

STABLY UNIFORM AFFINOIDS ARE SHEAFY

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ABSTRACT. We develop some of the foundations of affinoid pre-adic spaces without Noetherian or finiteness hypotheses. We give explicit examples of non-adic affinoid pre-adic spaces, and also a new condition ensuring that the structure presheaf on $\mathrm{Spa}(R, R^+)$ is a sheaf. This condition can be used to give a new proof that the spectrum of a perfectoid algebra is an adic space.

1. INTRODUCTION

Let k be a field complete with respect to a non-trivial non-archimedean norm $k \rightarrow \mathbf{R}_{\geq 0}$. If (R, R^+) is an affinoid k -algebra (in the sense of Definition 2.6(ii) of [Sch12]) then we can associate to it a certain topological space $X := \mathrm{Spa}(R, R^+)$ whose elements are certain valuations on R . This topological space – a so-called affinoid pre-adic space – has a natural presheaf of complete topological rings \mathcal{O}_X on it. The presheaf is known to be a sheaf if R satisfies certain finiteness conditions. For example it is a sheaf if R is a quotient of a Tate algebra $k\langle T_1, T_2, \dots, T_n \rangle$ (that is, the ring of power series which converge on the closed unit polydisc), and there are other finiteness conditions which also suffice to guarantee \mathcal{O}_X is a sheaf. These finiteness conditions are imposed very early on in [Hub96] (Assumption (1.1.1)), which is mainly concerned with the theory of étale cohomology in the context of rigid spaces. However, more recently Scholze has introduced the concept of a perfectoid k -algebra, for which these finiteness conditions essentially never hold. Scholze showed in Theorem 6.3(iii) of [Sch12] that for R perfectoid, \mathcal{O}_X is still a sheaf. His proof is delicate, involving a direct calculation in characteristic p and then some machinery (almost mathematics, tilting) to deduce the result in characteristic zero. However it would be technically useful in some applications to have a more general method. For example in Conjecture 2.16 of [Sch13] Scholze asks the following question: if X can be covered by rational subsets which are perfectoid, then is X perfectoid? This unfortunately turns out not to be true in this generality, because there are examples of locally perfectoid affinoid pre-adic spaces X where \mathcal{O}_X fails to be a sheaf and hence X cannot be perfectoid. This raises the general question of what extra assumptions one should put on X in order to hope that one can check that it is perfectoid via local calculations. However there seem to be no examples in the literature at all of affinoid k -algebras for which \mathcal{O}_X is not a sheaf, and in general the problem seems to be very poorly-understood (or at least poorly-documented¹).

In this paper we give some examples of affinoid k -algebras for which \mathcal{O}_X is not a sheaf, and show that the phenomenon is strongly linked to the issue that the set of power-bounded elements in an affinoid ring may not be bounded. On the other hand, we show that if every rational subset of $\mathrm{Spa}(R, R^+)$ has the property that all power-bounded elements are bounded, then \mathcal{O}_X is a sheaf (with no finiteness or perfectoid assumptions). Hopefully the result and the counterexamples can be used as a guide to how one should expect \mathcal{O}_X to behave in these situations which are now of interest because of Scholze’s breakthrough works.

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¹However, Mihara posted an example ([Mih14]) to the ArXiv while this paper was being typed up.

2. DEFINITIONS

The definition of an adic space is due to Huber (see Chapter 1 of [Hub96] for example), but we use §2 of [Sch12] as a convenient reference because we will restrict to the case of adic spaces over a complete field, where some simplifications occur (for example, the existence of a topologically nilpotent unit is guaranteed).

Throughout this paper, k will be a field complete with respect to a non-trivial non-archimedean norm $|\cdot| : k \rightarrow \mathbf{R}_{\geq 0}$, with integer ring \mathcal{O}_k , and $\varpi \in k^\times$ will satisfy $0 < |\varpi| < 1$. If R is a k -algebra and R_0 is an \mathcal{O}_k -subalgebra such that $kR_0 = R$, then R can be topologized by letting a basis of open sets be those of the form $r + \varpi^n R_0$ with $r \in R$ and $n \in \mathbf{Z}$. The resulting topological ring is called a *Tate k -algebra* in Definition 2.6 of [Sch12]. An element $r \in R$ is called *power-bounded* if there exists some $n \in \mathbf{Z}$ such that $r^m \in \varpi^{-n} R_0$ for all $m \in \mathbf{Z}_{\geq 0}$. The set R° of power-bounded elements is an open and integrally closed subring of R containing R_0 . If $R^+ \subseteq R^\circ$ now denotes an arbitrary open and integrally closed subring of R (for example $R^+ = R^\circ$) then the resulting pair (R, R^+) is called an *affinoid k -algebra*, and to this algebra one can associate a topological space $X = \text{Spa}(R, R^+)$ whose elements are (equivalence classes of) continuous valuations on R which are bounded by 1 on R^+ . The space X is furthermore endowed with a presheaf \mathcal{O}_X of complete topological rings, and the question this paper is mainly concerned with, is when this presheaf is a sheaf.

We refer to [Sch12] for careful definitions of X and \mathcal{O}_X ; we summarize here the facts that we will need. Firstly we mention some basic results about completions for which we could find no easily-accessible reference. If W is a k -vector space and W_0 is an \mathcal{O}_k -submodule whose k -span is W , then we can topologize W by letting $w + \varpi^n W_0$ ($w \in W$, $n \in \mathbf{Z}$) be a basis of open sets; then W becomes a topological group, and $\varpi^n W_0$ for $n \in \mathbf{Z}$ are a basis of open neighbourhoods of the origin. We will refer to this topology as “the topology on W induced from W_0 ”. We can complete W with respect to this topology; the completion, denoted \widehat{W} , is the limit $\varprojlim_n W/\varpi^n W_0$, endowed with the projective limit topology (the quotients $W/\varpi^n W_0$ have the discrete topology). There is a canonical map $i : W \rightarrow \widehat{W}$. If $U \subseteq W$ is an open subgroup then let \widehat{U} denote the closure of $i(U)$ in \widehat{W} (this is just notation, but it is reasonable because the closure of $i(U)$ is isomorphic to the completion of U in the sense of [Bou71], by [Bou71] II, §3.9 Corollaire 1).

Lemma 1. *In the situation described above:*

- (i) *The subgroups $\widehat{\varpi^n W_0}$ form a basis of open neighbourhoods of the origin of \widehat{W} .*
- (ii) *If U is an open subgroup of W then $i^{-1}(\widehat{U}) = U$.*
- (iii) *If R is a k -algebra and R_0 is an \mathcal{O}_k -subalgebra with $kR_0 = R$, then \widehat{R} is a topological ring with topology induced from the subring \widehat{R}_0 , and $\widehat{R}^\circ = (\widehat{R})^\circ$ (i.e., completion commutes with taking power-bounded elements).*

Proof. (i) The closure $\overline{\{0\}}$ of $\{0\} \subseteq W$ is easily checked to be $\bigcap_n \varpi^n W_0$, so the result follows by applying [Bou71] III §3.4 Proposition 7 to W modulo this closure, noting that any subgroup containing an open subgroup is open.

(ii) Clearly $i^{-1}(\widehat{U})$ contains U . Conversely, the universal property of the completion ([Bou71] III, §4.8) gives us a map $\widehat{W} \rightarrow W/U$ through which the canonical map $W \rightarrow W/U$ factors. The kernel K of $\widehat{W} \rightarrow W/U$ has the property that it contains $i(U)$ and hence its closure, but also that $i^{-1}(K) = U$. This shows $i^{-1}(\widehat{U}) = U$.

(iii) \widehat{R} and \widehat{R}_0 are rings by [Bou71] III, §6.5 and II, §3.9 Corollaire 1. The topology on \widehat{R} is induced from \widehat{R}_0 by [Bou71] III §3.4 Proposition 7. By part (i) $i(R^\circ)$ consists of power-bounded elements, and $(\widehat{R})^\circ$ is open so it contains \widehat{R}° . Conversely, say $\hat{r} \in \widehat{R}$ is power-bounded. Because $i(R)$ is dense in \widehat{R} and $(\widehat{R})^\circ$ is open, for any $n \geq 0$ we may find $r_n \in R$ such that $\hat{r} - i(r_n) \in \varpi^n \widehat{R}_0$, and one checks using the binomial theorem that $i(r_n)$ is power-bounded and hence (using (ii)) that r_n is too. Hence $\hat{r} \in \widehat{R}^\circ$. \square

We now return to our description of \mathcal{O}_X . The ring $\mathcal{O}_X(X)$ is not R , but the completion \widehat{R} of R with respect to the topology induced by R_0 , namely $\varprojlim_{n \geq 0} R/\varpi^n R_0$. Let us now describe

\mathcal{O}_X on certain open subsets of X . Choose $t \in R$. Then we can cover X by two open subsets $U := \{x : |t(x)| \leq 1\}$ and $V := \{x : |t(x)| \geq 1\}$ (where we make the standard abuse of notation: x is a valuation on R and $|t(x)|$ is just another way of writing $x(t)$). If \mathcal{O}_X is a sheaf of complete topological rings then in particular it is a sheaf of k -vector spaces and so the sequence

$$(*) \quad 0 \rightarrow \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U \cap V)$$

of k -vector spaces is exact. We will now describe these vector spaces and homomorphisms explicitly. The subsets U , V and $U \cap V$ are rational subsets of X , so it is not hard to compute \mathcal{O}_X on them directly.

Set $A = R$ and $A_0 = R_0[t]$. We topologize the ring A using A_0 as above. The space $\mathcal{O}_X(U)$ is the completion \widehat{A} of A with respect to A_0 . Set $B = R[1/t]$, the localization of R at the set $\{1, t, t^2, t^3, \dots\}$ obtained by inverting t . Let $\phi : R \rightarrow B$ denote the canonical map. We set $B_0 = \phi(R_0)[1/t]$ and topologize B using B_0 as above. The space $\mathcal{O}_X(V)$ is the completion \widehat{B} of B . Finally we set $C = B$ and $C_0 = \phi(R_0)[t, 1/t]$. The space $\mathcal{O}_X(U \cap V)$ is the completion \widehat{C} of C .

The abstract rings R and A coincide, but their topologies will not coincide in general. More precisely, $R_0 \subseteq A_0$ and hence the identity map $R \rightarrow A$ is continuous, but if $A_0 \not\subseteq \varpi^{-N}R_0$ for any $N \geq 0$ then the identity map $A \rightarrow R$ is not. (An example where this happens is $R = A = k[[T]]$, $R_0 = \mathcal{O}_k[[\varpi T]]$ and $t = T$ so $A_0 = \mathcal{O}_k[[T]]$.) Similarly, the identity map $B \rightarrow C$ is continuous but the identity map $C \rightarrow B$ may not be. Also, $\phi : R \rightarrow B$ is continuous as are the induced maps $\phi : R \rightarrow C$ and $\phi : A \rightarrow C$, but the induced map $A \rightarrow B$ may not be.

Define $\epsilon : R \rightarrow A \oplus B$ by $\epsilon(r) = (r, \phi(r))$, and define $\delta : A \oplus B \rightarrow C$ by $\delta(a, b) = b - \phi(a)$. One checks easily that the sequence of abstract k -vector spaces

$$(**) \quad 0 \rightarrow R \xrightarrow{\epsilon} A \oplus B \xrightarrow{\delta} C \rightarrow 0$$

is exact, and indeed it is naturally split, the map $C \rightarrow A \oplus B$ sending c to $(0, c)$ being a splitting. However if we topologize R , $A \oplus B$ and C using R_0 , $A_0 \oplus B_0$ and C_0 respectively, then ϵ and δ are continuous but the splitting may not be continuous.

The sequence $(*)$ whose exactness we care about consists of the first three arrows in the completion of the sequence $(**)$ with respect to the topologies defined by R_0 , $A_0 \oplus B_0$ and C_0 . The issue then, is whether taking completions can destroy left exactness.

Before we embark on a discussion of this, we recall the notion of strictness. A continuous map between topological groups $\psi : V \rightarrow W$ is called *strict* if the two topologies on $\psi(V)$, namely the quotient topology coming from V and the subspace topology coming from W , coincide. We see that $\delta : A \oplus B \rightarrow C$ is strict, because it is a continuous surjection and the image of $A_0 \oplus B_0$ is C_0 so δ is open. On the other hand, ϵ is strict iff $S_0 := A_0 \cap \phi^{-1}(B_0)$ is bounded in R , which is not always the case; we will see explicit examples of that later on.

The following lemma shows that exactness of $(*)$ is in fact equivalent to strictness of ϵ .

Lemma 2. *The following are equivalent:*

- (i) $(*)$ is exact,
- (ii) $(*)$ is exact and furthermore the map $\mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U \cap V)$ is surjective,
- (iii) there exists some $n \in \mathbf{Z}$ such that $\varpi^n(A_0 \cap \phi^{-1}(B_0)) \subseteq R_0$,
- (iv) ϵ is strict (and hence all maps in $(**)$ are strict).

Proof. Let S denote the ring R and define $S_0 := A_0 \cap \phi^{-1}(B_0)$. Topologize S using S_0 in the usual way. Then $\epsilon : S \rightarrow A \oplus B$ is strict and the identity map $R \rightarrow S$ is a continuous bijection. In particular, strictness of ϵ is equivalent to $R \rightarrow S$ being a homeomorphism, which is equivalent to R_0 being open in S . Hence (iii) and (iv) are equivalent.

Now (ii) implies (i) trivially. Furthermore, it is a general fact in this setting that for an exact sequence with all morphisms strict, its completion remains exact (see for example [Bou61] III.2.12, Lemme 2, or [BGR84] Corollary 1.1.9/6). Applying this to the strict surjection δ we deduce that $\mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U \cap V)$ is always surjective, and so (i) implies (ii). Furthermore, if (iv) holds then every map in $(**)$ is strict, so the completion of $(**)$ is still exact, and hence (iv) implies (ii).

It suffices to prove that (i) implies (iii). Note that the converse of the “strict implies completion exact” result used several times above is not true in general. (For example, if $R = A = k[T]$, $R_0 = \mathcal{O}_k[\varpi T]$ and $A_0 = \mathcal{O}_k[T]$, then $R \rightarrow A$ is injective and not strict, but the induced map $\widehat{R} \rightarrow \widehat{A}$ is still injective.) So assume (i). Then

$$0 \rightarrow S \rightarrow A \oplus B \rightarrow C \rightarrow 0$$

is exact and all the maps are strict, so the sequence remains exact under completion, and if furthermore (i) holds we deduce that the map $\widehat{R} \rightarrow \widehat{S}$ induced by the continuous map $R \rightarrow S$ must be a bijection. By the open mapping theorem (see for example [BGR84] §2.8.1 or [Bou81] Chapter I, §3.3, Théorème 1), $\widehat{R} \rightarrow \widehat{S}$ is open. In particular the image of \widehat{R}_0 must contain $\varpi^n \widehat{S}_0$ for some $n \in \mathbf{Z}$. Pulling back via the natural map $R \rightarrow \widehat{R}$ and using Lemma 1(ii) we conclude that R_0 must contain $\varpi^n S_0$, which is (iii). \square

This lemma is used in two ways in the sequel. In the next section we observe that if the power-bounded elements of R are bounded, then condition (iii) of the lemma follows (Corollary 4), and hence we get a criterion for checking the sheaf axiom for the cover $X = U \cup V$, which we can turn (Theorem 7) into a criterion for checking that the presheaf \mathcal{O}_X on an affinoid pre-adic space is a sheaf. As a consequence (Corollary 9) we get a new proof that \mathcal{O}_X is a sheaf if $X = \mathrm{Spa}(R)$ with R perfectoid.

In Section 4 we construct rings where part (iii) is violated, and use them to build examples of affinoid pre-adic spaces which are not adic.

3. A CRITERION FOR \mathcal{O}_X TO BE A SHEAF ON AN AFFINOID PRE-ADIC SPACE

As before, let R be a k -algebra and let R_0 be an \mathcal{O}_k -subalgebra such that $kR_0 = R$. As usual we topologize R by letting $\varpi^n R_0$ for $n \in \mathbf{Z}$ be a basis of open neighbourhoods of the identity. We recall that R° denotes the subring of power-bounded elements of R . The topological ring R is called *uniform* if R° is bounded, in other words if there exists some $n \in \mathbf{Z}$ such that $R^\circ \subseteq \varpi^{-n} R_0$. Examples of uniform rings include reduced affinoid algebras in Tate’s original sense (i.e., those which are topologically of finite type over k), and conversely any Tate k -algebra with a non-zero nilpotent element r such that $kr \not\subseteq R_0$ would be a non-uniform ring, as $kr \subseteq R^\circ$.

The key lemma we need in this section is that if an element of R is locally in R_0 then it is globally power-bounded. This sounds geometrically reasonable, and we now give a short elementary algebraic proof. We first remind the reader that every open cover an affinoid pre-adic space over k can be refined to a rational cover (see Lemma 8(i) below, and just before that lemma for the definition of a rational cover).

Lemma 3. *Let R be a Tate k -algebra. Let t_1, \dots, t_n in R such that $t_1 R + \dots + t_n R = R$. For each i , let $R[1/t_i]$ be the localization of R at the multiplicative set $\{1, t_i, t_i^2, \dots\}$ and $\phi_i : R \rightarrow R[1/t_i]$ the natural homomorphism. Then*

$$\bigcap_{i=1}^n \phi_i^{-1}(\phi_i(R_0)[t_1/t_i, \dots, t_n/t_i]) \subseteq R^\circ.$$

Proof. Suppose $r \in \bigcap_{i=1}^n \phi_i^{-1}(\phi_i(R_0)[t_1/t_i, \dots, t_n/t_i])$. For each i there is a homogeneous polynomial $f_i \in R_0[T_1, \dots, T_n]$ such that $\phi_i(r) = t_i^{-\deg(f_i)} \phi_i(f_i(t_1, \dots, t_n))$. Since $t_i^{\deg(f_i)} r - f_i(t_1, \dots, t_n) \in \ker(\phi_i)$, there exists $c_i \geq 0$ such that $t_i^{c_i} (t_i^{\deg(f_i)} r - f_i(t_1, \dots, t_n)) = 0$. So $t^{d_i} r = g_i(t_1, \dots, t_n)$ where $g_i = T_i^{c_i} f_i \in R_0[T_1, \dots, T_n]$ is homogeneous of degree $d_i = c_i + \deg(f_i)$.

Set $N = d_1 + \dots + d_n$. Take $\eta \in k^\times$ such that $\eta t_i \in R_0$ for all i . We will show, by induction on $m \geq 0$, that $\eta^N h(t_1, \dots, t_n) r^m \in R_0$ for every $h \in R_0[T_1, \dots, T_n]$ that is homogeneous of degree N and all $m \geq 0$. The case $m = 0$ is clear because $\eta t_i \in R_0$ for all i . Induction step: $m > 0$. It is sufficient to consider the case where h is a monomial, $h = T_1^{e_1} \dots T_n^{e_n}$. Since $e_1 + \dots + e_n = N = d_1 + \dots + d_n$, there is at least one i for which $e_i \geq d_i$. Without loss of generality we can assume that $i = 1$. Now $\eta^N t_1^{e_1} \dots t_n^{e_n} r^m = \eta^N t_1^{e_1 - d_1} t_2^{e_2} \dots t_n^{e_n} g_1(t_1, \dots, t_n) r^{m-1}$ and by the induction hypothesis this is in R_0 . This concludes the induction proof.

There exist $a_1, \dots, a_n \in R$ such that $a_1 t_1 + \dots + a_n t_n = 1$. Take $\theta \in k^\times$ such that $\theta a_i \in R_0$ for all i . Applying the above result to $h = (\theta a_1 T_1 + \dots + \theta a_n T_n)^N$ shows that $\eta^N \theta^N r^m \in R_0$ for all $m \geq 0$, and hence $r \in R^\circ$. \square

Corollary 4. *Let (R, R^+) be a uniform affinoid k -algebra, and let $X = \text{Spa}(R, R^+)$ be the associated affinoid pre-adic space. Let $t \in R$ and set $U = \{x \in X : |t(x)| \leq 1\}$ and $V = \{x \in X : |t(x)| \geq 1\}$. Then the sequence*

$$0 \rightarrow \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U \cap V) \rightarrow 0$$

is exact.

Proof. The conclusion of the corollary is condition (ii) of Lemma 2, so it suffices to verify condition (iii) of that lemma. Applying Lemma 3 with $t_1 = 1$ and $t_2 = t$ (ϕ_1 is the identity, $\phi_2 = \phi$) we deduce $A_0 \cap \phi^{-1}(B_0) \subseteq R^\circ$, and we can conclude because $R^\circ \subseteq \varpi^{-n} R_0$ for some $n \in \mathbf{Z}$ by uniformity. \square

Corollary 5. *If X is an affinoid pre-adic space, $f \in \mathcal{O}_X(X)$, and X has a cover by opens U_i such that $f|_{U_i} = 0$ for all i , then f is topologically nilpotent.*

Proof. By Lemma 8(i) we may assume the cover is rational. By Lemma 3 any locally zero element is power-bounded. Applying this to $\varpi^{-1} f$ we see that $\varpi^{-1} f$ is power-bounded and hence f is topologically nilpotent. \square

Remark 6. We will need Lemma 3 and Corollary 4 later, but Peter Scholze points out to us that Corollary 5 also follows easily from Theorem 1.3.1 of [Ber90].

We now give a new criterion for the presheaf \mathcal{O}_X on $\text{Spa}(R, R^+)$ to be a sheaf. Let us say that an affinoid k -algebra (R, R^+) is *stably uniform* if every rational subset $U \subseteq \text{Spa}(R, R^+)$ has the property that $\mathcal{O}_X(U)$ is uniform. This notion is independent of the choice of R^+ , so we can safely talk about a Tate k -algebra being uniform or stably uniform. We say that an affinoid pre-adic space over k is *stably uniform* if its ring of global sections is stably uniform. We remark that a Tate k -algebra R is uniform (resp. stably uniform) iff its completion is, by Lemma 1(iii).

Theorem 7. *Let (R, R^+) be a stably uniform affinoid k -algebra. Then $X := \text{Spa}(R, R^+)$ is an adic space, in other words, the presheaf \mathcal{O}_X on X is a sheaf of complete topological rings.*

Note that there are no finiteness hypotheses on R whatsoever. Before we embark upon the proof, let us remark that its deduction from Corollary 4 is, to a large extent, an application of standard machinery, although unfortunately we have found no single reference in the literature that fully covers our requirements. The following sources were of great use to us: §2 of Huber's paper [Hub94] (proving an analogous result for adic spaces under some Noetherian hypotheses), Chapter 8 of [BGR84] (proving Tate's acyclicity theorem for affinoid algebras topologically of finite type) and finally §8.2 of [Wed12]. As preparation we now consider two special types of covers of affinoid pre-adic spaces and some relationships between them.

Say $t_1, t_2, \dots, t_n \in R$ are elements of an affinoid k -algebra (R, R^+) such that the ideal they generate is all of R . Set $X = \text{Spa}(R, R^+)$, and for $1 \leq i \leq n$ define $U_i := \{x \in X : |t_j(x)| \leq |t_i(x)| \text{ for all } 1 \leq j \leq n\}$. Then each U_i is a rational subset of X and the union of the U_i is X . Such a cover is called a *rational cover*. If furthermore each $t_i \in R^\times$, the cover is called a *rational cover generated by units*.

Say $t_1, t_2, \dots, t_n \in R$ are elements of an affinoid k -algebra (R, R^+) . Set $X = \text{Spa}(R, R^+)$, and for each subset I of $\{1, 2, \dots, n\}$ define $U_I = \{x \in X : |t_i(x)| \leq 1 \text{ for } i \in I, |t_i(x)| \geq 1 \text{ for } i \notin I\}$. Then each U_I is a rational subset of X and the union of the 2^n sets U_I is X . Such a cover is called a *Laurent cover*.

Lemma 8. *Let X be an affinoid pre-adic k -space.*

- (i) (Huber) *For every open cover \mathcal{U} of X , there exists a rational cover \mathcal{V} of X which is a refinement of \mathcal{U} .*
- (ii) *For every rational cover \mathcal{U} of X , there exists a Laurent cover \mathcal{V} of X such that for every $V \in \mathcal{V}$, the cover $\{U \cap V : U \in \mathcal{U}\}$ of V is a rational cover generated by units.*

(iii) For every rational cover \mathcal{U} of X generated by units, there exists a Laurent cover \mathcal{V} of X which is a refinement of \mathcal{U} .

Proof. (i) See [Hub94] Lemma 2.6.

(ii) If \mathcal{U} is generated by $t_1, t_2, \dots, t_n \in R$ then by assumption there are $a_i \in R$ such that $\sum_i a_i t_i = 1$. Because $R_0 \subseteq R^+$ we can find some $\eta \in k^\times$ with $|\eta| < 1$ such that $\eta a_i \in R^+$ for all i . By definition, for all $x \in X$ and all $r \in R^+$ we have $|x(r)| \leq 1$; hence if there existed $x \in X$ such that $|t_i(x)| < |\eta|$ for all i then $\sum_i (\eta a_i) t_i = \eta$ gives a contradiction. In particular if $c = \eta^{-2}$ then for all $x \in X$ we have $|c|^{-1} < \max_i \{|t_i(x)|\}$. One checks easily, see for example the proof of [BGR84] Lemma 8.2.2/3, that the Laurent cover generated by the ct_i has the desired property.

(iii) This can be shown by the purely combinatorial argument in the proof of [BGR84] Lemma 8.2.2/4. \square

Proof of Theorem 7. First consider \mathcal{O}_X as a presheaf of abelian groups on X . We claim that any Laurent cover of any rational subset of X is \mathcal{O}_X -acyclic. Note that a rational subset of a stably uniform affinoid pre-adic space is again stably uniform. For $n = 1$ the claim is just Corollary 4 and the general case follows from this by induction on n using [BGR84] Corollary 8.1.4/4. The proof of [BGR84] Proposition 8.2.2/5, using Lemma 8(i)–(iii) in lieu of [BGR84] Lemmas 8.2.2/2–4, now shows that any cover by rational subsets of any rational subset of X is \mathcal{O}_X -acyclic. It follows that \mathcal{O}_X is a sheaf of abelian groups on the site whose objects are rational subsets of $\mathrm{Spa}(R, R^+)$ and whose covers are covers of rational subsets by rational subsets.

Since \mathcal{O}_X is a presheaf of rings and a sheaf of abelian groups, it is also a sheaf of rings on this site. We claim that it is even a sheaf of complete topological rings on this site. For this it suffices, by the first paragraph of §2 of [Hub94], to check that if $U = \cup_i U_i$ is a cover of a rational subset U by rational subsets, then the induced map $\mathcal{O}_X(U) \rightarrow \prod_i \mathcal{O}_X(U_i)$ is strict. By Lemma 8(i) there is a rational cover $U = \cup_j V_j$ that refines $U = \cup_i U_i$. Lemma 3 says that the induced map $\mathcal{O}_X(U) \rightarrow \prod_j \mathcal{O}_X(V_j)$ is strict and since this map factors through $\mathcal{O}_X(U) \rightarrow \prod_i \mathcal{O}_X(U_i)$ that map must be strict too.

We have established that \mathcal{O}_X is a sheaf of complete topological rings on the basis of rational subsets of X and by [Gro60] Chapter 0 (3.2.2) we deduce that \mathcal{O}_X is a sheaf of complete topological rings on X . \square

As a toy example of an application, we get a new proof of Theorem 6.3(iii) of [Sch12], that avoids the arguments of 6.10–6.14 of *loc. cit.*

Corollary 9. *If k is a perfectoid field, then the affinoid pre-adic space associated to a perfectoid k -algebra is an adic space.*

Proof. Perfectoid affinoid k -algebras are uniform (by definition) and hence stably uniform (by Corollary 6.8 of [Sch12]), so the theorem directly implies that the affinoid pre-adic space associated to a perfectoid affinoid k -algebra is an adic space. \square

We also deduce that under the stably uniform assumption, in characteristic p we can check that a ring is perfectoid locally.

Corollary 10. *If k is a perfectoid field of characteristic $p > 0$, and if A is a stably uniform complete Tate k -algebra such that $\mathrm{Spa}(A, A^+)$ has a rational cover by affinoids of the form $\mathrm{Spa}(R_i, R_i^+)$ with the R_i perfectoid k -algebras, then A is perfectoid.*

Proof. By Proposition 5.9 of [Sch12], it suffices to show that the p th power map $A \rightarrow A$ is surjective. So say $a \in A$. Let a_i denote the restriction of a to R_i ; then because R_i is perfectoid (and hence reduced) we know $a_i = (b_i)^p$ for a unique $b_i \in R_i$. A rational subspace of an affinoid perfectoid space is again perfectoid, by Theorem 6.3 of [Sch12], and hence the b_i agree on overlaps; Theorem 7 implies that the b_i glue together to give an element $b \in A$. Now $b^p - a$ is locally zero and hence zero (again by Theorem 7), and hence $b^p = a$. \square

4. COUNTEREXAMPLES

In this section we give various examples of affinoid k -algebras for which the structure presheaf is not a sheaf of complete topological rings (and is not even a sheaf of abelian groups). Let us say that an affinoid k -algebra (R, R^+) is *sheafy* if $X := \mathrm{Spa}(R, R^+)$ is an adic space (that is, if \mathcal{O}_X is a sheaf of complete topological rings). We remark here that as this paper was being written, a preprint of Tomoki Mihara appeared on the ArXiv [Mih14] with another example; Mihara's work was independent of ours.

The following lemma will be helpful for us when attempting to locate the power-bounded elements in polynomial rings (which are naturally graded).

Lemma 11. *Let R be a Tate k -algebra with topology defined by an \mathcal{O}_k -subalgebra R_0 . Say we are given a torsion-free (additive) abelian group G and a G -grading of R , that is, a decomposition $R = \bigoplus_{g \in G} R^{(g)}$ where the $R^{(g)}$ are k -subspaces of R satisfying $R^{(g)}R^{(h)} \subseteq R^{(g+h)}$. Suppose that R_0 is also graded by this grading, that is, $R_0 = \bigoplus_{g \in G} (R_0)^{(g)}$, with $(R_0)^{(g)} = R_0 \cap R^{(g)}$. Then R° is also graded by this grading.*

Proof. Say $r \in R^\circ$. Then $r = \sum_{i \in I} r_i$, with $I \subseteq G$ a finite subset and $r_i \in R^{(i)}$. It suffices to check that $r_i \in R^\circ$ for all i . We do this by induction on the size of I . If $|I| \leq 1$ the result is clear. For $|I| > 1$ we let H be the subgroup of G generated by I and observe that H is finitely-generated and torsion-free, and hence a free abelian group, so there is an injection $H \rightarrow \mathbf{R}$, giving us an ordering on I . Say i_0 is the smallest element of I with respect to this embedding. Write $r = r_0 + r_1$ with $r_0 = r_{i_0}$. Because $r \in R^\circ$ there is some N such that $r^n \in \varpi^{-N}R_0$ for all $n \geq 0$, and hence $r_0^n + r_1^n \in \varpi^{-N}R_0$, where $r_0^n \in R^{(ni_0)}$ and r_1^n is a sum of elements in $R^{(j)}$ for $j \in H$, $j > ni_0$. In particular r_0^n must be in $\varpi^{-N}(R_0)^{(ni_0)}$ and in particular $r_0^n \in \varpi^{-N}R_0$, hence $r_0 \in R^\circ$, and hence $r_1 \in R^\circ$ and we can apply the inductive hypothesis to r_1 , finishing the argument. \square

4.1. A finitely-generated non-sheafy k -algebra. As before, let k be a field complete with respect to a non-trivial non-archimedean (rank 1) valuation, let \mathcal{O}_k denote its ring of integers, and let us fix $\varpi \in k$ with $0 < |\varpi| < 1$.

Even if R is a Tate k -algebra which is finitely-generated as an abstract k -algebra, the subalgebra R_0 defining the topology might be sufficiently nasty to ensure that (R, R^+) is not sheafy. This is not surprising – indeed Rost's example of a non-sheafy ring (which is not a k -algebra) given at the end of §1 of [Hub94] is finitely-generated over \mathbf{Z} . We remark here that before [Mih14], Rost's example was the only example known to us in the literature of a non-sheafy ring.

Now, let R be the ring $k[T, T^{-1}, Z]/(Z^2)$ and let R_0 denote the \mathcal{O}_k -submodule of R with \mathcal{O}_k -basis $\varpi^{|n|}T^n$ and $\varpi^{-|n|}T^nZ$ ($n \in \mathbf{Z}$). (For the avoidance of doubt, here $|\cdot|$ denotes the ordinary absolute value on \mathbf{Z} .) One checks easily that R_0 is an \mathcal{O}_k -subalgebra of R and that $kR_0 = R$. We note in passing that R_0 is not Noetherian – indeed, the ideal $ZR \cap R_0$ of R_0 is easily checked to be not finitely-generated.

Proposition 12. *For the space $X := \mathrm{Spa}(R, R^\circ)$ the presheaf \mathcal{O}_X is not a sheaf. In particular, X is covered by $U := \{x \in X : |T(x)| \leq 1\}$ and $V := \{x \in X : |T(x)| \geq 1\}$ and the map $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ is not injective.*

Before we begin the proof, we briefly note two consequences. Firstly this proposition (positively) resolves the footnote just before Definition 2.16 in [Sch12]. Secondly, R is Noetherian, but it cannot be strongly Noetherian because \mathcal{O}_X is a sheaf for strongly Noetherian Tate k -algebras by Theorem 2.2 of [Hub94].

Proof. That X is covered by the opens U and V is obvious. By definition, $\mathcal{O}_X(X) = \varprojlim_n R/\varpi^n R_0$, the completion of R . Similarly, $\mathcal{O}_X(U) = \varprojlim_n R/\varpi^n R_0[T]$ and $\mathcal{O}_X(V) = \varprojlim_n R/\varpi^n R_0[T^{-1}]$. We claim that the map $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ is not injective, and this suffices to show that \mathcal{O}_X is not even a sheaf of abelian groups on X .

More precisely, we claim that $0 \neq Z \in \mathcal{O}_X(X)$ but that Z restricts to zero in both U and V . To verify the first assertion it suffices to observe that $kZ \not\subseteq R_0$, which is clear because $kZ \cap R_0 = \mathcal{O}_k Z$. To verify the second assertion it suffices to check that $kZ \subseteq R_0[T]$ and $kZ \subseteq R_0[T^{-1}]$; but

both of these are also clear because for $n \geq 0$ we have $\varpi^{-n}Z = \varpi^{-n}T^{-n}Z.T^n \in R_0[T]$ and $\varpi^{-n}Z = \varpi^{-n}T^nZ.T^{-n} \in R_0[T^{-1}]$. \square

Note that $\mathcal{O}_X(U)$ is the completion of $k[T, T^{-1}]$ with respect to the topology generated by the subring $\mathcal{O}_k[T, \varpi T^{-1}]$ so in fact U is isomorphic to the adic space associated to the annulus $\{|\varpi| \leq |T| \leq 1\}$. Similarly V is isomorphic to the adic space associated to the annulus $\{1 \leq |T| \leq |\varpi|^{-1}\}$; however, X is not the adic space associated to the annulus $\{|\varpi| \leq |T| \leq |\varpi|^{-1}\}$ as $\mathcal{O}_X(X)$ contains nilpotents.

The reader aware of Rost's example at the end of §1 of [Hub94] will be able to see that the key idea of the counterexample above is basically Rost's.

4.2. A non-perfectoid, locally perfectoid space. In this subsection we assume the characteristic of k is p , and that k is perfectoid (or equivalently that k is perfect). In this situation we can basically “perfectify” our previous example, and in this way construct an affinoid pre-adic space which is not adic (and in particular not perfectoid), but which is locally perfectoid. In particular we resolve Conjecture 2.16 of [Sch13] (negatively).

The details are as follows. We start by perfectifying the ring $k[T, T^{-1}]$, that is, we take the direct limit $\lim_{x \mapsto x^p} k[T, T^{-1}]$; we call this ring $k[T^{1/p^\infty}, T^{-1/p^\infty}]$. We then adjoin a nilpotent by setting $R = k[T^{1/p^\infty}, T^{-1/p^\infty}][Z]/(Z^2)$. Then R has a k -basis consisting of elements of the form T^n and $T^n Z$ for $n \in \mathbf{Z}[1/p]$. We let R_0 denote the \mathcal{O}_k -submodule of R with basis $\varpi^{|n|}T^n$ and $\varpi^{-|n|}T^n Z$ ($n \in \mathbf{Z}[1/p]$). Topologize R as usual by letting subsets of the form $r + aR_0$ ($r \in R, a \in k^\times$) be a basis.

Proposition 13. *The space $X := \mathrm{Spa}(R, R^\circ)$ is not an adic space, because \mathcal{O}_X is not a sheaf. However $X = U \cup V$ with $U := \{x \in X : |T(x)| \leq 1\}$ and $V := \{x \in X : |T(x)| \geq 1\}$ both perfectoid spaces.*

Proof. We have $\varpi^{-1}Z \notin R_0$ and hence $Z \neq 0$ in $\mathcal{O}_X(X)$. But as before $kZ \subseteq R_0[T]$ and $kZ \subseteq R_0[T^{-1}]$, and hence Z restricts to zero on both U and V , so again \mathcal{O}_X is not a sheaf.

Next observe that the completion of R with respect to the basis given by $r + aR_0[T]$, $r \in R$, $a \in k^\times$, is equal to the completion of $k[T^{1/p^\infty}, T^{-1/p^\infty}]$ with respect to the topology defined by the subring $\mathcal{O}_k[T^{1/p^\infty}, (\varpi/T)^{1/p^\infty}]$ (that is, the direct limit of $\mathcal{O}_k[T, \varpi/T]$ via $x \mapsto x^p$); from this we deduce that U is the p -finite affinoid perfectoid space associated to the annulus $\{|\varpi| \leq |T| \leq 1\}$; similarly V is perfectoid. \square

Scholze (personal communication) observes that R° in the lemma above is not bounded (as it contains the line kZ) and asks whether his Conjecture 2.16 becomes true under the additional assumption that the ring is uniform. Explicitly, if A is uniform and complete, and $\mathrm{Spa}(A, A^+)$ has a cover by rational subsets which are perfectoid, is A perfectoid? One might also ask whether the conjecture becomes true if A is assumed stably uniform, where the question becomes more accessible – indeed we resolved this in the characteristic p case in Corollary 10, and perhaps minor modifications of these arguments will also deal with the characteristic zero case.

4.3. An affinoid pre-adic space with a non-nilpotent locally zero element. We have seen examples of global sections of affinoid pre-adic k -spaces which are non-zero but locally zero. The examples we have seen so far were nilpotent, which is perhaps not surprising: by Corollary 5 any such example has to be topologically nilpotent. Here we give an example of a section which is locally zero but genuinely not nilpotent.

Set $R = k[T, T^{-1}, Z]$ and let R_0 be the \mathcal{O}_k -subalgebra generated by ϖT , ϖT^{-1} , and for $n \geq 1$ the elements $\varpi^{-n}T^{a(n)}Z$ and $\varpi^{-n}T^{-b(n)}Z$, where $a(n)$ and $b(n)$ are two sequences of positive integers both tending to infinity rapidly. More precisely, the following will suffice: set $a(1) = 1$ and then for $J \geq 1$ ensure that

$$b(J) > J^2 + J \max\{b(j) : 1 \leq j < J, a(i) : 1 \leq i \leq J\}$$

and for $I \geq 2$ ensure that

$$a(I) > I^2 + I \max\{b(j) : 1 \leq j < I, a(i) : 1 \leq i < I\}.$$

The sequence $a(1), b(1), a(2), b(2), \dots$ can be constructed recursively such that these inequalities are satisfied.

Proposition 14. *Let $X = \text{Spa}(R, R^\circ)$. Then $Z \in \mathcal{O}_X(X)$ is not nilpotent but vanishes on the subsets $U := \{x : |T(x)| \leq 1\}$ and $V := \{x : |T(x)| \geq 1\}$ that cover X .*

Proof. By construction $\varpi^{-n}Z \in R_0[T]$ and $\varpi^{-n}Z \in R_0[T^{-1}]$ for all $n \geq 1$, so Z is zero on U and V . To see that Z is not nilpotent on $\text{Spa}(R, R^\circ)$ we need to show that Z^e is non-zero in the completion \widehat{R} of R for any $e \geq 1$, so we need to verify that for all $e \in \mathbf{Z}_{\geq 1}$ there exists some $M(e) \in \mathbf{Z}_{\geq 0}$ such that $\varpi^{-M(e)}Z^e \notin R_0$.

The ring R_0 is graded by $\mathbf{Z} \times \mathbf{Z}$ (the powers of T and Z) and the given generators of R_0 are homogeneous. It follows from this that if $\varpi^{-M}Z^e \in R_0$ then $\varpi^{-M}Z^e$ will be an \mathcal{O}_k -linear sum of products of the given generators, where each of these products is of the form λZ^e (with $\lambda \in k^\times$). So it suffices to check that for any $e \geq 0$ there exists some bound $M(e) \in \mathbf{Z}_{\geq 0}$ such that if λZ^e is a product of the given generators of R_0 then $|\lambda| < |\varpi|^{-M(e)}$.

Set $\alpha_n = \varpi^{-n}T^{a(n)}Z$ and $\beta_n = \varpi^{-n}T^{-b(n)}Z$. Say λZ^e is a product of the given generators of R_0 , and let us consider which α_i and β_j occur in this product. There are two cases. If the product mentions only α_i and β_j for $i, j \leq e$, then (because of the coefficient of Z) the product can mention only e such elements, so $|\lambda| \leq |\varpi|^{-e^2}$. If, however, the product mentions some α_i or β_j with $i > e$ then we claim that $|\lambda| \leq 1$, and it suffices to prove this claim. Let I denote the largest i such that α_i is mentioned (with $I = 0$ if no α_i are mentioned), and let J denote the largest j such that β_j is mentioned (with $J = 0$ if no β_j is mentioned). Write $\lambda Z^e = (\varpi T)^\ell (\varpi T^{-1})^m (\varpi^{-\mu} T^\nu Z^e)$, with $\varpi^{-\mu} T^\nu Z^e$ a product of α s and β s. If $I \leq J$ then because $b(J) > J^2 + J \max\{b(j) : j < J, a(i) : i \leq J\}$ we see that $|\nu| \geq b(J) - (e-1) \max\{b(j) : j < J, a(i) : i \leq J\} > J^2$ whereas $\mu \leq (e-1)J < J^2$, and because one of ℓ and m must be bigger than $|\nu|$ to kill the power of T , we see $|\lambda| \leq 1$. A similar argument works in the case $I > J$, this time using the defining property of $a(I)$. \square

4.4. Exactness failing in the middle. Scholze (personal communication) asked whether one could construct an example of an affinoid pre-adic k -space for which \mathcal{O}_X fails to be a sheaf for a reason other than the existence of sections which are locally zero but non-zero. Here is such an example – an example where glueing fails.

If R is a Tate k -algebra that contains a k -basis $\{r_1, r_2, \dots\}$ and there exist non-negative integers $\{n_1, n_2, \dots\}$ with the property that the r_i are an \mathcal{O}_k -basis for R° , and $\varpi^{n_i} r_i$ are an \mathcal{O}_k -basis for R_0 , then $(\widehat{R})^\circ$ contains no line, because $(\widehat{R})^\circ = \widehat{(R^\circ)}$ by Lemma 1(iii). So Corollary 5 implies that if $X = \text{Spa}(R, R^\circ) = U \cup V$ then the map $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ must be injective (as the kernel is a k -vector space all of whose elements are power-bounded). Here however is an example where $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U \cap V)$ is not exact – there are global sections of U and V which agree on $U \cap V$ but which do not glue together to give a section on $U \cup V$.

Set $R = k[T, T^{-1}, Z_1, Z_2, \dots]$ and let R_0 be the free \mathcal{O}_k -submodule of R generated by elements $\varpi^d T^a \prod_i Z_i^{e_i}$ with $d, a \in \mathbf{Z}$ and $e_1, e_2, \dots \in \mathbf{Z}_{\geq 0}$ satisfying

- (i) if $\sum_i e_i = 0$ then $d = |a|$;
- (ii) if $\sum_i e_i = 1$ then $d = |a| - 2 \min\{\sum_i i e_i, |a|\}$;
- (iii) if $\sum_i e_i \geq 2$ then $d = |a| - 2 \sum_i i e_i$.

It is easily checked that the product of two such generators is in R_0 and that $1 \in R_0$, so R_0 is a ring. It is clear that $kR_0 = R$. Note that ϖT and ϖT^{-1} are in R_0 but not in ϖR_0 . Set $U = \{x \in X : |T(x)| \leq 1\}$ and $V = \{x \in X : |T(x)| \geq 1\}$ as usual; set $A = B = R$ and topologize them using $A_0 = R_0[T]$ and $B_0 = R_0[T^{-1}]$, so $\mathcal{O}_X(U) = \widehat{A}$ and $\mathcal{O}_X(V) = \widehat{B}$.

Proposition 15. *(\widehat{R}°) contains no line.*

Proof. By Lemma 11, R° is graded by the degrees of T, Z_1, Z_2, \dots . It is easy to check that $\varpi^d T^a \prod_i Z_i^{e_i}$ is in R° iff

- (i) if $\sum_i e_i = 0$ then $d \geq |a|$;
- (ii) if $\sum_i e_i \geq 1$ then $d \geq |a| - 2 \sum_i i e_i$.

This, together with the arguments above, shows that $(\widehat{R^\circ})$ contains no line. \square

Note that for all $n \geq 1$ we have $\varpi^{-n}Z_n = \varpi^{-n}T^{-n}Z_n.T^n \in R_0[T]$ and similarly $\varpi^{-n}Z_n \in R_0[T^{-1}]$ but $\varpi^{-1}Z_n \notin R_0$, so the map $R \rightarrow A \oplus B$ is not strict.

Now because $Z_n \in \varpi^n A_0$ we have that $\sum_i Z_i$ converges in \widehat{A} ; let a be the limit. Similarly it converges in \widehat{B} ; let b be the limit.

Proposition 16. *$a \in \widehat{A}$ and $b \in \widehat{B}$ agree on $U \cap V$, but cannot be glued to an element of \widehat{R} .*

Proof. That a and b agree on $U \cap V$ is obvious, because the image of a in \widehat{C} and the image of b in \widehat{C} both are the limit of $\sum_i Z_i$ in \widehat{C} .

Let $r \in \widehat{R}$. There is a Cauchy sequence r_1, r_2, \dots in R with limit r . For each $n \geq 1$, let $\rho_n : R \rightarrow k$ be the map that sends an element of R to the coefficient of Z_n of its Z_n graded piece. This map is continuous and therefore factors through a unique continuous map $\widehat{\rho}_n : \widehat{R} \rightarrow k$. Similarly we define $\widehat{\alpha}_n : \widehat{A} \rightarrow k$.

We claim that $\widehat{\rho}_1(r), \widehat{\rho}_2(r), \dots$ converges to zero. Let $M \geq 0$. There exists $I \geq 1$ such that $r_i - r_j \in \varpi^M R_0$ for all $i, j \geq I$. It follows that $\rho_n(r_i) - \rho_n(r_j) \in \varpi^M \mathcal{O}_k$ for all $i, j \geq I$ and all $n \geq 1$. Take $N \geq 1$ such that none of Z_N, Z_{N+1}, \dots occurs in r_I . For all $n \geq N$ we have $\rho_n(r_I) = 0$, so $\rho_n(r_i) \in \varpi^M \mathcal{O}_k$ for all $i \geq I$, so $\widehat{\rho}_n(r) \in \varpi^M \mathcal{O}_k$. This concludes the proof that $\widehat{\rho}_n(r)$ converges to zero.

It is easily seen that $\widehat{\alpha}_n(a) = 1$ for all $n \geq 1$. Since $\widehat{\rho}_n$ and $\widehat{\alpha}_n$ are compatible through $\widehat{R} \rightarrow \widehat{A}$, it follows that the image in \widehat{A} of r cannot be a . \square

4.5. A uniform space with a subspace containing a line of power-bounded elements.

We now give an example of a uniform space that is not stably uniform. See also [Mih14], which was written independently and appeared on the ArXiv a few weeks before this paper did.

Consider the free \mathcal{O}_k -submodule R_0 of $k[T, T^{-1}, Z]$ generated by $(\varpi T)^a (\varpi Z)^b$ with $b \geq 0$ and $a \geq -b^2$. It is easily verified that R_0 is also an \mathcal{O}_k -subalgebra; indeed if $a \geq -b^2$ and $a' \geq -(b')^2$ then $a + a' \geq -b^2 - (b')^2 \geq -(b + b')^2$ if $b, b' \geq 0$. Set $R = A = kR_0$ and topologize them using $R_0 \subseteq R$ and $A_0 = R_0[T] \subseteq A$.

Proposition 17. *The affinoid k -algebra (R, R°) is uniform, but not stably uniform. In particular, A° contains the non-zero line kZ .*

Proof. We claim that $R^\circ = R_0$. By Lemma 11 it suffices to check that for every $r = (\varpi T)^a (\varpi Z)^b \in R_0$ (with $b \geq 0$ and $a \geq -b^2$) and $\lambda \in k$, if $\lambda r \in R^\circ$ then $\lambda \in \mathcal{O}_k$. An elementary calculation shows that it then suffices to check that $\varpi^{-1}r^n \notin R_0$ for any $n \geq 1$, and this is easily checked.

Note that $T^{-1}Z = (\varpi T)^{-1}(\varpi Z) \in R_0$ and hence $Z \in A_0$, but $\varpi^{-1}Z \notin A_0$ (this is not hard to see, using the grading on A_0). However, for $n \geq 1$ we have $(\varpi^{-n}Z)^{n+1} = (\varpi T)^{-n^2-2n-1}(\varpi Z)^{n+1} T^{n^2+2n+1} \in A_0$ and hence $\varpi^{-n}Z \in A^\circ$ for all $n \geq 1$. \square

4.6. A uniform affinoid space which is non-sheafy.

Finally we give an example of a uniform affinoid X over k for which \mathcal{O}_X is not a sheaf. Let R_0 be the free \mathcal{O}_k -submodule of $k[P, P^{-1}, Q, Q^{-1}, T, T^{-1}, Z]$ generated by elements $\varpi^d P^p Q^q T^a Z^e$ with $d, p, q, a \in \mathbf{Z}$ and $e \in \mathbf{Z}_{\geq 0}$ satisfying the following conditions:

- (i) $d = \max\{p + q + a, p + q - a, p + a, q - a\}$;
- (ii) if $e = 0$ then $p \geq 0$ and $q \geq 0$;
- (iii) if $e = 1$ then $p \geq 0$ or $q \geq 0$.

One checks easily that the product of two such \mathcal{O}_k -module generators of R_0 is in R_0 and that $1 \in R_0$, and hence R_0 is a ring. Set $R = kR_0$.

Proposition 18. *The affinoid k -algebra (R, R°) is uniform, but for the space $X := \text{Spa}(R, R^\circ)$ the presheaf \mathcal{O}_X is not a sheaf. In particular, Z is non-zero on the subspace $W := \{x \in X : |P(x)| \leq 1 \text{ and } |Q(x)| \leq 1\}$ but vanishes on the subspaces $U := \{w \in W : |T(w)| \leq 1\}$ and $V := \{w \in W : |T(w)| \geq 1\}$ that cover W .*

Proof. One verifies using Lemma 11 that $R^\circ = R_0$.

The subspace W has global sections given by the completion of the ring $A = R$ with respect to the topology defined by $A_0 = R_0[P, Q]$. Now we have $\varpi^{-n}Q^{-2n}T^{-n}Z \in R_0$, so $\varpi^{-n}Z = \varpi^{-n}Q^{-2n}T^{-n}Z \cdot Q^{2n}T^n \in A_0[T]$, for all $n \geq 0$. Similarly $\varpi^{-n}Z \in A_0[T^{-1}]$ for all $n \geq 0$ and we deduce that Z vanishes on the subspaces U and V . However, $\varpi^{-1}P^{-m}Q^{-n}Z$ is not in R_0 for any $m, n \geq 0$ (indeed for $m, n > 0$ this is not even in R), so $\varpi^{-1}Z$ is not in $A_0 = R_0[P, Q]$ and so Z is a non-zero function on W . \square

REFERENCES

- [Ber90] Vladimir G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990. MR 1070709 (91k:32038)
- [BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis*, Grundlehren der Mathematischen Wissenschaften, vol. 261, Springer-Verlag, Berlin, 1984. MR 746961 (86b:32031)
- [Bou61] N. Bourbaki, *Éléments de mathématique. Fascicule XXVIII. Algèbre commutative. Chapitre 3: Graduations, filtrations et topologies. Chapitre 4: Idéaux premiers associés et décomposition primaire*, Actualités Scientifiques et Industrielles, vol. 1293, Hermann, Paris, 1961. MR 0171800 (30 #2027)
- [Bou71] ———, *Éléments de mathématique. Topologie générale. Chapitres 1 à 4*, Hermann, Paris, 1971. MR 0358652 (50 #11111)
- [Bou81] ———, *Éléments de mathématique. Espaces vectoriels topologiques. Chapitres 1 à 5*, Masson, Paris, 1981. MR 633754 (83k:46003)
- [Gro60] A. Grothendieck, *Éléments de géométrie algébrique (révisés avec la collaboration de Jean Dieudonné): I. Le langage des schémas*, Publ. Math. IHÉS **4** (1960), 5–228. MR 0163908 (29 #1207)
- [Hub94] R. Huber, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513–551. MR 1306024 (95k:14001)
- [Hub96] Roland Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, vol. E30, Friedr. Vieweg & Sohn, Braunschweig, 1996. MR 1734903 (2001c:14046)
- [Mih14] Tomoki Mihara, *On Tate acyclicity and uniformity of Berkovich spectra and adic spectra*, arXiv:1403.7856v1, 2014.
- [Sch12] Peter Scholze, *Perfectoid spaces*, Publ. Math. IHÉS **116** (2012), 245–313. MR 3090258
- [Sch13] ———, *Perfectoid spaces: A survey*, arXiv:1303.5948v1 (to appear in Proceedings of the 2012 conference on Current Developments in Mathematics), 2013.
- [Wed12] Torsten Wedhorn, *Adic spaces*, unpublished notes, 2012.

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