8 Lecture 8: More properties for Cont(A)

8.1 Introduction

The aim of this lecture is to start the proof of the crucial result that Cont(A), the continuous valuation spectrum of a Huber ring A, is a spectral space (and hence quasi-compact). Quasi-compactness of affinoid adic spaces will be a consequence of this fact.

8.2 Preliminaries on continuous valuations

We begin by addressing some loose ends concerning specialization within the set of continuous valuations.

Theorem 8.2.1 *Let A be a non-archimedean ring. If $v \in \mathrm{Cont}(A)$ then every horizontal specialization of v is continuous, as is every non-trivial vertical generization.*

We will only need the horizontal case in what follows. The non-triviality condition in the vertical case is unavoidable since every v admits the trivial valuation on $\mathrm{Frac}(A/\mathfrak{p}_v)$ as a vertical generization but this is continuous if and only if its support $\mathfrak{p}_v$ is open, which is to say if and only if v is a non-analytic point (see Definition 8.3.1 for a reminder of the notion of analyticity). For Tate rings we will see in Corollary 8.3.3 that *all* points in Cont(A) are analytic.

Proof. Consider a *continuous* valuation $v : A \to \Gamma_v \cup \{0\}$, and let $H \subset \Gamma_v$ be a convex subgroup.

Step 1: horizontal specialization. Assume $v(A)_{\leq 1} \subseteq H$, so as explained in Proposition 4.3.5 and the subsequent discussion we know that $v_{|_H} : A \to H \cup \{0\}$ given by

$$v_{|_H}(a) := \begin{cases} v(a), & v(a) \in H \\ 0 & \text{otherwise} \end{cases}$$

is a valuation on A. Let $w := v_{|_H}$; this is the general recipe for horizontal specializations of v.

Since $w(a) \leq v(a)$ for all $a \in A$, given any $\gamma \in H \subset \Gamma_v$ we have:

$$\{a \in A \mid v(a) < \gamma\} \subset \{a \in A \mid w(a) < \gamma\}$$

implying that the latter additive subgroup of A is open, so w is continuous.

Step 2: vertical generization. Notation being as in Definition 4.2.2, define $w := v_{/_H}$; this is a vertical generization of v (and w is non-trivial if and only if $H \neq \Gamma_v$ since $v(A - \mathfrak{p}_v)$ generates Γ_v). By construction, w is given by the composition

$$A \xrightarrow{v} \Gamma_v \cup \{0\} \longrightarrow (\Gamma_v/H) \cup \{0\}, \qquad v_{/_H} : A \to (\Gamma_v/H) \cup \{0\}$$

For any $\gamma \in \Gamma_v$ we have

$$\{a \in A \mid w(a) \leq \gamma H/H\} \supset \{a \in A \mid v(a) \leq \gamma\},$$

so the former is open as an additive subgroup of A. Hence, w is continuous *provided* that it is non-trivial (see Remark 5.3.2). $\square$

In general $\mathrm{Cont}(A)$ can fail to be stable under horizontal generization. We give an example using non-trivial valuations:

Example 8.2.2 For a field k let $A = k[x,y]$ equipped with the y-adic topology and let w be the valuation on A arising from the rank-2 valuation ring $k[y]_{(y)} + xA_{(x)}$ (i.e., the valuation arising from the y-adic discrete valuation on the residue field $k(y)$ of the discrete x-adic valuation on $K = \mathrm{Frac}(A)$). Then w is not continuous since the topologically nilpotent element y in A has value $w(y)$ that is not cofinal in the totally ordered group $\Gamma_w = x^{\mathbf{Z}} \times y^{\mathbf{Z}}$ (with lexicographical order); see Definition 8.4.5. Since $c\Gamma_w = \{1\}$, for *any* convex subgroup $H \subset \Gamma_w$ we can form the horizontal specialization $w_{|_H}$ valued in H. The subgroup $H = y^{\mathbf{Z}}$ is convex in Γ_w and the horizontal specialization $w_{|_H}$ corresponds to the y-adic discrete valuation on $A/xA = k[y]$ that *is* continuous since A is equipped with the y-adic topology (ensuring that small neighborhoods of 0 in A do reduce into small neighborhoods of 0 in $A/xA = k[y]$). Thus, the continuous $w_{|_H}$ has w as a non-continuous horizontal generization.

8.3 Analytic points: continuation

We now fulfill the promise made in Remark 6.2.2 to show that within the set of *analytic* points in $\mathrm{Cont}(A)$ for a Huber ring A, all specializations are vertical. Let's first recall the definition of analyticity:

Definition 8.3.1 Let A be a Huber ring. A point $v \in \mathrm{Cont}(A)$ is called an *analytic point* if its support $\mathfrak{p}_v \in \mathrm{Spec}(A)$ is not an open ideal of A. We denote the subspace of analytic points of $\mathrm{Cont}(A)$ by $\mathrm{Cont}(A)_{\mathrm{an}}$, endowed with the subspace topology.

We shall keep calling $\mathfrak{p}_v$ the support of a point $v \in \mathrm{Spv}(A)$, as usual. (The definition of analyticity can be made without requiring v to be continuous, but it is only of interest to us for continuous v.) An equivalent characterization of analyticity of v is that the trivial valuation on $A/\mathfrak{p}_v$ is *not* continuous since the trivial valuation with a given support is continuous if and only if its "open unit disc" is open yet that disc is exactly its support. Also, if $\mathfrak{p}$ is an open prime ideal then *all* valuations on A with support $\mathfrak{p}$ are continuous since $A/\mathfrak{p}$ is discrete.

Proposition 8.3.2 *Let A be a Huber ring, and choose a couple of definition (A_0, I). The point $v \in \mathrm{Spv}(A)$ is analytic if and only if $v(I) \neq 0$.*

Proof. The support $\mathfrak{p}_v$ of v is open in A if and only if it contains I^n for sufficiently large $n > 0$. But $\mathfrak{p}_v$ is radical, so this is equivalent to saying $\mathfrak{p}_v$ contains I, which means exactly $v(I) = 0$. □

Corollary 8.3.3 *Let A be a Tate ring. Then*

$$\mathrm{Cont}(A) = \mathrm{Cont}(A)_{\mathrm{an}}.$$

Proof. Let $u \in A$ be a topologically nilpotent unit. Then $v \in \mathrm{Spv}(A)$ cannot kill u, although $u^n \in I$ for all sufficiently large positive integers n, which implies $v(I) \neq 0$. By Proposition 8.3.2, we are done. □

Remark 8.3.4 The reader should now turn back to Example 8.2.2, and review it in light of Corollary 8.3.3.

Let A^{00} be the subset of A of topologically nilpotent elements. Beware that this is *not* an ideal of A; just consider the case $A = \mathbf{Q}_p$ (for which $A^{00} = p\mathbf{Z}_p$).

Observe that $v \in \mathrm{Spv}(A)$ is analytic if and only if there exists some $a \in A^{00}$ such that $v(a) \neq 0$. In other words, analyticity of points in the valuation spectrum of A can be checked on topologically nilpotent elements in A. This characterization is an easy consequence of the fact that for any (A_0, I) as usual, an element $a \in A$ is topologically nilpotent if and only if for sufficiently large positive integer n, we have $a^n \in A_0$ with nilpotent image in A_0/I.

Since topologically nilpotent elements are power-bounded, it follows that if a is topologically nilpotent then the open subring $A_0[a]$ is bounded and hence is a ring of definition. Thus, any two topologically nilpotent elements $a, b \in A$ lie in a common ring of definition A_1, with a^n and b^n lying in a finitely generated ideal of definition I_1 of A_1 for all large n. It then following easily that $(a+b)^m \in A_1$ for all large m and that for any $m > 0$ we have $(a+b)^e \in I_1^m$ for all large e. Hence, A^{00} is at least an additive subgroup of A, and it is also an A_0-submodule for any ring of definition A_0 of A. (Indeed, if $a \in A^{00}$ and $x \in A_0$ then for any $m > 0$ we have $a^e \in I^m$ for all large e, so $(ax)^e \in I^m x^e \subseteq I^m$ for all large e, so $ax \in A^{00}$ too.)

Proposition 8.3.5 *Let A be a Huber ring. For $a \in A$, if $a^n \in A^{00}$ for some $n > 0$ then $a \in A^{00}$.*

Proof. Pick a pair (A_0, I) as usual, and fix the choice of n. For each $m > 0$, if e is large (depending on m) then $(a^n)^e \in I^m$. Any large N has the form $ne + r$ with $0 \leq r < n$ and e large, so $a^N = a^r(a^n)^e \in a^r I^m$. Since r ranges through just finitely many values depending only on n (not on e or m), continuity of multiplication on A ensures that for any $m' > 0$ we have $a^r I^m \subset I^{m'}$ for all large m and any $0 \leq r < n$. Hence, $a \in A^{00}$. □

Structure of $\mathrm{Cont}(A)_{\mathrm{an}}$ inside $\mathrm{Cont}(A)$

Let A be a Huber ring. Fix a point v in $\mathrm{Cont}(A)$. We choose a couple of definition (A_0, I), and let

$$\{f_1, \ldots, f_n\}$$

be generators of I as an ideal of A_0. Assume v is analytic. Then we have

$$\mathfrak{p}_v \not\supseteq I = \sum_{i=1}^{n} A_0 f_i$$

by Proposition 8.3.2. This means that for some $i = 1, \ldots, n$, we must have $v(f_i) \neq 0$. Choose i_0 among such indexes such that $v(f_{i_0}) \in \Gamma_v$ is maximal. It follows

$$v(f_j) \leq v(f_{i_0}) \neq 0, \quad \text{for all } j.$$

We now let

$$T := T_1 = \{f_1, \ldots, f_n\},$$

so $T \cdot I^m$ is open for all m (as it is just I^{m+1}). The reader may wish to look back at Definition 6.3.3 and the discussion thereof.

By Proposition 7.4.2 (see also the discussion preceding it, as well as Proposition 6.3.5), we can endow the algebraic localization $A_{f_{i_0}}$ with a Huber ring structure given by

$$A\left(\frac{T}{f_{i_0}}\right) =: A\left(\frac{f_1}{f_{i_0}}, \ldots, \frac{f_n}{f_{i_0}}\right).$$

For all indexes i such that $v(f_i) \neq 0$, we have a natural map $A \to A(T/f_{i_0})$ which induces a commutative diagram of continuous maps:

$$\begin{array}{ccc} \mathrm{Cont}\left(A\left(\frac{T}{f_i}\right)\right) & \longrightarrow & \mathrm{Cont}(A) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spv}\left(A\left(\frac{T}{f_i}\right)\right) & \longrightarrow & \mathrm{Spv}\,(A) \end{array}$$

Let now v be a valuation on A which factors through $A_{f_{i_0}}$ *continuously.* In other words, v fits into the following commutative diagram of continuous maps, $\Gamma_v \cup \{0\}$ being endowed with the order topology:

$$\begin{array}{ccc} A & \xrightarrow{\ v\ } & \Gamma_v \cup \{0\} \\ \big\downarrow & \nearrow_{v_{i_0}} & \\ A\left(\frac{T}{f_{i_0}}\right) & & \end{array}$$

Keeping in mind the universal mapping property of $A(T/f_{i_0})$, as discussed in Proposition 7.4.2, continuity of v_{i_0} is *equivalent* to having

$$v(f_i/f_{i_0}) \leq 1 \quad \text{for all } i = 1, \ldots, n.$$

The nature of such v's can be translated into the following:

Proposition 8.3.6 *The image of the natural continuous map*

$$\mathrm{Cont}\left(A\left(\frac{T}{f_{i_0}}\right)\right) \to \mathrm{Cont}(A)$$

is the subspace of $\mathrm{Cont}(A)_{\mathrm{an}}$ *defined by*

$$\{v \in \mathrm{Cont}(A)_{\mathrm{an}} \mid v(f_j) \leq v(f_{i_0}) \text{ for all } j\}.$$

Remark 8.3.7 We remark once more that assuming v was an analytic point in $\mathrm{Cont}(A)$ was truly essential, as we need to have $v(f_{i_0}) \neq 0$. Moreover, f_j/f_{i_0} is seen as an element of the algebraic localization $A_{f_{i_0}}$, and hence as an "actual quotient", so $v(f_j/f_{i_0}) \leq 1$ if and only if $v(f_j) \leq v(f_{i_0}) \neq 0$.

In the above discussion, we fixed an analytic point $v \in \mathrm{Cont}(A)$, and chose i_0 from a subset of those indexes in $\{1, \ldots, n\}$ with the property that $v(f_{i_0})$ was maximal in $\Gamma_v \cup \{0\}$. Given this setup, we characterized the image of the natural map $\mathrm{Cont}\left(A\left(\frac{T}{f_{i_0}}\right)\right) \to \mathrm{Cont}(A)$. We can, instead, fix an index i, and given such characterization of the image of the above natural map, consider the subspace of $\mathrm{Cont}(A)_{\mathrm{an}}$:

$$\{v \in \mathrm{Cont}(A)_{\mathrm{an}} \mid v(f_j) \leq v(f_i)\}.$$

This is the image of

$$\mathrm{Cont}\left(A\left(\frac{T}{f_i}\right)\right) \to \mathrm{Cont}(A).$$

(This subspace of $\mathrm{Spv}(A)$ could of course be empty.) The upshot of Proposition 8.3.6 is:

Proposition 8.3.8 *Let A be a Huber ring, and $f_1, \ldots, f_n$ and T as before. We have:*

$$\operatorname{Cont}(A)_{\mathrm{an}} = \bigcup_{i=1}^{n} \operatorname{image}\left(\operatorname{Cont}\left(A\left(\frac{T}{f_i}\right)\right)\right)$$

where we have

$$\operatorname{image}\left(\operatorname{Cont}\left(A\left(\frac{T}{f_i}\right)\right)\right) = \operatorname{Spv}(A) \cap \operatorname{Spv}\left(A\left(\frac{T}{f_i}\right)\right).$$

Remark 8.3.9 We observe that $A\left(\frac{T}{f_i}\right)$ is Tate! Indeed, a ring of definition is given by the A_0-algebra generated by the quotients f_j/f_i inside $(A_0)_{f_i}$. Denote this A_0-algebra as $A_0[T/f_i]$. It is endowed with the $I \cdot A_0[T/f_i]$-adic topology, which coincides with the f_i-adic topology since $f_j = (f_j/f_i)f_i$ in $A_0[T/f_i]$. Informally, analytic points in $\operatorname{Cont}(A)$ are those points having a "neighbourhood" which is Tate. (This description of analytic points will later take on a more substantive meaning for affinoid adic spaces.)

Specializations among analytic points

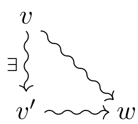
An important property of analytic points is that the only possible specializations (and so equivalently generizations) among them are vertical! One situation where this can be quite effective is for proving openness results via the criterion that a constructible subset of a spectral space is open if and only if it is stable under generization.

Proposition 8.3.10 *Let A be a Huber ring. Then all specializations inside* $\operatorname{Cont}(A)_{\mathrm{an}}$ *are* vertical. *In particular, if A is a Tate ring then all specializations in* $\operatorname{Cont}(A)$ *are vertical.*

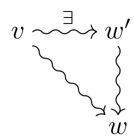
The conclusion for Tate rings follows from the rest by Corollary 8.3.3. The first part of Proposition 8.3.10 rests on an earlier result concerning specialization in the valuation spectrum, Theorem 4.2.2, whose statement we now recall:

Theorem 8.3.11 *Suppose $v \rightsquigarrow w$ is a specialization inside* $\operatorname{Spv}(A)$. *Then*

(1) *there exist a vertical specialization $v \rightsquigarrow v'$ and a horizontal specialization $v' \rightsquigarrow w$ so as:*



(2) *either there exists a horizontal specialization $v \rightsquigarrow w'$ and vertical specialization $w' \rightsquigarrow w$ so that*



or else

$$v(a) \leq 1 \quad \text{for all} \quad a \in A \quad \text{and} \quad v_{|A - \mathfrak{p}_w} = 1.$$

Proof of Proposition 8.3.10. We use the second situation in Theorem 8.3.11, so as a first step we rule out the second possibility there. In such a situation we have

$$\mathfrak{p}_w \supset \{x \in A \mid v(x) < 1\}.$$

The locus $\{v < 1\}$ is open, so w is continuous with open support, contradicting that w is analytic. Hence, there must exist $w' \in \mathrm{Spv}(A)$ as in the second situation of Theorem 8.3.11. Since $\mathfrak{p}_{w'} = \mathfrak{p}_w$ is not open and w' inherits continuity as a horizontal specialization of w (Theorem 8.2.1), we have $w' \in \mathrm{Cont}(A)_{\mathrm{an}}$. Thus, to prove that w is a vertical specialization of v it suffices to prove $w' = v$. Hence, we have reduced to showing that inside $\mathrm{Cont}(A)_{\mathrm{an}}$ there is no proper horizontal specialization.

Let $H \subset \Gamma_v$ be a convex subgroup containing $v(A)_{\geq 1}$. For such H, $v_{|_H}$ is a proper horizontal specialization of v, and it lies in $\mathrm{Cont}(A)$ by Theorem 8.2.1. It is enough to prove that if $v_{|_H}$ is analytic then $H = \Gamma_v$. Suppose to the contrary that there exists $\gamma \in \Gamma_v - H$, so by replacing γ with $1/\gamma$ if necessary we can assume $\gamma \leq 1$. By continuity of v, the set

$$\{a \in A \mid v(a) < \gamma\}$$

is open in A. For any $a \in A$ such that $v(a) < \gamma$, $v(a)$ cannot be in H because otherwise the comparisons $v(a) < \gamma \leq 1$ would force $\gamma \in H$ by convexity of H, a contradiction to how γ was chosen. It follows that $(v_{|_H})(a) = 0$ for any such a due to how $v_{|_H}$ is defined, so

$$\{a \in A \mid v(a) < \gamma\} \subset \mathrm{supp}(v_{|_H}).$$

This forces the support (an additive subgroup of A) to be open, contradicting the analyticity hypothesis. Hence, in such analytic cases there is no such γ, so $H = \Gamma_v$. □

8.4 $\mathrm{Cont}(A)$ of a Huber ring A is spectral

One of the main goals of next time will be to prove the following result:

Theorem 8.4.1 *Let A be a Huber ring. Then* $\mathrm{Cont}(A)$ *is a spectral subspace of* $\mathrm{Spv}(A)$.

Let us first briefly sketch the strategy. For any ring A whatsoever (no topology!) we are going to build a class of subspaces $\mathrm{Spv}(A, J)$ of $\mathrm{Spv}(A)$ attatched to "nice" ideals $J \subset A$ such that:

(1) the subspace $\mathrm{Spv}(A, J) \subset \mathrm{Spv}(A)$ is spectral for all "nice" J,

(2) for any Huber ring A, the ideal $A^{00} \cdot A$ is "nice" and $\mathrm{Cont}(A)$ is a pro-constructible set of $\mathrm{Spv}(A, A^{00} \cdot A)$ inside $\mathrm{Spv}(A)$.

From these two properties and the discussion following Definition 3.3.8, it is immediate that $\mathrm{Cont}(A)$ is spectral (hence quasi-compact) for any Huber ring A.

Remark 8.4.2 If A is Tate then $A^{00} \cdot A = A$, so in such cases $\mathrm{Cont}(A)$ will be pro-constructible in $\mathrm{Spv}(A, A)$. The case $J = A$ is therefore an important one!

Remark 8.4.3 Let A be a Huber ring, and (A_0, I) a couple of definition. Then clearly

$$\mathrm{rad}(A^{00} \cdot A) \supseteq \mathrm{rad}(I \cdot A),$$

and in fact equality holds since clearly $A^{00} \subseteq \mathrm{rad}(I \cdot A)$. In view of this equality of radicals, there is natural interest in the class of ideals J in A such that $\mathrm{rad}(J)$ is equal to the radical of a *finitely generated* ideal (e.g., $A^{00} \cdot A$ is one such J, and this will play an essential role in characterizing $\mathrm{Cont}(A)$ inside $\mathrm{Spv}(A)$ in almost "algebraic" terms).

Example 8.4.4 Consider

$$A := \mathbf{Z}_p\{x_1, \ldots, x_n\}/\mathfrak{a}$$

where $\mathbf{Z}_p\{x_1, \ldots, x_n\}$ is the ring of restricted power series (p-adic of a polynomial ring over $\mathbf{Z}_p$). Let $J := pA$. Observe that $A[1/p]$ is a $\mathbf{Q}_p$-affinoid algebra. By [dJ, Lemma 7.1.9] $\mathrm{Sp}(A[1/p])$ is in natural bijection with the set of closed points of $\mathrm{Spec}(A[1/p]) = \mathrm{Spec}(A) - V(J)$. This provides an "algebraic" description of the points of the rigid-analytic generic fiber (in the sense of Raynaud) for formal schemes topologically of finite type over $\mathbf{Z}_p$ (and likewise with $\mathbf{Q}_p$ replaced by any complete discretely-valued field).

The preceding example motivates the following construction that is pure commutative algebra. Fix an arbitrary commutative ring A and ideal $J \subset A$ such that

$$\mathrm{rad}(J) = \mathrm{rad}((a_1, \dots, a_n))$$

for some $a_1, \dots, a_n \in A$. (For example, we saw above that if A is Huber then $A^{00} \cdot A$ is such a J.) This is *exactly* the condition that the open subspace $\mathrm{Spec}(A) - V(J)$ of $\mathrm{Spec}(A)$ is quasi-compact, due to Proposition 3.2.1.

Definition 8.4.5 Let $v : A \to \Gamma_v \cup \{0\}$ be a valuation on A, and let $H \subset \Gamma_v$ be a convex subgroup. We say $\gamma \in \Gamma_v \cup \{0\}$ is *cofinal* with respect to H if for each $h \in H$ we have

$$\gamma^n < h$$

for all sufficiently large positive integers n.

Next time we will apply this definition for general such H, but for the remaining discussion today let's focus on the case $H = \Gamma_v$. We ask :

Question: When is every element of $v(J)$ cofinal with respect to Γ_v?

This is a "topological nilpotence" condition on elements of J relative to v (but keep in mind that A has no given topology here). Note that v lies over $\mathrm{Spec}(A) - V(J)$ if and only if $v(a_i) \neq 0$ for some i, an "analyticity" property of v relative to J.

Let us break up the possibilities for v into three cases, using $c\Gamma_v$ as defined in the discussion just above Proposition 4.3.6 (i.e., it is the minimal convex subgroup of Γ_v containing $v(A)_{\geq 1}$, and plays a distinguished role in the definition of horizontal specialization):

(i) v lies over $V(J)$ (i.e., $v(J) = 0$),

(ii) v lies over $\mathrm{Spec}(A) - V(J)$ and $v(J) \cap c\Gamma_v = \emptyset$,

(iii) v lies over $\mathrm{Spec}(A) - V(J)$ and $v(J) \cap c\Gamma_v \neq \emptyset$.

The serious fact, to be proved next time, is that in case (ii) all elements of $v(J)$ are cofinal with respect to Γ_v *precisely when* v has no proper horizontal specialization over $\mathrm{Spec}(A) - V(J)$; informally, this says $\mathfrak{p}_v$ looks like a "closed point" in $\mathrm{Spec}(A) - V(J)$ from the perspective of v-convex prime ideals (see Theorem 4.4.3). This motivates our interest in the property that v has no proper horizontal specializations over $\mathrm{Spec}(A) - V(J)$.

In case (i), obviously v has *no* specializations (horizontal or otherwise) over $\mathrm{Spec}(A) - V(J)$ and likewise all elements of $v(J) = 0$ are cofinal with respect to Γ_v.

Consider case (iii). The horizontal specialization $v_{|c\Gamma_v}$ does *not* kill J and so lies over $\mathrm{Spec}(A) - V(J)$. Thus, v has no proper horizontal specialization over $\mathrm{Spec}(A) - V(J)$ if and only if $c\Gamma_v = \Gamma_v$. Note that if $c\Gamma_v$ is a proper subgroup of Γ_v then there exists $\gamma \notin c\Gamma_v$ with $\gamma < 1$ and necessarily $\gamma < v(a)$ for $a \in J$ satisfying $v(a) \in c\Gamma_v$ (such a exists since we are in case (iii)) because otherwise we would have $v(a) \leq \gamma < 1$, contradicting the convexity of $c\Gamma_v$. In other words, in case (iii) a *necessary* condition for the cofinality property of all elements of $v(J)$ is that v has no proper horizontal specialization over $\mathrm{Spec}(A) - V(J)$ (though if v satisfies this necessary condition then we do not have a useful way to characterize when the cofinality holds).

The upshot is that the absence of proper horizontal specializations of v lying over the quasi-compact open subset $\mathrm{Spec}(A) - V(J)$ is "nearly always" equivalent to the property that all elements of $v(J)$ are cofinal with respect to Γ_v: it can only fail for v over $\mathrm{Spec}(A) - V(J)$ such that $c\Gamma_v = \Gamma_v$. The subset $\mathrm{Spv}(A, J) \subseteq \mathrm{Spv}(A)$ as defined next time will be exactly the set of $v \in \mathrm{Spv}(A)$ with no proper horizontal specialization over $\mathrm{Spec}(A) - V(J)$, but this description will *not* be the official definition. Hopefully the analogy with generic fibers of formal schemes as in Example 8.4.4 is now apparent. The spectrality of $\mathrm{Spv}(A, J)$ for commutative rings A and "nice" ideals J will not be an easy result, and it ultimately underlies the spectrality of $\mathrm{Cont}(A)$ for Huber rings A (and thus the spectrality of affinoid adic spaces).

References

[dJ] A.J. de Jong, "Crystalline Dieudonné theory via rigid and formal geometry", IHES **82** (1995), 5–96.