Math 396. Interior, closure, and boundary

We wish to develop some basic geometric concepts in metric spaces which make precise certain intuitive ideas centered on the themes of "interior" and "boundary" of a subset of a metric space. One warning must be given. Although there are a number of results proven in this handout, none of it is particularly deep. If you carefully study the proofs (which you should!), then you'll see that none of this requires going much beyond the basic definitions. We will certainly encounter some serious ideas and non-trivial proofs in due course, but at this point the central aim is to acquire some linguistic ability when discussing some basic geometric ideas in a metric space. Thus, the main goal is to familiarize ourselves with some very convenient geometric terminology in terms of which we can discuss more sophisticated ideas later on.

1. Interior and closure

Let X be a metric space and $A \subseteq X$ a subset. We define the *interior* of A to be the set

$$\operatorname{int}(A) = \{a \in A \,|\, \text{some } B_{r_a}(a) \subseteq A, r_a > 0\}$$

consisting of points for which A is a "neighborhood". We define the *closure* of A to be the set

$$\overline{A} = \{x \in X \,|\, x = \lim_{n \to \infty} a_n, \text{ with } a_n \in A \text{ for all } n\}$$

consisting of limits of sequences in A.

In words, the interior consists of points in A for which all nearby points of X are also in A, whereas the closure allows for "points on the edge of A". Note that obviously

$$\operatorname{int}(A) \subseteq A \subseteq \overline{A}.$$

We will see shortly (after some examples) that $\operatorname{int}(A)$ is the largest open set inside of A — that is, it is open and contains any open lying inside of A (so in fact A is open if and only if $A = \operatorname{int}(A)$) — while $\overline{A}$ is the smallest closed set containing A; i.e., $\overline{A}$ is closed and lies inside of any closed set containing A (so in fact A is closed if and only if $\overline{A} = A$).

Beware that we have to prove that the closure is actually closed! Just because we call something the "closure" does not mean the concept is automatically endowed with linguistically similarly-sounding properties. The proof won't be particularly deep, as we'll see.

Example 1.1. Let's work out the interior and closure of the "half-open" square

$$A = \{(x, y) \in \mathbf{R}^2 \,|\, -1 \leq x \leq 1,\ -1 < y < 1\} = [-1, 1] \times (-1, 1)$$

inside of the metric space $X = \mathbf{R}^2$ (the phrase "half-open" is purely intuitive; it has no precise meaning, but the picture should make it clear why I use this terminolgy). Intuitively, this is a square region whose horizontal edges are "left out". The interior of A should be $(-1, 1) \times (-1, 1)$ and the closure should be $[-1, 1] \times [-1, 1]$, as drawing a picture should convince you. Of course, we want to see that such conclusions really do follow from our precise definitions.

First we check that $\operatorname{int}(A)$ is correctly described. If $-1 < x < 1$ and $-1 < y < 1$ then for

$$r = \min(|-1 - x|, |1 - x|, |-1 - y|, |1 - y|) > 0$$

it is easy to check $B_r((x, y)) \subseteq (-1, 1) \times (-1, 1)$ (since a square box with side-length r contains the disc of radius r with the same center). Thus, $(-1, 1) \times (-1, 1) \subseteq A$ is an open subset of $X = \mathbf{R}^2$. To check it is the full interior of A, we just have to show that the "missing points" of the form $(\pm 1, y)$ do not lie in the interior. But for any such point $p = (\pm 1, y) \in A$, for any positive small $r > 0$ there is always a point in $B_r(p)$ with the same y-coordinate but with the x-coordinate either slightly larger than 1 or slightly less than -1. Such a point is not in A. Thus, $p \notin \operatorname{int}(A)$.

Now we check that $\overline{A} = [-1,1] \times [-1,1]$. Since convergence in $\mathbf{R}^2$ forces convergence in coordinates, to see

$$\overline{A} \subseteq [-1,1] \times [-1,1]$$

it suffices to check that $[-1,1]$ is closed in $\mathbf{R}$ (since certainly $A \subseteq [-1,1] \times [-1,1]$). But this is clear (either by using sequences or by explicitly showing its complement in $\mathbf{R}$ to be open). To see that $\overline{A}$ fills up all of $[-1,1] \times [-1,1]$, we have to show that each point in $[-1,1] \times [-1,1]$ can be obtained as a limit of a sequence in A. We just have to deal with points not in $A = [-1,1] \times (-1,1)$, since points in A are limits of constant sequences. That is, we're faced with studying points of the form $(x, \pm 1)$ with $x \in [-1,1]$. Such a point is a limit of a sequence (x, q_n) with $q_n \in (-1,1)$ having limit ± 1.

Example 1.2. What happens if we work with the same set A but view it inside of the metric space $X = A$ (with the Euclidean metric)? In this case $\mathrm{int}(A) = A$ and $\overline{A} = A$! Indeed, quite generally for any metric space X we have $\mathrm{int}(X) = X$ and $\overline{X} = X$. These are easy consequences of the definitions (check!). Likewise, the empty subset $\emptyset$ in any metric space has interior and closure equal to the subset $\emptyset$.

The moral is that one has to always keep in mind what ambient metric space one is working in when forming interiors and closures! One could imagine that perhaps our notation for interior and closure should somehow incorporate a designation of the ambient metric space. But just as we freely use the same symbols "+" and "0" to denote the addition and additive identity in any vector space, even when working with several spaces at once, it would simply make life too cumbersome (and the notation too cluttered) to always write things like $\mathrm{int}_X(A)$ or $(\overline{A})_X$. One just has to pay careful attention to what is going on so as to keep track of the ambient metric space with respect to which one is forming interiors and closures. The context will usually make it obvious what one is using as the ambient metric space, though if considering several ambient spaces at once it is sometimes helpful to use more precise notation such as $\mathrm{int}_X(A)$.

Theorem 1.3. *Let A be a subset of a metric space X. Then* $\mathrm{int}(A)$ *is open and is the largest open set of X inside of A (i.e., it contains all others).*

Proof. We first show $\mathrm{int}(A)$ is open. By its definition if $x \in \mathrm{int}(A)$ then some $B_r(x) \subseteq A$. But then since $B_r(x)$ is itself an open set we see that any $y \in B_r(x)$ has some $B_s(y) \subseteq B_r(x) \subseteq A$, which forces $y \in \mathrm{int}(A)$. That is, we have shown $B_r(x) \subseteq \mathrm{int}(A)$, whence $\mathrm{int}(A)$ is open.

If $U \subseteq A$ is an open set in X, then for each $u \in U$ there is some $r > 0$ such that $B_r(u) \subseteq U$, whence $B_r(u) \subseteq A$, so $u \in \mathrm{int}(A)$. This is true for all $u \in U$, so $U \subseteq \mathrm{int}(A)$. ■

Corollary 1.4. *A subset A in a metric space X is open if and only if $A = \mathrm{int}(A)$.*

Proof. By the theorem, $\mathrm{int}(A)$ is the unique largest open subset of X contained in A. But obviously A is open if and only if such a unique maximal open subset of X lying in A is actually equal to A (why?). This establishes the corollary. ■

We next want to show that the closure of a subset A in X is related to closed subsets of X containing A in a manner very similar to the way in which the interior of A is related to open subsets of X which lie inside of A. This goes along with the general idea that openness and closedness are "complementary" points of view (recall that a subset S in a metric space X is open (resp. closed) if and only if its complement $X - S$ is closed (resp. open)). It is actually more convenient for us to first show that closures and interiors have a complementary relationship, and to then use this to deduce our desired properties of closure from already-established properties of interior.

Theorem 1.5. *Let A be a subset of a metric space X. Then $X - \overline{A} = \mathrm{int}(X - A)$ and $X - \mathrm{int}(A) = \overline{X - A}$.*

Before proving this theorem, we illustrate with an example. Consider $X = \mathbf{R}^2$ with the usual metric, and let $A = [-1, 1] \times (-1, 1)$ be the "half-open" square as considered above. In this case, we have computed $\overline{A} = [-1, 1] \times [-1, 1]$ and $\mathrm{int}(A) = (-1, 1) \times (-1, 1)$. By drawing pictures of $X - A$ and of the complements of $\overline{A}$ and $\mathrm{int}(A)$, you should convince yourself intuitively that the assertions in this theorem make sense in this case.

Now we prove Theorem 1.5.

Proof. We begin by proving $X - \overline{A} = \mathrm{int}(X - A)$. If $x \in X$ is not in $\overline{A}$, there must exist some $B_{1/2^n}(x)$ not meeting A, for otherwise we'd have some $x_n \in B_{1/2^n}(x) \cap A$ for all n, so clearly $x_n \to x$, contrary to the fact that $x \notin \overline{A}$ is not a limit of a sequence of elements of A. This shows

$$X - \overline{A} \subseteq \mathrm{int}(X - A).$$

Conversely, if x is in the interior of $X - A$ then some $B_r(x)$ lies in $X - A$ and hence is disjoint from A. It follows that no sequence in A can possibly converge to x because for $\varepsilon = r > 0$ we'd run into problems (i.e., there's nothing in A within a distance of less than ε from x, since $B_\varepsilon(x) \subseteq X - A$).

Applying the *general* equality

$$X - \overline{A} = \mathrm{int}(X - A)$$

for arbitrary subsets A to X to the subset $X - A$ in the role of A, we get

$$X - \overline{X - A} = \mathrm{int}(A).$$

Taking complements of both sides within X yields

$$\overline{X - A} = X - \mathrm{int}(A),$$

as desired. ■

Corollary 1.6. *Let A be a subset of a metric space X. Then $\overline{A}$ is closed and is contained inside of any closed subset of X which contains A.*

Proof. Since the complement of $\overline{A}$ is equal to $\mathrm{int}(X - A)$, which we know to be open, it follows that $\overline{A}$ is closed. If Z is any closed set containing A, we want to prove that Z contains $\overline{A}$ (so $\overline{A}$ is "minimal" among closed sets containing A). But this is clear for several reasons. On the one hand, by definition every point $x \in \overline{A}$ is the limit of a sequence of elements in $A \subseteq Z$, so by closedness of Z such limit points x are also in Z. This shows $\overline{A} \subseteq Z$. On the other hand, one can argue by noting that passage to complement takes Z to an open set $X - Z$ contained inside of $X - A$, so by maximality this open $X - Z$ must lie inside the interior of $X - A$, which we have seen is the complement $X - \overline{A}$ of $\overline{A}$. Passage back to complements then gives

$$\overline{A} = X - (X - \overline{A}) = X - \mathrm{int}(X - A) \subseteq X - (X - Z) \subseteq Z,$$

as desired. ■

Corollary 1.7. *For subsets $A_1, \ldots, A_n$ in a metric space X, the closure of $A_1 \cup \cdots \cup A_n$ is equal to $\cup \overline{A}_i$; that is, the formation of a finite union commutes with the formation of closure.*

Proof. A closed set Z contains $\cup A_i$ if and only if it contains each A_i, and so if and only if it contains $\overline{A}_i$ for every i. Since $\cup \overline{A}_i$ is a finite union of closed sets, it is closed. We conclude that this closed set is minimal among all closed sets containing $\cup A_i$, so it is the closure of $\cup A_i$. ■

2. Further aspects of interior and closure

The "interior" and "closure" constructions have been seen to be well-behaved with respect to the formation of complements within a metric space. However, these notions are not well-behaved with respect to intersections within a metric space. Also, one cannot compute the closure of a set just from knowing its interior. For example, a set can have empty interior and yet have closure equal to the whole space: think about the subset $\mathbf{Q}$ in $\mathbf{R}$.

Here is one mildly positive result.

Theorem 2.1. *The formation of closures is local in the sense that if U is open in a metric space X and A is an arbitrary subset of X, then the closure of $A \cap U$ in X meets U in $\overline{A} \cap U$ (where $\overline{A}$ denotes the closure of A in X). In particular, if Z is closed in X then $U \cap \overline{Z \cap U} = Z \cap U$.*

Also if U is the interior of a closed set Z in X, then $\operatorname{int}(\overline{U}) = U$.

After proving the theorem, we'll present an interesting example of an open subset of a metric space which is *not* equal to the interior of its closure (and hence, by the second part of the theorem, cannot be expressed as the interior of any closed set at all). It is probably not immediately obvious to you how to find such open sets, since typical open sets one writes down in $\mathbf{R}$ or $\mathbf{R}^2$ tend to be the interior of their closures.

Proof. Since $\overline{A} \cap U$ is a closed set in U that contains $A \cap U$, for the first part of the theorem we need to prove that every point $x \in \overline{A} \cap U$ is a limit of a sequence of points $x_n \in A \cap U$. Since $x \in \overline{A}$ we can write $x = \lim x_n$ with $x_n \in A$. By hypothesis $x \in U$, so by the openness of U we must have some $B_r(x) \subseteq U$, and so since $x_n \to x$ by considering just sufficiently large n we have $x_n \in U$. Thus, for large n the sequence $\{x_n\}$ lies in $A \cap U$ and converges to x.

Now we assume that U is the interior of a closed set Z and we wish to prove U is the interior of $\overline{U}$. Since Z is a closed set containing U, it also contains the closure of U, and by openness of U the open subset U inside of $\overline{U}$ must lie inside the interior of $\overline{U}$. To summarize, we have

$$U \subseteq \operatorname{int}\overline{U} \subseteq \operatorname{int}Z = U,$$

so equality is forced throughout. ∎

Let's give a counterexample to the equality $\operatorname{int}(\overline{U}) = U$ if one only requires U to be an open subset of X (rather than even the interior of a closed set). The basic problem is that the closure of U can be quite a lot bigger than U. In fact, we'll find a rather "small" open subset $U \subseteq \mathbf{R}$ with closure equal to $\mathbf{R}$ (whose interior is $\mathbf{R}$, and hence larger than U).

Let $S \subseteq \mathbf{Q}$ denote the set of elements of the form $q = a/10^n$ with $a \in \mathbf{Z}$ and $n \geq 0$ (i.e., finite decimal expansions). We define $n(q) \geq 0$ to be the exponent of 10 in the denominator of q. In words, the base 10 decimal expansion of $q \in S$ is finite and (if $q \notin 10\mathbf{Z}$) begins on the right with a non-zero digit in the $10^{-n(q)}$-slot. Define U to be the union of intervals $B_{1/10^{n(q)+2}}(q)$ for $q \in S$. This union U is certainly open, as it is a union of open intervals. Try to draw a picture of where U meets $[0, 1]$; it's pretty tough, but after working out a bunch of intervals in U you'll get a sense for what U looks like: it's very "sparse", yet somehow all over the place since certainly $S \subseteq U$. The problem is that "most" elements of S have pretty big denominators, and the tiny interval from U surrounding a choice of $q \in S$ is *really* tiny (depending on how big the denominator of q is).

Since $q \in S$ has a decimal expansion which terminates at the $n(q)$th digit past the decimal point, all points in $B_{1/10^{n(q)+2}}(q)$ have a $10^{-n(q)-1}$th digit equal to either 0 or 9 (think about $.253 \pm .00000998$), but not $2, \ldots, 8$. In particular, if we consider real numbers whose fractional parts consist *entirely* of digits from 2 to 8, such numbers cannot lie in U. Actually, a lot more can't lie in U (as your picture should convince you), but one needs measure theory to give a precise description of just how sparse U is. In any case, we have at least shown that U is a proper open subset of $\mathbf{R}$. But every real number is a limit of a sequence in $S \subseteq U$, so the closure of U is equal to $\mathbf{R}$.

Let us conclude with considerations related to the local nature of closedness.

Theorem 2.2. *Let X be a metric space, and $A \subseteq X$ a subset. Let $\{U_i\}$ be an open cover of X. The set A is closed in X if and only if the subset $A \cap U_i$ is closed in U_i for all i.*

Proof. Replacing A with $X - A$, it is equivalent to say that A is open in X if and only if $A \cap U_i$ is open in U_i for all i. However, a subset of U_i is open in U_i if and only if it is open in X (as U_i is open in X), so it is equivalent to say that A is open in X if and only if $A \cap U_i$ if open in X for all i. The implication $\Rightarrow$ is clear, and the converse follows from the observation that A is the union of the overlaps $A \cap U_i$. ■

Note that it is crucial in the preceding theorem that the U_i's cover all of X, and not just A. For example, if $A = [0, 1)$ in $X = \mathbf{R}$ and we take $U = (-\infty, 1)$ and $V = (1, \infty)$ then the open sets U and V barely fail to cover X ($U \cup V = X - \{1\}$), and although $U \cap A = A$ is closed in U and $V \cap A = \emptyset$ is closed in V, clearly A is not closed in X. Sometimes when we are trying to analyze the geometry of a subset A inside of a metric space X, the best we can do is work locally near the points of A rather than locally near arbitrary points of X. In such cases we clearly cannot hope to prove that A is closed in X, and so the best we can hope to do is to verify the conditions in the following definition:

Definition 2.3. A subset A in a metric space X is *locally closed* if for all $a \in A$ there exists an open set $U_a \subseteq X$ containing a such that $U_a \cap A$ is closed in U_a.

The point of this definition is that the union of the U_a's may fail to equal X, though it does contain A. As an example, $A = [0, 1)$ is locally closed in $X = \mathbf{R}$: for every $a \in A$ distinct from 1 we can take $U_a = (0, 1)$ and for $a = 1$ we can take $U_a = (-1, 1)$. More interesting examples are $(-1, 1) \times \{0\}$ in $\mathbf{R}^2$ (using $U_a = (-1, 1) \times \mathbf{R}$ for all a) and any open subset U of a topological space X (taking $U_a = U$ for all $a \in U$).

The reason for interest in locally closed sets is that they naturally arise when trying to prove closedness of A in X in situations where one is only able to study the situation locally near elements of A. The point worth noting is that locally closed sets look closed if we replace the ambient set with a suitable open around A:

Theorem 2.4. *Let A be a subset of a metric space X. The subset A is locally closed in X if and only if there exists an open set $U \subseteq X$ containing A with A a closed subset of U; in other words, $A = C \cap U$ for a closed subset $C \subseteq X$.*

This theorem says that locally closed sets are precisely the overlap of an open set and a closed set.

Proof. If $A = C \cap U$ for closed C in X and open U in X then we verify the definition of local closedness by taking $U_a = U$ for all $a \in A$. Conversely, if A is locally closed in X then let $U = \cup_{a \in A} U_a$ where $U_a \subseteq X$ is an open set containing $a \in A$ such that $A \cap U_a$ is closed in U_a. Clearly A is a subset of the metric space U and the U_a's contitute an open covering of U. Thus, the local nature of closedness in metric spaces (applied to U) implies that A is closed in U. ■

3. Boundary

We now introduce a notion which sits somewhere between closure and interior: the boundary.

Definition 3.1. Let A be a subset of a metric space X. We define the *boundary* ∂A of A to be $\overline{A} - \mathrm{int}(A)$.

As with the concepts of interior and closure, the boundary depends on the ambient space (though we suppress this in the notation, lest things become unwieldy).

Example 3.2. If $A = [-1, 1] \times (-1, 1)$ inside of $X = \mathbf{R}^2$, then $\partial A = \overline{A} - \mathrm{int}(A)$ consists of points (x, y) on the edge of the unit square: it is equal to

$$(\{-1, 1\} \times [-1, 1]) \bigcup ([-1, 1] \times \{-1, 1\}),$$

as you should check (from our earlier determination of the closure and interior of A).

Example 3.3. Consider the subset $A = \mathbf{Q} \subseteq \mathbf{R}$. We then have $\mathrm{int}(A) = \emptyset$ because no non-empty open interval can fail to contain irrationals (i.e., to be contained inside of $A = \mathbf{Q}$), and $\overline{A} = \mathbf{R}$ since every real number is a limit of a sequence of rationals. Thus, in this case $\partial A = \mathbf{R}$. Of course, if we switched points of view and regarded $A = \mathbf{Q}$ as a subset of the metric space $X = \mathbf{Q}$, then we'd have $\mathrm{int}(A) = A$ (since the interior of a metric space is always equal to the whole space) and $\overline{A} = A$, so $\partial A = \emptyset$!

Just as it is geometrically reasonable that an open subset of a metric space is one which is equal to its own interior, a closed subset ought to be exactly one which contains its boundary. This is the first part of:

Theorem 3.4. *Let A be a subset of a metric space X. Then A is closed if and only if it contains ∂A, and in general*

$$\partial A = \overline{A} \cap \overline{X - A} = \partial(X - A).$$

If one again considers our friend the half-open square $A = [-1, 1] \times (-1, 1)$ in $\mathbf{R}^2$, it is instructive to recall our earlier determinations of the closures of A and $X - A$ and to see that, sure enough, their intersection is just what the boundary ought to be.

Proof. The boundary ∂A is *defined* as $\overline{A} - \mathrm{int}(A)$. Thus,

$$\overline{A} = \mathrm{int}(A) \bigcup \partial A \subseteq A \bigcup \partial A,$$

so when $\partial A \subseteq A$ we get $\overline{A} \subseteq A$ and therefore (the reverse inclusion being obvious) that $A = \overline{A}$, so A is indeed closed. Conversely, if A is closed then since $\partial A \subseteq \overline{A}$ by definition and $\overline{A} = A$ for closed A we get $\partial A \subseteq A$.

Once we establish that $\partial A = \overline{A} \cap \overline{X - A}$, then since the right side is unaffected by replacing A with $X - A$ everywhere (because $X - (X - A) = A$), it follows that $\partial A = \partial(X - A)$. As for verifying that ∂A is the intersection of the closures of A and $X - A$, we use the definition of ∂A to rewrite this as:

$$\overline{A} - \mathrm{int}(A) \stackrel{?}{=} \overline{A} \cap \overline{X - A}.$$

Since $\overline{A} - \mathrm{int}(A) = \overline{A} \cap (X - \mathrm{int}(A))$, it suffices to check that

$$X - \mathrm{int}(A) = \overline{X - A}.$$

But this was one of the "complementary" relationships we proved earlier between interiors and closures. ■

We conclude with a geometrically pleasing corollary.

Corollary 3.5. *Let A be a subset of a metric space X. Then X can be expressed as a disjoint union*

$$X = \mathrm{int}(A) \bigcup \partial A \bigcup \mathrm{int}(X - A).$$

In other words, every point of X satisfies exactly one of the following properties: it is interior to A, interior to $X - A$, or on the common boundary $\partial A = \partial(X - A)$.

The disjointness in this corollary "justifies" the idea that $\partial A = \partial(X - A)$ is sort of a "common interface" between A and $X - A$. For ugly subsets $A \subseteq X$ one can't take this intuition too seriously.

Proof. Since $X - \text{int}(A) = \overline{X - A}$ by an earlier theorem, the assertion of the corollary is exactly the statement

$$\overline{X - A} = \partial A \cup \text{int}(X - A)$$

with ∂A disjoint from $\text{int}(X - A)$. But by definition of boundary for $X - A$ we have a *disjoint* union decomposition

$$\overline{X - A} = \partial(X - A) \cup \text{int}(X - A).$$

Thus, it suffices to show $\partial A = \partial(X - A)$. But this latter equality was shown in the preceding theorem. ∎