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A generalization of formal schemes and rigid analytic varieties

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0 Introduction

In this paper we construct a natural category $\mathscr{A}$ of locally and topologically ringed spaces which contains both the category of locally noetherian formal schemes and the category of rigid analytic varieties as full subcategories. This category has applications in algebraic geometry and rigid analytic geometry.

The idea of the definition of the category $\mathscr{A}$ is the following. From a formal point of view there is a certain similarity in constructing formal schemes and rigid analytic varieties. In both cases one starts with a certain class of topological rings (the adic rings in formal geometry and Tate algebras in rigid geometry), defines to every topological ring of this class a locally and topologically ringed space, and glueing of such spaces give formal schemes or rigid analytic varieties.

There is a natural class of topological rings which contains both the noetherian adic rings and the Tate algebras and which suggests itself. Namely the class of topological rings which have an open adic subring with a finitely generated ideal of definition. We call such a ring f-adic.

The points of the formal scheme Spf$A$ associated with an adic ring A are the open prime ideals of A, and the points of the rigid analytic variety SpA associated with a Tate algebra A are the maximal ideals of A. In both cases one can consider the points as continuous valuations of A. (A valuation v: $A \to \Gamma_v \cup \{0\}$ of a topological ring A is called continuous if the mapping v is continuous with respect to the ring topology of A and the order-induced topology of $\Gamma_v \cup \{0\}$.) Namely, if $\mathfrak{p}$ is an open prime ideal of an adic ring A then the trivial valuation $v_\mathfrak{p}$ of A with $v_\mathfrak{p}(a) = 0$ iff $a \in \mathfrak{p}$ is continuous, and if $\mathfrak{p}$ is a maximal ideal of a Tate algebra A over a valued field k then the

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valuation $v_{\mathfrak{p}}$ of A extending the valuation of k and with $v_{\mathfrak{p}}(a) = 0$ iff $a \in \mathfrak{p}$ is continuous.

This suggests to consider, for every f-adic ring A, the set Spa$A$ of all continuous valuations of A. There is a natural topology on Spa$A$. The topological space Spa$A$ has been studied in [H1]. In this paper we will show that if A satisfies a certain condition of noetherianness then there is a natural sheaf $\mathcal{O}_A$ of topological rings on Spa$A$. All stalks of $\mathcal{O}_A$ are local rings. Every continuous ring homomorphism $A \to B$ between f-adic rings induces a morphism of locally and topologically ringed spaces (Spa$B$, $\mathcal{O}_B$) $\to$ (Spa$A$, $\mathcal{O}_A$). We call a locally and topologically ringed space which is locally isomorphic to some Spa$A$ = (Spa$A, \mathcal{O}_A$) adic.(Here we suppress some details of the definition of adic spaces).

The functor Spf$A \mapsto$ Spa$A$ from the category of noetherian affine formal schemes to the category of adic spaces extends in a natural way to a functor from the category of locally noetherian formal schemes to the category of adic spaces. Similarly, the functor Sp$A \mapsto$ Spa$A$ from the category of affinoid rigid analytic varieties to the category of adic spaces extends to a functor from the category of rigid analytic varieties to the category of adic spaces. Both functors are fully faithful.

In the papers [H2] and [H3] we will study the coherent cohomology and the étale cohomology of adic spaces. Concerning the coherent cohomology, the main results are Theorem A and B, (i.e., every coherent $\mathcal{O}_A$-module $\mathcal{F}$ on an adic space Spa$A$ is generated by its global sections and all higher cohomology groups H^i(Spa$A$, $\mathcal{F}$) vanish) and the proper coherence theorem, (i.e., all direct image sheaves $R^n f_* \mathcal{F}$ of a coherent $\mathcal{O}_X$-module under a proper morphism of adic spaces $f: X \to Y$ are coherent $\mathcal{O}_Y$-modules). Many of the basic results of the étale cohomology of schemes also hold for the étale cohomology of adic spaces (for example, proper and smooth base change theorem, purity, Poincaré duality).

The theory of adic spaces may be useful for problems in algebraic geometry and rigid analytic geometry. For example, in [H3] we will show that the étale topos of a rigid analytic variety X is canonically equivalent to the étale topos of the adic space associated with X. Therefore all the results mentioned above on the étale cohomology of adic spaces also hold for the étale cohomology of rigid analytic varieties. In general, it is much easier to work with the étale toposes of rigid analytic varieties. For example, using geometric points of an adic space X, one can easily describe the category of points of the étale topos $X_{\acute{e}t}^{\sim}$ of X. (One obtains that $X_{\acute{e}t}^{\sim}$ has enough points.) But it is very complicated to describe within the category of rigid analytic varieties the points of the étale topos of a rigid analytic variety. Another application of the category of adic spaces will be given in Sect. 5 of this paper where we analyze Mumford's construction of semi-abelian group schemes within the category of adic spaces.

Notations

This paper is based on [H1]. We recall some notations from [H1]. For more details and all unexplained notations we refer to [H1].

All our valuations are multiplicative. For a valuation $v: A \to \Gamma \cup \{0\}$ of a ring A the prime ideal $\mathrm{supp}(v) := \{a \in A \mid v(a) = 0\}$ of A is called the support of A.

The group of units of a ring A is denoted by A^*, and if R, S are subsets of A then $R \cdot S$ denotes the additive subgroup of A generated by $\{r \cdot s \mid r \in R, s \in S\}$.

For a topological ring A, the set of power-bounded elements of A is denoted by A° and the set of topologically nilpotent elements of A is denoted by $A^{\circ\circ}$. A ring of definition of a f-adic ring A is an open adic subring of A. A Tate ring is a f-adic ring which has a topologically nilpotent unit. An affinoid ring is a pair $A = (A^\triangleright, A^+)$ where $A^\triangleright$ is a f-adic ring and A^+ is a ring of integral elements of $A^\triangleright$, i.e. A^+ is a subring of $A^\triangleright$ which is open and integrally closed in $A^\triangleright$ and is contained in $(A^\triangleright)^\circ$. For an affinoid ring $A = (B, C)$, the pair of the completions $\hat{A} := (\hat{B}, \hat{C})$ is an affinoid ring which is called the completion of A. (In this paper complete always means complete and hausdorff). For an affinoid ring A, we put $\mathrm{Spa}A = \{v \mid v$ is a continuous valuation of $A^\triangleright$ with $v(a) \leqq 1$ for every $a \in A^+\}$ and equip $\mathrm{Spa}A$ with the topology generated by the sets $\{v \in \mathrm{Spa}A \mid v(a) \leqq v(b) \neq 0\}$ $(a, b \in A^\triangleright)$. A subset U of $\mathrm{Spa}A$ is called rational if there are elements $s_1, \ldots, s_n$ of $A^\triangleright$ and finite subsets $T_1, \ldots, T_n$ of $A^\triangleright$ such that $T_i \cdot A^\triangleright$ is open in $A^\triangleright$ for $i = 1, \ldots, n$ and $U = \bigcap_{i=1}^n \{v \in \mathrm{Spa}A \mid v(t) \leqq v(s_i) \neq 0$ for all $t \in T_i\}$. The rational subsets of $\mathrm{Spa}A$ form a basis of the topology of $\mathrm{Spa}A$ and are quasi-compact.

Contents

1 A presheaf on Spa$A$

In this section we define, for every affinoid ring $A = (A^\triangleright, A^+)$, a presheaf $\mathcal{O}_A$ of "analytic functions" on the topological space Spa$A$. The idea of the definition of $\mathcal{O}_A$ is the following. Let $U = R\left(\frac{T_1}{s_1}, \ldots, \frac{T_n}{s_n}\right) = \bigcap_{i=1}^n \{v \in \mathrm{Spa}A \mid v(t) \leqq v(s_i) \neq 0$ for all $t \in T_i\}$ be a rational subset of Spa$A$. Of course, every element of $A^\triangleright$ should give an analytic function on U, and the functions given by $s_1, \ldots, s_n$ should be invertible. So every element of the ring $(A^\triangleright)_{s_1 \cdot \ldots \cdot s_n} = A^\triangleright\left[\frac{1}{s_1}, \ldots, \frac{1}{s_n}\right]$ should induce a function on U. Furthermore, every function on U which can be "approximated" by functions from

$A^{\triangleright}\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$ should be an element of $\mathcal{O}_A(U)$. So we will define a ring topology on $A^{\triangleright}\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$, and then define $\mathcal{O}_A(U)$ to be the completion of $A^{\triangleright}\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$.

Lemma and definition. (i) *Let A be a* f*-adic ring, $s_1,\ldots,s_n$ elements of A and $T_1,\ldots,T_n$ subsets of A such that $T_i \cdot A$ is open in A for $i = 1,\ldots,n$. Let A_0 be a ring of definition of A and let I be an ideal of definition of A_0. Let T be the group topology of the localization $A_{s_1\cdot\ldots\cdot s_n} = A\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$ such that $\{I^n \cdot B \mid n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of* 0, *where B is the subring $A_0\left[\frac{t}{s_i} | i = 1,\ldots,n, t \in T_i\right]$ of $A_{s_1\cdot\ldots\cdot s_n}$. Then T is even a ring topology and independent of the choice of A_0 and I. The topological ring $(A_{s_1\cdot\ldots\cdot s_n}, T)$ is denoted by $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$. The completion of $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ is denoted by $A\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$.*

(ii) *Let $A = (A^{\triangleright}, A^+)$ be an affinoid ring, $s_1,\ldots,s_n$ elements of $A^{\triangleright}$ and $T_1,\ldots,T_n$ subsets of $A^{\triangleright}$ such that $T_i \cdot A^{\triangleright}$ is open in $A^{\triangleright}$ for $i = 1,\ldots,n$. Let C be the integral closure of $A^+\left[\frac{t}{s_i} \mid i = 1,\ldots,n, t \in T_i\right]$ in $A^{\triangleright}\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$. Then C is a ring of integral elements of $B := A^{\triangleright}\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$. The affinoid ring (B, C) is denoted by $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$. The completion of $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ is denoted by $A\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$.*

Proof. (i) By the subsequent Lemma 1.1 the set $T_i \cdot A_0$ is open in A, i.e., there exists a $n \in \mathbb{N}$ with $I^n \subseteq T_i \cdot A_0$. Then $\frac{t}{s_i} \cdot I^n \subseteq A_0\left[\frac{t}{s_i} \mid t \in T_i\right]$. Hence, for every $x \in A\left[\frac{1}{s_1},\ldots,\frac{1}{s_n}\right]$ and every neighbourhood U of 0 in the topology T, there exists a neighbourhood V of 0 in T with $x \cdot V \subseteq U$. Consequently T is a ring topology. Let A_1, A_2 be two rings of definition of A and I_1, I_2 ideals of definition of A_1, A_2. For every $m \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ with $I_1^k \subseteq I_2^m$, and then $I_1^k \cdot A_1\left[\frac{t}{s_i} \mid i = 1,\ldots,n, t \in T_i\right] \subseteq I_2^m \cdot A_2\left[\frac{t}{s_i} \mid i = 1,\ldots,n, t \in T_i\right]$. Hence T is independent of the choice of A_0 and I. □

Lemma 1.1 *Let A be a* f*-adic ring and T a subset of A such that $T \cdot A$ is open in A. Then, for every $n \in \mathbb{N}$ and every neighbourhood G of* 0 *in A, the set $T^n \cdot G$ is open in A.*

Proof. With $T \cdot A$ also $T^n \cdot A = (T \cdot A)^n$ is open. Let U be a subset of A and S a finite subset of U such that $\{U^m | m \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0 in $A, S \cdot U = U^2 \subseteq U$ and $U \subseteq T^n \cdot A$. We choose a finite subset R of A with $S \subseteq T^n \cdot R$, and a $k \in \mathbb{N}$ with $R \cdot U^k \subseteq G$. Then $U^{k+1} = S \cdot U^k \subseteq (T^n \cdot R) \cdot U^k = T^n \cdot (R \cdot U^k) \subseteq T^n \cdot G$. □

Our definition of $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ immediately implies

(1.2) *Universal property of* $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$. (i) Let A be a f-adic ring, $s_1,\ldots,s_n$ elements of A and $T_1,\ldots,T_n$ finite subsets of A such that $T_i \cdot A$ is open in A for $i=1,\ldots,n$. Then the topological ring $A\left(\frac{T_1}{s_n},\ldots,\frac{T_n}{s_n}\right)$ is f-adic, the canonical ring homomorphism $h\colon A \to A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ is continuous, $h(s_i) \in A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)^*$ and $\frac{h(t)}{h(s_i)} \in A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)^\circ$ for every $i=1,\ldots,n, t\in T_i$, and if $f\colon A\to B$ is a continuous ring homomorphism from A to a f-adic ring B such as that $f(s_i)\in B^*$ and $\frac{f(t)}{f(s_i)} \in B^\circ$ for every $i=1,\ldots,n, t\in T_i$ then there exists a unique continuous ring homomorphism $g\colon A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right) \to B$ with $f = g\circ h$.

(ii) Let A be an affinoid ring, $s_1,\ldots,s_n$ elements of $A^\rhd$ and $T_1,\ldots,T_n$ finite subsets of $A^\rhd$ such that $T_i\cdot A^\rhd$ is open in $A^\rhd$ for $i=1,\ldots,n$. Then the canonical ring homomorphism $h\colon A \to A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ from A to the affinoid ring $A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ is continuous, $h(s_i) \in \left(A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)^\rhd\right)^*$ and $\frac{h(t)}{h(s_i)} \in A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)^+$ for every $i=1,\ldots,n, t\in T_i$, and if $f\colon A\to B$ is a continuous ring homomorphism from A to an affinoid ring B such that $f(s_i)\in (B^\rhd)^*$ and $\frac{f(t)}{f(s_i)}\in B^+$ for every $i=1,\ldots,n, t\in T_i$, then there exists a unique continuous ring homomorphism $g\colon A\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)\to B$ with $f=g\circ h$.

As mentioned before, we want to define, for a rational subset $U = R\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ of the topological space $\mathrm{Spa}A$, $\mathcal{O}_A(U) = A^\rhd\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$. But we have to check that $\mathcal{O}_A(U)$ depends only on U and is independent of the choice of $s_1,\ldots,s_n$, $T_1,\ldots,T_n$. This is done in the next proposition using an idea of Tate in [T].

Proposition 1.3 *Let A be an affinoid ring and U a rational subset of* Spa$A$. *Then* (i) *There exists a continuous ring homomorphism $h\colon A\to F_A(U)$ from A to a complete affinoid ring $F_A(U)$ such that* $\mathrm{im}(\mathrm{Spa}(h))\subseteq U$ *and whenever $f\colon A\to B$ is a continuous ring homomorphism from A to a complete affinoid ring B with* $\mathrm{im}(\mathrm{Spa}(f))\subseteq U$ *then there is a unique continuous ring homomorphism $g\colon F_A(U)\to B$ with $f=g\circ h$.*
(ii) *Let $s_1,\ldots,s_n$ be elements of $A^\rhd$ and $T_1,\ldots,T_n$ finite subsets of $A^\rhd$ such that $T_i\cdot A^\rhd$ is open in $A^\rhd$ for $i=1,\ldots,n$ and $U = R\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$. Then the canonical ring homomorphism $A\to A\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$ satisfies the property of* (i).

Proof. Let $U = R\left(\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right)$ be as in (ii) and let $h\colon A\to A\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$ be the canonical ring homomorphism. It follows immediately from the definition of $A\left\langle\frac{T_1}{s_1},\ldots,\frac{T_n}{s_n}\right\rangle$ that $\mathrm{im}(\mathrm{Spa}(h))\subseteq U$. Let $f\colon A\to B$ be a continuous ring homomorphism from A to a complete affinoid ring B such that $\mathrm{im}(\mathrm{Spa}(f)) \subseteq U$. Then $v(f(s_i)) \neq 0$ for every $i = 1, \ldots, n, v \in \mathrm{Spa}B$, and hence the subsequent Lemma 1.4 implies that $f(s_i) \in (B^\triangleright)^*$ for $i = 1, \ldots, n$. Furthermore, we have $v\left(\frac{f(t)}{f(s_i)}\right) \leqq 1$ for every $i = 1, \ldots, n, t \in T_i$, $v \in \mathrm{Spa}B$, and hence by [H1, 3.3.i] $\frac{f(t)}{f(s_i)} \in B^+$ for every $i = 1, \ldots, n, t \in T_i$. Now the universal property (1.2. ii) says that there exists a unique continuous homomorphism $g: A\left\langle\frac{T_1}{s_1}, \ldots, \frac{T_n}{s_n}\right\rangle \to B$ with $f = g \circ h$. □

Lemma 1.4 *Let A be a complete affinoid ring and $\mathfrak{m}$ a maximal ideal of $A^\triangleright$. Then there exists a point $v \in \mathrm{Spa}A$ with $\mathfrak{m} = \mathrm{supp}(v)$.*

Proof. We equip $B^\triangleright := A^\triangleright/\mathfrak{m}$ with the quotient topology. Let $B^+ \subseteq B^\triangleright$ be the integral closure of A^+ is $B^\triangleright$. Then $B := (B^\triangleright, B^+)$ is an affinoid ring, and the image of the natural mapping $\mathrm{Spa}B \to \mathrm{Spa}A$ is the set of points $v \in \mathrm{Spa}A$ with $\mathfrak{m} = \mathrm{supp}(v)$. Hence we have to show $\mathrm{Spa}B \neq \emptyset$. The set of units of $A^\triangleright$ is open, since $1 + (A^\triangleright)^{\circ\circ}$ is open and every element of $1 + (A^\triangleright)^{\circ\circ}$ is a unit of $A^\triangleright$. Hence $\mathfrak{m}$ is closed in $A^\triangleright$, i.e., $B^\triangleright$ is Hausdorff. Now [H1, 3.6.i] implies that $\mathrm{Spa}B \neq \emptyset$. □

The ring homomorphism $A \to F_A(U)$ in (1.3.i) is uniquely determined up to unique isomorphism. For the following we fix, for every affinoid ring A and every rational subset U of $\mathrm{Spa}A$, a ring homomorphism $h_{A,U}: A \to F_A(U)$ which has the property of (1.3.i). We note some properties of these ring homomorphisms.

Lemma 1.5 *Let A be an affinoid ring and U a rational subset of* Spa$A$.

(i) *If V is a rational subset of* Spa$A$ *with $V \subseteq U$ then there is a unique continuous ring homomorphism $g: F_A(U) \to F_A(V)$ with $h_{A,V} = g \circ h_{A,U}$.*
(ii) *The mapping $g := \mathrm{Spa}(h_{A,U}) : \mathrm{Spa}F_A(U) \to \mathrm{Spa}A$ is a homeomorphism from* Spa$F_A(U)$ *onto U, and it induces an one-to-one correspondence between the rational subsets of* Spa$F_A(U)$ *and the rational subsets of* Spa$A$ *which are contained in U (i.e. if V is a rational subset of* Spa$F_A(U)$ *then $g(V)$ is a rational subset of* Spa$A$, *and if V is a rational subset of* Spa$A$ *then $g^{-1}(V)$ is a rational subset of* Spa$F_A(U)$).
(iii) *Put $B := F_A(U)$ and $g := \mathrm{Spa}(h_{A,U})$: $\mathrm{Spa}B \to \mathrm{Spa}A$. Let V be a rational subset of* Spa$A$ *with $V \subseteq U$. Then there exists a unique continuous ring homomorphism $r: F_A(V) \to F_B(g^{-1}(V))$ such that the following diagram is commutative*

$$\begin{array}{ccc} F_B(g^{-1}(V)) & \xleftarrow{r} & F_A(V) \\ {\scriptstyle h_{B,\,g^{-1}(V)}}\uparrow & & \uparrow {\scriptstyle h_{A,V}} \\ B & \xleftarrow[h_{A,U}]{} & A \end{array}$$

Furthermore, r is an isomorphism (i.e., r has a continuous inverse).

Proof. (i) follows from (1.3.i).
(ii) Let s be an element of $A^\triangleright$ and T a finite subset of $A^\triangleright$ such that $T \cdot A^\triangleright$ is open in $A^\triangleright$ and $U = R\left(\frac{T}{s}\right)$. By (1.3.ii) we can assume that $h_{A,U}$ is the canoni-

cal ring homomorphism $h: A \to A\left\langle\frac{T}{s}\right\rangle$. We factorize h into $A \xrightarrow{f} A\left(\frac{T}{s}\right) \xrightarrow{i} A\left\langle\frac{T}{s}\right\rangle$. Since f and i are adic, h is adic, and hence $g^{-1}(V)$ is a rational subset of $\mathrm{Spa}A\left\langle\frac{T}{s}\right\rangle$ for every rational subset V of $\mathrm{Spa}A$ (by [H1, 3.8.iv]). By [H1, 3.9] Spa (i) : $\mathrm{Spa}A\left\langle\frac{T}{s}\right\rangle \to \mathrm{Spa}A\left(\frac{T}{s}\right)$ is a homeomorphism and maps rational subsets to rational subsets. So it remains to show that $\mathrm{Spa}(f)\colon \mathrm{Spa}A\left(\frac{T}{s}\right) \to \mathrm{Spa}A$ is a homeomorphism from $\mathrm{Spa}A\left(\frac{T}{s}\right)$ onto U and maps rational subsets to rational subsets. It is evident from our definition of $A\left(\frac{T}{s}\right)$ that $\mathrm{Spa}(f)$ is a homeomorphism from $\mathrm{Spa}A\left(\frac{T}{s}\right)$ onto U. Let V be a rational subset of $\mathrm{Spa}A\left(\frac{T}{s}\right)$. We choose $\ell, g_1, \ldots, g_n \in A\left(\frac{T}{s}\right)^{\triangleright} = A^{\triangleright}\left[\frac{1}{s}\right]$ such that $V = \left\{v \in \mathrm{Spa}A\left(\frac{T}{s}\right) \mid v(g_i) \leqq v(\ell) \neq 0 \text{ for } i = 1, \ldots, n\right\}$. Multiplying $\ell, g_1, \ldots, g_n$ with a suitable power of s, we may assume that there exists an element d of $A^{\triangleright}$ and a finite subset C of $A^{\triangleright}$ with $\ell = f(d)$ and $\{g_1, \ldots, g_n\} = f(C)$. Since V is quasi-compact, there exists by [H1, 3.11] a neighbourhood E of 0 in $A^{\triangleright}$ with $v(f(e)) \leqq v(\ell)$ for every $v \in V$, $e \in E$. Let D be a finite subset of E such that $D \cdot A^{\triangleright}$ is open in $A^{\triangleright}$. Then we have the rational subset $W = R\left(\frac{C \cup D}{d}\right)$ of $\mathrm{Spa}A$, and $\mathrm{Spa}(f)(V) = U \cap W$. Hence $\mathrm{Spa}(f)(V)$ is rational in $\mathrm{Spa}A$.
(iii) The existence and the uniqueness of r follow from (1.3.i). By (i) there exists a unique continuous ring homomorphism $h: B \to F_A(V)$ with $h_{A,V} = h \circ h_{A,U}$. We have $\mathrm{im}(\mathrm{Spa}(h)) \subseteq g^{-1}(V)$ (since $\mathrm{im}(\mathrm{Spa}(h_{A,V})) \subseteq V$). Then by (1.3.i) there exists a unique continuous ring homomorphism $f: F_B(g^{-1}(V)) \to F_A(V)$ with $h = f \circ h_{B,\, g^{-1}(V)}$. The ring homomorphism f is the inverse of r. □

Let A be an affinoid ring. For every rational subset W of Spa A we have the affinoid ring $F_A(W) = (F_A(W)^{\triangleright}, F_A(W)^{+})$. If $V \subseteq W$ then we have by (1.5.i) a canonical continuous ring homomorphism $\rho_{V,W}: F_A(W)^{\triangleright} \to F_A(V)^{\triangleright}$. For every open subset U of $\mathrm{Spa}A$ we put

$$\mathcal{O}_A(U) = \varprojlim_{V} F_A(V)^{\triangleright},$$

where the projective limit is taken over all rational subsets V of $\mathrm{Spa}A$ contained in U and with respect to the ring homomorphisms $\rho_{V,W}$. We equip $\mathcal{O}_A(U)$ with the projective limit topology. Then $\mathcal{O}_A(U)$ is a complete topological ring. Furthermore, if V, W are open subsets of $\mathrm{Spa}A$ with $V \subseteq W$ then we have a canonical continuous ring homomorphism $\mathcal{O}_A(W) \to \mathcal{O}_A(V)$. Thus we have a presheaf $\mathcal{O}_A$ on $\mathrm{Spa}A$ with values in the category of complete topological rings.

For every $x \in \mathrm{Spa}A$ let $\mathcal{O}_{A,\,x} = \varinjlim_{x \in U} \mathcal{O}_A(U)$ be the stalk of $\mathcal{O}_A$ at x. (The inductive limit is taken in the category of rings.) Since the rational subsets of $\mathrm{Spa}A$ form a basis of the topology of $\mathrm{Spa}A$, we have

$$(1) \qquad \mathcal{O}_{A,x} = \varinjlim_{\substack{x \in U \\ U \text{ rational}}} \mathcal{O}_A(U).$$

For every rational subset U of $\text{Spa}A$ with $x \in U$ the valuation $x: A^\triangleright \to \Gamma_x \cup \{0\}$ extends uniquely to a continuous valuation $v_U: \mathcal{O}_A(U) = F_A(U)^\triangleright \to \Gamma_x \cup \{0\}$ (cf. (1.5.ii)). Then by (1), the valuations v_U define a valuation

$$v_x: \mathcal{O}_{A,x} \to \Gamma_x \cup \{0\}.$$

Thus we have a triple $(\text{Spa}A, \mathcal{O}_A, (v_x | x \in \text{Spa}A))$ consisting of a topological space $\text{Spa}A$, a presheaf $\mathcal{O}_A$ of complete topological rings on $\text{Spa}A$ and a family of valuations v_x on the stalks $\mathcal{O}_{A,x}$. For every open subset U of $\text{Spa}A$ we put

$$\mathcal{O}_A^+(U) = \{ f \in \mathcal{O}_A(U) \mid v_x(f) \leqq 1 \text{ for all } x \in U \}.$$

Then $\mathcal{O}_A^+$ is a presheaf of rings on $\text{Spa}A$. For every $x \in \text{Spa}A$ let $\mathcal{O}_{A,x}^+$ denote the stalk of $\mathcal{O}_A^+$ at x.

Proposition 1.6 (i) *For every $x \in \text{Spa}A$, the stalk $\mathcal{O}_{A,x}$ is a local ring and the maximal ideal of $\mathcal{O}_{A,x}$ is the support of v_x.*
(ii) *For every $x \in \text{Spa}A$ the stalk $\mathcal{O}_{A,x}^+$ is a local ring. We have $\mathcal{O}_{A,x}^+ = \{ f \in \mathcal{O}_{A,x} \mid v_x(f) \leqq 1\}$ with maximal ideal $\{ f \in \mathcal{O}_{A,x} \mid v_x(f) < 1\}$.*
(iii) *For every open subset U of $\text{Spa}A$ and every $f, g \in \mathcal{O}_A(U)$ the set $\{x \in U | v_x(f) \leqq v_x(g) \neq 0\}$ is open in $\text{Spa}A$.*
(iv) *For every rational subset U of $\text{Spa}A$ we have $\mathcal{O}_A(U) = F_A(U)^\triangleright$ and $\mathcal{O}_A^+(U) = F_A(U)^+$.*

Proof. (i) Let x be a point of $\text{Spa}A$, U an open subset of $\text{Spa}A$ with $x \in U$ and f an element of $\mathcal{O}_A(U)$ with $v_x(f) \neq 0$. We have to show that f is a unit in $\mathcal{O}_{A,x}$. Let W be a rational subset of $\text{Spa}A$ with $x \in W \subseteq U$. The valuation v_x of $\mathcal{O}_{A,x}$ gives by restriction to $\mathcal{O}_A(W)$ a continuous valuation v_W of $\mathcal{O}_A(W)$. Since $v_W(f) \neq 0$, there exists a finite subset T of the f-adic ring $F_A(W)^\triangleright = \mathcal{O}_A(W)$ such that $T \cdot F_A(W)^\triangleright$ is open in $F_A(W)^\triangleright$ and $v_W(t) \leqq v_W(f)$ for every $t \in T$. Then we have in $\text{Spa}F_A(W)$ the rational subset $V = R\left(\frac{T}{f}\right)$ with $v_W \in V$, and f is a unit in $F_B(V)^\triangleright$ where $B := F_A(W)$. Now (1.5.ii) and (1.5.iii) imply that there exists a rational subset S of $\text{Spa}A$ such that $x \in S \subseteq W$ and f is a unit in $\mathcal{O}_A(S)$.
(ii) With (iii) we obtain $\mathcal{O}_{A,x}^+ = \{ f \in \mathcal{O}_{A,x} \mid v_x(f) \leqq 1\}$. Then (i) implies that $\mathcal{O}_{A,x}^+$ is a local ring with maximal ideal $\{ f \in \mathcal{O}_{A,x} \mid v_x(f) < 1\}$.
(iii) We can assume that U is rational. The set $\{v \in \text{Spa}F_A(U) \mid v(f) \leqq v(g) \neq 0\}$ is open in $\text{Spa}F_A(U)$ by the definition of the topology of $\text{Spa}F_A(U)$. Now the assertion follows from (1.5.ii).
(iv) By definition we have $\mathcal{O}_A(U) = F_A(U)^\triangleright$. By (1.5.ii) the mapping $\text{Spa}F_A(U) \to \text{Spa}A$ is injective with image U. Hence $\mathcal{O}_A^+(U) = \{ f \in F_A(U)^\triangleright \mid v(f) \leqq 1 \text{ for all } v \in \text{Spa}F_A(U)\}$. Now [H1, 3.3.i] implies $\mathcal{O}_A^+(U) = F_A(U)^+$. □

In general, the presheaf $\mathcal{O}_A$ is not a sheaf of rings as the following example of M. Rost shows. Let $A^\triangleright = \mathbb{Z}[X_1, X_2, X_3]_{X_1 X_2}$ be the localization of the polynomial ring $\mathbb{Z}[X_1, X_2, X_3]$ by X_1X_2, and let B be the subring of $A^{\triangleright}$ generated by X_2, X_1X_2, $X_1^{-1}X_2$, $(X_1^nX_2^{-n}X_3 \mid n \in \mathbb{N})$, $(X_1^{-n}X_2^{-n}X_3 \mid n \in \mathbb{N})$. We equip $A^{\triangleright}$ with the group topology such that $\{X_2^nB \mid n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0. Then $A^{\triangleright}$ is a Tate ring. Let A^+ be an arbitrary ring of integral elements of $A^{\triangleright}$, and put $A = (A^{\triangleright}, A^+)$. The topological space Spa$A$ is covered by the open subsets $U = \{v \in \mathrm{Spa}A \mid v(X_1) \leqq 1\}$ and $V = \{v \in \mathrm{Spa}A \mid v(X_1) \geqq 1\}$. Let X be the element of $\mathcal{O}_A(\mathrm{Spa}A)$ given by X_3. Then $X|U = 0$ and $X|V = 0$, since X_3 lies in every neighbourhood of 0 in $A^{\triangleright}\left(\frac{1, X_1}{1}\right)$ and $A^{\triangleright}\left(\frac{1}{X_1}\right)$. But $X \neq 0$, since $X_3 \notin X_2B$ as a direct computation shows.

2 Adic spaces

Let X be a topological space and F a presheaf of complete topological rings on X (i.e., F is a presheaf of rings on X and, for every open subset $U \subseteq X$, the ring $F(U)$ is equipped with a complete ring topology such that all restriction homomorphisms $F(V) \to F(W)$ are continuous). Then F is a sheaf of complete topological rings (in the sense of [EGA*, 0.3.1]) if and only if F is a sheaf of rings and for every open covering $(U_i)_{i \in I}$ of an open subset U of X the natural mapping $F(U) \to \prod_{i \in I} F(U_i)$ is a topological embedding, where $\prod_{i \in I} F(U_i)$ carries the product topology.

We will use the following category $\mathscr{V}$. The objects are the triples $X = (X, \mathcal{O}_X, (v_x \mid x \in X))$, where X is a topological space, $\mathcal{O}_X$ is a sheaf of complete topological rings on X and $v_x \in \mathrm{Spv}\mathcal{O}_{X,x}$ is a valuation of the stalk $\mathcal{O}_{X,x}$. ($\mathcal{O}_{X,x}$ denotes the inductive limit $\varinjlim_{x \in U} \mathcal{O}_X(U)$ in the category of rings.) The morphisms $X \to Y$ are the pairs (f, φ), where $f: X \to Y$ is a continuous mapping and $\varphi: \mathcal{O}_Y \to f_*\mathcal{O}_Y$ is a morphism of sheaves of topological rings (i.e. φ is a morphism of sheaves of rings and, for every open subset $U \subseteq Y$, the mapping $\varphi_U: \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is continuous) such that, for every $x \in X$, the induced ring homomorphism $\varphi_x: \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X,x}$ is compatible with the valuations v_x and $v_{f(x)}$ (i.e., $v_{f(x)} = \mathrm{Spv}(\varphi_x)(v_x)$).

In the last paragraph we constructed to every affinoid ring A a triple $(\mathrm{Spa}A, \mathcal{O}_A, (v_x|x \in \mathrm{Spa}A))$. We saw that, in general, $\mathcal{O}_A$ is not a sheaf. But if $\mathcal{O}_A$ is a sheaf of topological rings then $(\mathrm{Spa}A, \mathcal{O}_A, (v_x|x \in \mathrm{Spa}A))$ is an object of $\mathscr{V}$ which we call the adic space associated with A.

Definition. *An* **affinoid adic space** *is an object of $\mathscr{V}$ which is isomorphic to the adic space associated with an affinoid ring. An* **adic space** *is an object $(X, \mathcal{O}_X, (v_x|x \in X))$ of $\mathscr{V}$ which is locally an affinoid adic space, i.e., every $x \in X$ has an open neighbourhood $U \subseteq X$ such that $(U, \mathcal{O}_X \mid U, (v_x \mid x \in U))$ is an affinoid adic space. A morphism $X \to Y$ between adic spaces X, Y is a morphism in $\mathscr{V}$.*

The main aim of this paragraph is to show that for two important classes of affinoid rings A the presheaf $\mathcal{O}_A$ on Spa$A$ is a sheaf of topological rings (Theorem 2.2). Furthermore, we will show that the morphisms Spa$A \to$ Spa$B$ between the adic space Spa$A$ and Spa$B$ associated with complete affinoid rings

A and B correspond bijectively to the continuous ring homomorphism $B \to A$ (Proposition 2.1). With these two results we will see in Sect. 4 that the category of adic spaces is large enough that it contains both the category of locally noetherian formal schemes and the category of rigid analytic varieties.

We begin with some general remarks on adic spaces. Let $(X, \mathcal{O}_X, (v_x|x \in X))$ be an adic space. Then $\mathcal{O}_X^+$ denotes the subsheaf of $\mathcal{O}_X$ with $\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \mid v_x(f) \leqq 1$ for all $x \in U\}$. $\mathcal{O}_X^+$ is a sheaf of rings. By (1.6.i, ii) the stalks $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x}^+$ of the sheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are local rings. Furthermore, if $f:(X, \mathcal{O}_X, (v_x \mid x \in X)) \to (Y, \mathcal{O}_Y, (v_y \mid y \in Y))$ is a morphism of adic spaces then f induces morphisms of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y^+)$. (In (2.1) we will prove a converse of this.)

Let $(\mathrm{Spa}A, \mathcal{O}_A, (v_x \mid x \in \mathrm{Spa}A))$ be the adic space associated with an affinoid ring A. Then $(U, \mathcal{O}_A \mid U, (v_x \mid x \in U))$ is an affinoid adic space for every rational subset U of $\mathrm{Spa}A$ (by 1.5.ii, iii)). Hence open subspaces of adic spaces are adic spaces. If X is an affinoid adic space then $(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ is an affinoid ring, and X is isomorphic to the adic space associated with $(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ (by (1.5.ii, iii) and (1.6.iv)).

Let $A = (A^\triangleright, A^+)$ be an affinoid ring and $\hat{A} = ((A^\triangleright)^\wedge, (A^+)^\wedge)$ the completion of A. Then $\mathcal{O}_A$ is a sheaf of topological rings on $\mathrm{Spa}A$ if and only if $\mathcal{O}_{\hat{A}}$ is a sheaf of topological rings on $\mathrm{Spa}\hat{A}$, and the adic spaces associated with A and $\hat{A}$ are isomorphic. This follows from (1.5.ii, iii) with $U = \mathrm{Spa}A$ (note $F_A(\mathrm{Spa}A) \cong \hat{A}$).

Proposition 2.1 (i) *Let $X = \mathrm{Spa}B$ and $Y = \mathrm{Spa}A$ be the adic spaces associated with affinoid rings B and A. Then every continuous ring homomorphism $A \to B$ induces in a canonical way a morphism of adic spaces $X \to Y$. Thus we have a mapping from the set of continuous ring homomorphisms $A \to B$ to the set of morphisms of adic spaces $\mathrm{Spa}B \to \mathrm{Spa}A$. If B is complete then this mapping is bijective.*
(ii) *Let X be an adic space and $Y = \mathrm{Spa}A$ the adic space associated with an affinoid ring A. Then there is a natural one-to-one correspondence between the set of morphisms $X \to Y$ and the set of continuous ring homomorphisms $A \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$.*
(iii) *Let $X = (X, \mathcal{O}_X, (v_x \mid x \in X))$ and $Y = (Y, \mathcal{O}_Y, (v_y \mid y \in Y))$ be adic spaces, and let $g = (f, \varphi)$: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of topologically ringed spaces (i.e., (f, φ) is a morphism of ringed spaces and, for every open subset $U \subseteq Y$, the ring homomorphism $\varphi_U: \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is continuous). Then g is a morphism of adic spaces if and only if the following two conditions are satisfied.*

(a) *$g: (X, \mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces.*
(b) *g induces a morphism of locally ringed space $(X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y^+)$ (i.e., the sheaf homomorphism φ: $\mathcal{O}_Y \to f_* \mathcal{O}_X$ satisfies $\varphi(\mathcal{O}_Y^+) \subseteq f_* \mathcal{O}_X^+$, and (f, ψ): $(X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y^+)$ is a morphism of locally ringed spaces, where ψ: $\mathcal{O}_Y^+ \to f_* \mathcal{O}_X^+$ is the restriction of φ).*

Proof. (i) Let $f: A \to B$ be a continuous ring homomorphism. We construct a morphism of adic spaces $s(f): X \to Y$. Let $g = \mathrm{Spa}(f): \mathrm{Spa}B \to \mathrm{Spa}A$ be the continuous mapping induced by f. By the universal property described in (1.3.i) there exists, for every rational subset U of $\mathrm{Spa}B$ and every rational subset V of $\mathrm{Spa}A$ with $g(U) \subseteq V$, a unique continuous ring homomorphism $\varphi_{V,U}: F_A(V) \to F_B(U)$ such that the following diagram commutes

$$\begin{array}{ccc} F_B(U) & \xleftarrow{\varphi_{V,U}} & F_A(V) \\ {\scriptstyle h_{B,U}} \uparrow & & \uparrow {\scriptstyle h_{A,V}} \\ B & \xleftarrow[f]{} & A\,. \end{array}$$

These $\varphi_{V,U}$ induce a morphism of sheaves of topological rings $\varphi: \mathcal{O}_A \to g_* \mathcal{O}_B$. For every $x \in \mathrm{Spa}B$ the induced ring homomorphism $\mathcal{O}_{A,g(x)} \to \mathcal{O}_{B,x}$ is compatible with the valuations $v_{g(x)}$ and v_x. Hence $s(f) := (g, \varphi): X \to Y$ is a morphism of adic spaces.

Now assume that B is complete. Let $f_1, f_2: A \to B$ be continuous ring homomorphisms with $s(f_1) = s(f_2)$. Let φ: $\mathcal{O}_A(\mathrm{Spa}A) \to \mathcal{O}_B(\mathrm{Spa}B)$ be the ring homomorphism induced by $s(f_1) = s(f_2)$, and let h_A: $A^{\triangleright} \to \mathcal{O}_A(\mathrm{Spa}A)$ and h_B: $B^{\triangleright} \to \mathcal{O}_B(\mathrm{Spa}B)$ be the canonical ring homomorphisms. Then by the construction of $s(f_1)$ and $s(f_2)$ we have $h_B \circ f_1 = \varphi \circ h_A$ and $h_B \circ f_2 = \varphi \circ h_A$. Since B is complete and therefore h_B is an isomorphism, we obtain $f_1 = f_2$. Let $r: X \to Y$ be a morphism of adic spaces. We have to show that there exists a continuous ring homomorphism $f: A \to B$ with $r = s(f)$. Let $\varphi: F_A(\mathrm{Spa}A) = (\mathcal{O}_A(\mathrm{Spa}A), \mathcal{O}_A^+(\mathrm{Spa}A)) \to (\mathcal{O}_B(\mathrm{Spa}B), \mathcal{O}_B^+(\mathrm{Spa}B)) = F_B(\mathrm{Spa}B)$ be the ring homomorphism induced by r. Since $h_{B,\mathrm{Spa}B}$ is an isomorphism, we have a continuous ring homomorphism $f: A \to B$ such that the following diagram commutes

$$(1) \qquad \begin{array}{ccc} F_B(\mathrm{Spa}B) & \xleftarrow{\varphi} & F_A(\mathrm{Spa}A) \\ {\scriptstyle h_{B,\mathrm{Spa}B}} \uparrow & & \uparrow {\scriptstyle h_{A,\mathrm{Spa}A}} \\ B & \xleftarrow[f]{} & A\,. \end{array}$$

Since for every $x \in \mathrm{Spa}B$ the ring homomorphism $\mathcal{O}_{A,r(x)} \to \mathcal{O}_{B,x}$ induced by r is compatible with the valuations $v_{r(x)}$ and v_x, the commutativity of (1) shows $r(x) = \mathrm{Spa}(f)(x)$ for every $x \in \mathrm{Spa}B$. Let $U = R\left(\frac{T}{d}\right)$ be a rational subset of $\mathrm{Spa}A$. Put $V := r^{-1}(U) = s(f)^{-1}(U) \subset \mathrm{Spa}B$. Let $h_U: A^{\triangleright} \to \mathcal{O}_A(U)$ and $h_V: B^{\triangleright} \to \mathcal{O}_B(V)$ be the canonical ring homomorphisms, and let α: $\mathcal{O}_A(U) \to \mathcal{O}_B(V)$ and β: $\mathcal{O}_A(U) \to \mathcal{O}_B(V)$ be the continuous ring homomorphisms induced by r and $s(f)$. Of course β makes the following diagram commutative

$$(2) \qquad \begin{array}{ccc} \mathcal{O}_B(V) & \longleftarrow & \mathcal{O}_A(U) \\ {\scriptstyle h_V} \uparrow & & \uparrow {\scriptstyle h_U} \\ B^{\triangleright} & \xleftarrow[f]{} & A^{\triangleright}. \end{array}$$

The commutativity of (1) implies that also α makes the diagram (2) commutative. Since d is invertible in $\mathcal{O}_A(U)$ and $A^{\triangleright}\left[\frac{1}{d}\right]$ is dense in $\mathcal{O}_A(U)$, we obtain $\alpha = \beta$. Hence $r = s(f)$.

(ii) To a morphism $f:X \to Y$ we assign the continuous ring homomorphism $A \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ which is the composition of the canonical ring homomorphism $A \to (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ and the ring homomorphism $(\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y)) \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ induced by f. Thus we have a mapping from the set of morphisms $X \to Y$ to the set of continuous ring homomorphisms $A \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ which is bijective by (i).
(iii) Assume that (a) and (b) are satisfied. We have to show that for every $x \in X$ the mapping φ_x: $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is compatible with the valuations $v_{f(x)}$ and v_x, i.e., $v_{f(x)} = \mathrm{Spv}(\varphi_x)(v_x)$. By (1.6.i) the maximal ideals of $\mathcal{O}_{Y,f(x)}$ and $\mathcal{O}_{X,x}$ are the supports of $v_{f(x)}$ and v_x. Hence by (a) both $v_{f(x)}$ and $\mathrm{Spv}(\varphi_x)(v_x)$ have the support $\mathfrak{m}_{f(x)} \subseteq \mathcal{O}_{Y,f(x)}$. Let $h: \mathcal{O}_{Y,f(x)} \to k := \mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)}$ be the canonical mapping, and let $A_{f(x)} \subset k$ and $A_x \subset k$ be the valuation rings to $v_{f(x)}$ and $\mathrm{Spv}(\varphi_x)(v_x)$. We have to show that $A_{f(x)} = A_x$. The rings $h^{-1}(A_{f(x)})$ and $h^{-1}(A_x)$ are local. By (1.6.ii) we have $h^{-1}(A_{f(x)}) = \mathcal{O}^+_{Y,f(x)}$ and $h^{-1}(A_x) = \varphi_x^{-1}(\mathcal{O}^+_{X,x})$. So (b) implies that $h^{-1}(A_x)$ dominates $h^{-1}(A_{f(x)})$. Then A_x dominates $A_{f(x)}$, and hence $A_x = A_{f(x)}$. □

For every Tate ring A let $A\langle X_1,\ldots,X_n\rangle$ denote the subring $\{\Sigma a_\nu X^\nu \in \hat{A}[[X_1,\ldots,X_n]] \mid (a_\nu)_{\nu\in\mathbb{N}_0^n}$ is a zero sequence in $\hat{A}\}$ of $\hat{A}[[X_1,\ldots,X_n]]$. We equip $A\langle X_1,\ldots,X_n\rangle$ with the group topology such that $\{U\langle X\rangle \mid U$ neighbourhood of 0 in $\hat{A}\}$ is a fundamental system of neighbourhoods of 0 in $A\langle X_1,\ldots,X_n\rangle$, where $U\langle X\rangle = \{\Sigma a_\nu X^\nu \in A\langle X_1,\ldots,X_n\rangle \mid a_\nu \in U$ for all $\nu\}$. Then $A\langle X_1,\ldots,X_n\rangle$ is a complete Tate ring. We say that A is **strongly noetherian** if $A\langle X_1,\ldots,X_n\rangle$ is noetherian for every $n \in \mathbb{N}_0$. Now we can state the main result of this paragraph.

Theorem 2.2 *Let A be an affinoid ring such that $A^\triangleright$ has a noetherian ring of definition or $A^\triangleright$ is a strongly noetherian Tate ring. Then $\mathcal{O}_A$ is a sheaf of complete topological rings on* Spa$A$. *Furthermore, $H^i(U, \mathcal{O}_A) = 0$ for every $i \in \mathbb{N}$ and every rational subset U of* Spa$A$.

We will even prove a slight generalization of (2.2), namely a theorem analogous to (2.2) for $\mathcal{O}_A$-premodules. For that we need some preparations.

Let A be a topological ring and M a finitely generated A-module. We call the A-module topology on M such that every A-module homomorphism $M \to N$ from M to a topological A-module N is continuous the natural A-module topology of M. If $m_1,\ldots,m_s$ is a system of generators of M over A then the set of all sets $\{x_1 m_1 + \ldots + x_s m_s \mid x_1,\ldots,x_s \in U\}$ with U a neighbourhood of 0 in A is a fundamental system of neighbourhoods of 0 in the natural A-module topology of M. Now assume that A is f-adic with ring of definition A_0 and ideal of definition I of A_0. Let M_0 be a A_0-submodule of M, and equip M_0 with the I-adic topology. Then M_0 is an open subspace of M in the natural A-module topology of M if and only if M_0 is open and bounded in M. (A subset $M_0 \subseteq M$ is called bounded if for every neighbourhood U of 0 in M there exists a neighbourhood V of 0 in A with $\{a \cdot m \mid a \in V, m \in M_0\} \subseteq U$.)

A mapping $f:X \to Y$ between topological spaces X and Y is called strict if f is continuous and the restriction $X \to f(X)$ is open.

Lemma 2.3 *Let A be a* f*-adic ring which has a noetherian ring of a definition. Let M and N be finitely generated A-modules equipped with their natural A-module topology. Then.*

(i) *Every A-module homomorphism $f: M \to N$ is strict.*

(ii) *If A is complete then M is complete and every A-submodule of M is closed in M.*

(iii) *The natural mapping $M \bigotimes_A \hat{A} \to \hat{M}$ is an isomorphism of topological $\hat{A}$-modules if we equip $M \bigotimes_A \hat{A}$ with its natural $\hat{A}$-module topology.*

Proof. (i) and (ii) follow from analogous results for finitely generated modules over noetherian adic rings, and (iii) is a consequence of (ii). □

Lemma 2.4 (i) *Let A be a topological ring which has a zero sequence $(a_n \mid n \in \mathbb{N})$ with $a_n \in A^*$ for every $n \in \mathbb{N}$ (for example, A a Tate ring). Let M and N be topological A-modules which are complete and have countable fundamental systems of neighbourhoods of 0. Then every continuous surjective A-module homomorphism $M \to N$ is open.*

(ii) *Let A be a complete noetherian Tate ring, and let M and N be finitely generated A-modules equipped with their natural A-module topologies. Then M is complete and every A-module homomorphism $M \to N$ is strict.*

Proof. In order to prove (i) one can take over without any change the proof of Banach's open mapping theorem (cf. [B1, I.3.3.]). (ii) follows from (i) with the methods of [BGR, 3.7]. □

Let A be an affinoid ring and let M be a finitely generated $A^\triangleright$-module. For every rational subset W of Spa$A$ we equip the $\mathcal{O}_A(W)$-module $M \bigotimes_{A^\triangleright} \mathcal{O}_A(W)$ with the natural $\mathcal{O}_A(W)$-module topology. Then, for rational subsets V, W with $V \subseteq W$, the natural mapping $\rho_{V,W}: M \bigotimes_{A^\triangleright} \mathcal{O}_A(W) \to M \bigotimes_{A^\triangleright} \mathcal{O}_A(V)$ is continuous. For every open subset U of Spa$A$ we put

$$(M \otimes \mathcal{O}_A)(U) := \varprojlim_{V} M \bigotimes_{A^\triangleright} \mathcal{O}_A(V),$$

where the projective limit is taken over all rational subsets V of Spa$A$ contained in U and with respect to the transition mappings $\rho_{V,W}$. We equip $(M \otimes \mathcal{O}_A)(U)$ with the projective limit topology. Then $(M \otimes \mathcal{O}_A)(U)$ is a topological $\mathcal{O}_A(U)$-module, and for $V \subseteq U$ the canonical mapping $(M \otimes \mathcal{O}_A)(U) \to (M \otimes \mathcal{O}_A)(V)$ is continuous. So we have a topological $\mathcal{O}_A$-premodule $M \otimes \mathcal{O}_A$. Now we can generalize (2.2) to the following theorem

Theorem 2.5 *Let A be an affinoid ring such that $A^\triangleright$ has a noetherian ring of definition or $A^\triangleright$ is a strongly noetherian Tate ring, and let M be a finitely generated $A^\triangleright$-module. Then $M \otimes \mathcal{O}_A$ is a sheaf of complete topological groups on* Spa$A$, *and $H^i(U, M \otimes \mathcal{O}_A) = 0$ for every $i \in \mathbb{N}$ and every rational subset U of* Spa$A$.

Proof. We distinguish the cases that $A^\triangleright$ has a noetherian ring of definition and that $A^\triangleright$ is a strongly noetherian Tate ring. In the first case our proof is motivated by Raynaud's paper [R], and in the second case our proof is similar to the proof of Tate's acyclicity theorem in [BGR].

Case I. $A^\triangleright$ has a noetherian ring of definition.

First we describe a construction which generalizes the blowing up of ideals. Let $\varphi: C \to B$ be a ring homomorphism and let J be a C-submodule of B. Then we have the graduated C-algebra $\bigoplus_{n\in\mathbb{N}_0} J^n$, where $J^0 := \varphi(C)$ and, for $n \geqq 1, J^n$ is the subgroup of B generated by $j_1 \cdot \ldots \cdot j_n$ with $j_1, \ldots, j_n \in J$. We have a canonical morphism of schemes $g: \mathrm{Proj}(\bigoplus_{n\in\mathbb{N}_0} J^n) \to \mathrm{Spec}\, C$. Let I be the ideal of B generated by J. Then the inclusion $\bigoplus_{n\in\mathbb{N}_0} J^n \to \bigoplus_{n\in\mathbb{N}_0} I^n$ defines a morphism of schemes $s: \mathrm{Proj}(\bigoplus_{n\in\mathbb{N}_0} I^n) \to \mathrm{Proj}(\bigoplus_{n\in\mathbb{N}_0} J^n)$ such that the following diagram commutes

$$\text{(I.1)} \qquad \begin{array}{ccc} X := \mathrm{Proj}(\bigoplus\limits_{n\in\mathbb{N}_0} I^n) & \longrightarrow & \mathrm{Spec} B \\ s\downarrow & & \downarrow \\ Y := \mathrm{Proj}(\bigoplus\limits_{n\in\mathbb{N}_0} J^n) & \underset{g}{\longrightarrow} & \mathrm{Spec} C \end{array}$$

One can easily check

(I.2)

(i) s is affine.

(ii) Let F be a set of generators of the C-module J. Let S be a nonempty finite subset of $F \subseteq J^1$ and let $t \in J^{|S|}$ be the product of the elements of S. Put $U = D_+(t) = \{\mathfrak{p} \in \mathrm{Proj}(\bigoplus_{n\in\mathbb{N}_0} J^n) | t \notin \mathfrak{p}\} \subseteq Y$. Then one can consider $\mathcal{O}_Y(U)$ and $\mathcal{O}_X(s^{-1}(U))$ as subrings of the localization B_t, and we have $\mathcal{O}_Y(U) = C[\frac{f}{s} \mid f \in F, s \in S]$ and $\mathcal{O}_X(s^{-1}(U)) = B[\frac{f}{s} \mid f \in F, s \in S]$.

(iii) Let H be an ideal of C such that $\mathrm{Spec}(\varphi): \mathrm{Spec} B \to \mathrm{Spec} C$ induces an isomorphism $\mathrm{Spec} B - V(H \cdot B) \to \mathrm{Spec} C - V(H)$ and $V(I) \subseteq V(H \cdot B)$. Then g induces an isomorphism $Y - g^{-1}(V(H)) \to \mathrm{Spec} C - V(H)$.

The essential step in our proof of case I is the following point.

(I.3)
Let B be an affinoid ring, let $f_0, \ldots, f_n$ be elements of $B^\rhd$ with $B^\rhd = f_0 B^\rhd + \ldots + f_n B^\rhd$, and let P be a finitely generated $B^\rhd$-module. We assume that $B^\rhd$ has a noetherian ring of definition. Put $\mathcal{F} := P \otimes \mathcal{O}_B, Z := \mathrm{Spa} B$, and $U_i := R\left(\frac{f_0, \ldots, f_n}{f_i}\right) \subseteq Z$ for $i = 0, \ldots, n$. Then the augmented Cech complex to $\mathcal{F}$ and the covering $\{U_0, \ldots, U_n\}$ of Z

$$(*) \qquad 0 \to \mathcal{F}(Z) \to \prod_{i_0} \mathcal{F}(U_{i_0}) \to \prod_{(i_0, i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1}) \to \ldots$$

is exact. Furthermore, if we equip all components of $(*)$ with their natural topologies then all differentials of $(*)$ are strict.

Proof of (I.3) Let $\tilde{P}$ be the quasi-coherent sheaf on $\mathrm{Spec} B^\rhd$ defined by P. We consider the augmented Cech complex to $\tilde{P}$ and the covering $\{D(f_0), \ldots, D(f_n)\}$ of $X = \mathrm{Spec} B^\rhd$

$$(**) \qquad 0 \to \tilde{P}(X) \to \prod_{i_0} \tilde{P}(D(f_{i_0})) \to \prod_{(i_0,\, i_1)} \tilde{P}(D(f_{i_0}) \cap D(f_{i_1})) \to \ldots$$

Put $F = \{f_0, \ldots, f_n\}$. We equip $\tilde{P}(X) = P$ with the natural $B^\rhd$-module topology. For every $(f_{i_0}, \ldots, f_{i_k}) \in F^{k+1}$, the underlying ring of the topological ring $B^{\triangleright}\left(\frac{F}{f_{i_0}},\ldots,\frac{F}{f_{i_k}}\right)$ is the localization of $B^{\triangleright}$ by $f_{i_0}\cdot\ldots\cdot f_{i_k}$. So $\tilde{P}(D(f_{i_0})\cap\ldots\cap D(f_{i_k})) = P\bigotimes_{B^{\triangleright}} B^{\triangleright}\left(\frac{F}{f_{i_0}},\ldots,\frac{F}{f_{i_k}}\right)$, and hence $\tilde{P}(D(f_{i_0})\cap\ldots\cap D(f_{i_k}))$ is a finitely generated $B^{\triangleright}\left(\frac{F}{f_{i_0}},\ldots,\frac{F}{f_{i_k}}\right)$-module. We equip $\tilde{P}(D(f_{i_0})\cap\ldots\cap D(f_{i_k}))$ with the natural $B^{\triangleright}\left(\frac{F}{f_{i_0}},\ldots,\frac{F}{f_{i_k}}\right)$- module topology, and $\prod_{(i_0,\ldots,i_k)\in F^{k+1}}\tilde{P}(D(f_{i_0})\cap\ldots\cap D(f_{i_k}))$ with the product topology. Then every differential of $(**)$ is continuous. Let $(**)^{\wedge}$ be the completion of the complex $(**)$. By (2.3. iii), $(**)^{\wedge}$ is the complex $(*)$. The complex $(**)$ is exact. If all differentials of $(**)$ are strict then by [B2, III.2.12 Lemma 2] all differentials of $(**)^{\wedge}$ are strict and $(**)^{\wedge}$ is exact. So it is sufficient to show that all differentials of $(**)$ are strict.

Let C be a noetherian ring of definition of $B^{\triangleright}$, and let J be the C-submodule of $B^{\triangleright}$ generated by F. Since $B^{\triangleright} = f_0B^{\triangleright}+\ldots+f_nB^{\triangleright}$, we have by (I.1) a commutative diagram

$$\begin{array}{ccc} & & \operatorname{Spec}B^{\triangleright} = X \\ & \swarrow^{s} & \downarrow h \\ Y := \operatorname{Proj}\left(\bigoplus_{n\in\mathbb{N}_0} J^n\right) & \xrightarrow[g]{} & \operatorname{Spec}C\,. \end{array}$$

Let N be a finitely generated C-submodule of P which generates the $B^{\triangleright}$-module P, and let $\tilde{N}$ be the coherent sheaf on Spec$C$ defined by N. The inclusion $N\to P$ induces a morphism of sheaves $\tau:\tilde{N}\to h_*(\tilde{P})$. The composition of $g^*(\tau): g^*(\tilde{N})\to g^*h_*(\tilde{P}) = g^*g_*s_*(\tilde{P})$ with the adjunction morphism $g^*g_*s_*(\tilde{P})\to s_*(\tilde{P})$ gives a morphism of sheaves $\sigma: g^*(\tilde{N})\to s_*(\tilde{P})$. The image $\mathscr{G}$ of σ is a coherent sheaf on Y, since $g^*(\tilde{N})$ is coherent and $s_*(\tilde{P})$ is quasi-coherent.

Let I be an ideal of definition of C. For every $i\in\mathbb{N}$ let $K_i^{\bullet}$ be the augmented Cech complex to the sheaf $I^i\mathscr{G}$ and the covering $\{D_+(f_0),\ldots,D_+(f_n)\}$ of Y. Since $D(f_k) = s^{-1}(D_+(f_k))$ for $k = 0,\ldots,n$, we can identify the complex $(**)$ with the augmented Cech complex $K_0^{\bullet}$ to the sheaf $s_*(\tilde{P})$ and the covering $\{D_+(f_0),\ldots,D_+(f_n)\}$ of Y. Since $s_*(\tilde{P})\supseteq\mathscr{G}\supseteq I\mathscr{G}\supseteq I^2\mathscr{G}\supseteq\ldots$, we have a sequence of subcomplexes $K_0^{\bullet}\supseteq K_1^{\bullet}\supseteq K_2^{\bullet}\supseteq\ldots$. For every $i\in\mathbb{N}_0$ let $d_i^p: K_i^p\to K_i^{p+1}$ be the differentials of the complex $K_i^{\bullet}$. Then we have

(I.3.1)

(i) Given $p\in\mathbb{Z}$ and $u\in\mathbb{N}$ then there exists a $v\in\mathbb{N}_0$ with $\operatorname{im}(d_u^p)\supseteq\ker(d_v^{p+1})$.

(ii) For every $p\in\mathbb{Z}$, the set $\{K_i^p\mid i\in\mathbb{N}\}$ is a fundamental system of neighbourhoods of 0 in K_0^p.

Proof. (i) The assertion is trivial if $p<0$. So we fix a $p\in\mathbb{N}_0$. For every $x\in I^r$ and $y\in H^{p+1}(Y,I^u\mathscr{G})$ let $x\odot y\in H^{p+1}(Y,I^{u+r}\mathscr{G})$ be the image of y under the mapping $H^{p+1}(Y,I^u\mathscr{G})\to H^{p+1}(Y,I^{u+r}\mathscr{G})$ induced by the x-multiplication $I^u\mathscr{G}\to I^{u+r}\mathscr{G}$. Then by [EGA, III. 3.3.2] there exists a $k\in\mathbb{N}$ such that

(1) $H^{p+1}(Y,I^{u+r}\mathscr{G}) = I^r\odot H^{p+1}(Y,I^u\mathscr{G})$ for every $u\geqq k$ and $r\geqq 0$.

It is enough to proof (I.3.1. i) for $u \geqq k$. We fix a $u \geqq k$. By [H1, 3.7], Spec$B^{\triangleright} - V(I \cdot B^{\triangleright}) \to$ Spec$C - V(I)$ is an isomorphism. Hence (I.2. iii) implies that $Y - g^{-1}(V(I)) \to$ Spec$C - V(I)$ is an isomorphism. The sheaf $H^{p+1}(Y,I^u\mathscr{G})^{\sim}$ on Spec$C$ associated with the finitely generated C-module $H^{p+1}(Y,I^u\mathscr{G})$ is equal to the higher direct image sheaf $R^{p+1}g_*(I^u\mathscr{G})$. Hence the restriction of $H^{p+1}(Y,I^u\mathscr{G})^{\sim}$ to Spec $C - V(I)$ vanishes. Then according to [EGA*, I.6.8.4] there exists a $t \in \mathbb{N}$ with

(2) $I^t H^{p+1}(Y,I^u\mathscr{G}) = 0$.

We show $\mathrm{im}(d_u^p) \supseteq \ker(d_{u+t}^{p+1})$. Notice that $H^n(K_i^\bullet) = H^n(Y,I^i\mathscr{G})$ for every $n,i \in \mathbb{N}$. Let $a \in \ker(d_{u+t}^{p+1})$ be given, and let $\bar{a} \in H^{p+1}(Y,I^{u+t}\mathscr{G})$ be the cohomology class represented by a. By (1) there exist $x_1,\dots,x_w \in I^t$ and $\bar{y}_1,\dots,\bar{y}_w \in H^{p+1}(Y,I^u\mathscr{G})$ with $\bar{a} = x_1 \odot \bar{y}_1 + \dots + x_w \odot \bar{y}_w$. Let $y_1,\dots,y_w$ be elements of $\ker(d_u^{p+1})$ which represent the cohomology classes $\bar{y}_1,\dots,\bar{y}_w$. Then $a - (x_1y_1 + \dots + x_wy_w) \in \mathrm{im}(d_{u+t}^p) \subseteq \mathrm{im}(d_u^p)$. By (2) we have $x_jy_j \in \mathrm{im}(d_u^p)$ for $j = 1,\dots,w$. Hence $a \in \mathrm{im}(d_u^p)$.

(ii) Let S be a non empty subset of F. Then $C[\frac{f}{s} \mid s \in S,\ f \in F]$ is a ring of definition of the f-adic ring $B^{\triangleright}(\frac{F}{s} \mid s \in S)$, and $L := I \cdot C[\frac{f}{s} \mid s \in S,\ f \in F]$ is an ideal of definition of $C[\frac{f}{s} \mid s \in S,\ f \in F]$. Let G be the image of $N \bigotimes_C C[\frac{f}{s} \mid s \in S,\ f \in F]$ in $P \bigotimes_{B^{\triangleright}} B^{\triangleright}(\frac{F}{s} \mid s \in S)$. Then $\{L^i \cdot G \mid i \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0 in the natural $B^{\triangleright}(\frac{F}{s} \mid s \in S)$-module topology of $P \bigotimes_{B^{\triangleright}} B^{\triangleright}(\frac{F}{s} \mid s \in S)$. By (I.2.ii) we have $\mathscr{O}_Y(\bigcap_{s\in S} D_+(s)) = \mathscr{O}_Y(D_+(\prod_{s\in S} s)) = C[\frac{f}{s} \mid s \in S, f \in F]$. Hence in case $p \geqq 0$ the assertion of (I.3.1.ii) follows immediately from our construction of the sheaf $\mathscr{G}$. Now we assume $p = -1$. By construction of $\mathscr{G}$ we have $I^iN \subseteq \Gamma(Y,I^i\mathscr{G})$ for every $i \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed. We have to show that there exists a $i \in \mathbb{N}$ with $\Gamma(Y,I^i\mathscr{G}) \subseteq I^kN$. By [EGA, III.3.3.2] there exists a $r \in \mathbb{N}$ such that $\Gamma(Y,I^{r+s}\mathscr{G}) = I^s\Gamma(Y,I^r\mathscr{G})$ for every $s \in \mathbb{N}$. Since $\Gamma(Y,I^r\mathscr{G})$ is a finitely generated C-submodule of P and I^kN is an open C-submodule of P, there exists a $t \in \mathbb{N}$ with $I^t\Gamma(Y,I^r\mathscr{G}) \subseteq I^kN$. Hence $\Gamma(Y,I^{r+t}\mathscr{G}) \subseteq I^kN$.

Now we can show that all differentials of the complex $(**) = K_0^\bullet$ are strict. Let $p \in \mathbb{Z}$ be given. Applying (I.3.1.ii) to K_0^{p+1}, we see that $\mathrm{im}(d_0^p) \cap \ker(d_i^{p+1}) = \mathrm{im}(d_0^p) \cap K_i^{p+1}$ is a neighbourhood of 0 in $\mathrm{im}(d_0^p)$ for every $i \in \mathbb{N}$. Now (I.3.1.i, ii) shows that $d_0^p : K_0^p \to \mathrm{im}(d_0^p)$ is open. This concludes the proof of (I.3).

By (1.5) we have

(I.4)
Let U be a rational subset of Spa$A$. Put $B = F_A(U)$ and $P = M \bigotimes_{A^{\triangleright}} B^{\triangleright}$. Then there exists a homeomorphism f: Spa$B \to U$ and a morphism of presheaves $\varphi: (M \otimes \mathscr{O}_A) \mid U \to f_*(P \otimes \mathscr{O}_B)$ such that f induces a bijection between the set of the rational subsets of Spa$B$ and the set of the rational subsets of Spa$A$ contained in U, and, for every rational subset V of Spa$A$ with $V \subseteq U$, the mapping $\varphi(V): (M \otimes \mathscr{O}_A)(V) \to (P \otimes \mathscr{O}_B)(f^{-1}(V))$ is an isomorphism of topological groups.

Note that, for every rational subset U of Spa$A$, $F_A(U)^{\triangleright}$ is complete and has a noetherian ring of definition. Hence, applying (I.3), (I.4) and the subsequent Lemma 2.6, we obtain

(I.5)

Let $(V_j)_{j\in J}$ be an open covering of a rational subset U of Spa$A$. Then there exist rational subsets $U_1,\ldots,U_n$ of Spa$A$ such that $U=\bigcup_{i=1}^n U_i$, every U_i is contained in some V_j, and the augmented Cech complex to $\mathscr{F}:=M\otimes\mathcal{O}_A$ and the covering $(U_i\mid i=1,\ldots,n)$ of U

$$0\to\mathscr{F}(U)\to\prod_{i_0}\mathscr{F}(U_{i_0})\to\prod_{(i_0,\,i_1)}\mathscr{F}(U_{i_0}\cap U_{i_1})\to\ldots$$

is exact and has strict differentials.

Now we are ready to finish the proof of (2.5) in our first case. We have to show that $M\otimes\mathcal{O}_A$ is a sheaf of complete topological groups and that, for every rational subset U of Spa$A$ and every $i\in\mathbb{N}, H^i(U,M\otimes\mathcal{O}_A)=0$. By (2.3.ii), $(M\otimes\mathcal{O})(U)$ is complete for every rational subset U of Spa$A$, and then $(M\otimes\mathcal{O})(U)$ is complete for every open subset U of Spa$A$ (by definition of $M\otimes\mathcal{O}_A$). So we have a presheaf of complete topological groups. In order to show that $M\otimes\mathcal{O}_A$ is a sheaf of complete topological groups it is enough to show that, for every rational subset U of Spa$A$ and every covering $(U_i)_{i\in I}$ of U by rational subsets of Spa$A$, the sequence

$$(+)\quad 0\longrightarrow(M\otimes\mathcal{O}_A)(U)\xrightarrow{\ d\ }\prod_i(M\otimes\mathcal{O}_A)(U_i)\longrightarrow\prod_{(i,\,j)}(M\otimes\mathcal{O}_A)(U_i\cap U_j)$$

is exact and d is strict [EGA*, 0.3.2.2]. Then it is easy to see that it is even sufficient to show that, for every rational subset U of Spa$A$ and every covering $(V_j)_{j\in J}$ of U by rational subsets of Spa$A$, there exists a refinement $(U_i)_{i\in I}$ of $(V_j)_{j\in J}$ by rational subsets U_i of Spa$A$ such that (+) is exact and d is strict. But this is covered by (I.5). Likewise by (I.5) we have $\check{H}^i(U,M\otimes\mathcal{O}_A)=0$ for every $i\in\mathbb{N}$ and every rational subset U of Spa$A$. Then [G, 3.8 Corollary 4] implies $H^i(U,M\otimes\mathcal{O}_A)=0$ for every $i\in\mathbb{N}$ and every rational subset U of Spa$A$.

Case II. $A^{\triangleright}$ is a strongly noetherian Tate ring.

In the following two points (II.1) and (II.2) let B be an affinoid ring such that $B^{\triangleright}$ is a strongly noetherian Tate ring.

(II.1)

(i) Let $B^{\triangleright}\langle X,X^{-1}\rangle$ be the ring of all formal series $\sum_{n\in\mathbb{Z}}b_nX^n$ such that $b_n\in(B^{\triangleright})^{\wedge}$ for every $n\in\mathbb{Z}$ and, for every neighbourhood U of 0 in $(B^{\triangleright})^{\wedge}, b_n\notin U$ for only finitely many $n\in\mathbb{Z}$. Then the $B^{\triangleright}$-algebras $B^{\triangleright}\langle X,X^{-1}\rangle$ and $B^{\triangleright}\langle X,Y\rangle/(1-XY)$ are canonically isomorphic.

(ii) For $i=1,\ldots,m$ let $T_i=\{f_{i1},\ldots,f_{in(i)},g_i\}$ be a finite subset of $B^{\triangleright}$ with $B^{\triangleright}=T_i\cdot B^{\triangleright}$. Put $C=B^{\triangleright}\langle X_{ij}\mid i=1,\ldots,m,j=1,\ldots,n(i)\rangle$ and let I be the ideal of C generated by $\{f_{ij}-g_iX_{ij}\mid i=1,\ldots,m,j=1,\ldots,n(i)\}$. Then the topological $B^{\triangleright}$-algebras $B^{\triangleright}\left\langle\frac{T_1}{g_1},\ldots,\frac{T_m}{g_m}\right\rangle$ and C/I are canonically isomorphic.

(iii) For every rational subset U of $\mathrm{Spa}B$, the Tate ring $\mathcal{O}_B(U)$ is strongly noetherian.

(iv) For every rational subset U of $\mathrm{Spa}B, \mathcal{O}_B(U)$ is flat over $\mathcal{O}_B(\mathrm{Spa}B)$.

Proof. (i) We equip $B^{\triangleright}\langle X,X^{-1}\rangle$ with the group topology such that the sets $\{\sum_{n\in\mathbb{Z}} b_nX^n \in B^{\triangleright}\langle X,X^{-1}\rangle \mid b_n \in U \text{ for every } n \in \mathbb{Z}\}$ (U neighbourhood of 0 in $(B^{\triangleright})^{\wedge}$) form a fundamental system of neighbourhoods of 0 in $B^{\triangleright}\langle X,X^{-1}\rangle$. Then $B^{\triangleright}\langle X,X^{-1}\rangle$ is a complete Tate ring. By (2.4.ii), the ideal $(1-XY)$ is closed in $B^{\triangleright}\langle X,Y\rangle$ and hence $B^{\triangleright}\langle X,Y\rangle/(1-XY)$ is a complete Tate ring. Both ring homomorphisms $B^{\triangleright} \to (B^{\triangleright}\langle X,X^{-1}\rangle;X)$ and $B^{\triangleright} \to (B^{\triangleright}\langle X,Y\rangle/(1-XY);X)$ are universal with respect to continuous ring homomorphisms from $B^{\triangleright}$ to complete Tate rings with a distinguished unit u such that u and u^{-1} are power-bounded (cf. (3.3.i)).

(ii) By (2.4.ii), I is closed in C, and hence C/I is a complete f-adic ring. Every g_i is a unit in C/I, since $B^{\triangleright} = T_i \cdot B^{\triangleright}$. Now it is easily seen that both ring homomorphisms $B^{\triangleright} \to B^{\triangleright}\left\langle \frac{T_1}{g_1},\ldots,\frac{T_m}{g_m}\right\rangle$ and $B^{\triangleright} \to C/I$ satisfy the same universal property, cf. (1.2) and (3.3.i).

(iii) follows from (3.4.i, ii).

(iv) We proceed as in the proof of [FP, III.7.10]. First we show that it is enough to prove the assertion only for some special rational subsets U. For that we use (1.5). Namely let $f_1,\ldots,f_n,g$ be elements of $B^{\triangleright}$ with $U = \{x \in \mathrm{Spa}B \mid x(f_i)\leqq x(g)\neq 0 \text{ of } i = 1,\ldots,n\}$. Since U is quasi-compact, there exists by [H1, 3.11] a unit s of $B^{\triangleright}$ with $x(s)\leqq x(g)$ for every $x \in U$. Put $Y_1 := \{x \in \mathrm{Spa}B \mid 1\leqq x\left(\frac{g}{s}\right)\}$. Then $g \mid Y_1$ is a unit of $\mathcal{O}_B(Y_1)$, and so we can define inductively rational subsets $Y_2,\ldots,Y_{n+1}$ of $Y_0 := \mathrm{Spa}B$ with $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_{n+1} = U$ by $Y_k = \{x \in Y_{k-1} \mid x\left(\frac{f_{k-1}}{g}\right) \leqq 1\}$ $(k = 2,\ldots,n+1)$. Then the restriction $h\colon \mathcal{O}_B(\mathrm{Spa}B) \to \mathcal{O}_B(U)$ factorizes into $h = h_n \circ \ldots \circ h_0$, where $h_i\colon \mathcal{O}_B(Y_i) \to \mathcal{O}_B(Y_{i+1})$ is the restriction. Using (2.4.ii) and (ii) one can prove with the ideas of [FP, III.7.8, 7.9] that, for every complete affinoid ring D such that $D^{\triangleright}$ is noetherian and Tate and for every $f \in D^{\triangleright}$, the restrictions $\mathcal{O}_D(\mathrm{Spa}D) \to \mathcal{O}_D(U_1)$ and $\mathcal{O}_D(\mathrm{Spa}D) \to \mathcal{O}_D(U_2)$ are flat, where $U_1 = \{x \in \mathrm{Spa}D \mid x(f)\geqq 1\}$ and $U_2 = \{x \in \mathrm{Spa}D \mid x(f)\leqq 1\}$. Hence by (iii), all ring homomorphisms h_i are flat.

(II.2)

Let f be an element of $B^{\triangleright}$, and put $U_1 = \{x \in \mathrm{Spa}B \mid x(f)\leqq 1\}$ and $U_2 = \{x \in \mathrm{Spa}B \mid x(f)\geqq 1\}$. Then the augmented Cech complex to $\mathcal{O}_B$ and the covering $\{U_1,U_2\}$ of $\mathrm{Spa}B$

$$0 \longrightarrow \mathcal{O}_B(\mathrm{Spa}\ B) \xrightarrow{\varepsilon} \mathcal{O}_B(U_1) \times \mathcal{O}_B(U_2) \xrightarrow{\delta} \mathcal{O}_B(U_1 \cap U_2) \longrightarrow 0$$

is exact.

Proof. By (II.1.iv) the ring homomorphism ε is flat. With (1.4) we obtain that ε is faithfully flat and hence injective. By (II.1.i, ii) we have $\mathcal{O}_B(U_1) = B^{\triangleright}\langle X\rangle/(f-X), \mathcal{O}_B(U_2) = B^{\triangleright}\langle Y\rangle/(1-fY)$ and $\mathcal{O}_B(U_1 \cap U_2) = B^{\triangleright}\langle X,Y\rangle/(f-X,1-fY) = B^{\triangleright}\langle X,Y\rangle/(f-X,1-XY) = B^{\triangleright}\langle X,X^{-1}\rangle/(f-X)$. Now an easy computation shows that δ is surjective and $\mathrm{im}(\varepsilon) = \ker(\delta)$ (cf. [BGR, 8.2.3]).

(II.3)
Let $(V_j)_{j\in J}$ be an open covering of a rational subset U of Spa$A$. Then there exist rational subsets $U_1,\ldots,U_n$ of Spa$A$ such that $U = \bigcup_{i=1}^n U_i$, every U_i is contained in some V_j, and the augmented Cech complex to $M \otimes \mathcal{O}_A$ and the covering $(U_i|i = 1,\ldots,n)$ of U is exact.

Proof. By (II.1.iv) it is sufficient to prove (II.3) for $M = A^{\triangleright}$ (use the alternating Cech complexes). (I.4) holds also in our case II. Hence by (II.1.iii) we can assume that $A^{\triangleright}$ is complete and $U =$ Spa$A$. Then by the subsequent lemma we can assume $(V_j)_{j\in J} = (R\left(\frac{f_0,\ldots,f_m}{f_i}\right)|i = 0,\ldots,m)$ with some $f_0,\ldots,f_m \in A^{\triangleright}$ with $A^{\triangleright} = f_0A^{\triangleright} + \ldots + f_mA^{\triangleright}$. Now using (II.2) one can prove (II.3) with the reasonings of [BGR, 8.2.2].

(II.3) implies that $M \otimes \mathcal{O}_A$ is a sheaf of groups and $H^i(U, M \otimes \mathcal{O}_A) = 0$ for every $i \in \mathbb{N}$ and every rational subset U of Spa$A$. By (2.4.ii) and (II.1.iii), $(M \otimes \mathcal{O}_A)(U)$ is complete for every rational subset U of Spa$A$. Then $(M \otimes \mathcal{O}_A)(U)$ is complete for every open subset U of Spa$A$. It remains to show that $M \otimes \mathcal{O}_A$ is a sheaf of topological groups. Let $(U_i)_{i\in I}$ be an open covering of an open subset U of Spa$A$. We have to show that $\varepsilon\colon (M \otimes \mathcal{O}_A)(U) \to \prod_{i\in I} (M \otimes \mathcal{O}_A)(U_i)$ is strict. By [EGA*, 0.3.2.2] we may assume that U and all U_i are rational, and since U is quasi-compact we may assume that I is finite. im(ε) is closed in $\prod_{i\in I} (M \otimes \mathcal{O}_A)(U_i)$, since im$(\varepsilon)$ is the kernel of $\prod_{i\in I} (M \otimes \mathcal{O}_A)(U_i) \to \prod_{i,\, j\in I} (M \otimes \mathcal{O}_A)(U_i \cap U_j)$. Now (2.4.i) implies that ε is strict. □

Lemma 2.6 *Let A be a complete affinoid ring, and let $(V_j)_{j\in J}$ be an open covering of* Spa$A$. *Then there exist $f_0,\ldots,f_n \in A^{\triangleright}$ such that $A^{\triangleright} = f_0A^{\triangleright} + \ldots + f_nA^{\triangleright}$ and, for every $i \in \{0,\ldots,n\}$, the rational subset $R\left(\frac{f_0,\ldots,f_n}{f_i}\right)$ is contained in some V_j.*

Proof. Let $x \in$ Spa$A \subseteq$ Spv$A^{\triangleright}$. Let $c\Gamma_x$ be the characteristic subgroup of Γ_x as defined in [H1]. Let y be the valuation $x \mid c\Gamma_x$ of $A^{\triangleright}$. Then $\Gamma_y = c\Gamma_y$ and $y \in$ Spa$A$. We choose a V_j with $y \in V_j$. Since $\Gamma_y = c\Gamma_y$, there exists by the proof of [H1, 2.6.ii] an element $s \in A^{\triangleright}$ and a finite subset T of $A^{\triangleright}$ such that $1 \in T$ and $y \in R\left(\frac{T}{s}\right) \subseteq V_j$. We have $x \in R\left(\frac{T}{s}\right)$, since x is a generalization of y. Using that Spa$A$ is quasi-compact, we obtain that the there exist elements $s_1,\ldots,s_n \in A^{\triangleright}$ and finite subsets $T_1,\ldots,T_n$ of $A^{\triangleright}$ such that $1 \in T_i$ for every $i = 1,\ldots,n$ and $(R\left(\frac{T_i}{s_i}\right)|i = 1,\ldots,n)$ is a covering of Spa$A$ refining $(V_j)_{j\in J}$. We may assume $s_i \in T_i$ for $i = 1,\ldots,n$.

Now we can follow the proof of [FP, III.2.5]. Put $T = \{t_1 \cdot \ldots \cdot t_n | t_i \in T_i$ for $i = 1,\ldots,n\}$ and $S = \{t_1 \cdot \ldots \cdot t_n | t_i \in T_i$ for $i = 1,\ldots,n$ and $t_i = s_i$ for at least one $i \in \{1,\ldots,n\}\}$. We note the following trivial properties

(1) For every $t_1 \in T_1,\ldots,t_n \in T_n$ we have $R\left(\frac{T}{t}\right) = \bigcap_{i=1}^n R\left(\frac{T_i}{t_i}\right)$ with $t := t_1 \cdot \ldots \cdot t_n$.

(2) For every $i \in \{1,\ldots,n\}$, Spa$A = \bigcup_{t\in T_i} R\left(\frac{T_i}{t}\right)$.

By (1), $R\left(\frac{T}{s}\right)$ is contained in some $R\left(\frac{T_i}{s_i}\right)$ for every $s \in S$. (1) and (2) imply

(3) $\mathrm{Spa}A = \bigcup_{s \in S} R\left(\frac{T}{s}\right)$.

From (1.4) and (3) we obtain $A^{\triangleright} = S \cdot A^{\triangleright}$. Furthermore, (3) implies $R\left(\frac{T}{s}\right) = R\left(\frac{S}{s}\right)$ for every $s \in S$. Hence every $R\left(\frac{S}{s}\right)$ is contained in some V_j. □

3 Adic morphisms, morphisms of finite type and fibre products

The following definition turns out to be very useful.

Definition. *A point x of an adic space X is called* **analytic** *if there exists an open neighbourhood U of x in X such that $\mathcal{O}_X(U)$ has a topologically nilpotent unit. We put $X_{\mathrm{a}} = \{x \in X | x \text{ is analytic}\}$ and $X_{\mathrm{na}} = X - X_{\mathrm{a}}$. If every point of X is analytic then X is called analytic.*

Remark 3.1 If $X = \mathrm{Spa}A$ is the adic space to an affinoid ring A then $x \in X$ is analytic if and only if $\mathrm{supp}(x) \in \mathrm{Spec}A^{\triangleright}$ is not open in $A^{\triangleright}$. (So the above definition coincides with the definition of an analytic point in [H1].)

Proof. Let x be a point of $\mathrm{Spa}A$ such that $\mathrm{supp}(x)$ is open in $A^{\triangleright}$, and let U be an open neighbourhood of x in $\mathrm{Spa}A$. We have to show that $\mathcal{O}_X(U)$ has no topologically nilpotent unit. Let V be a rational subset of $\mathrm{Spa}A$ with $x \in V \subseteq U$. Put $\mathfrak{p} = \{f \in \mathcal{O}_X(V) \mid v_x(f) = 0\}$. Then $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_X(V)$ with $\mathfrak{p} \cap A^{\triangleright} = \mathrm{supp}(x)$. Since $\mathrm{supp}(x)$ is open in $A^{\triangleright}$, it follows immediately from the definition of $\mathcal{O}_X(V)$ that $\mathfrak{p}$ is open $\mathcal{O}_X(V)$. Hence every topologically nilpotent element of $\mathcal{O}_X(V)$ lies in $\mathfrak{p}$. But since $\mathfrak{p}$ contains no unit of $\mathcal{O}_X(V)$, $\mathcal{O}_X(V)$ has no topologically nilpotent unit. Now let x be a point of $\mathrm{Spa}A$ such that $\mathrm{supp}(x)$ is not open in $A^{\triangleright}$. Then there exists a topologically nilpotent element s of $A^{\triangleright}$ with $x(s) \neq 0$. Put $U = \{y \in \mathrm{Spa}A \mid y(s) \neq 0\}$. The U is an open neighbourhood of x in $\mathrm{Spa}A$, and the image of s in $\mathcal{O}_X(U)$ is a topologically nilpotent unit of $\mathcal{O}_X(U)$. □

In [H1] we called a ring homomorphism $f: A \to B$ between f-adic rings adic if there exist rings of definition A_0, B_0 of A, B and an ideal of definition I of A_0 such that $f(A_0) \subseteq B_0$ and $f(I) \cdot B_0$ is an ideal of definition of B_0.

Definition. *A morphism $f: X \to Y$ between adic space is called* **adic** *if for every $x \in X$ there exist an open affinoid neighbourhood U of x in X and an open affinoid subspace V of Y such that $f(U) \subseteq V$ and the ring homomorphism between* f*-adic rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ induced by f is adic.*

(3.1) and [H1, 3.8] imply

Proposition 3.2 *Let $f: X \to Y$ be a morphism of adic spaces. Then*

(i) *If f is adic then, for all open affinoid subspaces U and V of X and Y with $f(U) \subseteq V$, the ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is adic.*

(ii) *f is adic if and only if $f(X_{\mathrm{a}}) \subseteq Y_{\mathrm{a}}$.*

(iii) $f(X_{\mathrm{na}}) \subseteq Y_{\mathrm{na}}$.

In order to define morphisms of finite type between adic spaces we first introduce ring homomorphisms of topologically finite type between f-adic rings and affinoid rings. Let A be a f-adic ring and let $M_1, \ldots, M_n$ be finite subsets of A such that $M_i \cdot A$ is open in A for every $i = 1, \ldots, n$. We put

$A\langle X\rangle_M = A\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n} := \{\sum a_\nu X^\nu \in \hat{A}[[X_1, \ldots, X_n]] \mid$ for every neighbourhood U of 0 in $\hat{A}, a_\nu \notin M_1^{\nu_1} \cdot \ldots \cdot M_n^{\nu_n} \cdot U$ for only finitely many $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n\}$.

Obviously $A\langle X\rangle_M$ is a subgroup of $\hat{A}[[X_1, \ldots, X_n]]$. We equip $A\langle X\rangle_M$ with the group topology such that the sets $\{\sum a_\nu X^\nu \in A\langle X\rangle_M \mid a_\nu \in M_1^{\nu_1} \cdot \ldots \cdot M_n^{\nu_n} \cdot U$ for every $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n\}$ (U a neighbourhood of 0 in $\hat{A}$) from a fundamental system of neighbourhoods of 0. Using (1.1), one can easily see that $A\langle X\rangle_M$ is a subring of $\hat{A}[[X_1, \ldots, X_n]]$, and moreover $A\langle X\rangle_M$ is a complete f-adic ring. The natural ring homomorphism $A \to A\langle X\rangle_M$ is continuous.

We put

$$A\langle X\rangle = A\langle X_1, \ldots, X_n\rangle := A\langle X_1, \ldots, X_n\rangle_{\{1\},\ldots,\{1\}}.$$

We say that a ring homomorphism $f: A \to B$ from A to a complete f-adic ring B is of **topologically finite type** if there exist $n \in \mathbb{N}_0$, finite subsets $M_1, \ldots, M_n$ of A with $M_i \cdot A$ open in A for $i = 1, \ldots, n$ and a surjective, continuous, open ring homomorphism $g: A\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n} \to B$ such that $f = g \circ h$, where $h: A \to A\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n}$ is the natural ring homomorphism. One can easily check

Lemma 3.3 (i) (Universal property of $A\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n}$) *Let $h: A \to A\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n}$ be the natural ring homomorphism. Then $h(m)X_i$ is power-bounded in $A\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n}$ for every $i \in \{1, \ldots, n\}$ and $m \in M_i$, and if $f: A \to B$ is a continuous ring homomorphism from A to a complete f-adic ring B and $b_1 \ldots, b_n$ are elements of B such that $f(m)b_i$ is power-bounded in B for every $i \in \{1, \ldots, n\}$ and $m \in M_i$ then there exists a unique continuous ring homomorphism $g: A\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n} \to B$ with $f = g \circ h$ and $b_i = g(X_i)$ for $i = 1, \ldots, n$.*

(ii) *Let $f: A \to B$ be a ring homomorphism from A to a complete f-adic ring B. Then the following conditions are equivalent*

(a) *f is of topologically finite type*

(b) *f is adic, there exists a finite subset M of B such that $A[M]$ is dense in B, and there exist rings of definition A_0, B_0 of A, B and a finite subset N of B_0 such that $f(A_0) \subseteq B_0$ and $A_0[N]$ is dense in B_0.*

(iii) *If A is Tate then a ring homomorphism $f: A \to B$ from A to a complete f-adic ring B is of topologically finite type if and only if f factors through a surjective, continuous and open ring homomorphism $g: A\langle X_1, \ldots, X_n\rangle \to B$ for some $n \in \mathbb{N}_0$.*

(iv) *Let B, C be complete f-adic rings and $f: A \to B$ and $g: B \to C$ continuous ring homomorphisms. If f and g are of topologically finite type then $g \circ f$ is of topologically finite type, and if $g \circ f$ is of topologically finite type then g is of topologically finite type.*

Corollary 3.4 (i) *Let $s_1,\dots,s_n$ be elements of A and $T_1,\dots,T_n$ finite subsets of A such that $T_i \cdot A$ is open for $i = 1,\dots,n$. Then the natural ring homomorphism $A \to A\left\langle \frac{T_1}{s_1},\dots,\frac{T_n}{s_n}\right\rangle$ is of topologically finite type.*
(ii) *If A is Tate then the following conditions are equivalent*

(a) *A is strongly noetherian.*
(b) *Every Tate ring of topologically finite type over A is noetherian.*
(c) *Every Tate ring of topologically finite type over A is strongly noetherian.*

(iii) *Let B be a complete f-adic ring of topologically finite type over A. If A has a noetherian ring of definition then also B has a noetherian ring of definition.*

Proof. Applying (3.3.ii), (i) follows immediately from the definition of $A\left\langle \frac{T_1}{s_1},\dots,\frac{T_n}{s_n}\right\rangle$. (ii) follows from (3.3.iii,iv), and (iii) follows from (3.3.ii). □

Let k be a complete, non-archimedean, valued field. Then k is a Tate ring. In rigid analytic geometry one calls the complete topological k-algebras which are quotients of some $k\langle X_1,\dots,X_n\rangle$ affinoid algebra or Tate algebras over k. We call them, according to (3.3.iii), Tate rings of topologically finite type over k.

Let $A = (A^{\triangleright},A^{+})$ be an affinoid ring, and let $M_1,\dots,M_n$ be finite subsets of $A^{\triangleright}$ such that $M_i \cdot A^{\triangleright}$ is open in $A^{\triangleright}$ for $i = 1,\dots,n$. Then $B := \{\sum a_\nu X^\nu \in A^{\triangleright}\langle X\rangle_M \mid a_\nu \in M_1^{\nu_1}\cdot\ldots\cdot M_n^{\nu_n}\cdot (A^+)^\wedge$ for every $\nu = (\nu_1,\dots,\nu_n) \in \mathbb{N}_0^n\}$ is a subring of $A^{\triangleright}\langle X\rangle_M$. The integral closure C of B in $A^{\triangleright}\langle X\rangle_M$ is a ring of integral elements of $A^{\triangleright}\langle X\rangle_M$. The affinoid ring $(A^{\triangleright}\langle X\rangle_M, C)$ is denoted by $A\langle X\rangle_M = A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n}$. We put $A\langle X_1,\dots,X_n\rangle := A\langle X_1,\dots,X_n\rangle_{\{1\},\dots,\{1\}}$.

A ring homomorphism $f: B \to C$ between affinoid rings is called a quotient mapping if $f: B^{\triangleright} \to C^{\triangleright}$ is surjective, continuous and open and C^+ is the integral closure of $f(B^+)$ in $C^{\triangleright}$. A ring homomorphism $f: A \to B$ from A to a complete affinoid ring B is called of **topologically finite type** if there exist a $n \in \mathbb{N}_0$, finite subsets $M_1,\dots,M_n$ of $A^{\triangleright}$ with $M_i \cdot A^{\triangleright}$ open in $A^{\triangleright}$ for $i = 1,\dots,n$ and a quotient mapping $g: A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n} \to B$ such that $f = g\circ h$, where $h: A \to A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n}$ is the natural homomorphism of affinoid rings. Then we have analogously to (3.3)

Lemma 3.5 (i) (Universal property of $A\langle X_1,\dots,X_n\rangle_{M_1,\dots,M_n}$) *Let $h: A \to A\langle X\rangle_M$ be the natural ring homomorphism. Then $h(m)X_i \in (A\langle X\rangle_M)^+$ for every $i \in \{1,\dots,n\}$ and $m \in M_i$, and if $f: A \to B$ is a continuous ring homomorphism from A to a complete affinoid ring B and $b_1,\dots,b_n$ are elements of $B^{\triangleright}$ such that $f(m)b_i \in B^+$ for every $i \in \{1,\dots,n\}$ and $m \in M_i$, then there exists a unique continuous ring homomorphism $g: A\langle X\rangle_M \to B$ with $f = g \circ h$ and $g(X_i) = b_i$ for $i = 1,\dots,n$.*
(ii) *Let $f: A \to B$ be a ring homomorphism from A to a complete affinoid ring B. Then the following conditions are equivalent*

(a) *f is of topologically finite type*

(b) *$f: A^{\triangleright} \to B^{\triangleright}$ is of topologically finite type and there exists an open subring C of B^+ such that B^+ is integral over C, $f(A^+) \subseteq C$ and $f: A^+ \to C$ is of topologically finite type.*

In particular, for elements $s_1, \ldots, s_n$ of $A^{\triangleright}$ and finite subsets $T_1, \ldots, T_n$ of $A^{\triangleright}$ with $T_i \cdot A^{\triangleright}$ open in $A^{\triangleright}$ for $i = 1, \ldots, n$, the natural ring homomorphism $A \to A\left\langle \frac{T_1}{s_1}, \ldots, \frac{T_n}{s_n} \right\rangle$ is of topologically finite type.

(iii) *If A is Tate then a ring homomorphism $f: A \to B$ from A to a complete affinoid ring B is of topologically finite type if and only if f factors through a quotient mapping $A\langle X_1, \ldots, X_n \rangle \to B$ for some $n \in \mathbb{N}_0$.*

(iv) *Let B, C be complete affinoid rings and $f: A \to B$ and $g: B \to C$ continuous ring homomorphisms. If f and g are of topologically finite type then $g \circ f$ is of topologically finite type, and if $g \circ f$ is of topologically finite type then g is of topologically finite type.*

Definition. *Let $f: X \to Y$ be a morphism between adic spaces. We say f is* **locally of finite type** *if for every $x \in X$ there exist an open affinoid neighbourhood U of x in X and an open affinoid subspace V of Y such that $f(U) \subseteq V$ and the morphism between affinoid rings $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ induced by f is of topologically finite type. If f is quasi-compact (i.e., for every quasi-compact open subset U of Y, $f^{-1}(U)$ is quasi-compact) and locally of finite type then f is called of* **finite type**.

If $f: X \to Y$ is locally of finite type then to every neighbourhood U' of a point $x \in X$ and every neighbourhood V' of $f(x)$ there exist open affinoid neighbourhoods U and V of x and $f(x)$ in U' and V' such that $f(U) \subseteq V$ and $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is of topologically finite type (3.5.ii, iv). In [H2] we will show

Proposition 3.6 *Let X, Y be adic spaces and $f: X \to Y$ a morphism locally of finite type. Let U and V be open affinoid subspaces of X and Y with $f(U) \subseteq V$. We assume that $\mathcal{O}_Y(V)$ has a noetherian ring of definition or is a strongly noetherian Tate ring. Then the morphism of affinoid rings $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is of topologically finite type.*

Now we consider fibre products. In (3.7) we consider fibre products in the category of adic spaces and in (3.8) we consider the "fibre product" $S \times_Y X$ of an adic space S and a scheme X both living over a scheme Y.

Proposition 3.7 *Let X, Y, S be adic spaces and $f: X \to S$ and $g: Y \to S$ morphisms. We assume that f is locally of finite type, g is adic, and every point $y \in Y$ has an open affinoid neighbourhood U in Y such that $\mathcal{O}_Y(U)$ has a noetherian ring of definition or is a strongly noetherian Tate ring. Then there exists in the category of adic spaces the fibre product $X \times_S Y$ of X and Y over S. The projection $X \times_S Y \to Y$ is locally of finite type and the projection $X \times_S Y \to X$ is adic.*

Proof. We can assume that $S = \operatorname{Spa} A, Y = \operatorname{Spa} B, X = \operatorname{Spa} C$ are affinoid, $B^{\triangleright}$ has a noetherian ring of definition or is a strongly noetherian Tate ring, and g and f are induced by ring homomorphisms $\lambda: A \to B$ and $\mu: A \to C$ such that λ is adic and μ is of topologically finite type. Then there exist finite subsets

$M_1, \ldots, M_n$ of $A^\triangleright$ such that $M_i \cdot A^\triangleright$ is open in $A^\triangleright$ and μ factors through a quotient mapping $\sigma: A\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n} \to C$. Since λ is adic, $\lambda(M_i) \cdot B^\triangleright$ is open in $B^\triangleright$ for $i = 1, \ldots, n$. Let $\lambda': A\langle X_1, \ldots, X_n\rangle_{M_1,\ldots,M_n} \to B\langle X_1, \ldots, X_n\rangle_{\lambda(M_1),\ldots,\lambda(M_n)} =: E$ be the natural extension of λ. We equip $D^\triangleright := E^\triangleright/\lambda'(\ker\sigma) \cdot E^\triangleright$ with the f-adic topology such that the canonical mapping $\pi: E^\triangleright \to D^\triangleright$ is continuous and open. Let D^+ be the integral closure of $\pi(E^+)$ in $D^\triangleright$. Then D^+ is a ring of integral elements of $D^\triangleright$. We have a commutative diagram of affinoid rings $(D := (D^\triangleright, D^+))$

$$(*) \qquad \begin{array}{ccc} D & \longleftarrow & C \\ \uparrow & & \uparrow \\ B & \longleftarrow & A \end{array}$$

which is cocartesian in the category of complete affinoid rings (3.5.i). By (3.4.ii, iii) $D^\triangleright$ has a noetherian ring of definition or is a strongly noetherian Tate ring. So by (2.2) we have an adic Spa$D$ associated with D. The commutative diagram induced by $(*)$

$$\begin{array}{ccc} \mathrm{Spa}\,D & \longrightarrow & \mathrm{Spa}\,C \\ \downarrow & & \downarrow \\ \mathrm{Spa}\,B & \longrightarrow & \mathrm{Spa}\,A \end{array}$$

is cartesian in the category of adic spaces by (2.1.i). □

Proposition 3.8 *Let _ denote the forgetful functor from the category of adic spaces to the category of locally ringed spaces which assigns to an adic space $(X, \mathcal{O}_X, (v_x \mid x \in X))$ the locally ringed space $(X, \mathcal{O}_X)$.*

Let X, Y be schemes, S an adic space, and $f: X \to Y$ and $g: \underline{S} \to Y$ morphisms of locally ringed spaces. We assume that f is locally of finite type and that every $s \in S$ has an open affinoid neighbourhood U in S such that $\mathcal{O}_S(U)$ has a noetherian ring of definition or is a strongly noetherian Tate ring. Then there exist an adic space R, a morphism of adic spaces $p: R \to S$ and a morphism of locally ringed spaces $q: \underline{R} \to X$ such that the diagram

$$\begin{array}{ccc} \underline{R} & \xrightarrow{q} & X \\ {\scriptstyle \underline{p}}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{S} & \xrightarrow[g]{} & Y \end{array}$$

commutes and the following universal property is satisfied. If U is an adic space, $u: U \to S$ a morphism of adic spaces and $v: \underline{U} \to X$ a morphism of locally ringed spaces with $g \circ \underline{u} = f \circ v$ then there exists a unique morphism of adic spaces $w: U \to R$ with $u = p \circ w$ and $v = q \circ \underline{w}$.

The morphism $p: R \to S$ is locally of finite type. We call R the fibre product of X and S over Y and denote it by $X \times_Y S$. Every morphism $f: X_1 \to X_2$ between schemes locally of finite type over Y induces by the universal property a morphism $X_1 \times_Y S \to X_2 \times_Y S$ of adic spaces over S which we denote by $f_{(S)}$.

Proof. We may assume that $Y = \mathrm{Spec}B, X = \mathrm{Spec}B[X_1, \ldots, X_n]/I, S = \mathrm{Spa}A, A^\triangleright$ has a noetherian ring of definition or is a strongly noetherian Tate ring, and g is induced by a ring homomorphism $\lambda: B \to A^\triangleright$. Let E be a finite set of topologically nilpotent elements of $A^\triangleright$ such that $E \cdot A^\triangleright$ is open. For every

$k \in \mathbb{N}$ let $A(k) := A\langle X_1,\ldots,X_n\rangle_{E(k),\ldots,E(k)}$ with $E(k) := \{e_1 \cdot \ldots \cdot e_k \mid e_i \in E$ for $i = 1,\ldots,k\}$, and let $\lambda_k: B[X_1,\ldots,X_n] \to A(k)^{\triangleright}$ be the extension of λ with $\lambda_k(X_i) = X_i$ for $i = 1,\ldots,n$. Let $\pi_k: A(k)^{\triangleright} \to A(k)^{\triangleright}/\lambda_k(I) \cdot A(k)^{\triangleright} =: A_k^{\triangleright}$ be the natural mapping. We equip $A_k^{\triangleright}$ with the f-adic topology such that π_k is continuous and open. Then $A_k^{\triangleright}$ has a noetherian ring of definition or is a strongly noetherian Tate ring (3.4.ii, iii). Let A_k^+ be the integral closure of $\pi_k(A(k)^+)$ in $A_k^{\triangleright}$. Then $A_k := (A_k^{\triangleright}, A_k^+)$ is an affinoid ring. By (2.2) we have an adic space R_k associated with A_k. For $k \leqq h$, let $\varphi_{kh}: R_k \to R_h$ be the morphism which is induced by the continuous A-homomorphism $A_h \to A_k$ with $\pi_h(X_i) \mapsto \pi_k(X_i)$ for $i = 1,\ldots,n$ (3.5.i). Then φ_{kh} is an isomorphism of R_k onto the rational subset $\{x \in R_h \mid v_x(e\pi_h(X_i)) \leqq 1$ for every $e \in E(k), i = 1,\ldots,n\}$ of R_h. Hence there exists in the category of adic spaces over S the inductive limit R of the system $(R_k, \varphi_{kh} \mid k,h \in \mathbb{N})$. Let $p: R \to S$ be the structure morphism. The ring homomorphisms $\pi_k \circ \lambda_k: B[X_1,\ldots,X_n] \to A_k^{\triangleright}$ induce morphisms of locally ringed spaces $(R_k)_{-} \to X$ which glue together to a morphism of locally ringed spaces $q: \underline{R} \to X$. We have $g \circ \underline{p} = f \circ q$. Using (2.1.i) and (3.5.i), one can easily check that R, p, q satisfy the universal property. □

Lemma 3.9 (i) *Let*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & S \end{array}$$

be a cartesian square as in (3.7). *Then, for every $x \in X$ and $y \in Y$ with $f(x) = g(y)$, there exists a $z \in X \times_S Y$ with $x = q(z)$ and $y = p(z)$.*

(ii) *Let*

$$\begin{array}{ccc} \underline{R} & \xrightarrow{q} & X \\ {\scriptstyle \underline{p}}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{S} & \xrightarrow[g]{} & Y \end{array}$$

be the commutative diagram of (3.8). *Then, for every $x \in X$ and $s \in S$ with $f(x) = g(s)$, there exists a $r \in R$ with $x = q(r)$ and $s = p(r)$.*

Proof. (i) We use the notations of the proof of (3.7). Let A_0, B_0, C_0 be rings of definition of $A^{\triangleright}$, $B^{\triangleright}$, $C^{\triangleright}$ with $\lambda(A_0) \subseteq B_0$ and $\mu(A_0) \subseteq C_0$, and let I be an ideal of definition of A_0. Let F be the image of $B_0 \otimes_{A_0} C_0$ in $E^{\triangleright} := B^{\triangleright} \otimes_{A^{\triangleright}} C^{\triangleright}$, and let E^+ be the integral closure of the image of $B^+ \otimes_{A^+} C^+$ in $E^{\triangleright}$. We equip $E^{\triangleright}$ with the group topology such that $\{I^n \cdot F | n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0. Then $E^{\triangleright}$ is a f-adic ring and E^+ is a ring of integral elements of $E^{\triangleright}$. The completion of the affinoid ring $E = (E^{\triangleright}, E^+)$ is denoted by $B \hat{\otimes}_A C$. The commutative diagram

$$\begin{array}{ccc} B\hat{\otimes}_A C & \longleftarrow & C \\ \uparrow & & \uparrow \\ B & \longleftarrow & A \end{array}$$

is cocartesian in the category of complete affinoid rings. Hence $X \times_S Y = \mathrm{Spa} B\hat{\otimes}_A C = \mathrm{Spa} E$.

Let x be a point of X and y a point of Y with $f(x) = g(y)$. We consider x and y as valuations of $C^{\triangleright}$ and $B^{\triangleright}$. There exists a valuation v of $E^{\triangleright}$ which lies over x and y. Let $\langle\Gamma_x\rangle$ and $\langle\Gamma_y\rangle$ be the convex hulls of the value groups Γ_x and Γ_x of x and y in the value group Γ_v of v. Put $H := \langle\Gamma_x\rangle \cup \langle\Gamma_y\rangle$ (hence $H = \langle\Gamma_x\rangle$ or $H = \langle\Gamma_y\rangle$). Then H contains the characteristic subgroup $c\Gamma_v$ of v (cf. [H1, Sect. 2]). Therefore we have the valuation $w := v|H$ of $E^\triangleright$ (cf. [H1, Sect. 2]). Also w lies over x and y. We have $w(e) \leqq 1$ for every $e \in E^+$, and $w(i)$ is cofinal in Γ_w for every $i \in I$ (since $x(i)$ is cofinal in Γ_x and $y(i)$ is cofinal in Γ_y). Hence $w \in \mathrm{Spa}E$.

(ii) We may assume $Y = \mathrm{Spec}B$, $X = \mathrm{Spec}C$, $S = \mathrm{Spa}A$, and f and g are given by ring homomorphisms $B \to C$ and $B \to A^\triangleright$. Let $\mathfrak{p} \in \mathrm{Spec}C$ be a prime ideal of C and $v \in \mathrm{Spa}A$ a valuation of $A^\triangleright$ with $\mathfrak{p} \cap B = \mathrm{supp}(v) \cap B$. We show that there exists a $r \in R$ with $\mathfrak{p} = q(r)$ and $v = p(r)$. We choose a prime ideal $\mathfrak{q}$ of $C \otimes_B A^\triangleright$ with $\mathfrak{p} = \mathfrak{q} \cap C$ and $\mathrm{supp}(v) = \mathfrak{q} \cap A^\triangleright$, and a valuation r of $C \otimes_B A^\triangleright$ such that $\mathfrak{q} = \mathrm{supp}(r)$, r extends the valuation v of $A^\triangleright$ and the value groups Γ_v and Γ_r of v and r have the same divisible hull (in particular, Γ_r is the convex hull of Γ_v in Γ_r). Let A_0 be a ring of definition of $A^\triangleright$, I a finitely generated ideal of definition of A_0, and $C' \subseteq C$ a finite set of generators of C over B. We choose a $k \in \mathbb{N}$ such that $r(c \otimes i) \leqq 1$ for every $c \in C'$ and $i \in I^k$. Then let D_0 denote the subring $A_0[c \otimes i | c \in C', i \in I^k]$ of $D^\triangleright := C \otimes_B A^\triangleright$, and let D^+ denote the integral closure of $A^+[c \otimes i | c \in C', i \in I^k]$ in $D^\triangleright$. We equip $D^\triangleright$ with the group topology such that $\{I^n \cdot D_0 | n \in \mathbb{N}\}$ is a fundamental system of neighbourhoods of 0. Then $D^\triangleright$ is a f-adic ring and D^+ is a ring of integral elements of $D^\triangleright$. We have $r \in \mathrm{Spa}(D^\triangleright, D^+)$, and $\mathrm{Spa}(D^\triangleright, D^+)$ is an open subspace of R. Hence r is an element of R with $\mathfrak{p} = q(r)$ and $v = p(r)$. $\square$

4 Formal schemes, rigid analytic varieties and adic spaces

In this paragraph we construct a functor $t: \mathscr{F} \to \mathscr{A}$ form the category $\mathscr{F}$ of locally noetherian formal schemes to the category $\mathscr{A}$ of adic spaces, and a functor $r_k: \mathscr{R}_k \to \mathscr{A}$ from the category of rigid analytic varieties over a complete non-archimedean valued field k to the category of adic spaces. The idea of the definition of t and r_k is the following. First we notice that if A is a noetherian adic ring or a Tate ring of topologically finite type over k then, for every ring A^+ of integral elements of A, we have an adic space $\mathrm{Spa}(A, A^+)$ associated with the affinoid ring (A, A^+) (2.2). So we can define t and r_k on the affine objects of the categories $\mathscr{F}$ and $\mathscr{R}_k$, namely if $X = \mathrm{Spf}A$ then $t(X) := \mathrm{Spa}(A, A)$ and if $X = \mathrm{Sp}A$ then $r_k(X) := \mathrm{Spa}(A, A^\circ)$. We will see that every open subset U of an affine object X of $\mathscr{F}$ (resp. $\mathscr{R}_k$) induces in a natural way an open subset $a(U)$ of $t(X)$ (resp. $r_k(X)$), and every morphism $f: U_1 \to U_2$ between two open subspaces of affine objects X_1, X_2 of $\mathscr{F}$ (resp. $\mathscr{R}_k$) induces in a natural way a morphism of adic spaces $a(f): a(U_1) \to a(U_2)$. Then it is obvious how one has to define t and r_k on general objects, namely if an object X of $\mathscr{F}$ (resp. $\mathscr{R}_k$) is obtained by glueing affine objects $(X_i | i \in I)$ along isomorphisms $\varphi_{ji}: U_{ij} \to U_{ji}$ then $t(X)$ (resp. $r_k(X)$) is defined to be the adic space obtained by glueing $(t(X_i) | i \in I)$ (resp. $(r_k(X_i) | i \in I)$) along the isomorphisms $a(\varphi_{ji})$: $a(U_{ij}) \to a(U_{ji})$.

In the following proposition we construct the functor t. If X is an adic space then we equip, for every open subset U of X, $\mathcal{O}_X^+(U)$ with the subspace topology of $\mathcal{O}_X(U)$. Then $(X, \mathcal{O}_X^+)$ is a locally and topologically ringed space.

Proposition 4.1 *Let X be a locally noetherian formal scheme. Then there exist an adic space $t(X)$ and a morphism of locally and topologically ringed spaces $\pi = \pi_X: (t(X), \mathcal{O}_{t(X)}^+) \to (X, \mathcal{O}_X)$ such that the following universal property is satisfied. If Z is an adic space and $\lambda: (Z, \mathcal{O}_Z^+) \to (X, \mathcal{O}_X)$ is a morphism of locally and topologically ringed spaces then there exists a unique morphism between adic space $f: Z \to t(X)$ such that the diagram*

$$\begin{array}{ccc} (Z, \mathcal{O}_Z^+) & & \\ \mu \downarrow & \searrow^{\lambda} & \\ (t(X), \mathcal{O}_{t(X)}^+) & \underset{\pi}{\longrightarrow} & (X, \mathcal{O}_X) \end{array}$$

commutes, where μ is the morphism induced by f.

If $f: X \to Y$ is a morphism between locally noetherian formal schemes then by the universal property there exists a unique morphism between adic spaces $t(f): t(X) \to t(Y)$ such that the diagram

$$\begin{array}{ccc} (t(X), \mathcal{O}_{t(X)}^+) & \overset{\pi_X}{\longrightarrow} & (X, \mathcal{O}_X) \\ \mu \downarrow & & \downarrow f \\ (t(Y), \mathcal{O}_{t(Y)}^+) & \underset{\pi_Y}{\longrightarrow} & (Y, \mathcal{O}_Y) \end{array}$$

commutes, where μ is the morphism induced by $t(f)$. So we have a functor t from the category of locally noetherian formal schemes to the category of adic spaces.

Proof. We may assume that $X = \mathrm{Spf} A$ is affine. Then we put $t(X) := \mathrm{Spa}(A, A)$. By the following point (1), the identity $A \to A$ induces a morphism of locally and topologically ringed spaces $\pi: (t(X), \mathcal{O}_{t(X)}^+) \to (X, \mathcal{O}_X)$. By (2.1.ii) and (1), $t(X)$ and π satisfy the universal property.

(1) Let Y be an adic space. Then the morphisms of locally and topologically ringed spaces $(Y, \mathcal{O}_Y^+) \to (X, \mathcal{O}_X)$ correspond bijectively to the continuous ring homomorphisms $\mathcal{O}_X(X) \to \mathcal{O}_Y^+(Y)$.

Proof. We may assume that $Y = \mathrm{Spa}(B, B^+)$ is the adic space associated with a complete affinoid ring (B, B^+). Let a continuous ring homomorphism $\varphi: \mathcal{O}_X(X) \to \mathcal{O}_Y^+(Y)$ be given. We will construct a morphism of locally and topologically ringed spaces $f: (Y, \mathcal{O}_Y^+) \to (X, \mathcal{O}_X)$ such that $f^*: \mathcal{O}_X(X) \to \mathcal{O}_Y^+(Y)$ is equal to φ. For every $y \in Y$ put $g(y) = \{a \in \mathcal{O}_X(X) | v_y(\varphi(a)) < 1\}$. Then $g(y)$ is an open prime ideal of $\mathcal{O}_X(X) = A$. Hence we have a mapping $g: Y \to X$. This mapping is continuous, since, for every $s \in \mathcal{O}_X(X), g^{-1}(\{x \in X | s(x) \neq 0\}) = \{y \in Y | 1 \leqq v_y(\varphi(a))\}$.

Let $s \in \mathcal{O}_X(X)$ be given. Put $U = \{x \in X | s(x) \neq 0\}$. Then $\mathcal{O}_X(U) = A\langle \frac{1}{s} \rangle$ and $\mathcal{O}_Y^+(g^{-1}(U)) = B^+\langle \frac{1}{\varphi(s)} \rangle$. Hence by (1.2.i) there exists a unique continuous ring homomorphism $\varphi_U: \mathcal{O}_X(U) \to \mathcal{O}_Y^+(g^{-1}(U))$ such that the diagram

$$\begin{array}{ccc} \mathcal{O}_Y^+(g^{-1}(U)) & \xleftarrow{\varphi_U} & \mathcal{O}_X(U) \\ \uparrow & & \uparrow \\ \mathcal{O}_Y^+(Y) & \xleftarrow[\varphi]{} & \mathcal{O}_X(X) \end{array}$$

commutes. The φ_U define a morphism of sheaves $\psi: \mathcal{O}_X \to g_*\mathcal{O}_Y^+$. It remains to show that, for every $y \in Y$, the ring homomorphism $\psi_y: \mathcal{O}_{X,g(y)} \to \mathcal{O}_{Y,y}^+$ induced by ψ is local. Let $\mathfrak{m}_y$ and $\mathfrak{m}_{g(y)}$ be the maximal ideals of $\mathcal{O}_{Y,y}^+$ and $\mathcal{O}_{X,g(y)}$, and let $h: \mathcal{O}_X(X) \to \mathcal{O}_{X,g(y)}$ be the natural ring homomorphism. Then by definition of g we have $h^{-1}(\mathfrak{m}_{g(y)}) = h^{-1}(\psi_y^{-1}(\mathfrak{m}_y))$ which implies $\mathfrak{m}_{g(y)} = \psi_y^{-1}(\mathfrak{m}_y)$. □

We call a subcategory $\mathscr{C}$ of the category $\mathscr{A}$ of adic spaces saturated if it satisfies the following properties. If $X \in \mathscr{C}$ and $U \subseteq X$ is an open subspace then $U \in \mathscr{C}$; if $X \in \mathscr{A}$ has an open covering $(U_i)_{i \in I}$ with $U_i \in \mathscr{C}$ for every $i \in I$ then $X \in \mathscr{C}$; if $X \in \mathscr{A}$ is isomorphic to a $Y \in \mathscr{C}$ then $X \in \mathscr{C}$; $\mathscr{C}$ is a full subcategory of $\mathscr{A}$.

Proposition 4.2 (i) *The functor t: $\mathscr{F} \to \mathscr{A}$ from the category $\mathscr{F}$ of locally noetherian formal schemes to the category of adic spaces is fully faithful. A morphism f: $X \to Y$ in $\mathscr{F}$ is adic* [EGA*, I.10.12] *resp. locally of finite type* [EGA*, I.10.13] *if and only if $t(f)$: $t(X) \to t(Y)$ is adic resp. locally of finite type.*
(ii) *Let $\mathscr{C}$ be the smallest subcategory of $\mathscr{A}$ such that $t(\mathscr{F}) \subseteq \mathscr{C}$ and $\mathscr{C}$ is saturated. Then the objects of $\mathscr{C}$ are the adic spaces X with the property that every $x \in X$ has an open affinoid neighbourhood U such that $\mathcal{O}_X(U)$ has a noetherian ring of definition A such that $\mathcal{O}_X(U)$ is finitely generated over A and $\mathcal{O}_X^+(U)$ is the integral closure of A in $\mathcal{O}_X(U)$.*

Proof. (i) For every locally noetherian formal scheme Z we have the locally ringed space $Z_0 := (Z_t, \mathcal{O}_{t(Z)}^+ | Z_t)$ with $Z_t = \{z \in t(Z) | v_z$ is a trivial valuation$\} \subseteq t(Z)$. The morphism of locally ringed spaces $j_Z: Z_0 \to Z$ induced by π_Z is an isomorphism. Let X, Y be locally noetherian formal schemes. Let $f: t(X) \to t(Y)$ be a morphism of adic spaces. Then $f(X_t) \subseteq Y_t$, and hence f induces a morphism of locally ringed spaces $f_0: X_0 \to Y_0$. Let $g: X \to Y$ be the morphism of locally ringed spaces with $g \circ j_X = j_Y \circ f_0$. Then g is a morphism of formal schemes (since X is locally noetherian), and it is easily seen that $f = t(g)$. Let $g_1, g_2: X \to Y$ be morphisms of formal schemes with $t(g_1) = t(g_2) =: f$. We have $g_1 \circ j_X = j_Y \circ f_0$ and $g_2 \circ j_X = j_Y \circ f_0$ which implies $g_1 = g_2$. Hence the functor t is fully faithful.

Let $f: X \to Y$ be a morphism between locally noetherian formal schemes. (3.3.ii) shows that a ring homomorphism $g: A \to B$ between noetherian complete adic rings is of topologically finite type (in the sense as defined in Sect. 3) if and only if g factors through a continuous and open mapping $A\langle X_1, \ldots, X_n \rangle \to B$. Then (3.2.i) and (3.6) imply that f is adic (resp. locally of finite type) if and only if $t(f)$ is adic (resp. locally of finite type).

(A more detailed proof would show that one can avoid the use of (3.2.i) and (3.6)).

(ii) Let $\mathscr{D}$ be the full subcategory of $\mathscr{A}$ consisting of the objects with the property described in (ii). Then $t(\mathscr{F}) \subseteq \mathscr{D}$ and $\mathscr{D}$ is saturated. Let $\mathscr{E}$ be a subcategory of $\mathscr{A}$ such that $t(\mathscr{F}) \subseteq \mathscr{E}$ and $\mathscr{E}$ is saturated. We have to show $\mathscr{D} \subseteq \mathscr{E}$. Let A be an affinoid ring such that there exist a noetherian ring of definition B of $A^{\triangleright}$ and a finite subset $E = \{e_1, \ldots, e_n\}$ of $A^{\triangleright}$ such that $A^{\triangleright} = B[E]$ and A^+ is the integral closure of B in $A^{\triangleright}$. We show that every point x of Spa$A$ has an open neighbourhood U in Spa$A$ with $U \in \mathscr{E}$. We assume that $v_x(e_i) > 1$ for $i = 1, \ldots, s$ and $v_x(e_i) \leqq 1$ for $i = s+1, \ldots, n$. We choose $b_1, \ldots, b_r \in B$ such that $\{b_1, \ldots, b_r\} \cdot B$ is an ideal of definition of B and $v_x(b_i e_j) \leqq 1$ for all $i = 1, \ldots, r, j = 1, \ldots, s$. Let C be the subring $B[e_{s+1}, \ldots, e_n]\left[\frac{1}{e_1}, \ldots, \frac{1}{e_s}\right]$ of the localization $A^{\triangleright}_{e_1, \ldots, e_s}$. We equip C with the adic topology such that $\{b_1, \ldots, b_r\} \cdot C$ is an ideal of definition. Let U be the rational subset $\{y \in \mathrm{Spa}A \mid 1 \leqq v_y(e_i)$ for $i = 1, \ldots, s$ and $v_y(e_i) \leqq 1$ for $i = s+1, \ldots, n$ and $v_y(b_i e_j) \leqq 1$ for $i = 1, \ldots, r, j = 1, \ldots, s\}$ of Spa$A$. Then $x \in U$ and U is isomorphic to the rational subspace $\{y \in \mathrm{Spa}(C, C) \mid v_y(b_i) \leqq v_y\left(\frac{1}{e_j}\right) \neq 0$ for $i = 1, \ldots, r, j = 1, \ldots, s\}$ of Spa(C, C). □

Let k be a field equipped with the complete topology of a rank 1 valuation of k. Let $\mathscr{R}_k$ be the category of rigid analytic varieties over k as defined in [BGR, 9.3.1]. Let $\mathscr{A}_k$ be the category of adic spaces over Spa(k, k°). (Every object of $\mathscr{A}_k$ is analytic in the sense of Sect. 3). We will construct a natural functor $r_k : \mathscr{R}_k \to \mathscr{A}_k$. For every $X \in \mathscr{R}_k$ let $|X|$ be the Grothendieck topology of X. Then $(|X|, \mathcal{O}_X)$ is a ringed site [SGA, IV.13.1], and every morphism $f : X \to Y$ in $\mathscr{R}_k$ induces a morphism of ringed sites $\bar{f} : (|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$ [SGA, IV.13.3]. Similarly, for every $X \in \mathscr{A}_k$ let $|X|$ be the topology underlying X. Then $(|X|, \mathcal{O}_X)$ is a ringed site, and every morphism $f : X \to Y$ in $\mathscr{A}_k$ induces a morphism of ringed sites $\bar{f} : (|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$. If $X = \mathrm{Spa}(k, k^\circ)$ and $Y = \mathrm{Sp}k$ then $(|X|, \mathcal{O}_X) = (|Y|, \mathcal{O}_Y)$, and we denote this ringed site by S. If X is an object of $\mathscr{R}_k$ or $\mathscr{A}_k$ then $(|X|, \mathcal{O}_X)$ is a ringed site over S, and if f is a morphism in $\mathscr{R}_k$ or $\mathscr{A}_k$ then $\bar{f}$ is a morphism of ringed sites over S, i.e, we have functors from the categories $\mathscr{R}_k$ and $\mathscr{A}_k$ to the category of ringed sites over S. In the following proposition we construct the functor $r_k : \mathscr{R}_k \to \mathscr{A}_k$.

Proposition 4.3 *To every $X \in \mathscr{R}_k$ there exist a $r_k(X) \in \mathscr{A}_k$ and a morphism $\rho = \rho_X : (|r_k(X)|, \mathcal{O}_{r_k(X)}) \to (|X|, \mathcal{O}_X)$ of ringed sites over S such that $r_k(X)$ is locally of finite type over* Spa(k, k°)*, ρ is locally coherent* [SGA, VI.3.7] *and the following universal property is satisfied. If Z is an adic space locally of finite type over* Spa(k, k°) *and $\lambda : (|Z|, \mathcal{O}_Z) \to (|X|, \mathcal{O}_X)$ is a locally coherent morphism between ringed sites over S then there exists a unique morphism $f : Z \to r_k(X)$ in $\mathscr{A}_k$ such that the diagram of ringed sites*

$$\begin{array}{ccc} (|Z|, \mathcal{O}_Z) & \searrow^{\lambda} & \\ \bar{f} \downarrow & & \\ (|r_k(X)|, \mathcal{O}_{r_k(X)}) & \xrightarrow[\rho]{} & (|X|, \mathcal{O}_X) \end{array}$$

commutes.

If $f: X \to Y$ is a morphism in $\mathcal{R}_k$ then $\bar{f}: (|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$ is locally coherent, and hence by the universal property there exists a unique morphism $r_k(f): r_k(X) \to r_k(Y)$ in $\mathcal{A}_k$ such that the diagram

$$\begin{array}{ccc} (|r_k(X)|, \mathcal{O}_{r_k(X)}) & \xrightarrow{\rho_X} & (|X|, \mathcal{O}_X) \\ \overline{r_k(f)} \downarrow & & \downarrow \bar{f} \\ (|r_k(Y)|, \mathcal{O}_{r_k(Y)}) & \xrightarrow[\rho_Y]{} & (|Y|, \mathcal{O}_Y) \end{array}$$

commutes. Thus we have a functor $r_k: \mathcal{R}_k \to \mathcal{A}_k$.

Proof. We may assume that $X = \mathrm{Sp}A$ is the affinoid rigid analytic variety associated with a Tate ring A of topologically finite type over k. We put $r_k(X) := \mathrm{Spa}(A, A^\circ)$. By the subsequent lemma $r_k(X)$ is of finite type over $\mathrm{Spa}(k, k^\circ)$. By [H1, 4.3] there is a natural one-to-one correspondence between the rational subsets of $\mathrm{Sp}A$ and the rational subsets of $\mathrm{Spa}(A, A^\circ)$. This correspondence extends in a unique way to a morphism of sites $m: |r_k(X)| \to |X|$. m is coherent. For every rational subset U of X and corresponding rational subset V of $r_k(X)$ there exists a unique A-algebra homomorphism $\Theta_U: \mathcal{O}_X(U) \to \mathcal{O}_{r_k(X)}(V)$, and Θ_U is bijective. The Θ_U's induce an isomorphism of sheaves $\Theta: \mathcal{O}_X \to m_* \mathcal{O}_{r_k(X)}$. Then $\rho = (m, \Theta): (|r_k(X)|, \mathcal{O}_{r_k(X)}) \to (|X|, \mathcal{O}_X)$ is a morphism of ringed sites over S. We check the universal property. Let Z be an adic space locally of finite type over $\mathrm{Spa}(k, k^\circ)$ and $\lambda: (|Z|, \mathcal{O}_Z) \to (|X|, \mathcal{O}_X)$ a locally coherent morphism between ringed sites over S. There exists a unique morphism of ringed sites $\mu: (|Z|, \mathcal{O}_Z) \to (|r_k(X)|, \mathcal{O}_{r_k(X)})$ with $\lambda = \rho \circ \mu$. Let $g: (Z, \mathcal{O}_Z) \to (r_k(X), \mathcal{O}_{r_k(X)})$ be the morphism of ringed spaces whose associated morphism of ringed sites is μ (cf. [SGA, IV.4.2.3]). We have to show that g is a morphism in $\mathcal{A}_k$. We know that g is a morphism of ringed spaces over k and that for every open quasi-compact subset U of Z the restriction $g: U \to r_k(X)$ is a quasi-compact mapping. Since the k-algebra homomorphisms between Tate rings of topologically finite type over k are continuous [BGR, 6.1.3], we see that $\mathcal{O}_{r_k(X)} \to g_* \mathcal{O}_Z$ is a morphism of topological sheaves. It remains to show that, for every $z \in Z$, the ring homomorphism $g_z^*: \mathcal{O}_{r_k(X), g(z)} \to \mathcal{O}_{Z,z}$ is compatible with the valuations $v_{g(z)}$ and v_z. For that we note the following property

(1) Let $U \subseteq r_k(X)$ be open, $f \in \mathcal{O}_{r_k(X)}(U)$, $a \in k^*$. We consider the sets $U_1 = \{x \in U | v_x(f) < v_x(a)\}, U_2 = \{x \in U | v_x(f) \leqq v_x(a)\}, V_1 = \{x \in g^{-1}(U) | v_x(g^*(f)) < v_x(a)\}, V_2 = \{x \in g^{-1}(U) | v_x(g^*(f)) \leqq v_x(a)\}$. Then $V_1 = g^{-1}(U_1)$ and $V_2 = g^{-1}(U_2)$.

Proof. Let M be the set of points $z \in Z$ such that the residue class field of $\mathcal{O}_{Z,z}$ is finite over k. For every $z \in M$ the ring homomorphism $g_z^*: \mathcal{O}_{r_k(X), g(z)} \to \mathcal{O}_{Z,z}$ is compatible with the valuations $v_{g(z)}$ and v_z, since g_z^* is local and the valuation ring k° of k extends uniquely to every finite extension field of k. Hence $V_1 \cap M = g^{-1}(U_1) \cap M$ and $V_2 \cap M = g^{-1}(U_2) \cap M$. Then [H1, 4.3] and the subsequent lemma imply $V_1 = g^{-1}(U_1)$ and $V_2 = g^{-1}(U_2)$.

Let $z \in Z$ be given. We consider the ring homomorphism $g_z^*: \mathcal{O}_{r_k(X), g(z)} \to \mathcal{O}_{Z,z}$. Let $\mathfrak{m}_{g(z)}$ and $\mathfrak{m}_z$ be the maximal ideals of $\mathcal{O}_{r_k(X), g(z)}$ and $\mathcal{O}_{Z,z}$. Then by (1), $g_z^*(\mathfrak{m}_{g(z)}) = g_z^*(\bigcap_{a \in k^*} \{f \in \mathcal{O}_{r_k(X), g(z)} | v_{g(z)}(f) \leqq v_{g(z)}(a)\}) \subseteq \bigcap_{a \in k^*} \{f \in \mathcal{O}_{Z,z} | v_z(f) \leqq v_z(a)\} = \mathfrak{m}_z$, i.e., g_z is a local ring homomorphism.

Furthermore, (1) implies that if f is an element of $\mathcal{O}_{r_k(X),g(z)}$ with $v_{g(z)}(f) \leqq 1$ (resp. $v_{g(z)}(f) < 1$) then $v_z(g_z^*(f)) \leqq 1$ (resp. $v_z(g_z^*(f)) < 1$). This shows that g_z^* is compatible with the valuations $v_{g(z)}$ and v_z. $\square$

Lemma 4.4 *Let A be a Tate ring of topologically finite type over k. Then there exists a unique ring A^+ of integral elements of A such that $k^\circ \subseteq A^+$ and the ring homomorphism of affinoid rings $(k, k^\circ) \to (A, A^+)$ is of topologically finite type, namely $A^+ = A^\circ$.*

Proof. Let A^+ be a ring of integral elements of A such that $f: (k, k^\circ) \to (A, A^+)$ is of topologically finite type. By (3.5.iii) f factors through a quotient mapping $g: (k, k^\circ)\langle X_1, \ldots, X_n\rangle \to (A, A^+)$. We have $(k, k^\circ)\langle X_1, \ldots, X_n\rangle^+ = k^\circ\langle X_1, \ldots, X_n\rangle = k\langle X_1, \ldots, X_n\rangle^\circ$. Since $g: k\langle X_1, \ldots, X_n\rangle \to A$ is surjective, A° is the integral closure of $g(k\langle X_1, \ldots, X_n\rangle^\circ)$ in A [BGR, 6.4.3]. Hence $A^+ = A^\circ$. This also shows that $(k, k^\circ) \to (A, A^\circ)$ is of topologically finite type. $\square$

For an object X of $\mathscr{A}_k$ or $\mathscr{R}_k$ let Shv(X) denote the topos of the site $|X|$. Before (4.2) we defined the saturated subcategories of $\mathscr{A}$. Analogously we define the saturated subcategories of $\mathscr{A}_k$.

Proposition 4.5 (i) *For every $X \in \mathscr{R}_k$, the morphism of ringed toposes $(\mathrm{Shv}(r_k(X)), \mathcal{O}_{r_k(X)}) \to (\mathrm{Shv}(X), \mathcal{O}_X)$ induced by the morphism of ringed sites $\rho_X: (|r_k(X)|, \mathcal{O}_{r_k(X)}) \to (|X|, \mathcal{O}_k)$ is an equivalence of ringed toposes.*
(ii) *The functor $r_k: \mathscr{R}_k \to \mathscr{A}_k$ is fully faithful.*
(iii) *Let $\mathscr{C}$ be the smallest subcategory of $\mathscr{A}_k$ such that $r_k(\mathscr{R}_k) \subseteq \mathscr{C}$ and $\mathscr{C}$ is saturated. Then the objects of $\mathscr{C}$ are the adic spaces locally of finite type over* Spa(k, k°).
(iv) *The functor $r_k: \mathscr{R}_k \to \mathscr{A}_k$ restricts to an equivalence between the category of quasi-separated rigid analytic varieties over k and the category of quasi-separated adic spaces locally of finite type over* Spa(k, k°).
(v) *For every $X \in \mathscr{R}_k$, the adic space $r_k(X)$ is affinoid if and only if X is an affinoid rigid analytic variety over k.*

Proof. (i)–(iv) follow from the proof of (4.3) (cf. [H1, 4.6] for (i)). Let $X \in \mathscr{R}_k$ such that $Y := r_k(X)$ is an affinoid adic space. By (3.6) the morphism of affinoid rings $(k, k^\circ) \to (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))$ is of topologically finite type. Then by (4.4) $\mathcal{O}_Y^+(Y) = \mathcal{O}(Y)^\circ$. The adic space $r_k(X)$ is isomorphic to $\mathrm{Spa}(\mathcal{O}_Y(Y), \mathcal{O}_Y(Y)^\circ) = r_k(\mathrm{Sp}\mathcal{O}_Y(Y))$ (over Spa(k, k°)). Then by (ii) X is isomorphic to $\mathrm{Sp}\mathcal{O}_Y(Y)$. $\square$

Remark 4.6 (i) Let X be a scheme locally of finite type over k. One associates with X a rigid analytic variety X^{an} [BGR, 9.3.4]. The identity $k \to k$ induces a morphism of locally ringed spaces $\mathrm{Spa}(k, k^\circ)_- \to \mathrm{Spec}\,k$. Then by (3.8) we have the adic space $X \times_{\mathrm{Spec}\,k} \mathrm{Spa}(k, k^\circ)$ over Spa(k, k°). It follows from our construction of the functor r_k and of $X \times_{\mathrm{Spec}\,k} \mathrm{Spa}(k, k^\circ)$ that

$$r_k(X^{\mathrm{an}}) \cong X \times_{\mathrm{Spec}k} \mathrm{Spa}(k, k^\circ),$$

where the isomorphism is an isomorphism over Spa(k, k°).
(ii) Let V be a complete discrete valuation ring of rank 1, and let k be the quotient field of V. In [R] there is constructed a functor $s: \mathscr{D} \to \mathscr{R}_k$ from the category $\mathscr{D}$ of formal schemes locally of finite type over Spf$V$ to the category

$\mathscr{R}_k$ of rigid analytic varieties over k. Let $\mathscr{E}$ be the category whose objects are the locally noetherian formal schemes and whose morphisms are the adic morphisms between formal schemes. By (3.2.ii) and (4.2.i) we have the functor $t_a: \mathscr{E} \to \mathscr{A}, X \to t(X)_a$ which assigns to every $X \in \mathscr{E}$ the open subspace $t(X)_a$ of analytic points of the adic space $t(X)$. Let $u: \mathscr{D} \to \mathscr{E}$ be the natural functor. Then the diagram

$$\begin{array}{ccc} \mathscr{D} & \xrightarrow{s} & \mathscr{R}_k \\ u\downarrow & & \downarrow r_k \\ \mathscr{E} & \xrightarrow[t_a]{} & \mathscr{A} \end{array}$$

commutes up to isomorphism, i.e., the functors $r_k \circ s$ and $t_a \circ u$ are isomorphic.
(iii) We use the notations of (ii). Let $f: X \to Y := \operatorname{Spf} V$ be a formal scheme locally of finite type over $\operatorname{Spf} V$. We consider the morphism of adic spaces $t(f): t(X) \to t(Y)$. The topological space Z underlying $t(Y)$ is the Sierpinski space, i.e., Z has an open generic point η (i.e., $\{\eta\}$ is open and $\overline{\{\eta\}} = Z$) and Z consists of two points. We have $\{\eta\} = t(Y)_a$, and the fibre $t(f)^{-1}(\eta)$ is the open subspace of analytic points of $t(X)$. Raynaud calls the rigid analytic variety $s(X)$ the generic fibre of f. So the diagram in (ii) says that the functor r_k transforms the generic fibre $s(X)$ of f into the generic fibre $t(f)^{-1}(\eta)$ of $t(f)$, i.e., $r_k(s(X)) \cong t(f)^{-1}(\eta)$.
(iv) If A is an adic noetherian ring then the identity $A \to A$ induces a morphism of locally ringed spaces $\operatorname{Spa}(A, A)_- \to \operatorname{Spec} A$. More general, if $\hat{X}$ is the formal completion of a locally noetherian scheme X along a closed subset $X' \subseteq X$ then there is a natural morphism of locally ringed spaces $\rho = \rho_{\hat{X}}: \underline{t(\hat{X})} \to X$. Namely, if $(X_i | i \in I)$ is an open affine covering of X and $\hat{X}_i$ denotes the formal completion of X_i along $X' \cap X_i$ then $(t(\hat{X}_i) | i \in I)$ is an open covering of $t(\hat{X})$ and the morphisms $\underline{t(\hat{X}_i)} \to X_i$ just described glue together to a morphism $\rho: \underline{t(\hat{X})} \to X$.

Let $f: X \to Y$ be a morphism between locally noetherian schemes, and let $X' \subseteq X$ and $Y' \subseteq Y$ be closed subsets with $f(X') \subseteq Y'$. Then f induces a morphism $\hat{f}: \hat{X} \to \hat{Y}$ between the formal completions of X and Y along X' and Y'. The diagram

$$(*) \qquad \begin{array}{ccc} \underline{t(\hat{X})} & \xrightarrow{\rho_{\hat{X}}} & X \\ \underline{t(\hat{f})}\downarrow & & \downarrow f \\ \underline{t(\hat{Y})} & \xrightarrow[\rho_{\hat{Y}}]{} & Y \end{array}$$

is commutative. If f is locally of finite type then, by (3.8), $(*)$ induces a morphism of adic spaces

$$\varphi: t(\hat{X}) \to X \times_Y t(\hat{Y}).$$

If $X' = f^{-1}(Y')$ then the following holds.

(a) $\operatorname{im}(\varphi) = \{x \in X \times_Y t(\hat{Y}) | x$ has a specialization y in $X \times_Y t(\hat{Y})$ such that v_y is a trivial valuation $\}$.
(b) φ is a local isomorphism.
(c) If f is separated then φ is an open embedding.
(d) If f is proper then φ is an isomorphism.

5 Mumford's construction of semi-abelian schemes

Starting from a lattice Y in the generic fibre of a split torus $\tilde{G}$ over an adic noetherian integral domain A, Mumford constructs in [M] a semi-abelian group scheme G over A. In this paragraph we interpretate this construction in the category of adic spaces over $\mathrm{Spa}(A, A)$. The adic geometry has two advantages. First, G is not a quotient of $\tilde{G}$ in the category of schemes over $\mathrm{Spec}A$, whereas Y induces in a natural way a closed adic subgroup L of the adic group $\tilde{G}^{\mathrm{ad}}$ associated with $\tilde{G}$ and the adic group G^{ad} associated with G is the quotient of $\tilde{G}^{\mathrm{ad}}$ by L in the category of adic spaces over $\mathrm{Spa}(A, A)$. Secondly, the construction of G is rather complicated, whereas the construction of G^{ad} as the quotient of $\tilde{G}^{\mathrm{ad}}$ by L is very easy.

First we recall the construction of rigid analytic tori in our context of adic spaces. Let R be an adic space. Let Y be an abelian group, and for every $y \in Y$, let U_y be an open subspace of R such that $U_0 = R, U_{-y} = U_y$ and $U_y \cap U_{y'} \subseteq U_{y+y'}$ for all $y, y' \in Y$. Then the direct sum

$$L := \coprod_{y \in Y} U_y$$

is in a natural way an adic group over R. Let $\mathbb{G}_m^r$ be the torus $\mathrm{Spec}\ \mathbb{Z}[T_1, T_1^{-1}, \ldots, T_r, T_r^{-1}]$. Then according to (3.8).

$$\mathbb{G}_{m,R}^r := \mathbb{G}_m^r \times_{\mathrm{Spec}\,\mathbb{Z}} R$$

is an adic group over R. The images of $T_1, \ldots, T_r \in \mathcal{O}_{\mathbb{G}_m^r}(\mathbb{G}_m^r)$ under the ring homomorphism $\mathcal{O}_{\mathbb{G}_m^r}(\mathbb{G}_m^r) \to \mathcal{O}_{\mathbb{G}_{m,R}^r}(\mathbb{G}_{m,R}^r)$ are also denoted by $T_1, \ldots, T_r$. Let $f: L \to \mathbb{G}_{m,R}^r$ be a homomorphism of adic groups over R. If $\mu: \mathbb{G}_{m,R}^r \times_R \mathbb{G}_{m,R}^r \to \mathbb{G}_{m,R}^r$ denotes the multiplication of $\mathbb{G}_{m,R}^r$ then $\mu \circ (f \times \mathrm{id}_{\mathbb{G}_{m,R}^r}): L \times_R \mathbb{G}_{m,R}^r \to \mathbb{G}_{m,R}^r$ is an action of L on $\mathbb{G}_{m,R}^r$.

Proposition 5.1 *Assume that for every quasi-compact open subset U of $\mathbb{G}_{m,R}^r$ the set $\{y \in Y | U_y \cap f^{-1}(U) \neq \emptyset\}$ is finite (for example, this is satisfied if f is quasi-compact) and that Y is torsion-free. Then in the category of adic spaces over R the quotient $\pi: \mathbb{G}_{m,R}^r \to Z$ of $\mathbb{G}_{m,R}^r$ by L exists. π is surjective and a local isomorphism. (Hence Z is locally of finite type over R). There exists a unique adic group structure on Z such that π is a homomorphism of adic groups over R.*

Proof. Let $p: \mathbb{G}_{m,R}^r \to R$ be the structure morphism. For every $y \in Y$, $f|U_y$ is a section of p over U_y. Let $s_y: p^{-1}(U_y) \to p^{-1}(U_y)$ be the translation induced by $f|U_y$. Assume

(1) There exists an open covering $\{V_t | t \in T\}$ of $\mathbb{G}_{m,R}^r$ such that $V_t \cap s_y(V_t \cap p^{-1}(U_y)) = \emptyset$ for every $t \in T$ and every $y \in Y - \{0\}$.

Then one can construct the quotient of $\mathbb{G}_{m,R}^r$ by L as follows. For $t, t' \in T$ and $y \in Y$ put $V_{t,t',y} := V_t \cap s_y^{-1}(V_{t'} \cap p^{-1}(U_y))$. Then by (1), $V_{t,t',y} \cap V_{t,t',y'} = \emptyset$ for $y \neq y'$. We put $V_{t,t'} := \cup_{y \in Y} V_{t,t',y} \subseteq V_t$. Since $s_y(V_{t,t',y}) = V_{t',t,-y}$, the $s_y (y \in Y)$ define an isomorphism $s_{t,t'}: V_{t,t'} \to V_{t',t}$. Let Z be the adic space obtained by glueing together the V_t, $t \in T$ along the isomorphisms $s_{t,t'}$. Since the V_t cover $\mathbb{G}^r_{m,R}$, we have a natural morphism $\pi\colon \mathbb{G}^r_{m,R} \to Z$. It is easily seen that π is the quotient of $\mathbb{G}^r_{m,R}$ by L and that π has the properties stated in (5.1).

So it remains to show (1). Let U be an open affinoid subset of R. Let $H = \{h_1, \ldots, h_n\}$ be a finite set of topologically nilpotent elements of $\mathcal{O}_R(U)$ such that $H \cdot \mathcal{O}_R(U)$ is open in $\mathcal{O}_R(U)$. Put $L = \{1, \ldots, n\} \times \{1, \ldots, r\}$. The open subset $W := \{x \in p^{-1}(U) | v_x(h_i T_j) \leqq 1$ and $v_x(h_i T_j^{-1}) \leqq 1$ for every $(i,j) \in L\}$ of $\mathbb{G}^r_{m,R}$ is quasi-compact. Hence the set $K := \{y \in Y | U_y \cap f^{-1}(W) \neq \emptyset\}$ is finite. Put $k := |K|$ and $V := \{x \in p^{-1}(U) | v_x(h_i T_j^k) \leqq 1$ and $v_x(h_i T_j^{-k}) \leqq 1$ for every $(i,j) \in L\}$. Then we have

(2) $U_y \cap f^{-1}(V) = \emptyset$ for every $y \in Y - \{0\}$.

Indeed, assume $U_y \cap f^{-1}(V) \neq \emptyset$ for some $y \in Y - \{0\}$. Choose a $x \in U_y \cap f^{-1}(V)$. Then for $i = 0, \ldots, k$, $f(ix) = if(x) \in W$ and $ix \in U_{iy}$. Since Y is torsion-free, we obtain $|K| \geqq k + 1$, contradiction.

For every family $A = ((a_{ij}, b_{ij}, c_{ij}, d_{ij}) | (i,j) \in L)$ with $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{N}_0$ and $a_{ij} + c_{ij} = b_{ij} + d_{ij} + 2$ for every $(i,j) \in L$ we put $V_A = \{x \in p^{-1}(U) | v_x(h_i^{a_{ij}} T_j^{2k}) \leqq v_x(h_i^{b_{ij}}) \neq 0$ and $v_x(h_i^{c_{ij}} T_j^{-2k}) \leqq v_x(h_i^{d_{ij}}) \neq 0$ for every $(i,j) \in L\}$. Now (1) follows from the following point

(3) $V_A \cap s_y(V_A \cap p^{-1}(U_y)) = \emptyset$ for every $y \in Y - \{0\}$ and the V_A's cover $p^{-1}(U)$.

Proof of (3) Assume there exists a $y \in Y - \{0\}$ with $V_A \cap s_y(V_A \cap p^{-1}(U_y)) \neq \emptyset$. Let $f_1, \ldots, f_r$ be the units of $\mathcal{O}_R(U_{-y})$ with $s^*_{-y}(T_j) = f_j T_j$ for $j = 1, \ldots, r$. Then $V_A \cap s_y(V_A \cap p^{-1}(U_y)) = V_A \cap s^{-1}_{-y}(V_A \cap p^{-1}(U_{-y})) = \{x \in p^{-1}(U \cap U_{-y}) | v_x(h_i^{a_{ij}} T_j^{2k}) \leqq v_x(h_i^{b_{ij}}) \neq 0, v_x(h_i^{c_{ij}} T_j^{-2k}) \leqq v_x(h_i^{d_{ij}}) \neq 0, v_x(h_i^{a_{ij}} f_j^{2k} T_j^{2k}) \leqq v_x(h_i^{b_{ij}}) \neq 0, v_x(h_i^{c_{ij}} f_j^{-2k} T_j^{-2k}) \leqq v_x(h_i^{d_{ij}}) \neq 0$ for every $(i,j) \in L\}$. Let z be an element of $V_A \cap s_y(V_A \cap p^{-1}(U_y))$. Since $a_{ij} + c_{ij} = b_{ij} + d_{ij} + 2$ for every $(i,j) \in L$, we conclude $v_z(h_i f_j^k) \leqq 1$ and $v_z(h_i f_j^{-k}) \leqq 1$ for every $(i,j) \in L$ which means $p(z) \in U_{-y} \cap f^{-1}(V)$, in contradiction to (2).

Let $x \in p^{-1}(U)$ be given. We look for a family A with $x \in V_A$. Let $i \in \{1, \ldots, n\}$ be fixed. If $v_x(h_i) = 0$ then we put $a_{ij} = c_{ij} = 1$ and $b_{ij} = d_{ij} = 0$ for $j \in \{1, \ldots, r\}$. If $v_x(h_i) \neq 0$ then we choose, for every $j \in \{1, \ldots, r\}$, such $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{N}_0$ that $a_{ij} + c_{ij} = b_{ij} + d_{ij} + 2$ and $v_x(h_i)^{c_{ij} - d_{ij}} \leqq v_x(T_j^{2k}) \leqq v_x(h_i)^{b_{ij} - a_{ij}}$. (This is possible since $v_x(T_j^{2k}) \neq 0$ and $v_x(h_i)$ is cofinal in the value group of v_x.) Then for this $A = ((a_{ij}, b_{ij}, c_{ij}, d_{ij}) | (i,j) \in L)$ we have $x \in V_A$. □

Now we specify R, Y, L, f. Let A be a noetherian normal integral domain which is complete with respect to an adic topology, and let K be the quotient field of A. S denotes the affine scheme Spec$A$ and $\tilde{G}$ the torus $\mathbb{G}^r_m \times_{\mathrm{Spec}\mathbb{Z}} S = \mathrm{Spec}A[T_1, T_1^{-1}, \ldots, T_r, T_r^{-1}]$. Let $Y \subseteq \tilde{G}(K) = (K^*)^r$ be a torsion-free finitely generated subgroup of rank r which admits a polarization in the sense of [M, 1.2]. To $\tilde{G}$ and Y Mumford constructs in [M] a semi-abelian scheme G over Spec$A$.

For every $y = (y_1, \ldots, y_r) \in Y$ we put $V_y = \{\mathfrak{p} \in \operatorname{Spec} A | y_i \in (A_\mathfrak{p})^* \text{ for } i = 1, \ldots, r\}$. Then V_y is an open subset of $\operatorname{Spec} A$, and $V_0 = S, V_y = V_{-y}$ and $V_y \cap V_{y'} \subseteq V_{y+y'}$ for all, $y, y' \in Y$. Hence the direct sum

$$H := \coprod_{y \in Y} V_y$$

is in a natural way a group over S. For every $y = (y_1, \ldots, y_r) \in Y$, let $g_y : V_y \to \tilde{G}$ be the S-morphism with $g_y^*(T_i) = y_i \in \mathcal{O}_S(V_y)$ for $i = 1, \ldots, r$. Then

$$g := \coprod_{y \in Y} g_y : H \to \tilde{G}$$

is a homomorphism of groups over S.

Let R denote the adic space $\operatorname{Spa}(A, A)$. The identity $A \to A$ induces a morphism of locally ring spaces $\varphi : \underline{R} \to S$. By (3.8), φ induces the functor $X \mapsto X \times_S R$ from the category of schemes locally of finite type over S to the category of adic spaces locally of finite type over R. According to (4.6.i) this functor corresponds to the functor "associated rigid analytic variety" from the category of schemes locally of finite type over a non-archimedean field k to the category of rigid analytic varieties over k. So we call, for every scheme X locally of finite type over $S, X \times_S R$ the associated adic space and denote it by X^{ad}.

$$\begin{aligned}
\tilde{G}^{\mathrm{ad}} &:= \tilde{G} \times_S R = \mathbb{G}^r_{m,R} \\
G^{\mathrm{ad}} &:= G \times_S R \\
H^{\mathrm{ad}} &:= H \times_S R = \coprod_{y \in Y} \varphi^{-1}(V_y) \\
g^{\mathrm{ad}} &:= g_{(R)} : H^{\mathrm{ad}} \to \tilde{G}^{\mathrm{ad}} .
\end{aligned}$$

By the universal property in (3.8), $\tilde{G}^{\mathrm{ad}}, G^{\mathrm{ad}}$ and H^{ad} are adic groups over R and g^{ad} is a homomorphism of adic groups over $\underline{R}$. Although one considers G as the 'quotient' of $\tilde{G}$ by Y, G is not a quotient of $\tilde{G}$ in the category of schemes over S, nor does there exist a nontrivial morphism $\tilde{G} \to G$ of schemes over S. But in the category of adic spaces the situation is much better. We have the following result.

Theorem 5.2 (i) *There is a natural homomorphism $\lambda : \tilde{G}^{\mathrm{ad}} \to G^{\mathrm{ad}}$ of adic groups over R.*
(ii) *g^{ad} is quasi-compact. Hence by* (5.1), *in the category of adic spaces over R the quotient $\pi : \tilde{G}^{\mathrm{ad}} \to Z$ of $\tilde{G}^{\mathrm{ad}}$ by H^{ad} exists, and Z is in a natural way an adic group over R.*
(iii) *There exists a unique morphism of adic spaces $\mu : Z \to G^{\mathrm{ad}}$ such that the diagram*

$$\begin{array}{ccc}
 & \tilde{G}^{\mathrm{ad}} & \\
{}^{\pi}\swarrow & & \searrow^{\lambda} \\
Z & \xrightarrow[\mu]{} & G^{\mathrm{ad}}
\end{array}$$

commutes. μ is an isomorphism of adic groups over R.

Remark 5.3 (i) Let $p:(\tilde{G}^{\text{ad}})_- = (\tilde{G} \times_S R)_- \to \tilde{G}$ be the projection, and, for every $y \in Y \subseteq \tilde{G}(K) \subseteq \tilde{G}$, let $\overline{\{y\}}$ be the closure of $\{y\}$ in $\tilde{G}$. Then $\cup_{y\in Y}\overline{\{y\}}$ is not closed in $\tilde{G}$ but $p^{-1}(\cup_{y\in Y}\overline{\{y\}})$ is closed in $\tilde{G}^{\text{ad}}$, and $p^{-1}(\cup_{y\in Y}\overline{\{y\}})$ is the image of the mapping $g^{\text{ad}}: H^{\text{ad}} \to \tilde{G}^{\text{ad}}$.
(ii) Using sheaves of ideals of the structure sheaves, one can define in the usual way closed subspaces of adic spaces (cf. [H2]). In the situation of (5.1), the morphism $f: L \to \mathbb{G}^r_{m,R}$ is a locally closed embedding, i.e., f factors through an isomorphism onto a closed subspace of an open subspace of $\mathbb{G}^r_{m,R}$. Then by (i) and (5.2.ii), $g^{\text{ad}}: H^{\text{ad}} \to \tilde{G}^{\text{ad}}$ is a closed embedding, and so we can consider H^{ad} as a closed adic subgroup of $\tilde{G}^{\text{ad}}$. Then (5.2) says that G^{ad} is the quotient of $\tilde{G}^{\text{ad}}$ by a closed adic subgroups of $\tilde{G}^{\text{ad}}$.

Proof. Let X be the character group of $\tilde{G}$. Since we have already introduced the coordinates $T_1, \ldots, T_r$ of $\tilde{G}$, we identify X with $\mathbb{Z}^r$. For $y = (y_1, \ldots, y_r) \in Y \subseteq (K^*)^r$ and $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r$ we put $y^n = y_1^{n_1} \cdot \ldots \cdot y_r^{n_r} \in K^*$ and $T^n := T_1^{n_1} \cdot \ldots \cdot T_r^{n_r} \in \mathcal{O}_{\tilde{G}}(\tilde{G})$. Let $\phi: Y \to X$ be a polarization of Y.

(ii) and (5.3.i) follow from the following two points.

(a) For every $y \in Y, g_y: V_y \to \tilde{G}$ is a closed embedding.
(b) For every quasi-compact set U of $\tilde{G}^{\text{ad}}$, the set $\{y \in Y \mid \varphi^{-1}(V_y) \cap (g^{\text{ad}})^{-1}(U) \neq \emptyset\}$ is finite.

to (a): Let $y \in Y$ be given. By definition of ϕ, we have $h := y^{\phi(y)} \in A$. By [M, 1.4], $y^\alpha \in A_h$ for every $\alpha \in \mathbb{Z}^r$. In particular, for $\alpha = e_i = (0, \ldots, 1, \ldots, 0)$ we obtain $y_i = y^{e_i} \in A_h$ and $y_i^{-1} = y^{-e_i} \in A_h$, hence $y_i \in (A_h)^*$. This shows $D(h) \subseteq V_y$. As a section of the structure morphism $l: \tilde{G} \to S$ over V_y, g_y is a closed embedding into $l^{-1}(V_y)$. It remains to show that $g_y(V_y)$ is closed in $\tilde{G}$. As $D(y^{\phi(y)}) \subseteq V_y$, the zero set $V(T^{\phi(y)} - y^{\phi(y)}) \subseteq \tilde{G}$ of the global function $T^{\phi(y)} - y^{\phi(y)} \in \mathcal{O}_{\tilde{G}}(\tilde{G})$ is contained in $l^{-1}(V_y)$. Since $g_y(V_y) \subseteq V(T^{\phi(y)} - y^{\phi(y)})$, we conclude that $g_y(V_y)$ is closed in $\tilde{G}$.

to (b): Let U be a quasi-compact subset of $\tilde{G}^{\text{ad}}$. Let $a_1, \ldots, a_n$ be a set of generators of the ideal $A^{\circ\circ}$ of A. We choose a $k \in \mathbb{N}$ such that $U \subseteq V := \{x \in \tilde{G}^{\text{ad}} | v_x(a_i^k T_j) \leqq 1$ and $v_x(a_i^k T_j^{-1}) \leqq 1$ for every $(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, r\}\}$, and for every $y \in Y$ we choose a $n_y \in \mathbb{N}$ with $n_y > t \cdot r \cdot k$, where t is the maximum of the absolute values of the components of $\phi(y) \in \mathbb{Z}^r$. By [M, 1.3] there exists a finite subset Q of Y such that for every $z \in Y - Q$ there is a $y \in Y - \{0\}$ with $z^{\phi(y)} \in (y^{\phi(y)})^{n_y} \cdot A$. Since $y^{\phi(y)} \in A^{\circ\circ}$ for every $y \in Y - \{0\}$, we obtain, for every $z \in Y - Q$ and every $x \in \varphi^{-1}(V_z), v_x(z^{\phi(y)}) \leqq (\max\{v_x(a_1), \ldots, v_x(a_k)\})^{n_y}$ which implies min $\left\{v_x(z_j), \frac{1}{v_x(z_j)}\right\} < (\max\{v_x(a_1), \ldots, v_x(a_n)\})^k$ for some $j \in \{1, \ldots, r\}$. Hence $\varphi^{-1}(V_z) \cap (g^{\text{ad}})^{-1}(V) = \emptyset$ for every $z \in Y - Q$.

We sketch the proof of (i) and (iii). First we introduce a notation. If $g: M \to N$ is a morphism of ringed spaces and $I \subseteq \mathcal{O}_N$ is a sheaf of ideals then we put $\bar{g}(I) := \text{im}(g^*(I) \to \mathcal{O}_M)$ and $V(I) := \{x \in N \mid I_x \neq \mathcal{O}_{N,x}\}$. We briefly remind of the construction of G. Let $\tilde{P} \to S$ be a relatively complete model of $\tilde{G}$ with respect to the periods Y and the polarization ϕ [M, 2.1]. We assume that $\tilde{P}$ is separated. For every $y \in Y$, one has a S-morphism, $S_y: \tilde{P} \to \tilde{P}$ such that $y \mapsto S_y$ is an action of Y on $\tilde{P}$. Let $\tilde{\mathfrak{B}}$ be the formal completion of $\tilde{P}$ along $A^{\circ\circ}$. The action of Y on $\tilde{P}$ induces an action of Y on $\tilde{\mathfrak{B}}$. Let $h: \tilde{\mathfrak{B}} \to \mathfrak{B}$ be the quotient of $\tilde{\mathfrak{B}}$ by Y in the category of formal schemes over Spf A as constructed in [M], and let P be the projective S-scheme whose formal completion along $A^{\circ\circ}$ is equal to $\mathfrak{B}$. Let $a: \tilde{\mathfrak{B}} \to \tilde{P}$ and $b: \mathfrak{B} \to P$ be the natural morphisms of locally ringed spaces. $\tilde{G}$ is an open subspace of $\tilde{P}$. Let $I \subseteq \mathcal{O}_P$ be the coherent sheaf of ideals such that $\bar{h}(\bar{b}(I)) = \bar{a}(J) \subseteq \mathcal{O}_{\tilde{\mathfrak{B}}}$, where $J \subseteq \mathcal{O}_{\tilde{P}}$ is the sheaf of functions which vanish on the closed subset $\tilde{P} - \cup_{y \in Y} S_y(\tilde{G})$ of $\tilde{P}$. Then G is the open subspace $P - V(I)$ of P.

By (4.1), we have a morphism of locally ringed spaces $\pi: (t(\mathfrak{B}), \mathcal{O}^+_{t(\mathfrak{B})}) \to (\mathfrak{B}, \mathcal{O}_{\mathfrak{B}})$. Composing π with the natural morphism of ringed spaces $\underline{t(\mathfrak{B})} = (t(\mathfrak{B}), \mathcal{O}_{t(\mathfrak{B})}) \to (t(\mathfrak{B}), \mathcal{O}^+_{t(\mathfrak{B})})$, we obtain a morphism of ringed spaces $e: \underline{t(\mathfrak{B})} \to \mathfrak{B}$. We put $\mathcal{J} := \bar{e}(\bar{b}(I)) \subseteq \mathcal{O}_{t(\mathfrak{B})}$. By (4.6iv), we have natural morphisms $c: t(\tilde{\mathfrak{B}}) \to \tilde{P}^{\mathrm{ad}}$ and $d: t(\mathfrak{B}) \to P^{\mathrm{ad}}$ of adic spaces over R. For every $y \in Y$, the morphism $S_y: \tilde{P} \to \tilde{P}$ induces morphisms $T_y: t(\tilde{\mathfrak{B}}) \to t(\tilde{\mathfrak{B}})$ and $U_y: \tilde{P}^{\mathrm{ad}} \to \tilde{P}^{\mathrm{ad}}$. Then $y \mapsto T_y$ and $y \mapsto U_y$ are actions of Y on $t(\tilde{\mathfrak{B}})$ and $\tilde{P}^{\mathrm{ad}}$. Since $\tilde{G}$ and G are open subspaces of $\tilde{P}$ and P, we consider $\tilde{G}^{\mathrm{ad}}$ and G^{ad} as open subspaces of $\tilde{P}^{\mathrm{ad}}$ and P^{ad}. One can show

(1) (a) The morphism $t(h): t(\tilde{\mathfrak{B}}) \to t(\mathfrak{B})$ is the quotient of $t(\tilde{\mathfrak{B}})$ by Y in the category of adic spaces.

(b) The morphism $c: t(\tilde{\mathfrak{B}}) \to \tilde{P}^{\mathrm{ad}}$ is Y-equivariant.

(c) The morphism $d: t(\mathfrak{B}) \to P^{\mathrm{ad}}$ induces by restriction an isomorphism d': $t(\mathfrak{B}) - V(\mathcal{J}) \to G^{\mathrm{ad}}$.

(d) The morphism $c: t(\tilde{\mathfrak{B}}) \to \tilde{P}^{\mathrm{ad}}$ induces by restriction an isomorphism $c': t(\tilde{\mathfrak{B}}) - t(h)^{-1}(V(\mathcal{J})) \to \cup_{y \in Y} T_y(\tilde{G}^{\mathrm{ad}})$.

By (1.c) and (1.d) there exists a morphism $\sigma: \cup_{y \in Y} T_y(\tilde{G}^{\mathrm{ad}}) \to G^{\mathrm{ad}}$ such that the diagram

$$\begin{array}{ccc} t(\tilde{\mathfrak{B}}) - t(h)^{-1}(V(\mathcal{J})) & \xrightarrow[\sim]{c'} & \cup_{y \in Y} T_y(\tilde{G}^{\mathrm{ad}}) \\ {\scriptstyle t(h)} \downarrow & & \downarrow {\scriptstyle \sigma} \\ t(\mathfrak{B}) - V(\mathcal{J}) & \xrightarrow[d']{\sim} & G^{\mathrm{ad}} \end{array}$$

commutes. We put $\lambda := \sigma \mid \tilde{G}^{\mathrm{ad}}: \tilde{G}^{\mathrm{ad}} \to G^{\mathrm{ad}}$. By (1.a) and (1.b), σ is the quotient of $\cup_{y \in Y} T_y(\tilde{G}^{\mathrm{ad}})$ by Y. From this one can deduce that λ is the quotient of $\tilde{G}^{\mathrm{ad}}$ by H^{ad}. Hence there exists an isomorphism $\mu: Z \to G^{\mathrm{ad}}$ of adic spaces over R with $\lambda = \mu \circ \pi$. (Since π is a local isomorphism, there exists at most one morphism of adic spaces $\tau: Z \to G^{\mathrm{ad}}$ with $\lambda = \tau \circ \pi$.) It remains to show that λ is a homomorphism of adic groups over R. For that we have to analyze the construction of the group structure of G in [M]. Let $\tilde{G}^{\wedge}$ and $G^{\wedge}$ be the formal completions of $\tilde{G}$ and G along $A^{\circ\circ}$. Mumford's construction of G gives a natural isomorphism of formal schemes $\rho: \tilde{G}^{\wedge} \to G^{\wedge}$. By (4.6.iv) we have open embeddings $m: t(\tilde{G}^{\wedge}) \to \tilde{G}^{\mathrm{ad}}$ and $n: t(G^{\wedge}) \to G^{\mathrm{ad}}$. One can show

(2) The diagram

$$\begin{array}{ccc} t(\tilde{G}^{\wedge}) & \xrightarrow{m} & \tilde{G}^{\mathrm{ad}} \\ {\scriptstyle t(\rho)}\downarrow & & \downarrow{\scriptstyle \lambda} \\ t(G^{\wedge}) & \xrightarrow[n]{} & G^{\mathrm{ad}} \end{array}$$

commutes.

Now let $(\tilde{G}_i, Y_i, \phi_i, \tilde{P}_i), i = 1,2$ be two tori over S together with groups of periods, polarizations and relatively complete models. Let G_i be the associated semi-abelian schemes over S, and let $\lambda_i: \tilde{G}_i^{\mathrm{ad}} \to G_i^{\mathrm{ad}}$ be the morphisms of adic spaces as constructed above. Let $\tilde{\alpha}: \tilde{G}_1 \to \tilde{G}_2$ be a homomorphism of groups over S with $\tilde{\alpha}(Y_1) \subseteq Y_2$. Then by [M, 4.6] there exists a S-morphism $\alpha: G_1 \to G_2$ such that the diagram

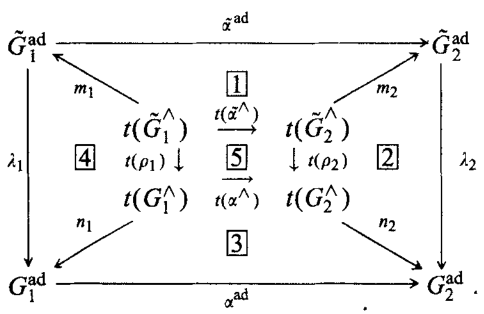
(3)
$$\begin{array}{ccc} \tilde{G}_1^{\wedge} & \xrightarrow{\tilde{\alpha}^{\wedge}} & \tilde{G}_2^{\wedge} \\ {\scriptstyle \rho_1}\downarrow & & \downarrow{\scriptstyle \rho_2} \\ G_1^{\wedge} & \xrightarrow[\alpha^{\wedge}]{} & G_2^{\wedge} \end{array}$$

commutes. This implies
(4) The diagram

$$\begin{array}{ccc} \tilde{G}_1^{\mathrm{ad}} & \xrightarrow{\tilde{\alpha}^{\mathrm{ad}}} & \tilde{G}_2^{\mathrm{ad}} \\ {\scriptstyle \lambda_1}\downarrow & & \downarrow{\scriptstyle \lambda_2} \\ G_1^{\mathrm{ad}} & \xrightarrow[\alpha^{\mathrm{ad}}]{} & G_2^{\mathrm{ad}} \end{array}$$

commutes.

Proof. We consider the diagram



The diagrams 1 and 3 commute by functoriality, 2 and 4 commute by (2), and 5 commutes by (3). Then the identity principle for adic morphisms imply $\lambda_2 \circ \tilde{\alpha}^{\mathrm{ad}} = \alpha^{\mathrm{ad}} \circ \lambda_1$.

(4) together with the construction of the group structure of G in [M, 4.8] shows that $\lambda: \tilde{G}^{\mathrm{ad}} \to G^{\mathrm{ad}}$ is a homomorphism of adic groups over R. □

References

[B1] Bourbaki, N.: Topological vectorspaces. Berlin Heidelberg New York: Springer 1987

[B2] Bourbaki, N.: Commutative Algebra. Paris: Hermann 1972

[BGR] Bosch, S., Güntzer, U., Remmert, R.: Non-archimedean analysis. Berlin Heidelberg New York: Springer 1984

[EGA] Grothendieck, A., Dieudonne, J.: Eléments de Géométrie Algébrique III. Publ. Math., Inst. Hautes Études Sci. **20** (1964)
[EGA*] Grothendieck, A., Dieudonne, J.: Eléments de Géométrie Algébrique I. Berlin Heidelberg New York: Springer 1971
[FP] Fresnel, J., Put, M. van der: Géométrie analytique rigide et applications. Boston Basel: Birkhäuser 1981
[G] Grothendieck, A.: Sur quelques points d'algèbre homologique. Tôhoku Math. J. **9**, 119–221 (1957)
[H1] Huber, R.: Continuous valuations. Mathematische Z. **212**, 455–477 (1993)
[H2] Huber, R.: Coherent sheaves on adic spaces. (in preparation)
[H3] Huber, R.: Étale cohomology of rigid analytic varieties and adic spaces. (pre-print)
[M] Mumford, D.: An analytic construction of degenerating abelian varieties over complete rings. Compos. Math. **24**, 239–272 (1972)
[R] Raynaud, M.: Géométrie analytique rigide d'apres Tate, Kiehl, ... Mém. Soc. Math. Fr. **39–40**, 319–327 (1974)
[SGA] Artin, M., Grothendieck, A., Verdier, J.L.: Théorie des Topos et Cohomologie Étale des Schémas. Berlin Heidelberg New York: Springer 1972
[T] Tate, J.: Rigid analytic spaces. Invent. Math. **12**, 257–289 (1971)