace Σ' and f with $\Sigma'_{\le r}$ and $f_{\le r}$ to get to the case when the index r' of $f(x_0)$ is also equal to r.

By Whitney's extension theorem (see §6), we can shrink U so that f extends to a V'-valued C^1-map $\widetilde{f}$ on an open neighborhood of $0 \in V$. Clearly $D\widetilde{f}(0) : V \to V'$ must coincide with $Df(0)$ (since a basis of V is given by tangent vectors to line segments lying in the sector Σ on which $\widetilde{f}$ recovers f), and so $D\widetilde{f}(0)$ is an isomorphism. Hence, we can apply the usual inverse function theorem to deduce that $\widetilde{f}$ is a local C^1-isomorphism between open neighborhoods of the origins in V and V'. How does this local C^1-inverse behave with respect to the r-sectors Σ and Σ'? In Step 1 we encountered the same sort of problem, and there it was solved (for index-1 points) by connectivity considerations.

We have arranged above that $\phi(i) = i$ for $1 \le i \le r$. An open box centered at 0 meets $V - (H_1 \cup \cdots \cup H_r)$ in an open set whose connected components are given by the evident 2^r systems of strict inequalities ($t_i < 0$ or $t_i > 0$ for each $1 \le i \le r$), so it follows from topological considerations with connectivity that the local inverse to $\widetilde{f}$ must carry Σ' near 0 back into Σ near 0 (make sure you understand this step; draw pictures for $r = 2$ and $n = 3$, and generalize the connectivity trick used in the case of index-1 points). Hence, we get a local C^1-inverse to f as maps between open neighborhoods of the respective origins in the sectors Σ and Σ'. Exactly as in the case of index-1 points, it follows formally that the C^1-inverse must be a C^p map. ■

Corollary 5.2 (Implicit function theorem with corners)**.** *Let V, V', and V'' be finite-dimensional vector spaces and let $\Sigma \subseteq V$, $\Sigma' \subseteq V'$, and $\Sigma'' \subseteq V''$ be sectors, so $\Sigma \times \Sigma' \subseteq V \times V'$ is a sector. Let*

$U \subseteq \Sigma$ and $U' \subseteq \Sigma'$ be open subsets and let $f : U \times U' \to \Sigma''$ be a C^p map with $1 \le p \le \infty$. Assume that $f(U \times \partial U') \subseteq \partial\Sigma''$, where $\partial U'$ and $\partial\Sigma''$ denote the singular loci of points with positive index.

Let $(a, a') \in U \times U'$ be a point and let $a'' = f(a, a')$. If the map $Df(a, a') : V \times V' \to V''$ induces an isomorphism from V' onto V'', then for sufficiently small connected open sets $U_0 \subseteq U$ around a there exists a unique continuous map $g : U_0 \to U'$ such that $g(a) = a'$ and $f(x, g(x)) = a''$, and moreover g is C^p.

Proof. Since the inverse function theorem has been proved in the C^p case with corners, the proof of the implicit function theorem (via the inverse function theorem) as in Math 296 carries over *verbatim* by simply replacing the role of the old inverse function theorem with its refinement that allows for corners. The main point is to check that the boundary condition in the inverse function theorem with corners is satisfied in the context where it is applied to prove the implicit function theorem. The method of the proof is to build an auxiliary map $\Phi : U \times U' \to U \times \Sigma''$ over the identity map on U, and it has to be checked that Φ carries $\partial(U \times U')$ into $\partial(U \times \Sigma'')$. Since

$$\partial(\Sigma \times \Sigma') = (\partial\Sigma \times \Sigma') \cup (\Sigma \times \partial\Sigma')$$

inside of $V \times V'$, we have $\partial(U \times U') = (\partial U \times U') \cup (U \times \partial U')$. Likewise,

$$\partial(U \times \Sigma'') = (\partial U \times \Sigma'') \cup (U \times \partial\Sigma'').$$

Since Φ lies over the identity map on U, it certainly carries $\partial U \times U'$ into $\partial U \times \Sigma''$. Hence, the only non-trivial boundary condition is that Φ carries $U \times \partial U'$ into $U \times \partial\Sigma''$, and this really says that the map $U \times U' \to \Sigma''$ induced by the composite of Φ with the projection $p_2 : U \times \Sigma'' \to \Sigma''$ carries $U \times \partial U'$ into $\partial\Sigma''$. But the definition of Φ is rigged so that $p_2 \circ \Phi = f$, so our assumption $f(U \times \partial U') \subseteq \partial\Sigma''$ is exactly what we need to verify this hypothesis for the application of the inverse function theorem with corners. ∎

6. Whitney's extension theorem

This long section is devoted to giving an "elementary" proof of the following special case of a general theorem of Whitney:

Theorem 6.1 (Whitney)**.** *Let V and V' be finite-dimensional nonzero vector spaces over $\mathbf{R}$ and let $\Sigma \subseteq V$ be a sector. Let $U \subseteq \Sigma$ be an open subset and $x_0 \in U$ a point. Fix $0 \le p \le \infty$.*

Any C^p map $f : U \to V'$ locally extends to a C^p map on an open neighborhood of x_0 in V. That is, there exists an open set $\widetilde{U} \subseteq V$ around x_0 and a C^p-map $\widetilde{f} : \widetilde{U} \to V'$ such that $\widetilde{f}|_{U \cap \widetilde{U}} = f$.

Whitney focused on the case $1 \le p \le \infty$ (as the case $p = 0$ is really a topological problem that was solved in vast generality by Urysohn), and he proved a lot more: he allowed U to be a rather more general kind of locally closed set in $\mathbf{R}^n$; for us, the restriction to open sets in sectors eliminates the serious geometric complications in the general case. Whitney also proved a global result that avoided having to shrink U (in our notation, this corresponds to finding $\widetilde{U}$ containing U as a closed subset), but this global result can be deduced from the local one by a "standard" argument using partitions of unity. For applications in the proofs of the inverse and implicit function theorems with corners, the local C^1 case for open sets in sectors will suffice.

We refer the reader to Appendix A in the book "Transversal mappings and flows" for an exposition of the general case. This appendix gives a proof that is a bit more efficient than the one in Whitney's pioneering original paper, but unfortunately it has some annoying typos. It assumes fluency with Taylor's theorem in terms of multilinear higher derivative maps, and is written in the language of calculus on Banach spaces (but such extra generality can be ignored without impacting the argument). Strictly speaking, Whitney's theorem as it is traditionally stated (e.g., in the book

"Transversal mappings and flows") is for a class of maps that looks different from the class of C^p maps, but the uniformity of the error estimate in the higher-dimensional Taylor formula ensures that (for $p \geq 1$) C^p maps on sectors in our sense *do* satisfy the hypotheses of Whitney's theorem in its traditional formulation. Whitney's 1934 proof is concrete and explicit, but looks like very tough reading.

Proof. We will have to use two entirely different methods: one method (based on the technique of averaged reflections) will apply to the C^p case for finite p and a second (entirely different) method will have to be used in the C^∞ case; neither method is applicable in the other case.

Let r be the index of x_0 in Σ. Since the problem is local near x_0, upon (harmlessly) making an additive translation to put x_0 at the origin and shrinking U around x_0, we may choose suitable linear coordinates so that we can suppose $V = \mathbf{R}^n$, $x_0 = 0$, $\Sigma = [0,\infty)^r \times \mathbf{R}^{n-r}$, and $U = [0,1)^r \times (-1,1)^{n-r}$. Our problem is local at the origin, so by multiplying f by the restriction to Σ of a C^∞ function on $\mathbf{R}^n$ that equals 1 on $[-1/2,1/2]^n$ and is compactly supported inside of $(-1,1)^n$, we may assume that f extends to all of Σ. Hence, we can assume $U = \Sigma$.

We first treat the C^p case for $0 \leq p < \infty$. (Keep in mind that the C^1 case is all that is required for the C^s versions of the inverse and implicit function theorems with corners for any $1 \leq s \leq \infty$.) In this case we will extend f to a C^p mapping from $\mathbf{R}^n$ to V'. The case $r = 0$ is trivial, so we assume $1 \leq r \leq n$. We let $x_1, \dots, x_n$ be the standard coordinate functions on $\mathbf{R}^n$. Pick any number $p' \geq p$, and let $c_0, \dots, c_{p'} \in \mathbf{R}$ be numbers satisfying $\sum c_j = 1$, to be chosen more precisely later; we allow the generality $p' \geq p$ (rather than $p' = p$) so as to not upset a later inductive step. Let $b_0, \dots, b_{p'} > 0$ be pairwise distinct positive numbers (such as $b_j = j + 1$). Consider the function $\widetilde{f}$ on $\Sigma' = \mathbf{R} \times [0,\infty)^{r-1} \times \mathbf{R}^{n-r}$ defined by

$$\widetilde{f}(x_1, \dots, x_n) = \begin{cases} f(x_1, \dots, x_n), & x_1 \geq 0 \\ \sum_{j=0}^{p'} c_j f(-b_j x_1, x_2, \dots, x_n), & x_1 < 0. \end{cases}$$

Roughly speaking, $\widetilde{f}$ is defined at any point $\xi = (a_1, \dots, a_n)$ on the side of the hyperplane $x_1 = 0$ opposite Σ by "averaging" (with weighted coefficients c_j adding to 1) the values of f at the points $(-b_j a_1, a_2, \dots, a_n)$ that lie on the other side of the wall and on the unique line passing through ξ and perpendicular to the wall. For example, if $p' = p = 0$ then $\widetilde{f}$ literally is the reflection of f through the wall $x_1 = 0$. Note also that because $\sum c_j = 1$, at points $\xi \in \Sigma' \cap \{x_1 = 0\}$ the formula in the second case of the definition of $\widetilde{f}$ returns the value $f(\xi)$ when it is evaluated at ξ. Hence, we may also take this second case as defining $\widetilde{f}$ on the region $\Sigma' \cap \{x_1 \leq 0\}$. We seek conditions on the c_j's so that $\widetilde{f}$ is a C^p map on the sector $\Sigma' = \mathbf{R} \times [0,1)^{r-1} \times \mathbf{R}^{n-r}$ that has only $r-1$ "faces". (This well-chosen averaging of values at reflected points is called the method of *Lions reflections*, named after the famous French mathematician J-L. Lions, father of 1994 Fields Medalist P-L. Lions, but probably it is due to earlier mathematicians.) If we arrange for $\widetilde{f}$ to be C^p for suitable c_j's, then induction on r will complete the argument.

Step 1. Away from $\Sigma' \cap \{x_1 = 0\}$ we claim that without any conditions on the c_j's the map $\widetilde{f}$ is C^p, and with just the condition $\sum c_j = 1$ we claim that $\widetilde{f}$ is continuous on Σ'. Clearly $\widetilde{f} = f$ on the set $\Sigma' \cap \{x_1 > 0\} = \mathrm{int}_{\Sigma'}(\Sigma)$, so it is C^p there since f is C^p on Σ, and on the *open* subset $\Sigma' \cap \{x_1 < 0\}$ in Σ' the map $\widetilde{f}$ is a finite sum of C^p maps to V'. More precisely, the maps in the sum are each a composite of three steps: the C^∞ reflection $(x_1, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$ that carries $\Sigma' \cap \{x_1 < 0\}$ over into the set $\Sigma' \cap \{x_1 > 0\}$ that lies on the *interior* of Σ relative to Σ', a positive scaling (some b_j) on the first coordinate, and the C^p map f on Σ. Hence, to verify the C^p property we may focus our attention at points in $\Sigma' \cap \{x_1 = 0\}$ (but we need to keep track of what is happening on opens in Σ' around these points in order to check existence and continuity for various partial derivatives).

Fix a point $\xi = (0, a_2, \ldots, a_n) \in \Sigma' \cap \{x_1 = 0\}$. We first show that $\widetilde{f}$ is continuous at ξ precisely because of the condition $\sum c_j = 1$. To see this, consider the sequential criterion for continuity. Let $\{\xi_m\}$ be a sequence in Σ' converging to ξ, so we need $\widetilde{f}(\xi_m) \to f(\xi)$. The part of the sequence in $\Sigma = \Sigma' \cap \{x_1 \geq 0\}$ offers no difficulties, as $\widetilde{f}|_\Sigma = f$ and we assumed f to be continuous on Σ. Hence, by breaking up the sequence into two subsequences, the points in Σ and the points not in Σ, we may restrict our attention to the second case and so we can assume $x_1(\xi_m) < 0$ for all m. In this case, the definition of $\widetilde{f}$ via reflection through $\{x_1 = 0\}$ lets us restate the continuity problem as follows: we have a sequence of points $\xi_m = (a_{1,m}, \ldots, a_{n,m})$ $(m \geq 1)$ in the original sector Σ converging to a point $\xi = (0, a_2, \ldots, a_n)$, and we want

$$\sum_{j=0}^{p'} c_j f(b_j a_{1,m}, a_{2,m}, \ldots, a_{n,m}) \to f(0, a_2, \ldots, a_n) = f(\xi)$$

in V' as $m \to \infty$. But this convergence is happening inside of the sector Σ on which f is continuous, and since ξ has vanishing first coordinate it follows that for all $0 \leq j \leq p'$ we have

$$(b_j a_{1,m}, a_{2,m}, \ldots, a_{n,m}) \to (0, a_2, \ldots, a_n) = \xi$$

as $m \to \infty$. Thus, $f(b_j a_{1,m}, a_{2,m}, \ldots, a_{n,m}) \to f(\xi)$, and hence the finite sums in question converge to $\sum_{j=0}^{p'} c_j f(\xi) = f(\xi)$ since $\sum_{j=0}^{p'} c_j = 1$. This establishes the continuity of $\widetilde{f}$ on all of Σ'.

Step 2. So far we have not had to impose any conditions on the c_j's beyond $\sum c_j = 1$, but we claim that the system of $p+1$ inhomogeneous linear equations in the c_j's,

$$\sum_{j=0}^{p'} (-b_j)^i c_j = 1 \tag{6.1}$$

for $0 \leq i \leq p$, is sufficient to ensure that $\widetilde{f}$ is C^p (and it fact it is also necessary if we consider varying f, but we don't care about necessity here). Note that the case $i = 0$ is the condition $\sum c_j = 1$, and that such a system of linear equations in the c_j's can always be solved because its $(p+1) \times p'$ matrix of coefficients has independent rows (and thus column rank $p+1$) due to the fact that it is a submatrix of the van der Monde matrix $((-b_j)^i)_{0 \leq i,j \leq p'}$ that is invertible (the $-b_j$'s are pairwise distinct).

These linear conditions on the c_j's are "universal" in the sense that they have nothing at all to do with the map $f : \Sigma \to V$; this will be crucial for our proof that the conditions in (6.1) are sufficient to imply that $\widetilde{f}$ is a C^p map. To prove the sufficiency of these linear conditions on the c_j's, we shall induct on $p \leq p'$ with p' and $b_0, \ldots, b_{p'}$ considered to be fixed and f allowed to vary over all C^p mappings from Σ to V'. (Put another way, we could have fixed $p' \geq 0$ at the beginning of the proof and have aimed to prove the theorem for all $p \leq p'$ by induction on p using fixed distinct positive $b_0, \ldots, b_{p'}$ and averaging at $p'+1$ reflected points.) The case $p = 0$ has already been settled for all f, so we may now assume $p > 0$ (so $p' \geq 1$), and we may assume that sufficiency is already known for *all* C^{p-1} maps from Σ to V'. (Note that $p - 1 \leq p'$.)

For $2 \leq i \leq n$, the formation of ∂_{x_i} commutes with linear operations in x_1 (and in particular its computation is done with x_1 held fixed), so it is clear that $\partial_{x_i}\widetilde{f} : \Sigma' \to V'$ exists and is related to

$\partial_{x_i} f$ *exactly* as $\widetilde{f}$ is related to f, namely

$$\partial_{x_i}\widetilde{f}(x_1,\dots,x_n) = \begin{cases} \partial_{x_i} f(x_1,\dots,x_n), & x_1 \geq 0 \\ \sum_{j=0}^{p'} c_j \partial_{x_i} f(-b_j x_1, x_2, \dots, x_n), & x_1 < 0. \end{cases}$$

Since the maps $\partial_{x_i} f : \Sigma \to V'$ are of class C^{p-1}, by the inductive hypothesis it follows that for $2 \leq i \leq n$ each $\partial_{x_i}\widetilde{f} : \Sigma' \to V'$ is a mapping of class C^{p-1}. To complete the argument, we have to analyze the existence of $\partial_{x_1}\widetilde{f} : \Sigma' \to V'$ and whether or not it is a mapping of class C^{p-1}. (Once this is verified then we will have proved that all first-order partials of $\widetilde{f} : \Sigma' \to V'$ exist and are of class C^{p-1}, so therefore $\widetilde{f}$ is of class C^p, as desired.)

We shall prove that $\partial_{x_1}\widetilde{f}$ does exist at all points of Σ' and that it is related to the C^{p-1} mapping $\partial_{x_1} f : \Sigma \to V'$ as $\widetilde{f}$ is related to the C^p mapping f, with the same (pairwise distinct and positive) b_j's but with the modification that p is replaced with $p-1$ and c_j is replaced with $-b_j c_j$ (where $\sum_j -(b_j c_j) = 1$ by (6.1)). That is, we claim that at all points $(a_1,\dots,a_n) \in \Sigma'$,

$$\partial_{x_1}\widetilde{f}(a_1,\dots,a_n) = \begin{cases} \partial_{x_1} f(a_1,\dots,a_n), & a_1 \geq 0 \\ \sum_{j=0}^{p'} (-b_j c_j) \partial_{x_1} f(-b_j a_1, a_2, \dots, a_n), & a_1 < 0. \end{cases}$$

Since $(-b_0 c_0, \dots, -b_{p'} c_{p'})$ *does* satisfy the system of p inhomogeneous linear conditions

$$\sum_{j=0}^{p-1} (-b_j)^i X_j = 1$$

in p unknowns for $0 \leq i \leq p-1$, the inductive hypothesis (now applied with the C^{p-1} mapping $\partial_{x_1} f$!) would thereby ensure that $\partial_{x_1}\widetilde{f}$ is also a map of class C^{p-1}, as desired. The proposed formula for $\partial_{x_1}\widetilde{f}$ (including its existence) is obvious on the open subsets $\Sigma' \cap \{x_1 > 0\}$ and $\Sigma' \cap \{x_1 < 0\}$ in Σ'. Thus, once again the only real problem is the case $a_1 = 0$. Since the definition of ∂_{x_1} for a V'-valued mapping on $\Sigma' = \mathbf{R} \times [0,\infty)^{r-1} \times \mathbf{R}^{n-r}$ only depends on the restriction of the function to lines where $x_2,\dots,x_n$ are held fixed, and the existence assertion and the proposed formula for $\partial_{x_1}\widetilde{f}$ likewise only depend on the restriction of $\widetilde{f}$ to such lines. Our problem is now reduced to one for the C^p maps from $[0,\infty)$ to V' given by $x \mapsto f(x, a_2, \dots, a_n)$.

Step 3. For $p \geq 1$ consider a C^p mapping $f : [0,\infty) \to V'$ and define $\widetilde{f} : \mathbf{R} \to V'$ by

$$\widetilde{f}(x) = \begin{cases} f(x), & x \geq 0 \\ \sum_{j=0}^{p'} c_j f(-b_j x), & x < 0, \end{cases}$$

with $p' \geq 1$. Assume $\sum_{j=0}^{p'} c_j (-b_j)^i = 1$ for $0 \leq i \leq p$. We claim that $\widetilde{f}$ is differentiable on $\mathbf{R}$ and

$$\widetilde{f}'(x) = \begin{cases} f'(x), & x \geq 0 \\ \sum_{j=0}^{p'} (-b_j c_j) f'(-b_j x), & x < 0. \end{cases}$$

This will certainly complete our analysis of $\partial_{x_1}\widetilde{f}$ in the multivariable setup on Σ' above. By choosing a basis for V', we get component functions $(f_1,\dots,f_N)$ for f and $(\widetilde{f}_1,\dots,\widetilde{f}_N)$ for $\widetilde{f}$, and each $\widetilde{f}_j$ is related to f_j exactly as $\widetilde{f}$ is related to f (except that $\widetilde{f}_j$ and f_j take values in $\mathbf{R}$ rather than in V'). The problem of differentiability and computing its value at a point are componentwise problems, and the proposed formula for the derivative of $\widetilde{f}$ in terms of f' is also well-behaved with respect to passage to component functions. Hence, we may now suppose $V' = \mathbf{R}$, thereby putting ourselves in the usual setting of one-variable calculus.

The situation at $x \neq 0$ is clear, and so we just have to show that $\widetilde{f}'(0)$ exists and is given by $f'(0)$. In one-variable calculus, we learned that if an $\mathbf{R}$-valued function on an open interval is differentiable away from one point and its derivatives away from that point have a limiting value L at the point, then the function is necessarily differentiable at the point with derivative equal to L (see Theorem 7 in Chapter 11 of Spivak's *Calculus*). Thus, our problem is to prove that the established formulas for $\widetilde{f}'(x)$ for $x > 0$ and $x < 0$ have a common limiting value $f'(0)$ as $x \to 0$. That is, we want the limits

$$\lim_{x \to 0^+} f'(x), \quad \lim_{x \to 0^-} \sum_{j=0}^{p'} (-b_j c_j) f'(-b_j x)$$

to exist and equal $f'(0)$. Since f is a C^1 function on $[0, \infty)$, the first limit exists and equals $f'(0)$. Thus, the problem is to prove that as $x \to 0^+$, we have

$$\sum_{j=0}^{p'} (-b_j c_j) f'(b_j x) \to f'(0).$$

Since $\sum_{j=0}^{p'} -b_j c_j = 1$, it suffices to show $f'(b_j x) \to f'(0)$ for each $0 \leq j \leq p'$. But this latter limit again follows from the fact that f' is continuous on $[0, \infty)$. This completes the induction on $p \leq p'$ (for an arbitrary but fixed p'), and so settles the case of C^p mappings for $0 \leq p \leq \infty$.

Step 4. Now we turn to the C^∞ case. The above method does not work here, as the system of conditions on the b_j's and c_j's would be an infinite system of infinite series identities, and difficult convergence problems arise (e.g., the signs of the c_j's cannot be controlled). However, the C^∞ case offers a tool not available in the C^p case for finite p: we can try to use the higher partials of f in the construction. This is forbidden in the C^p case for finite p because in such cases the higher derivatives of f generally have lower order of differentiability that f and thus the use of such derivatives ruins any chance of making extended maps that have the same order of differentiability as f. Our method will be a "parametric" generalization of the classical 1-variable theorem of E. Borel that constructed $h \in C^\infty(\mathbf{R})$ with any pre-specified Taylor coefficients at the origin. (The "parametric" aspect for us is the intervention of $x_2, \dots, x_n$.)

We have $\Sigma = [0, \infty)^r \times \mathbf{R}^{n-r}$ and a C^∞ mapping $f : \Sigma \to V'$ with $0 \leq r \leq n$. We want to extend f to a C^∞ mapping $B \to V'$ with B an open set around the origin in $\mathbf{R}^n$. By multiplying f against (the restriction to Σ of) a C^∞ function on $\mathbf{R}^n$ that is supported near the origin and equal to 1 near the origin, we may assume f is compactly supported. In this case, we shall make a compactly supported C^∞ extension of f to all of $\mathbf{R}^n$. The case $r = 0$ is trivial, so we may assume $1 \leq r \leq n$. Let $\Sigma' = \mathbf{R} \times [0, \infty)^{r-1} \times \mathbf{R}^{n-r}$. We will extend f to a compactly supported C^∞ map $\widetilde{f} : \Sigma' \to V'$. Thus, an induction on r would then complete the proof. Our problem is now of the same shape as in the C^p case for finite p, except that we must make sure our output has compact support (as the construction on Σ' will use the compactness of the support of f on Σ, so the induction on r won't work unless this property is preserved at each step).

Let $\varphi \in C^\infty(\mathbf{R})$ satisfy $\varphi(t) = 1$ for $|t| \leq 1/2$ and $\varphi(t) = 0$ for $|t| \geq 1$, so the higher derivatives $\varphi^{(j)}$ vanish on $[-1/2, 1/2]$ for all $j > 0$. For $j \geq 0$, define $c_j : \Sigma' \cap \{x_1 = 0\} = [0, \infty)^{r-1} \times \mathbf{R}^{n-r} \to V'$ by

$$c_j(\xi) = \frac{(\partial_{x_1}^j f)(\xi)}{j!},$$

so c_j is a compactly-supported C^∞ mapping since f is compactly supported and C^∞ on $\Sigma = [0, \infty)^r \times \mathbf{R}^{n-r}$; we shall write c_j as a function of $x_2, \dots, x_n$. Define $\widetilde{f}_j : \Sigma' \to V'$ as follows:

$\widetilde{f}_0(x_1,\ldots,x_n) = c_0(x_2,\ldots,x_n) = f(0,x_2,\ldots,x_n)$ and for $j>0$

$$\widetilde{f}_j(x_1,\ldots,x_n) = \varphi(A_j x_1)x_1^j c_j(x_2,\ldots,x_n) \tag{6.2}$$

with constants $A_j \geq j$ to be determined. Note that $\widetilde{f}_j$ is visibly compactly supported and C^∞ on Σ' for all $j \geq 0$. More specifically, all $\widetilde{f}_j$'s are supported in a common compact set, such as $[-1,1]\times K$ with K equal to where $\Sigma' \cap \{x_1 = 0\} = \Sigma \cap \{x_1 = 0\}$ meets a large hypercube in Σ whose interior in Σ contains the compact support of f.

Lemma 6.2. *Fix a norm $\|\cdot\|$ on V'. For $j \geq 1$ there exists $A_j \geq j$ such that $\|\partial_{x_1}^i \widetilde{f}_j(\sigma')\| \leq 1/2^j$ for all $\sigma' \in \Sigma'$ and $0 \leq i \leq j-1$. Explicitly, it suffices to take*

$$A_j \geq \max(j, \theta_j(1 + \sup_{\xi\in\Sigma'\cap\{x_1=0\}} \|\partial_{x_1}^j f(\xi)\|)),$$

with $\theta_j = 1 + \max_{0\leq k\leq j-1}\sup_{\mathbf{R}}|\varphi^{(k)}|$ a universal constant depending only on j and φ but not on f (nor on n or r).

Proof. Fix $j \geq 1$. For $0 \leq i \leq j-1$, the Leibniz rule for higher derivatives of products gives

$$\partial_{x_1}^i \widetilde{f}_j = \left(\sum_{k=0}^{i} C_{j,k,i} A_j^k \varphi^{(k)}(A_j x_1) x_1^{j-(i-k)}\right)\cdot c_j(x_2,\ldots,x_n)$$

with $C_{j,k,i} = \binom{i}{k}\cdot\prod_{\alpha=0}^{i-k-1}(j-\alpha) > 0$ a universal constant arising from factorials (if $i-k=0$ then $C_{j,k,i}=1$). Let

$$C_j = \max_{0\leq k\leq i\leq j-1} C_{j,k,i}$$

(this is a universal constant depending only on j), and let $M_j = M_j(f) \geq 0$ be the supremum of the compactly supported and continuous function $\xi \mapsto \|c_j(\xi)\|$ on $\Sigma' \cap \{x_1 = 0\}$. Since the formula for $\partial_{x_1}^i \widetilde{f}$ has a sum of $i+1$ terms and $i+1 \leq j$, it suffices to find $A_j \geq j$ such that

$$A_j^k|\varphi^{(k)}(A_j x_1)x_1^{j-(i-k)}| \leq \frac{1}{j(1+M_j(f))C_j 2^j}$$

for all $x_1 \in \mathbf{R}$ and all $0 \leq k \leq i \leq j-1$. Thus, it suffices to show more generally that for any $\varepsilon > 0$ there exists $A_j = A_j(\varepsilon) \geq j$ such that

$$A_j^k|\varphi^{(k)}(A_j x_1)x_1^{j-(i-k)}| \leq \varepsilon$$

for all $x_1 \in \mathbf{R}$ and all $0 \leq k \leq i \leq j-1$; taking $\varepsilon = 1/(jC_j 2^j(1+M_j(f)))$ then gives the desired result.

Regardless of what $A_j \geq j \geq 1$ we consider, $x_1 \mapsto \varphi^{(k)}(A_j x_1)$ is a continuous function on $\mathbf{R}$ that vanishes outside of $[-1/A_j, 1/A_j]$, and so we just need verify the desired inequality on $[-1/A_j, 1/A_j]$. Let B_j be a positive upper bound on the compactly supported functions $x_1 \mapsto |\varphi^{(k)}(x_1)|$ on $\mathbf{R}$ for $0 \leq k \leq j-1$, so it suffices to have $A_j^k B_j |x_1|^{j-(i-k)} \leq \varepsilon$ for all $|x_1| \leq 1/A_j$ and all $0 \leq k \leq i \leq j-1$. The maximum values are attained at $x_1 = \pm 1/A_j$, so the condition is $A_j^{i-j} \leq \varepsilon/B_j$ for $0 \leq i \leq j-1$. Provided $A_j \geq j$, it suffices to have $1/A_j \leq \varepsilon/B_j$. Hence, we take $A_j = \max(j, B_j/\varepsilon)$. ∎

We now define $A_0 = 0$, so (6.2) is valid for $j = 0$, and for $j \geq 1$ we choose

$$A_j = \max(j, \theta_j(1 + \sup_{\substack{\xi\in\Sigma'\cap\{x_1=0\}\\ i_2+\cdots+i_n\leq j}} \|\partial_{x_1}^j \partial_{x_2}^{i_2}\ldots\partial_{x_n}^{i_n} f(\xi)\|)) \geq \max(j, \theta_j \cdot \sup_{\xi\in\Sigma'\cap\{x_1=0\}} \|\partial_{x_1}^j f(\xi)\|).$$

By Lemma 6.2 with $i = 0$, for $j \geq 1$ we have $\|\widetilde{f}_j(\sigma')\| \leq 1/2^j$ for all $\sigma' \in \Sigma'$. Thus, $\widetilde{f} = \sum_{j \geq 0} \widetilde{f}_j$ is a uniformly convergent sum of continuous mappings from Σ' to V', and hence $\widetilde{f}$ is a continuous mapping from Σ' to V'. Since $A_j \to \infty$, near any particular point in $\Sigma' \cap \{x_1 \neq 0\}$ the sum defining $\widetilde{f}$ is a *finite* sum, so $\widetilde{f}$ is certainly C^∞ near there and its iterated partials may be computed as the sum of the corresponding partials of the $\widetilde{f}_j$'s. Since the $\widetilde{f}_j$'s share a common compact support, clearly $\widetilde{f}$ has compact support.

Step 5. We claim that $\widetilde{f}$ is a C^∞ map on Σ' and that at each point of $\Sigma' \cap \{x_1 = 0\} = \Sigma \cap \{x_1 = 0\}$ its iterated partials coincide with those of $f : \Sigma \to V'$. Granting this, the C^∞ mappings

$$\widetilde{f} : \Sigma' \cap \{x_1 \leq 0\} \to V', \;\; f : \Sigma' \cap \{x_1 \geq 0\} \to V'$$

on "adjacent" r-sectors in $\mathbf{R}^n$ have identical iterated partials at all points of $\Sigma' \cap \{x_1 = 0\}$. Thus, the map $\Sigma' \to V'$ defined by f for $x_1 \geq 0$ and $\widetilde{f}$ for $x_1 \leq 0$ is C^∞, and so it would thereby solve our problem, due to:

Lemma 6.3. *Let V be a finite-dimensional $\mathbf{R}$-vector space and let $\ell_1, \dots, \ell_s$ be linearly independent in $V^\vee$ with $s \geq 1$. Pick $c_1, \dots, c_s \in \mathbf{R}$ and let*

$$\Sigma_+ = \{\ell_1 \geq c_1, \ell_2 \geq c_2, \dots, \ell_s \geq c_s\}, \;\; \Sigma_- \{\ell_1 \leq c_1, \ell_2 \geq c_2, \dots, \ell_s \geq c_s\}$$

be s-sectors in V. Let $f_\pm : \Sigma_\pm \to V'$ be C^m mappings to a finite-dimensional vector space, with $0 \leq m \leq \infty$. If f_+ and f_- agree on the "common face" $\Sigma_0 = \Sigma_+ \cap \Sigma_-$ and $D^j f_+(x) = D^j f_-(x)$ as multilinear mappings from V^j to V' for all $1 \leq j \leq m$ (meaning $1 \leq j < \infty$ if $m = \infty$) then on the $(s-1)$-sector $\Sigma = \Sigma_+ \cup \Sigma_- = \{\ell_2 \geq c_2, \dots, \ell_s \geq c_s\}$ the map $f : \Sigma \to V'$ defined by $f_\pm$ on $\Sigma_\pm$ is a C^m mapping.

In terms of a linear coordinate system on V including the ℓ_j's, the equality of total derivative mappings of orders $\leq m$ is the same as the condition of all equaity for iterated partials of orders $\leq m$. We have essentially already proved this lemma in our proof of Whitney's theorem for C^p mappings with finite p, but we give the details for the convenience of the reader.

Proof. By making a linear translation and choosing a linear coordinate system on V including the ℓ_j's, we reduce ourselves to the following problem on $V = \mathbf{R}^n$ with $1 \leq s \leq n$. Let

$$\Sigma = \{x_2 \geq 0, \dots, x_s \geq 0\}$$

(so $\Sigma = \mathbf{R}^n$ if $s = 1$), and let $\Sigma_\pm = \Sigma \cap \{\pm x_1 \geq 0\}$. We are given C^m mappings $f_\pm : \Sigma_\pm \to V'$ whose partials up to order m coincide on $\Sigma \cap \{x_1 = 0\}$. (In particular, in order 0 this says f_+ and f_- coincide on $\Sigma \cap \{x_1 = 0\}$.) We want to prove that the mapping $f : \Sigma \to V'$ defined by $f_\pm$ on $\Sigma_\pm$ is C^m. It certainly suffices to treat the case $0 \leq m < \infty$, and so we may induct on m. The only difficulty is at points on $\Sigma \cap \{x_1 = 0\}$, as all other points on Σ are in the interior of either Σ_+ or Σ_- with respect to Σ. The case $m = 0$ is obvious via the sequential criterion for continuity, and so we can assume $m > 0$ and that the result is known in the case of C^{m-1} mappings.

It suffices to prove that the first-order partials of f exist at all points of Σ and are of class C^{m-1} (as this certainly implies f is of class C^m). For each $1 \leq i \leq n$ the partial derivative $\partial_{x_i} f_\pm$ on $\Sigma_\pm$ is of class C^{m-1}, and by hypothesis $\partial_{x_i} f_+$ and $\partial_{x_i} f_-$ agree on $\Sigma \cap \{x_1 = 0\}$. Hence, by induction these "glue" to define a C^{m-1} mapping $f_i : \Sigma \to V'$. We just have to prove that $\partial_{x_i} f$ exists on Σ and is equal to f_i. At points not in $\Sigma \cap \{x_1 = 0\}$ this is clear, so choose $\xi \in \Sigma \cap \{x_1 = 0\}$. For $2 \leq i \leq n$, the existence and evaluation of $\partial_{x_i} f(\xi)$ only depends on the restriction of f to the hyperplane slice $\Sigma \cap \{x_1 = 0\}$ (in the sense that it may be computed in terms of this restriction alone) because such a partial is computed with all variables except for the ith one fixed. This restriction of f agrees with the restriction of the mapping $f_\pm$ that is C^m on $\Sigma_\pm$, and so this restriction of f is certainly also C^m on $\Sigma \cap \{x_1 = 0\}$ (with $m \geq 1$). Hence, for $2 \leq i \leq n$ the partial $\partial_{x_i} f(\xi)$ indeed exists and is equal to $\partial_{x_i} f_\pm(\xi) = f_i(\xi)$, as desired.

Finally, we consider the situation for $\partial_{x_1} f(\xi)$. Being a problem with an x_1-partial at a point $\xi = (0, a_2, \ldots, a_n)$, we may restrict ourselves to the line with $x_i = a_i$ for $2 \leq i \leq n$. Thus, we are faced with a "one variable" problem:

$$f_\pm : \{t \in \mathbf{R} \mid \pm t \geq 0\} \to V'$$

is a pair of C^m mappings whose derivatives up to order $m \geq 1$ agree at the origin, and we want the "glued" function $f : \mathbf{R} \to V'$ to be differentiable at the origin with $f'(0)$ equal to the common value $f'_\pm(0) \in V'$. Choosing a basis of V' and passing to component functions brings us to a problem in 1-variable calculus: we have a function $f : \mathbf{R} \to \mathbf{R}$ whose restrictions to $\{t \leq 0\}$ and $\{t \geq 0\}$ are C^m with $m \geq 1$ and the restrictions

$$f_- : (-\infty, 0] \to \mathbf{R}, \;\; f_+ : [0, \infty) \to \mathbf{R}$$

have a common (one-sided) derivative L at the origin. We want f to be differentiable at 0 with $f'(0) = L$. Since $f_\pm$ is C^1 on its domain, $f'_\pm(t) \to f'_\pm(0) = L$ as $t \to 0^\pm$. Thus, we may again use the result from Spivak's *Calculus* concerning the existence of (and value for) a derivative at a point when a function is differentiable on a punctured open neighborhood in $\mathbf{R}$ with derivative that admits a two-sided limiting value at the point. ■

Step 6. Returning to our main program, it remains to analyze the evaluation and differentiability properties of the continuous $\widetilde{f}$ on Σ'; our problem (as explained before the preceding lemma) is to prove that $\widetilde{f}$ is C^∞ on Σ' and that its iterated partials at points of $\Sigma' \cap \{x_1 = 0\} = \Sigma \cap \{x_1 = 0\}$ agree with those of $f : \Sigma \to V'$. As we have seen, all of the problems are at points on $\Sigma' \cap \{x_1 = 0\}$: to see the C^∞ property at these points (considered as points in Σ') and to check that the iterated partials of $\widetilde{f}$ at these points agree with those of f. Near all other points, $\widetilde{f}$ has already been seen to be C^∞ with the series $\sum_{j \geq 0} \widetilde{f}_j$ defining $\widetilde{f}$ locally finite and therefore amenable to iterated partial differentiation termwise.

For any point $\xi \in \Sigma' \cap \{x_1 = 0\}$ we have

$$\widetilde{f}(\xi) = \sum_{j \geq 0} \widetilde{f}_j(\xi) = \widetilde{f}_0(\xi) = f(\xi)$$

since $\widetilde{f}_j(0, x_2, \ldots, x_n) = 0$ for $j > 0$. We need to study how each of the operators $\partial_{x_1}, \ldots, \partial_{x_n}$ interact with the formation of $\widetilde{f}$ from the $\widetilde{f}_j$'s. We first study ∂_{x_1}. By Lemma 6.2, for any $i \geq 1$ the sum $\sum_{j \geq 0} \partial^i_{x_1} \widetilde{f}_j$ is uniformly convergent, and so the theorem on uniform convergence of termwise derivatives implies that the uniformly convergent sum $\widetilde{f} = \sum_{j \geq 0} \widetilde{f}_j$ is infinitely differentiable with respect to x_1, and that $\partial^i_{x_1} \widetilde{f}$ is continuous and equal to the uniformly convergent sum $\sum_{j \geq 0} \partial^i_{x_1} \widetilde{f}_j$ for all $i \geq 0$. By the definition of $\widetilde{f}_j$ we see that at points on $\Sigma' \cap \{x_1 = 0\}$ the partial $\partial^i_{x_1} \widetilde{f}_j$ vanishes for $j \neq i$ (keep in mind that $\varphi = 1$ on $[-1/2, 1/2]$, so the higher derivatives of φ vanish on $[-1/2, 1/2]$) and it has value $\partial^i_{x_1} f$ for $j = i$. Hence,

$$(\partial^i_{x_1} \widetilde{f})|_{\Sigma' \cap \{x_1 = 0\}} = (\partial^i_{x_1} f)|_{\Sigma \cap \{x_1 = 0\}} \tag{6.3}$$

on the common domain $\Sigma' \cap \{x_1 = 0\} = \Sigma \cap \{x_1 = 0\}$ for all $i \geq 0$.

Observe that up to now we have not required A_j to be as large as we have made it; we have only used that it is as large as required by Lemma 6.2 (for f). The inclusion of partials of f with respect to $x_2, \ldots, x_n$ in the definition of A_j will now come into play. Consider the problem of existence and continuity for iterated partials of f with respect to $x_2, \ldots, x_n$. By the theorem on termwise differentiability under a uniform convergence hypothesis on the sum of derivatives, in order that $\partial_{x_i}\widetilde{f}$ exist for a fixed $1 \le i \le n$ it suffices for the sum $\sum_{j\ge0}\partial_{x_i}\widetilde{f}_j$ to be uniformly convergent. For $2 \le i \le n$, an inspection of the definitions of c_j and $\widetilde{f}_j$ shows that $\partial_{x_i}\widetilde{f}_j$ is related to $\partial_{x_i}f$ exactly as $\widetilde{f}_j$ is related to f, except that we have to determine if the constants A_j are "big enough" to ensure uniform convergence for $\sum\partial_{x_i}\widetilde{f}_j$. Provided $j \ge 1$, the definition of A_j includes $\partial_{x_1}^j\partial_{x_i}f$ on $\Sigma' \cap \{x_1 = 0\}$. Thus, our A_j as defined above is "big enough" in the sense of Lemma 6.2 for $\partial_{x_i}f$ in the role of f. We conclude that $\partial_{x_i}\widetilde{f}$ exists for $2 \le i \le n$ and it may be computed by termwise differentiation: it is equal to $\sum_{j\ge0}\partial_{x_i}\widetilde{f}_j$, with this latter sum uniformly convergent (and hence is *continuous* on Σ').

More generally, for any sequence $\{i_1, \dots, i_N\}$ in $\{2, \dots, n\}$ consider the problem of existence of $\partial_{x_{i_N}}\dots\partial_{x_{i_1}}\widetilde{f}$ and whether it can be computed by termwise on $\sum\widetilde{f}_j$. For $j \ge N$, the definition of A_j includes

$$\partial_{x_1}^j\partial_{x_{i_N}}\dots\partial_{x_{i_1}}f$$

on $\Sigma' \cap \{x_1 = 0\}$, so by inducting on N and using Lemma 6.2 for $\partial_{x_{i_N}}\dots\partial_{x_{i_1}}f$ and $j \ge N$ it follows that $\partial_{x_{i_N}}\dots\partial_{x_{i_1}}\widetilde{f}$ exists and is given by termwise differentiation:

$$\partial_{x_{i_N}}\dots\partial_{x_{i_1}}\widetilde{f} = \sum_{j\ge0}\partial_{x_{i_N}}\dots\partial_{x_{i_1}}\widetilde{f}_j,$$

with this sum uniformly convergent (strictly speaking, Lemma 6.2 applied here just gives uniform convergence for the summation over $j \ge N$, but adding in finitely many more times does not affect uniform convergence). Thus, $\widetilde{f}$ admits iterated partials of all orders with respect to $x_2, \dots, x_n$ and all such partials may be computed by termwise differentiation of $\sum\widetilde{f}_j$ with the resulting sums again uniformly convergent (and hence *continuous* on Σ').

More specifically, if ∂ is a composite of finitely many partial derivative operators with respect to $x_2, \dots, x_n$, then we have proved $\partial\widetilde{f} = \sum\widetilde{(\partial f)}_j$ with this sum uniformly convergent, and with the definition of $\widetilde{(\partial f)}_j$ using the constants A_j that are "big enough" for ∂f in Lemma 6.2 (at least for $j \ge 1$ as large as the order N of the differential operator ∂). The upshot is that after applying such an operator ∂, we may use our earlier study of $\partial_{x_1}^i\widetilde{f}$, but with ∂f replacing f, to conclude that $\partial_{x_1}^i\partial\widetilde{f}$ exists for all $i \ge 0$ and it may be computed termwise:

$$\partial_{x_1}^i\partial\widetilde{f} = \sum_{j\ge0}\partial_{x_1}^i\partial\widetilde{f}_j \tag{6.4}$$

with this summation uniformly convergent and hence continuous on Σ'. Beware that we have not yet proved that $\widetilde{f}$ admits continuous iterated partials when the x_1-partials are not all clumped "at the end". But the terms in the sum (6.4) are *insensitive* to switching the order of partial differentiation! Thus, in view of the uniformity of the convergence in (6.4), by inducting on the order of differential operators we may now run the process in reverse to conclude that $\widetilde{f}$ does admit all iterated partials of all orders and that these may be computed termwise as uniformly convergent (and hence continuous) sums. Thus, $\widetilde{f} = \sum_{j\ge0}\widetilde{f}_j$ is indeed a C^∞ mapping and its partials may be computed termwise as uniformly convergence sums.

The ability to compute partials of $\widetilde{f} = \sum\widetilde{f}_j$ termwise now lets us work out the Taylor coefficients of $\widetilde{f}$ at points on $\Sigma' \cap \{x_1 = 0\}$. Since we may compute the partials in any order we please, we shall compute $\partial_{x_1}^{i_1}\dots\partial_{x_n}^{i_n}\widetilde{f}$ for $i_1, \dots, i_n \ge 0$. The application of $\partial = \partial_{x_2}^{i_2}\dots\partial_{x_n}^{i_n}$ amounts to replacing f with ∂f in the construction (and noting that the constants A_j we have chosen are "big enough" for

Lemma 6.2 applied to ∂f, at least if $j \geq i_1 + \cdots + i_n$). Thus, up to renaming ∂f as f and noting that the general computation of $\partial^i_{x_1} \widetilde{f}$ by termwise operations on the $\widetilde{f}_j$'s only requires constants A_j as large as in Lemma 6.2 (at least for j sufficiently large), we can reduce our problem to that of comparing $\partial^i_{x_1} \widetilde{f}$ and $\partial^i_{x_1} f$ at points $\xi \in \Sigma' \cap \{x_1 = 0\}$. This comparison of x_1-partials was verified in general in (6.3), so we are done. ■