Math 396. Completions

We wish to develop some of the basic properties of complete metric spaces. We begin with a review of some definitions, prove some basic lemmas, and then give the general construction of completions and formulate universal properties thereof.

1. Definitions

Let X be a metric space with metric ρ. Recall that we say a sequence $\{x_n\}$ in X is *Cauchy* if, for all $\varepsilon > 0$, we have $\rho(x_n, x_m) < \varepsilon$ for $n, m \geq N_\varepsilon$ for some suitably large integer N_ε. We know that convergent sequences must be Cauchy, but the converse is very false. More specifically, although finite-dimensional normed vector spaces over $\mathbf{R}$ are complete, for infinite-dimensional spaces this breaks down. For example, it can be shown the quotient of the space of Riemann-integrable functions on $[a, b]$ modulo the subspace of those with "absolute integral 0" has a natural structure of normed space for which there are lots of non-convergent Cauchy sequences (this is not entirely trivial to establish).

In the development of calculus, it is crucial that Cauchy sequences in $\mathbf{R}$ be convergent. One needs to observe that the theory of $\mathbf{R}$ is rather special because we measure distance in $\mathbf{R}$ using $\mathbf{R}$ itself. Hence, the formulation of the completeness property for $\mathbf{R}$ is necessarily a little peculiar (we can either speak in terms of Cauchy sequences in *ordered* fields F, using "positive" $\varepsilon \in F$, or in terms of the least upper bound axiom). In any case, in the foundations of calculus (e.g., Math 295) it was shown that the field $\mathbf{R}$ occupies a distinguished role in mathematics: it is the *unique* ordered field satisfying the least upper bound axiom, in the strong sense that any two such fields are *uniquely* isomorphic to each other in a manner which respects the field structure (and order structure compatibility comes along for free).

Since $\mathbf{R}$ is so special, and plays a key role in the very *definition* of metric spaces (it's where the metric takes values!), what we are about to do is really most naturally imagined as a procedure which takes place after the theory of $\mathbf{R}$ has been set up. That said, if you think back to the construction of $\mathbf{R}$ from $\mathbf{Q}$ in terms of Cauchy sequences of rationals then you will recognize a strong similarity between that construction for "filling in the holes in $\mathbf{Q}$" and what we do below to "fill in the holes" in non-complete metric spaces. The important point to appreciate at the start is that the existence of a theory of $\mathbf{R}$ is a necessary prerequisite to get the theory of metric spaces off the ground. Thus, one can't quite use Theorem 2.3 below to construct $\mathbf{R}$ (in fact, the proof of Theorem 2.3 uses Lemma 2.10, whose proof in turn makes essential use of the convergence of Cauchy sequences in $\mathbf{R}$).

The take-off point for everything is:

Definition 1.1. We say that (X, ρ) is *complete* if all Cauchy sequences in X are convergent.

It is by no means obvious a priori whether every metric space X can be realized inside of a complete metric space X'. We actually seek such an X' which is "as small as possible". This corresponds to the idea that although $\mathbf{C}$ contains $\mathbf{Q}$ and is complete as a metric space, it is really $\mathbf{R}$ and not $\mathbf{C}$ than one gets when "filling in the holes" in $\mathbf{Q}$. In order to make precise such minimality, we make the:

Definition 1.2. Let X' be a metric space and $X \subseteq X'$ a subset. We say that X is *dense* in X' if every element of X' is a limit of a sequence of elements in X.

For example, $\mathbf{Q}$ is dense in $\mathbf{R}$, and more generally $\mathbf{Q}^n$ is dense in $\mathbf{R}^n$ but $\mathbf{Q}$ is not dense in $\mathbf{C}$. An equivalent formulation of the condition of density is given by:

Lemma 1.3. *Let X' be a metric space, and X a subset of X'. Then X is dense in X' if and only if $X \cap U \neq \emptyset$ for all non-empty opens U in X', and also if and only if the closure $\overline{X}$ of X in X' coincides with all of X'.*

Proof. First assume that X is dense in X' in the sense of Definition 1.2. Let U be a non-empty open in X'. We will prove that U meets X. Choose $u \in U$, so some ball $B_r(u)$ lies entirely inside of U. But since (by hypothesis) every element of X' (such as u) is a limit of a sequence in X, we conclude that $u = \lim x_n$ for some sequence $\{x_n\}$ in X. In particular, for large n we have $\rho(u, x_n) < r$, so $x_n \in B_r(x) \subseteq U$ for large n. Hence, U meets X (as all x_n lie in X).

Conversely, assuming that X meets every non-empty open, we will prove that every $x' \in X'$ has the form $x' = \lim x_n$ with $x_n \in X$ for all n. Well, $B_{1/n}(x')$ is an open which is non-empty (it contains x'!), so it meets X. Thus, we can choose $x_n \in B_{1/n}(x') \cap X$ for all n. Since $\rho(x_n, x') < 1/n$, it is clear that the sequence $\{x_n\}$ in X has limit x'. This proves the equivalence of denseness and the property of meeting all non-empty opens.

It remains to check that $X \cap U \neq \emptyset$ for all non-empty open U in X' if and only if $\overline{X} = X'$. We give a purely topological proof (and we urge the reader to give an alternative proof in terms of limits of sequences in the context of metric spaces). Since non-empty opens in X' are exactly the complements of proper closed subsets in X', and $X \cap U$ is non-empty if and only if X is not contained in $X' - U$, we conclude that $X \cap U$ is non-empty for all non-empty open U of X' if and only if X is not contained in a proper closed subset of X', which is to say that the unique minimal closed subset $\overline{X}$ containing X is the entire space X'. ■

Remark 1.4. The equivalence of the last two conditions in the lemma was proved for X' an arbitrary topological space, and so for general topological spaces X' and subsets X we take these equivalent conditions as the *definition* of density for X in X' (as we have just proved that in the setting of metric spaces it is equivalent to the more intuitive notion of density as was just defined in terms of sequences).

We will show below that any metric space can be realized as a dense subset of a complete space. The importance of dense subsets is due to the following two results.

Theorem 1.5. *Let $f, g : X' \rightrightarrows Y$ be continuous maps between two metric spaces, and $X \subseteq X'$ a dense subset. If $f(x) = g(x)$ for all $x \in X$, then $f = g$.*

For example, a *continuous* map $f : [a, b] \to \mathbf{R}$ (with $a < b$) is uniquely determined by the values $f(q)$ for $q \in [a, b] \cap \mathbf{Q}$. Of course, this is false without continuity required.

Proof. Choose $x' \in X'$. By hypothesis, $x' = \lim x_n$ with $x_n \in X$. Thus, $f(x_n) = g(x_n)$ in Y for all n. Passing to the limit, the continuity of f and g yields

$$f(x') = \lim f(x_n) = \lim g(x_n) = g(x').$$

■

Beware that it is *not* true that a continuous function $f : X \to Y$ on a dense subset $X \subseteq X'$ necessarily extends to a continuous function $X' \to Y$. For example, we can take $X' = Y = \mathbf{R}$, $X = \mathbf{R} - \{0\}$, and $f(x) = 1/x$. However, if we impose a mild condition to rule out "blowing up" and we assume that the target space has "no holes" (i.e., is complete) then things work well:

Theorem 1.6. *Let X' and Y be metric spaces and let X be a dense subset of X'. Assume that Y is complete.*

Let $f : X \to Y$ be a map, and assume f is "bounded" in the sense that

$$\rho_Y(f(x_1), f(x_2)) \leq M\rho_{X'}(x_1, x_2)$$

for all $x_1, x_2 \in X$, with $M > 0$ some constant; in particular, f is visibly continuous (use $\delta = \varepsilon/M$). Then there is a unique way to fill in a commutative triangle (with the top row the isometric inclusion)

$$\begin{array}{ccc} X & \longrightarrow & X' \\ & \searrow^{f} & \downarrow{\scriptstyle f'} \\ & & Y \end{array}$$

with a continuous f'. Moreover, $\rho_Y(f'(x'_1), f'(x'_2)) \leq M\rho_{X'}(x'_1, x'_2)$ for all $x'_1, x'_2 \in X'$.

This is *false* if we drop the completeness assumption on Y. For example, if we take $X' = \mathbf{R}$, $X = Y = \mathbf{Q}$, and f the identity map on $\mathbf{Q}$, there is no way to extend this to a continuous map $f' : \mathbf{R} = X' \to Y = \mathbf{Q}$. This theorem shows that, in the presence of completeness, under mild boundedness conditions we can do constructions by working with a dense subset. As one application, this result can be used to construct a theory of (continuous) exponential functions $x \mapsto a^x$ (for fixed $a > 0$) by interpolation through the elementary case of $x \in \mathbf{Q}$ (provided that one proves a suitable boundedness theorem for a^x as a function of $x \in \mathbf{Q}$).

Proof. Let's first note that if any such f' exists, it is uniquely determined. Indeed, this is immediate from Theorem 1.5. The interesting part is to actually construct f'. This goes as follows. Choose any $x' \in X'$, so $x' = \lim x_n$ for some sequence $\{x_n\}$ in X. If such a continuous f' is going to exist, then we'd better have that $f'(x')$ is the limit of the sequence $\{f'(x_n)\} = \{f(x_n)\}$. In other words, we are motivated to "define"

$$f'(x') = \lim f(x_n)$$

where $\{x_n\}$ is a sequence in X with limit x'. Of course, this is not a valid definition until we show two things:

- the limit $\lim f(x_n)$ actually exists;
- the limit depends only on $x' \in X'$ and not also on the specific choice of sequence $\{x_n\}$ in X with limit x' (as there are billions and billions of such sequences for a fixed x').

The existence of the limit will rest on the boundedness assumption and the completeness of Y, and the independence of the choice of $\{x_n\}$ will use the boundedness assumption on f. Since Y is *complete*, a sequence in Y converges if and only if it is Cauchy. Thus, we want $\{f(x_n)\}$ to be a Cauchy sequence. But

$$\rho_Y(f(x_n), f(x_m)) \leq M\rho_{X'}(x_n, x_m)$$

and since $\{x_n\}$ has a limit (namely x') it is a Cauchy sequence in X'. That is, for all $\varepsilon > 0$ there exists N_ε such that when $n, m \geq N_\varepsilon$ then $\rho(x_n, x_m) < \varepsilon$. For the same large n and m we then get

$$\rho_Y(f(x_n), f(x_m)) < M\varepsilon.$$

Replacing ε with ε/M in this calculation yields the Cauchy property of $\{f(x_n)\}$ as desired.

To show that $\lim f(x_n) = \lim f(z_n)$ for any two sequences $\{x_n\}$ and $\{z_n\}$ in X with limit x' (and hence our "definition" of $f'(x') \in Y$ is valid after all), we use boundedness. Namely, we have

$$\rho_Y(f(x_n), f(z_n)) \leq M\rho_{X'}(x_n, z_n)$$

and $\rho_{X'}(x_n, z_n) \to 0$ since $x_n \to x'$ and $z_n \to x'$. Thus, $\rho_Y(f(x_n), f(z_n)) \to 0$ as $n \to \infty$. If we let $L = \lim f(x_n)$ and $L' = \lim f(z_n)$, then by *continuity* of the map

$$\rho_Y : Y \times Y \to \mathbf{R}$$

(see HW 2), we conclude

$$\rho_Y(L, L') = \lim \rho_Y(f(x_n), f(z_n)) = 0,$$

whence $L = L'$ as desired.

This completes the proof that the only possible definition of our desired continuous $f' : X' \to Y$ actually makes sense, and we just have to show that this f' satisfies $f'(x) = f(x)$ for $x \in X$ and that

$$\rho_Y(f'(x'_1), f'(x'_2)) \leq M\rho_{X'}(x'_1, x'_2) \tag{1.1}$$

for all $x'_1, x'_2 \in X'$ (which also forces f' to be continuous, by the usual $\delta = \varepsilon/M$ trick). Since any $x \in X$ is the limit of the constant sequence $\{x, x, \dots\}$, it is clear that $f'(x) = f(x)$ for all $x \in X$.

Now we check (1.1). By definition, if we choose sequences $\{x_n^{(1)}\}$ and $\{x_n^{(2)}\}$ in X with $x_n^{(j)} \to x'_j$ in X' as $n \to \infty$, then we have

$$f'(x'_j) = \lim f(x_n^{(j)}).$$

Thus, by the continuity of the metric function ρ_Y, we have

$$\rho_Y(f'(x'_1), f'(x'_2)) = \lim_{n\to\infty} \rho_Y(f(x_n^{(1)}), f(x_n^{(2)})).$$

By hypothesis on f, we have

$$\rho_Y(f(x_n^{(1)}), f(x_n^{(2)})) \leq M\rho_{X'}(x_n^{(1)}, x_n^{(2)}) \tag{1.2}$$

for all n, and passage to the limit on (1.2) yields the inequality

$$\rho_Y(f'(x'_1), f'(x'_2)) \leq M\rho_{X'}(x'_1, x'_2),$$

where we have also used the continuity of the metric function $\rho_{X'}$ to compute the limit on the right side. ■

2. Filling in the holes

Let X be a metric space. If it is not complete, can we somehow "fill in the holes" in a systematic way? That is, can we realize X as a *dense* subset of a *complete* metric space (with compatible metrics, of course). More specifically, we seek an *isometry* (i.e., a metric-preserving map)

$$\iota : X \to X'$$

with dense image and X' complete (note that isometries are always injective). For example, when $X = \{x \in \mathbf{Q} \mid |x| < 1\}$ with the usual metric, we expect to recover $X' = \{x \in \mathbf{R} \mid |x| < 1\} = (-1, 1)$. Also, if $X = V$ is a normed vector space, we would like such an X' to also have a compatible structure of normed vector space (not just metric space).

A notable application of this is the normed vector space $R[a, b]/Z[a, b]$ of Riemann-integrable functions on $[a, b]$ modulo those satisfying $\int_a^b |f| = 0$. It turns out that this quotient space, which is of fundamental importance, has a natural structure of normed vector space but (by a non-trivial argument) it is *not* complete. Can we enhance it to a complete normed vector space? More interestingly, can this be done in such a way that the elements in the "completed" space can again be interpreted as essentially being *functions* (much like elements of $R[a, b]$)? The answer to this latter question is affirmative (as you will learn in your study of measure theory), and is basic in modern analysis.

Our aim is two-fold: for arbitrary metric spaces X we will prove the existence and (in a very precise sense) uniqueness of a *complete* space X' containing X as a dense subset, and we will show that when X is a normed vector space then this abstract X' also has such extra structure in a unique (compatible) way. The space X' (along with the data of the isometric injection $\iota : X \to X'$!) is then called the *completion* of X; that is, one should think of the pair (X', ι) and not merely the bare space X'. To get things started, let's show that there can be at most one solution to our problem of creating limits for Cauchy sequences.

Lemma 2.1. *Let X be a metric space and $\iota_1 : X \to X_1'$, $\iota_2 : X \to X_2'$ two isometric injections of X into complete metric spaces, with $\iota_j(X) \subseteq X_j'$ a dense subset. Then there exists unique continuous maps*

$$f : X_2' \to X_1', \quad g : X_1' \to X_2'$$

making commutative triangles.

$$\begin{array}{ccc} X & \xrightarrow{\iota_1} & X_1' \\ & \searrow^{\iota_2} & \downarrow f \\ & & X_2' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\iota_2} & X_2' \\ & \searrow^{\iota_1} & \downarrow g \\ & & X_1' \end{array}$$

Moreover, f and g are inverse to each other and are isometries.

Thus, there is a unique way to isometrically identify the metric spaces X_1' and X_2' in a manner which respects how X sits inside of each.

Proof. We'll use the same letter ρ to denote the metric on all of our spaces. This should not cause confusion.

Since the map $\iota_2 : X \to X_2'$ is an isometry and X is dense inside of X_1' via the isometry ι_1, we can apply Theorem 1.6 to the map ι_2 with $M = 1$ to deduce that there exists a unique continuous map $g : X_1' \to X_2'$ with $g \circ \iota_1 = \iota_2$, and in fact

$$\rho(g(x_1'), g(x_1'')) \leq \rho(x_1', x_1'').$$

Likewise, there is a unique continuous map $f : X_2' \to X_1'$ with $f \circ \iota_2 = \iota_1$, and in fact

$$\rho(f(x_2'), f(x_2'')) \leq \rho(x_2', x_2'').$$

It remains to show that the continuous f and g are inverses, and that in fact they are isometries. Since $g \circ f : X_2' \to X_2'$ and $f \circ g : X_1' \to X_1'$ are continuous and act as the *identity* on the *dense* subset $\iota_j(X) \subseteq X_j'$ determined by X, it follows from Theorem 1.5 that $g \circ f = \mathrm{id}_{X_2'}$ and $f \circ g = \mathrm{id}_{X_1'}$. Thus, for $x_1', x_1'' \in X_1'$ and $x_2', x_2'' \in X_2'$ we compute

$$\rho(x_1', x_1'') = \rho(f(g(x_1')), f(g(x_1''))) \leq \rho(g(x_1'), g(x_1''))$$

and likewise

$$\rho(x_2', x_2'') \leq \rho(f(x_2'), f(x_2'')).$$

These are the reverse inequalities to what we saw above, so f and g are isometries. ■

Definition 2.2. Let X be a metric space. A *completion* of X is a pair (X', ι_X) where X' is a complete metric space and $\iota : X \to X'$ is an isometry with dense image.

In practice we usually suppress mention of ι and simply refer to the space X' as being a completion (with the specification of how X sits in X' via ι being understood from context). For example, $\mathbf{R}^n$ is a completion of $\mathbf{Q}^n$ and more generally if Y' is any complete metric space and $Y \subseteq Y'$ is a dense subset (given the induced metric), then Y' is a completion of Y via the isometric inclusion map $Y \to Y'$. Lemma 2.1 makes precise the sense in which a completion of any X, if one exists, is unique up to *unique* isometric isomorphism.

The main theorem on completions is that they exist! The proof even constructs one explicitly. The method is quite similar to the technique of construction of $\mathbf{R}$ from $\mathbf{Q}$ via equivalence classes of Cauchy sequences, though as with $\mathbf{R}$ one doesn't really ever (or rather, should never) use the construction when proving things about the completion: one should only use its characterizing property as a complete metric space containing X isometrically as a dense subset. Nevertheless, one should become familiar with the construction method, as it is a useful technique in many situations.

Theorem 2.3. *Let X be a metric space. Then a completion of X exists.*

The proof constructs a very specific completion, so one might call it *the* completion of X (as opposed to *a* completion, which is to say some totally abstract thing). But what matters is its characterizing property: it has "no holes", (i.e., is complete) and it is "minimal" with this property (i.e., X is dense inside of it). Observe that the proof of uniqueness of completions in Lemma 2.1 uses exactly these two properties.

Proof. The proof will occupy the rest of this section, and will take a long time. Let's first motivate what is to happen before we give the construction. Pretend for a moment that we have a completion $X \hookrightarrow X'$ in hand. We want to describe points of X' in terms of points of X, and to describe the metric on X' in terms of the metric on X. We'll then try to use the resulting description to *define* X' in the general case and we will check that such a definition satisfies all of the required properties.

By the denseness of X in X', every $x' \in X'$ can be written in the form $x' = \lim x_n$ for a sequence $\{x_n\}$ in X. If we let ρ denote the metric on X and ρ' be the metric on the X' which we're assuming we have, then $\rho(x_n, x_m) = \rho'(x_n, x_m)$. Since a convergent sequence is always Cauchy, it follows that $\{x_n\}$ is a Cauchy sequence in X (though it lacks a limit there if $x' \notin X$). Thus, we can describe any $x' \in X'$ as a limit (in X'!) of a Cauchy sequence in X.

Now the same point $x' \in X'$ can be expressed as the limit of lots of Cauchy sequences in X. If we're given two such sequences $\{x_n\}$ and $\{z_n\}$, can we detect purely within X when these sequences have the same limit in X'? Sure: since $x_n \to x$ and $z_n \to x$, we have

$$0 \leq \rho(x_n, z_n) = \rho'(x_n, z_n) \leq \rho'(x_n, x') + \rho'(x', z_n) \to 0 + 0 = 0,$$

so in fact $\rho(x_n, z_n) \to 0$ as $n \to \infty$. Observe conversely that if $\{x_n\}$ is a Cauchy sequence in X with limit $x' \in X'$ then for any Cauchy sequence $\{z_n\}$ of X satisfying $\rho(x_n, z_n) \to 0$ we have

$$0 \leq \rho'(z_n, x') \leq \rho'(z_n, x_n) + \rho'(x_n, x') = \rho(z_n, x_n) + \rho'(x_n, x') \to 0 + 0 = 0,$$

so in fact $z_n \to x'$ also. This motivates the following considerations.

Forget that we have assumed a completion is known to exist. We will construct it with our bare hands out of X.

Definition 2.4. Let C_X denote the set of Cauchy sequences in the given metric space X.

A typical element in C_X is denoted $\{x_n\}$.

Definition 2.5. We say that two elements $\{x_n\}, \{z_n\} \in C_X$ are *equivalent* if $\rho(x_n, z_n) \to 0$. This will be denoted $\{x_n\} \sim \{z_n\}$, for for emphasis we use the terminology "$\sim$-equivalence".

Lemma 2.6. *The relation $\sim$ is an equivalence relation on C_X. That is,*

- $c \sim c$;
- $c \sim c' \Leftrightarrow c' \sim c$;
- $c \sim c',\ c' \sim c'' \Rightarrow c \sim c''$.

Proof. It is trivial to check that for $\{x_n\}, \{z_n\} \in C_X$, we have $\{x_n\} \sim \{x_n\}$ and moreover that $\{x_n\} \sim \{z_n\}$ if and only if $\{z_n\} \sim \{x_n\}$. The transitivity property follows from the triangle inequality. Indeed, if $\{x_n\} \simeq \{z_n\}$ and $\{z_n\} \sim \{y_n\}$ then

$$0 \leq \rho(x_n, y_n) \leq \rho(x_n, z_n) + \rho(z_n, y_n) \to 0 + 0 = 0,$$

so $\{x_n\} \sim \{y_n\}$. ■

Definition 2.7. We write $C_X/\sim$ to denote the set of $\sim$-equivalence classes in C_X.

Unlike the case of vector space quotients, there is no additive structure on the set $C_X/\sim$ (as there's none on C_X either). It is (for now) just a set, though the idea of its formulation is similar to that of quotient space in linear algebra. In the special case $X = \mathbf{Q}$ with the usual metric, the set $C_{\mathbf{Q}}/\sim$ is exactly the set of real numbers (i.e., this is exactly how one constructs $\mathbf{R}$ from $\mathbf{Q}$).

In view of our initial motivation, the set $C_X/\sim$ should "recreate" the completion of X, at least as a set. Note that there is a natural map of sets

$$\iota_X : X \to C_X/\sim$$

which assigns to each $x \in X$ the $\sim$-equivalence class of the constant sequence $\{x, x, \dots\}$ (which is trivially a Cauchy sequence!). The map ι_X is injective. Indeed, if $\iota_X(x) = \iota_X(x')$ then

$$\{x, x, \dots\} \sim \{x', x', \dots\},$$

so $\rho(x, x') \to 0$. This forces $x = x'$, as desired.

We now must enhance the structure of $C_X/\sim$ by giving it a metric (with respect to which we'll easily see ι_X is an isometry with dense image). The tricky part will then be to prove that this metric space is *complete*. We first define a "pseudo-metric" on the set C_X. Let's first define what we mean by "pseudo-metric":

Definition 2.8. A *pseudo-metric* d on a set S is a function $d : S \times S \to \mathbf{R}$ such that $d(s, s') \geq 0$ for all $s, s' \in S$, $d(s, s) = 0$ for all $s \in S$, and

$$d(s, s'') \leq d(s, s') + d(s', s'')$$

for all $s, s', s'' \in S$.

The only difference between a metric and a pseudo-metric is that a pseudo-metric might satisfy $d(s, s') = 0$ for some pair $s, s' \in S$ with $s \neq s'$.

Consider two elements $\{x_n\}, \{z_n\} \in C_X$. If we actually had a completion of X in which $\{x_n\}$ has limit ℓ and $\{z_n\}$ has limit ℓ', then by continuity of the metric function on the completion it would follow that the distance between ℓ and ℓ' is equal to $\lim \rho(x_n, z_n)$. This motivates us to make the following definition.

Definition 2.9. For $\{x_n\}, \{z_n\} \in C_X$, we define the *pseudo-distance* between them to be

$$d(\{x_n\}, \{z_n\}) = \lim_{n\to\infty} \rho(x_n, z_n).$$

Clearly $\sim$-equivalent elements in C_X can be distinct in C_X but (by definition of $\sim$!) their pseudo-distance is 0 (so this is not an honest "distance" function). Of course, in order for Definition 2.9 to be acceptable we must prove that the sequence $\{\rho(x_n, z_n)\}$ in $\mathbf{R}$ automatically has a limit for *any* $\{x_n\}, \{z_n\} \in C_X$. Once this point is settled, we will then check that the pseudo-distance d is a pseudo-metric on C_X, and we will then use this to define an actual metric on $C_X/\sim$ of the type we seek.

Lemma 2.10. *If $\{x_n\}$ and $\{z_n\}$ are Cauchy sequences in X, then $\{\rho(x_n, z_n)\}$ is a convergent sequence in $\mathbf{R}$.*

Proof. How are we to show that the sequence of distances $\{\rho(x_n, z_n)\}$ in $\mathbf{R}$ has a limit, where $\{x_n\}$ and $\{z_n\}$ are two arbitrary Cauchy sequences in X? Without the crutch of a larger space (which we do not yet have) in which we can work with limits, we are unable to work with a statement of the form "$\rho(x_n, z_n) \to \rho(\lim x_n, \lim z_n)$". If we can prove that the numbers $\rho(x_n, z_n) \in \mathbf{R}$ form a *Cauchy sequence*, then by the completeness of $\mathbf{R}$ the desired limit must exist!

In order to verify the Cauchy property for $\{\rho(x_n, z_n)\}$, we simply use the estimate

$$\rho(x_n, z_n) \leq \rho(x_n, x_m) + \rho(x_m, z_m) + \rho(z_m, z_n)$$

to get

$$\rho(x_n, z_n) - \rho(x_m, z_m) \leq \rho(x_n, x_m) + \rho(z_m, z_n),$$

and with the roles of n and m reversed we get

$$\rho(x_m, z_m) - \rho(x_n, z_n) \leq \rho(x_m, x_n) + \rho(z_n, z_m).$$

In other words,

$$|\rho(x_n, z_n) - \rho(x_m, z_m)| \leq \rho(x_n, x_m) + \rho(z_m, z_n).$$

For any $\varepsilon > 0$, each term on the right side is $< \varepsilon/2$ for large n and m, whence the Cauchy property of $\{\rho(x_n, z_n)\}$ in $\mathbf{R}$ falls out. ■

Now we have shown that our attempted definition of pseudo-distance

$$d : C_X \times C_X \to \mathbf{R}$$

on C_X is meaningful. In order that this be useful, we need to check that it is a pseudo-metric and satisfies some other properties:

Lemma 2.11. *Consider the pseudo-distance function d as above. The following conditions hold.*

- *For all $c, c' \in C_X$, we have $d(c, c') \geq 0$ with $d(c, c') = 0$ if and only if $c \sim c'$.*
- *For all $c, c' \in C_X$, we have $d(c, c') = d(c', c)$.*
- *For all $c, c', c'' \in C_X$, we have*

$$d(c, c'') \leq d(c, c') + d(c', c'').$$

- *If $x_1, x_2 \in X$ and $c_j = \iota_X(x_j) = \{x_j, x_j, \dots\} \in C_X$, then*

$$d(c_1, c_2) = \rho(x_1, x_2).$$

Proof. Let $c = \{x_n\}$, $c' = \{x'_n\}$ be two Cauchy sequences in X. By definition,

$$d(c, c') = \lim \rho(x_n, x'_n).$$

From this it is obvious that the first two asserted properties holds. Likewise, the fourth property is obvious. As for the triangle inequality, if $c'' = \{x''_n\}$ is a third Cauchy sequence, then

$$\rho(x_n, x''_n) \leq \rho(x_n, x'_n) + \rho(x'_n, x''_n)$$

for all n. Passing to the limit on n, we get

$$d(c, c'') \leq d(c, c') + d(c', c'')$$

as desired. ■

One important consequence of this lemma is:

Corollary 2.12. *For $c_1, c_2, c'_1, c'_2 \in C_X$ with $c_j \sim c'_j$ we have $d(c_1, c_2) = d(c'_1, c'_2)$. In other words, the pseudo-metric d on C_X respects $\sim$-equivalence.*

Proof. It suffices to show

$$d(c_1, c_2) = d(c_1', c_2), \quad d(c_1', c_2) = d(c_1', c_2').$$

Thus, we just have to show that replacing one of the two variables with a $\sim$-equivalent sequence has no effect on pseudo-distance. By symmetry of d, it suffices to study what happens for a fixed second variable. That is, we want $d(c_1, c) = d(c_1', c)$ if $c_1 \sim c_1'$. By the triangle inequality,

$$d(c_1, c) \leq d(c_1, c_1') + d(c_1', c) = d(c_1', c)$$

since $d(c_1, c_1') = 0$. Switching the roles of c_1 and c_1' in this calculation yields the reverse inequality. ■

Recall our earlier observation that *if* we were handed a completion of X and ℓ, ℓ' are two points in the completion which are expressed as $\ell = \lim x_n$ and $\ell' = \lim x_n'$ with two Cauchy sequences $\{x_n\}$ and $\{x_n'\}$ in X, then the distance between ℓ and ℓ' is equal to $\lim \rho(x_n, x_n')$. This motivates us to now make the:

Definition 2.13. Define $X' = C_X/\sim$ and define the function $\rho' : X' \times X' \to \mathbf{R}$ by

$$\rho'(\xi_1, \xi_2) = d(c_1, c_2)$$

where $c_1, c_2 \in C_X$ are respective representative elements for $\xi_1, \xi_2 \in X' = C_X/\sim$.

This definition of ρ' in terms of the pseudo-metric on representatives in C_X is a well-posed definition in view of Corollary 2.12. That is, $\rho'(\xi_1, \xi_2)$ really does depend only on the elements $\xi_j \in X'$ and not on the specific choices of representative Cauchy sequences used in Definition 2.13 (i.e., different choices of $c_j \in C_X$ representing $\xi_j \in X'$ would give the same answer). Since we compute ρ' by using the pseudo-distance d, the properties of d in Lemma 2.11 immediately yield exactly the following result:

Lemma 2.14. *The function $\rho' : X' \times X' \to \mathbf{R}$ is a metric, and for all $x_1, x_2 \in X$ we have*

$$\rho'(\iota_X(x_1), \iota_X(x_2)) = \rho(x_1, x_2).$$

We've now succeeded in making a metric space X' along with an isometric injection $\iota_X : X \to X'$, and it remains to show that $\iota_X(X) \subseteq X'$ is dense and that the metric space X' is *complete*. The denseness of the image of ι_X will be trivial, and the proof of completeness will require what is called the "diagonal trick".

First let us establish that ι_X has dense image. To do this, fix a choice of $\xi \in X'$. We must express ξ in the form $\xi = \lim \iota_X(x_n)$ for some sequence $\{x_n\}$ in X. There's a natural thing to try, in view of how the construction of X' was motivated. *Choose* a representative Cauchy sequence $\{x_n\} \in C_X$ for the $\sim$-equivalence class $\xi \in X'$. Let's show that the sequence of elements

$$\iota_X(x_n) \in \iota_X(X) \subseteq X'$$

has limit ξ! If we go back to the *definition* of the metric ρ' on X', for any fixed positive integer m_0 we have

$$\rho'(\xi, \iota_X(x_{m_0})) = d(\{x_n\}, \{x_{m_0}, x_{m_0}, \dots\}) = \lim_{n \to \infty} \rho(x_n, x_{m_0}).$$

What we need to prove is that for any $\varepsilon > 0$, there exists some large integer N_ε such that $\rho'(\xi, \iota_X(x_{m_0})) < \varepsilon$ whenever $m_0 \geq N_\varepsilon$. Well, since $\{x_n\}$ is *Cauchy* in X, there exists some $M_\varepsilon > 0$ such that whenever $n, m \geq M_\varepsilon$ we have $\rho(x_n, x_m) < \varepsilon$. Thus, if we *fix* $m_0 \geq M_\varepsilon$ then we get

$$\lim_{n \to \infty} \rho(x_n, x_{m_0}) \leq \varepsilon.$$

In other words, $\rho'(\xi, \iota_X(x_{m_0})) \leq \varepsilon$ for all $m_0 \geq M_\varepsilon$. That's exactly what we wanted (up to having $\leq \varepsilon$ instead of $< \varepsilon$, but just use $\varepsilon/2$ instead). This completes the verification that the isometry $\iota_X : X \to X'$ has dense image.

Now we come to the last and most important point: we must prove that X' with the metric ρ' is *complete*. This is a little subtle, because we think of elements of $X' = C_X/\sim$ as equivalence classes of Cauchy sequences in X, so when contemplating a Cauchy sequence $\{\xi_n\}$ in X' we're really dealing with sequences of sequences, so to speak. It can make the head spin if one loses track of the definitions.

Choose a Cauchy sequence $\{\xi_n\}$ in X'. We need to find a limit for it in X'. In order to keep the notation under control, we choose a representative Cauchy sequence in X for $\xi_n \in X' = C_X/\sim$, and denote it $\{x_{n,1}, x_{n,2}, \dots\}$. You should imagine the ξ_n's as arranged in the form of an infinite stack of horizontal rows, with the representative sequence $\{x_{1,m}\}$ (for ξ_1) on top, then $\{x_{2,m}\}$ on the next row, etc. all the way down. We will find the desired $\xi = \lim \xi_n \in X' = C_X/\sim$ by staring at this doubly infinite array of x_{ij}'s.

In order to find a limit for the Cauchy sequence, we will simplify life tremendously by recalling that in any metric space at all, a Cauchy sequence converges if and only if some subsequence converges. Thus, we can always replace our given $\{\xi_n\}$ sequence in X' with any desired subsequence and may focus attention on finding a limit for the subsequence.

With this point noted, by the Cauchy property of $\{\xi_1, \xi_2, \dots\}$ we can find integers $N_1, N_2, \dots$ so that if $n, m \geq N_j$ then $\rho'(\xi_n, \xi_m) \leq 1/j$. We can also assume without loss of generality that $N_1 \leq N_2 \leq \dots$ by recursively enlarging the N_j's if necessary.

Define a sequence of integers $n_1, n_2, \dots$ as follows: we set $n_1 = N_1$ and $n_j = \max(n_{j-1}, N_j) + 1$ for all $j > 1$. It is easy to check that $n_1 < n_2 < \dots$ and $n_j \geq N_j$ for all j, so since $\min(N_i, N_j) = N_{\min(i,j)}$ we get

$$\rho'(\xi_{n_i}, \xi_{n_j}) \leq 1/\min(i, j).$$

Thus, if we work instead with the subsequence

$$\{\xi_{n_1}, \xi_{n_2}, \dots\},$$

(which makes sense since $n_1 < n_2 < \dots$), then upon renaming ξ_{n_k} as ξ_k and renaming $x_{n_k,j}$ as $x_{k,j}$, we may assume

$$\rho'(\xi_i, \xi_j) \leq 1/\min(i, j)$$

for all $i, j \geq 1$. This will simplify some subsequent calculations.

Since $\xi_i = \{x_{i,1}, x_{i,2}, \dots\}$, by *definition* of ρ' we conclude from the condition

$$\rho'(\xi_i, \xi_j) \leq 1/\min(i, j)$$

that

$$\lim_{m \to \infty} \rho(x_{i,m}, x_{j,m}) \leq 1/\min(i, j). \tag{2.1}$$

Now we make one further simplifying observation. Since the entire problem of finding $\lim \xi_n$ is intrinsic to the ξ_n's in X', we are always free to make other choices of representative Cauchy sequences for the ξ_n's when doing calculations of ρ'-values. To exploit this, we first need to record:

Lemma 2.15. *Let $\{x_n\} \in C_X$ be a Cauchy sequence in X. If $\{x_{i_1}, x_{i_2}, \dots\}$ is any subsequence, then this is Cauchy and is $\sim$-equivalent to $\{x_n\}$. In other words, passage to a subsequence does not cause one to leave a given $\sim$-equivalence class in C_X.*

Proof. We need to show that $\rho(x_n, x_{i_n}) \to 0$ as $n \to \infty$. Pick $\varepsilon > 0$, so there exists N_ε such that when $n, m \geq N_\varepsilon$ we have $\rho(x_n, x_m) < \varepsilon$. In particular, if $n \geq N_\varepsilon$ and $m \geq n$ then also $m \geq N_\varepsilon$ and we get $\rho(x_n, x_m) < \varepsilon$. But certainly $i_n \geq n$ for all n (why?), so $0 \leq \rho(x_n, x_{i_n}) < \varepsilon$ for $n \geq N_\varepsilon$. ∎

We now improve the choice of representatives for the ξ_n's as follows. For each n, replace $\{x_{n,1}, x_{n,2}, \dots\}$ by the tail part of this sequence beyond which point all terms $x_{n,i}$, $x_{n,j}$ are within a distace of $1/n$ from each other. Making such a "better" choice of representatives (this being OK by Lemma 2.15), we put ourselves in the case where, for each n, we have

$$\rho(x_{n,i}, x_{n,j}) \leq 1/n \tag{2.2}$$

for *all* $i, j \geq 1$. Also recall the condition (2.1). Now we create $\lim \xi_n \in X' = C_X/\sim$ out of thin air, by proving the following lemma:

Lemma 2.16. *The sequence $\{x_{n,n}\}$ in X is Cauchy, and hence is an element in C_X.*

Proof. Choose $\varepsilon > 0$. We need to estimate $\rho(x_{n,n}, x_{m,m})$ for large n and m. We want to prove that if n and m are large enough (depending on ε), then $\rho(x_{n,n}, x_{m,m}) \leq \varepsilon$. We only consider $n, m \geq 1/\varepsilon$, so by (2.2) at least

$$\rho(x_{n,i}, x_{n,j}), \rho(x_{m,i}, x_{m,j}) \leq \varepsilon$$

for *all* $i, j \geq 1$. Since

$$\lim_{k\to\infty} \rho(x_{n,k}, x_{m,k}) = \rho'(\xi_n, \xi_m) \leq 1/\min(n, m) \leq \varepsilon,$$

for a large k_0 we have $\rho(x_{n,k_0}, x_{m,k_0}) \leq \varepsilon + \varepsilon = 2\varepsilon$. Thus, by the triangle inequality,

$$\rho(x_{n,n}, x_{m,m}) \leq \rho(x_{n,n}, x_{n,k_0}) + \rho(x_{n,k_0}, x_{m,k_0}) + \rho(x_{m,k_0}, x_{m,m}) \leq \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon$$

for $n, m \geq 1/\varepsilon$. Now working with $\varepsilon/4$ (or $\varepsilon/8 \dots$) everywhere gives what we need. ■

This lemma now motivates us to consider the *Cauchy sequence* $\{x_{n,n}\}$ in X, and more specifically to consider the element $\xi \in X'$ which is the equivalence class of this sequence. This device is called the "diagonal trick" for evident reasons, though it must be stressed that we are *not* forming the "diagonal sequence" $\{x_{n,n}\}$ on the *original* choices of representatives. Recall that we made two big changes from the initial setup at the start:

- we passed to a subsequence of the originally given $\{\xi_n\}$;
- for choices of representatives of each ξ_n, we passed to a tail of each such representative.

These two modifications were rigged exactly to make the proof of Lemma 2.16 go through without any mess. If we had tried to work with the original choices, there would have been a lot of double and triple subscripts as we kept track of subsequences of subsequences, and rather than the clean diagonal formula $\{x_{n,n}\}$ we would have had to use a rather more complicated construction $\{x_{n,\phi(n)}\}$ where $\phi(n)$ has to be well-chosen for each n so that various quantities rapidly approach 0 relative to other quantities. It would have been a big mess to have taken such an approach.

We shall now prove that the $\sim$-equivalence class ξ of $\{x_{n,n}\}$ is the limit of the Cauchy sequence $\{\xi_n\}$ in X', where ξ_n has representative $\{x_{n,1}, x_{n,2}, \dots\}$. The proof will be simplicity itself. Pick $\varepsilon > 0$. We just need to estimate the distance $\rho'(\xi_n, \xi)$ and hope to show it is $< \varepsilon$ for large enough n. Well, by definition

$$\rho'(\xi_n, \xi) = \lim_{k\to\infty} \rho(x_{n,k}, x_{k,k}).$$

We have the estimate

$$\rho(x_{n,k}, x_{k,k}) \leq \rho(x_{n,k}, x_{n,n}) + \rho(x_{n,n}, x_{k,k}) \leq 1/n + \rho(x_{n,n}, x_{k,k})$$

by (2.2). Also, since $\{x_{1,1}, x_{2,2}, x_{3,3}, \dots\}$ has been shown to be Cauchy in Lemma 2.16, there is some large M_ε so that when $n, k \geq M_\varepsilon$ then $\rho(x_{n,n}, x_{k,k}) < \varepsilon/2$. Thus, for any $n \geq M_\varepsilon$ and any $k \geq n$ (so $k \geq M_\varepsilon$) we have

$$\rho(x_{n,k}, x_{k,k}) < 1/n + \varepsilon/2.$$

For $n \geq \max(2/\varepsilon, M_\varepsilon)$, we get
$$\rho(x_{n,k}, x_{k,k}) < \varepsilon$$
for all $k \geq n$. Passing to the limit on k, we obtain $\rho'(\xi_n, \xi) \leq \varepsilon$ for $n \geq \max(2/\varepsilon, M_\varepsilon)$. This proves $\xi_n \to \xi$ in X', thereby completing the verification that X' is indeed a complete metric space. ■

For emphasis, let's record a special case of an earlier result.

Theorem 2.17. *Let $f : X \to Y$ be a bounded map of metric spaces (i.e., $\rho_Y(f(x_1), f(x_2)) \leq M\rho_X(x_1, x_2)$ for all $x_1, x_2 \in X$ and some fixed $M > 0$). Let X' and Y' be completions of X and Y. Then there exists a unique continuous map $f' : X' \to Y'$ fitting into a commutative square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle \iota_Y} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

Proof. The map $F = \iota_Y \circ f : X \to Y'$ is a bounded map to a complete space, so it uniquely factors through the completion X' (by Theorem 1.6), which is to say that F has the form $f' \circ \iota_X$ for a unique continuous $f' : X' \to Y'$. Moreover, Theorem 1.6 ensures that f' is even bounded with the same bound that works for f. ■

3. Completions of vector spaces

We now examine how the completion construction on metric spaces behaves in the context of normed vector spaces and inner product spaces over $\mathbf{R}$. Let V denote a normed vector space over $\mathbf{R}$. The norm $\|\cdot\|$ on V endows it with a structure of metric space. Let $\iota_V : V \to V'$ be a completion of V as a metric space. The theorem we want to prove is:

Theorem 3.1. *The completion V' of V as a metric space admits a unique structure of normed vector space with respect to which the norm $\|\cdot\|'$ on V' restricts to the norm $\|\cdot\|$ on V. Moreover, the metric on V' is induced by the norm $\|\cdot\|'$.*

Before we prove Theorem 3.1, we record some consequences. Let $T : V \to W$ be a bounded linear map between normed vector spaces. If we let $M = \|T\|$ denote the operator norm on T, then clearly T is bounded in the sense of Theorem 2.17, so we get a unique continuous (even bounded!) map $T' : V' \to W'$ between completions with T' extending T.

Corollary 3.2. *With the above notation, the map T' is linear and it has the same finite operator norm as T. Moreover, if T is an isometry then so is T'.*

Proof. For linearity, we want $T'(cv') = cT'(v')$ and $T'(v' + v'') = T'(v') + T'(v'')$ for all $c \in \mathbf{F}$ and $v', v'' \in V'$. We can write $v' = \lim v_n$ for a sequence $\{v_n\}$ in V, and likewise $v'' = \lim \widetilde{v}_n$ for a sequence $\{\widetilde{v}_n\}$ in V. Thus, as in any normed vector space, we have (in V')
$$v_n + \widetilde{v}_n \to v' + v'', \;\; cv_n \to cv'.$$
Since T' is continuous and T' restricts to T on V, we get
$$T'(cv') = \lim T(cv_n) = \lim c \cdot T(v_n) = c \cdot \lim T(v_n) = c \cdot T'(v')$$
and
$$T'(v' + v'') = \lim T(v_n + \widetilde{v}_n) = \lim T(v_n) + \lim T(\widetilde{v}_n) = T'(v') + T'(v'').$$
This gives the linearity of T', and in Theorem 2.17 we saw that the extended map on completions is bounded by the same bound which works for the given continuous map.

It follows that

$$\|T'(v') - T'(v'')\|_{W'} \leq \|T\|\|v' - v''\|_{V'}$$

for all $v', v'' \in V'$, so using $v'' = 0$ we see $\|T'\| \leq \|T\| < \infty$. But we also have

$$\|T'\| = \sup_{\|v'\|_{V'}=1} \|T'(v')\|_{W'}, \ \ \|T\| = \sup_{\|v\|_V=1} \|T(v)\|_W.$$

The first supremum is taken over a superset of the second one, and at $v \in V \subseteq V'$ the corresponding terms in the supremum are the same. Hence, the supremum defining $\|T'\|$ is at least as large as $\|T\|$. This give the reverse inequality $\|T'\| \geq \|T\|$, whence these two operator norms coincide.

Finally, if T is an isometry and we choose $v' \in V'$, then to show that $\|T'(v')\|_{W'} = \|v'\|_{V'}$ we write $v' = \lim v_n$ with $v_n \in V$. Continuity of T' gives $T'(v') = \lim T'(v_n) = \lim T(v_n)$, and by continuity of norm functions we then deduce

$$\|T'(v')\|_{W'} = \lim \|T'(v_n)\|_{W'} = \lim \|T(v_n)\|_W = \lim \|v_n\|_V = \|v'\|_{V'}.$$

■

The other corollary of interest is the following:

Corollary 3.3. *Let V be an inner product space The completion V' of V admits a unique inner product which restricts to the given one on V, and moreover it induces the unique norm on V' which restricts to the given one on V.*

An orthogonal linear map $T : V \to W$ between two such spaces has induced map $T' : V' \to W'$ on completions which is also orthogonal.

Proof. Since the inner product can be recovered from the norm, and in fact the mere existence of such an inner product inducing a given norm is equivalent to the parallelogram law (if you are unfamiliar with this fact, either figure our a proof or skip to the next paragraph for an alternative method), the entire corollary follows immediately from the preceding corollary once we check that the parallelogram law for the norm on V forces the parellelogram law for the norm on V'. This follows by chasing limits on the parellelogram law in V (and expressing various vectors in the parellelogram law on V' as limits of sequences of vectors in V).

We note that it is not really necessary to appeal to the parallelogram law. Namely, the same methods use to prove the previous corollary adapt without serious difficulty (but a little tedium) to the setting of inner product spaces (as opposed to the more general setting of normed vector spaces). ■

Now it is finally time to prove Theorem 3.1.

Proof. We first focus on existence, and then handle uniqueness. For the linear structure on V', if $v', v'' \in V'$ are expressed as limits of respective Cauchy sequences $\{v_n\}$ and $\{\widetilde{v}_n\}$ in V, then we leave it as an easy exercise to check that the sequences

$$\{v_n + \widetilde{v}_n\}, \ \ \{c \cdot v_n\}$$

are again *Cauchy* in V, and hence have limits in V'. We want to call these limits $v' + v''$ and cv' respectively. It is a straightfoward exercise to check that these "definitions" are independent of the choice of $\{v_n\}$ and $\{\widetilde{v}_n\}$, hence are well-posed, and satisfy the axioms to be an $\mathbf{R}$-vector space (use passage to the limit on various vector space identities in V). It is obvious that this vector space structure on V' recovers the given one on V, so V is a subspace of V'. This takes care of the existence of the desired linear structure on V'.

Now we turn to the existence of a norm compatible with the linear structure just defined on V' and also compatible with the given norm on V. If ρ' denotes the metric on V', we define

$\|v'\| = \rho'(v', 0)$ for $v' \in V'$. Of course, if $v \in V$ then this is just the given norm on V (due to the role of the given *norm* on V in the *definition* of the metric on V).

Using continuity properties of metrics and norms on V, it is a simple exercise in passage to the limit to deduce that $\|\cdot\|$ as just defined on the *vector space* V' satisfies the requirements to be a norm. To prove that $\rho'(v', v'') \stackrel{?}{=} \|v' - v''\| \stackrel{\text{def}}{=} \rho'(v' - v'', 0)$ for all $v', v'' \in V'$, we again use passage to the limit from V and the fact that metrics are continuous. This settles the existence part of the theorem.

Now we turn to the uniqueness part. Suppose that V' is endowed with *some* normed vector space structure restricting to that given on V (*without* assuming that the norm actually recovers the metric on this completion). Since norms are *continuous* on any normed vector space (such as V'), it follows that if $v' \in V'$ is a limit of a sequence $\{v_n\}$ in V, then $\{v_n\}$ is *Cauchy* in V and the norm of v' is the limit of the sequence $\{\|v_n\|\}$ in $\mathbf{R}$. Hence, the norm on V' is uniquely determined.

In particular, the norm on V' *must* coincide with the one constructed in the proof of the existence part above, whence it *must* induce the canonical metric on V' (as a completion of the metric space V). That is, convergence in V' with respect to the abstract norm we presently consider is *exactly the same* as convergence with respect to the canonical metric on V'. Hence, we can freely discuss limits of sequences in V' without any confusion, even prior to knowing whether the linear structure on V' is uniquely determined.

Since the maps $V' \times V' \to V'$ and $\mathbf{R} \times V' \to V'$ encoding the linear structure are continuous relative to *any* norm structure on V', we can use passage to the limit from V to deduce that when $v' = \lim v_n$ and $v'' = \lim \widetilde{v}_n$, then $v' + v''$ is the limit of the vectors $v_n + \widetilde{v}_n \in V$ and $c \cdot v' = \lim c \cdot v_n$ (for $c \in \mathbf{R}$). This shows that the entire linear structure on V' is also uniquely determined when we demand that it admit a norm structure extending that on V. ■

4. Changing the metric

We conclude with a discussion of a subtle point. Suppose that ρ_1 and ρ_2 are two metrics on X which are bounded with respect to each other (i.e., $a\rho_1 \le \rho_2 \le A\rho_1$ for some $a, A > 0$, so $A^{-1}\rho_2 \le \rho_1 \le a^{-1}\rho_2$). Such a pair of metrics induce the same concepts of open set, closed set, Cauchy sequence, and limit on X. If we let X'_j denote the completion of X with respect to ρ_j, it is only natural to expect that in some sense these completions are "the same". In fact, since Cauchyness with respect to ρ_1 is equivalent to Cauchyness with respec to ρ_2, we see that both metrics give rise to the same set C of Cauchy sequences in X. Moreover, it is trivial to check that the equivalence relation $\sim$ on C is the same for the two metrics. In other words, the underlying set of our completion construction $C/\sim$ is literally *the same* for both metrics.

The map $\iota_X : X \to C/\sim$ given by assigning to each $x \in X$ the $\sim$-equivalence class of the constant sequence $\{x, x, \dots\}$ is likewise the same regardless of which metric we use. If we look back at the construction, then we see that the metrics ρ'_1 and ρ'_2 defined on $C/\sim$ to make completions for (X, ρ_1) and (X, ρ_2) respectively are in fact bounded with respect to one another on $C/\sim$ (using the same constants $a, A > 0$ which serve to bound ρ_1 and ρ_2 with respect to each other on X). We summarize these conclusions in the following theorem, for which we give a pure thought proof which does not depend (in contrast to the preceding considerations) on the actual construction process for completions:

Theorem 4.1. *Let X be a set and ρ_1, ρ_2 two metrics on X which are bounded with respect to each other. Let $\iota_j : X \to X'_j$ be a completion of X for the metric ρ_j. Then there exist unique continuous* *maps* $g : X_1' \to X_2'$ *and* $f : X_2' \to X_1'$ *making commutative triangles*

$$\begin{array}{ccc} X & \xrightarrow{\iota_1} & X_1' \\ & {\scriptstyle \iota_2}\searrow & \downarrow {\scriptstyle g} \\ & & X_2' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\iota_2} & X_2' \\ & {\scriptstyle \iota_1}\searrow & \downarrow {\scriptstyle f} \\ & & X_1' \end{array}$$

and in fact f *and* g *are inverse to each other. Moreover,* f *and* g *are bounded maps.*

Proof. The method of proof of Lemma 2.1 adapts to the present situation essentially verbatim, for both existence and uniqueness. The only change is that we replace "isometry" with "bounded map" everywhere (and so we have to introduce some positive constants in various inequalities relating metrics). ■