

1. OVERVIEW

Let E_1, \dots, E_n be C^p vector bundles over a C^p premanifold with corners X , $0 \leq p \leq \infty$. We want to define the direct sum vector bundle $E_1 \oplus \dots \oplus E_n$ to be “the vector bundle whose x -fiber is $E_1(x) \oplus \dots \oplus E_n(x)$ for all $x \in X$.” To make this precise, we first give a slightly esoteric-looking definition and then we will see it has the desired fibers. To really certify the correctness of our construction, we will then check that it enjoys mapping properties analogous to those satisfied by direct sums of vector spaces.

Consider the \mathcal{O} -module direct sum $\oplus \underline{E}_j = \underline{E}_1 \oplus \dots \oplus \underline{E}_n$ whose “value” on an open set U is the $\mathcal{O}(U)$ -module direct sum of the $\mathcal{O}(U)$ -modules $\underline{E}_j(U) = E_j(U)$ over all j . The *direct sum* vector bundle $E = \oplus E_j$ is $V_{\oplus \underline{E}_j}$. This makes sense because $\oplus \underline{E}_j$ is locally free of finite rank. Indeed, for each $x \in X$ there exists an open U_j around x on which E_j admits a trivializing frame (i.e., elements $s_{1,j}, \dots, s_{r_j,j} \in E_j(U_j)$ defining a bundle isomorphism $E_j|_{U_j} \simeq U_j \times \mathbf{R}^{r_j}$), and so for the open $U_x = \cap U_j$ around x we see that if $U' \subseteq U_x$ is an open subset then the $\mathcal{O}(U')$ -module $\oplus \underline{E}_j(U') = \oplus E_j(U')$ is a free module on the basis of $s_{ij}|_{U'}$'s (using the $\mathcal{O}(U')$ -linear inclusion of $E_j(U')$ into $E_1(U') \oplus \dots \oplus E_n(U')$ via inclusion onto the j th factor). That is, the elements

$$s_{ij}|_{U'} \in E_j(U') \subseteq E_1(U') \oplus \dots \oplus E_n(U') = (\underline{E}_1 \oplus \dots \oplus \underline{E}_n)(U')$$

for opens $U' \subseteq U_x$ are an $\mathcal{O}(U')$ -module basis; since the U_x 's are an open cover of X as x varies, this says exactly that the \mathcal{O} -module $\oplus \underline{E}_j$ is locally free of finite rank.

Does this fancy-looking definition have the desired fibers? On fibers over $x \in X$ we have $V_{\mathcal{M}}(x) = \mathcal{M}(x)$ as \mathbf{R} -vector spaces for any locally free \mathcal{O} -module \mathcal{M} with finite rank (essentially by how the vector bundle $V_{\mathcal{M}} \rightarrow X$ was *constructed*). Thus, for all $x \in X$ we have

$$(\oplus E_j)(x) = (\oplus \underline{E}_j)(x) = \oplus (\underline{E}_j(x)) = \oplus E_j(x),$$

with the second isomorphism arising from the general fact that formation of fibers of an \mathcal{O} -module commutes with formation of direct sums:

Lemma 1.1. *Let $\mathcal{M}_1, \dots, \mathcal{M}_n$ be \mathcal{O} -modules. There is a natural isomorphism*

$$(\mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_n)(x) \simeq \mathcal{M}_1(x) \oplus \dots \oplus \mathcal{M}_n(x)$$

as \mathbf{R} -vector spaces.

Proof. For notational ease, we treat the case $n = 2$; the general case goes via the same procedure. Thus, we work with two \mathcal{O} -modules \mathcal{M} and \mathcal{M}' . The key is to produce an \mathcal{O}_x -linear isomorphism $\phi_x : (\mathcal{M} \oplus \mathcal{M}')_x \simeq \mathcal{M}_x \oplus \mathcal{M}'_x$. To make such an isomorphism, observe that for opens U around x we have

$$(\mathcal{M} \oplus \mathcal{M}')(U) = \mathcal{M}(U) \oplus \mathcal{M}'(U)$$

by *definition* of $\mathcal{M} \oplus \mathcal{M}'$, and so a typical germ $\mu \in (\mathcal{M} \oplus \mathcal{M}')_x$ is represented by a pair $(U; s, s')$ with $U \subseteq X$ an open around x and elements $s \in \mathcal{M}(U)$ and $s' \in \mathcal{M}'(U)$. These represent elements in \mathcal{M}_x and \mathcal{M}'_x respectively, and replacing $(U; s, s')$ with another representative for μ gives the same pair of elements in \mathcal{M}_x and \mathcal{M}'_x . Hence, this defines a map of sets

$$\phi_x : (\mathcal{M} \oplus \mathcal{M}')_x \rightarrow \mathcal{M}_x \oplus \mathcal{M}'_x$$

that one checks is \mathcal{O}_x -linear.

To see that ϕ_x is surjective, note that any element of $\mathcal{M}_x \oplus \mathcal{M}'_x$ has the form (m, m') with m represented by some pair (U, s) and m' represented by some pair (U', s') with opens U, U' around x and elements $s \in \mathcal{M}(U)$ and $s' \in \mathcal{M}'(U)$. Working up to equivalence around x allows us to

replace U and U' here with their open overlap $U \cap U'$, so for representatives of m and m' we may take them to be on the same open set around x . But in the case $U = U'$ we have the element $(U; s, s') \in (\mathcal{M} \oplus \mathcal{M}')(U)$ whose associated germ in $(\mathcal{M} \oplus \mathcal{M}')_x$ is sent under ϕ_x to (m, m') . Hence, ϕ_x is surjective. Injectivity is similar: if $\phi_x(\mu_1) = \phi_x(\mu_2)$ then for representatives $(U_1; s_1, s'_1)$ and $(U_2; s_2, s'_2)$ of μ_1 and μ_2 we have that $[(U_1, s_1)]_x = [(U_2, s_2)]_x$ in \mathcal{M}_x and $[(U_1, s'_1)]_x = [(U_2, s'_2)]_x$ in \mathcal{M}'_x , so $s_1|_W = s_2|_W$ in $\mathcal{M}(W)$ for some open $W \subseteq U_1 \cap U_2$ around x and $s'_1|_{W'} = s'_2|_{W'}$ in $\mathcal{M}'(W)$ for some open $W' \subseteq U_1 \cap U_2$ around x . Hence, for the open $U = W \cap W' \subseteq U_1 \cap U_2$ around x we have $(s_1, s'_1)|_U = (s_2, s'_2)|_U$ in $(\mathcal{M} \oplus \mathcal{M}')(U)$, whence the associated germs μ_1 and μ_2 in $(\mathcal{M} \oplus \mathcal{M}')_x$ are equal as desired. This completes the construction of the \mathcal{O}_x -linear isomorphism ϕ_x .

Since ϕ_x is an \mathcal{O}_x -linear isomorphism, it carries the submodule $\mathfrak{m}_x \cdot (\mathcal{M} \oplus \mathcal{M}')_x$ in $(\mathcal{M} \oplus \mathcal{M}')_x$ over to the submodule $\mathfrak{m}_x \cdot (\mathcal{M}_x \oplus \mathcal{M}'_x) = \mathfrak{m}_x \mathcal{M}_x \oplus \mathfrak{m}_x \mathcal{M}'_x$ (explain the equality!) in $\mathcal{M}_x \oplus \mathcal{M}'_x$. Thus, it induces an isomorphism of quotients

$$(\mathcal{M} \oplus \mathcal{M}')_x \simeq (\mathcal{M}_x \oplus \mathcal{M}'_x) / (\mathfrak{m}_x \mathcal{M}_x \oplus \mathfrak{m}_x \mathcal{M}'_x) \simeq \mathcal{M}(x) \oplus \mathcal{M}'(x)$$

as modules over $\mathcal{O}_x / \mathfrak{m}_x = \mathbf{R}$ (i.e., as vector spaces over \mathbf{R}). ■

Intuitively speaking, the vector bundle $\oplus E_j \rightarrow X$ is a “gluing” of the fiberwise direct sums $\oplus E_j(x)$ over all $x \in X$. Our aim in this handout is to show that that E behaves like a direct sum of vector spaces. To make this precise, we first record the general properties of direct sums of vector spaces as they are used in linear algebra:

Theorem 1.2. *Let V_1, \dots, V_n be finitely many vector spaces over a field F . Let $V = \oplus V_j$ be the direct sum, viewed as an F -vector space in the usual manner.*

If $T_j : V_j \rightarrow V'$ are F -linear maps to an F -vector space V' , there is a unique linear map $T : V \rightarrow V'$ whose composite with the standard inclusion $V_j \rightarrow V$ is T_j ; explicitly, $T(v_1, \dots, v_n) = \sum T_j(v_j)$.

If $T_j : V' \rightarrow V_j$ are F -linear maps from an F -vector space V' , there is a unique linear map $T' : V' \rightarrow V$ whose composite with the standard projection $V \rightarrow V_j$ is T_j ; explicitly, $T'(v') = (T_1(v'), \dots, T_n(v'))$ for all $v' \in V'$.

Proof. For the first claim, the map set-theoretically given by the indicated formula is the only one that can possibly work (in view of how the inclusions $i_j : V_j \rightarrow V$ are defined, and the fact that $(v_1, \dots, v_n) = \sum_j i_j(v_j)$). We just have to check that this map is F -linear. This follows from the linearity of the T_j 's and the definition of the F -vector space structure on $V = \oplus V_j$. For the second claim, the definition of T' is again the only one that can possibly work, and we just have to check that it is F -linear. This linearity is immediate from the assumed linearity of the T_j 's and the definition of the F -vector space structure on $V = \oplus V_j$. ■

Our goal is to show that the direct sum $\oplus E_j$ considered in the category of C^p vector bundles over X enjoys similar mapping properties with respect to C^p vector bundles morphisms from the E_j 's to a C^p vector bundle, as well as from a C^p vector bundle to each of the E_j 's.

The basic idea, which will recur over and over again in later applications of “linear algebra operations” to vector bundles, is this: we consider vector bundles whose fibers admit a concrete description (much as $\oplus E_j$ has fibers identified with direct sums of fibers $E_j(x)$), and we show that certain maps induce fiber maps that behave “as expected” in terms of the concrete fibral descriptions. There are other ways to approach the problem of giving an intrinsic characterization of operations on vector bundles, but to follow such alternative routes would necessitate a very long digression into sheaf theory and would not be helpful for us here.

2. MAPPING PROPERTIES

We treat bundle analogues of the two properties of direct sums for vector spaces as stated above. We first need a lemma.

Lemma 2.1. *Let $E_j \rightarrow X$ be C^p vector bundles over X , and let $E = \oplus E_j$. Choose j_0 . There are unique maps $E_{j_0} \rightarrow E$ and $E \rightarrow E_{j_0}$ that, on fibers over each $x \in X$, are the usual inclusions and projections*

$$E_{j_0}(x) \rightarrow E(x) = \oplus E_j(x), \quad \oplus E_j(x) = E(x) \rightarrow E_{j_0}(x).$$

Proof. We are specifying the linear maps on fibers, so the only problem is to prove that such set-theoretic maps over X are C^p . We may work locally over X (why?), and so we can assume the E_j 's are all trivial. Let r_j be the rank of E_j , and $s_{1,j}, \dots, s_{r_j,j} \in E_j(X)$ be sections defining a C^p isomorphism $X \times \mathbf{R}^n \simeq E_j$ as C^p vector bundles over X . We likewise view the collection of all s_{ij} 's as a global frame for E .

Define maps

$$E_{j_0} \simeq X \times \mathbf{R}^{r_{j_0}} \rightarrow X \times (\mathbf{R}^{r_1} \times \dots \times \mathbf{R}^{r_n}) \simeq E$$

using the matrix for the standard inclusion of $\mathbf{R}^{r_{j_0}}$ into the j_0 th factor of $\mathbf{R}^{r_1} \times \dots \times \mathbf{R}^{r_n}$. One checks that the resulting bundle morphism $E_{j_0} \rightarrow E$ has the desired effect on x -fibers for any $x \in X$. The case of the ‘‘projection’’ maps $E \rightarrow E_{j_0}$ goes similarly. ■

Let $T_j : E_j \rightarrow E'$ be bundle morphisms over X to a C^p vector bundle $\pi' : E' \rightarrow X$.

Theorem 2.2. *There is a unique bundle morphism $E \rightarrow E'$ over X such that on fibers over $x \in X$ it is the map $\oplus E_j(x) \rightarrow E'(x)$ given by $(v_1, \dots, v_n) \mapsto \sum T_j(x)(v_j)$.*

In words, to map $\oplus E_j$ to a vector bundle E' over X , we just have to say how each E_j maps to E' . Indeed, by Lemma 2.1 the theorem says that any bundle morphism $\oplus E_j \rightarrow E'$ is uniquely determined by its composite with the ‘‘bundle inclusions’’ $E_{j_0} \rightarrow \oplus_j E_j$ for all j_0 , and that conversely any collection of bundle maps $T_{j_0} : E_{j_0} \rightarrow E'$ for varying j_0 arise in this way.

Proof. The map is specified on fibers, so the only problem is to verify that it is a C^p mapping. This problem is local over X , so we may assume that the E_j 's are all trivial and that E' is trivial. Choose trivializations for all of these, and consider $E = \oplus E_j$ as trivialized via the combined collection of trivializing sections for all of the E_j 's (viewed as X -sections of E via $E_j \rightarrow E$). Hence, T_j is described by a matrix whose entries are C^p functions on X . Stacking these matrices next to each other gives a big ‘‘direct sum’’ matrix that defines the desired map (with C^p matrix-entry functions, so it is a C^p mapping). ■

Let $T_j : E' \rightarrow E_j$ be bundle morphisms over X from a C^p vector bundle $\pi' : E' \rightarrow X$.

Theorem 2.3. *With notation as above, there is a unique bundle morphism $E' \rightarrow E$ over X such that on fibers over $x \in X$ it is the map*

$$v' \mapsto (T_1(x)(v'), \dots, T_m(x)(v')) \in \oplus E_j(x).$$

In words, to map a vector bundle E' to $\oplus E_j$ over X is ‘‘the same’’ as to say how to map E' to each E_j . That is, a bundle morphism $E' \rightarrow \oplus E_j$ is uniquely determined by its composite with the ‘‘projections’’ $\oplus E_j \rightarrow E_{j_0}$ for all j_0 , and any collection of bundle morphisms $T'_{j_0} : \oplus E_j \rightarrow E_{j_0}$ for varying j_0 arise in this way.

Proof. As in the preceding proof, we may work locally over X so that the E_j 's and E' are all trivial. The same matrix methods carries over (now stacking them on top of each other). ■

Example 2.4. Let $f_j : E_j \rightarrow E'_j$ be maps of C^p vector bundles over X . We want to define the *direct sum mapping*

$$\oplus f_j : \oplus E_j \rightarrow \oplus E'_j$$

by the condition that on x -fibers it is the usual direct sum mapping

$$E_1(x) \oplus \cdots \oplus E_n(x) \rightarrow E'_1(x) \oplus \cdots \oplus E'_n(x)$$

given by

$$(v_1, \dots, v_n) \mapsto (f_1|_x(v_1), \dots, f_n|_x(v_n))$$

for all $x \in X$ (with $f_j|_x : E_j(x) \rightarrow E'_j(x)$ the map induced on x -fibers by f_j). Set-theoretically there is no problem: we have just specified the mapping on fibers. But is it a C^p mapping?

Let $E = \oplus E_j$ and $E' = \oplus E'_j$, so there are natural bundle mappings

$$E_{j_0} \xrightarrow{f_j} E'_{j_0} \rightarrow E'$$

for all j_0 , and by Theorem 2.2 these “arise” from a unique bundle mapping

$$E = \oplus E_j \rightarrow E',$$

and passing to fibers the statement of Theorem 2.2 ensures that the map is exactly the direct sum of the fiber maps $f_j|_x : E_j(x) \rightarrow E'_j(x)$.