7 Lecture 7: Rational domains, Tate rings and analytic points

7.1 Introduction

The aim of this lecture is to topologize localizations of Huber rings, and prove some of their properties. We will discuss an important example, and at the very end introduce the notion of "analytic points" in the continuous valuation spectrum of a non-archimedean ring.

7.2 Completions of Huber rings: complements

Let A be a non-archimedean ring. Recall that we say A is *complete* if it is separated (i.e., Hausdorff) and all Cauchy nets converge. If A has a countable base of open subsets around 0 then this is equivalent to saying that all Cauchy *sequences* converge. We stress that separatedness is part of the definition.

Remark 7.2.1 Huber rings do have a countable base of open subsets around 0, given by the positive integer powers of an ideal of definition.

Given A, we can form, in general, its (Hausdorff) "completion" as an additive topological group:

$$A^\wedge = \varprojlim_{U \subset A} A/U$$

with U running over the open additive subgroups of A, and where $A^\wedge$ is endowed with the inverse limit topology, for which there is a unique multiplication making it into a topological A-algebra. Let us show that this really deserves the name "completion":

Proposition 7.2.2 *Let A be a non-archimedean topological ring, and $A^\wedge$ its completion as above. Then the topological A-algebra $A^\wedge$ is complete as a Hausdorff topological space and is initial among pairs (B, h) consisting of a non-archimedean rings B whose underlying topological space is complete and a continuous ring homomorphism $h : A \to B$.*

Proof. Let A' be the completion of A as a topological space. For each open additive subgroups U of A as use in the formation of $A^\wedge$, we observe A/U is discrete, hence complete. By the universal property of completion (in the sense of topological spaces) the natural map

$$A \to A/U$$

extends uniquely to a natural map

$$A' \to A/U$$

which induces a unique map

$$A' \to A^\wedge.$$

Let $J \subset A$ denote the closure of $\{0\}$, so J is an ideal of A (why?) and A/J with the quotient topology is a topological group whose identity point is closed, so it is Hausdorff. It is clear that any continuous additive map from A to a Hausdorff topological group must kill J, and likewise any U as above contains J (since U is closed, as for open subgroups of any topological group). Thus, for computing both A' and $A^\wedge$ we may replace A with A/J so that A is Hausdorff. In particular, $\cap_U U = \{0\}$, so $A \to A^\wedge$ is injective. Thus, the map $A \to A'$ is also injective, and the copies of A thereby made inside A' and $A^\wedge$ are dense. Since A is Hausdorff, A' can be built as the space of equivalence classes of Cauchy nets indexed by U's as above, from which it is easy to check that the identification between

the dense copies of A inside A' and $A^\wedge$ extends uniquely to an isomorphism of topological spaces $A' \simeq A^\wedge$. In particular, $A^\wedge$ is complete as a topological space.

Any continuous ring map $u : A \to B$ where B is a complete non-archimedean ring will factor uniquely through $A^\wedge$ as a map of topological spaces, and by density of the image of A and the Hausdorff property of B we see that this factorization $A^\wedge \to B$ is necessarily both additive and multiplicative (hence a map of topological rings). $\qquad\square$

Example 7.2.3 Recall that if A is Huber, with couple of definition (A_0, I), then $A^\wedge$ is separated by design, and Huber, with couple of definition $(A_0^\wedge, I \cdot A_0^\wedge)$ where $A_0^\wedge$ is the usual commutative algebra I-adic completion of A_0. The reader can refer to the notes for Lecture 5 for further details (also see [SP, Lemma 10.91.7]). We stress once again that A may not be separated, although $A^\wedge$ always is.

Given a collection T of finite nonempty subsets $T_1, \ldots, T_m$ in a non-archimedean topological ring A such that $T_i \cdot U$ is open for each i and open additive subgroup U of A, let us recall how the topology on $A[X]_T$ $(= A[X_1, \ldots, X_m]$ ring-theoretically) was constructed. For any open additive subgroup U of A we define

$$U[TX] := \left\{ \sum_{\text{finite}} a_I X^I \mid a_I \in T^I \cdot U \right\}$$

and use these as a base of neighbourhoods of 0 in $A[X]$, thus topologizing this as an additive group, and it is a topological ring precisely due to the openness hypothesis on each $T_i \cdot U$; this topological ring is denoted $A[X]_T$. (See Definition 6.3.4 and the discussion thereafter for further details.)

Recall that $A[X]_T$ is *initial* among non-archimedean A-algebras B equipped with $b_1, \ldots, b_m$ in B having the property that the elements

$$t_{ij} b_i \in B$$

are power-bounded for all $t_{ij} \in T_i$, and for all $i = 1, \ldots, m$. We shall next discuss a completed version of $A[X]_T$.

7.3 Completed relative Tate algebras

Definition 7.3.1 Let A be a complete non-archimedean ring (it is always understood that when A is complete, it is also separated). We define

$$A\langle X \rangle_T := \{ \sum a_I X^I \in A[\![X]\!] \mid \text{for all open } U \subset A, a_I \in T^I \cdot U \text{ for all but finitely many } I \}.$$

We remark that in the above definition, the condition "for all but finitely many I" depends on U.

Example 7.3.2 Let A a k-affinoid algebra (k a non-archimedean field), $T_i = \{c_i\}$ for some units $c_i \in A^\times$ (e.g., $c_i \in k^\times$) for each i. Then as a topological $A[X]$-algebra we see that

$$A\langle X \rangle_T = A\langle c_1 X_1, \ldots, c_m X_m \rangle$$

is the relative Tate algebra for the polyradii c_i.

Clearly $A\langle X \rangle_T$ is an $A[X]$-submodule of $A[\![X]\!]$. We want to topologize $A\langle X \rangle_T$ by specifying a base of open additive subgroups, given the datum of a base $\{U\}$ of open neighbourhoods of 0 in A.

We define

$$U\langle TX \rangle := \left\{ \sum a_I X^I \in A\langle X \rangle_T \mid a_I \in T^I \cdot U, \text{ for all } I \right\}$$

and use these additive subgroups as a base of open neighbourhoods of zero to topologize $A\langle X \rangle_T$ as an addtive topological group.

Lemma 7.3.3 *The subset $A\langle X\rangle_T \subset A[\![X]\!]$ is a* subring *and the natural inclusion*

$$A[X]_T \hookrightarrow A\langle X\rangle_T$$

is the completion of $A[X]_T$ as well as a topological embedding.

Remark 7.3.4 We remark that the topology on $A\langle X\rangle_T$ has nothing to do with any topology on $A[\![X]\!]$!

Proof. Arguing using multiplication by $a \in A$ and X^{I_0}, for $I_0 \in \mathbf{N}^n$, we see $A\langle X\rangle_T \subset A[\![X]\!]$ is an $A[X]$-submodule, and the subspace topology on $A[X]$ coming from $A\langle X\rangle_T$ is the one defining $A[X]_T$. If $U, V \subset A$ are open additive subgroups such that $U \cdot U \subset V$, then it is easily checked that

$$U\langle TX\rangle \cdot U\langle TX\rangle \subset V\langle TX\rangle$$

which implies $A\langle X\rangle_T$ is a subring of $A[\![X]\!]$. From this everything follows easily, and we refer the reader to [Wed, 5.49(3)] for complete details. $\qquad\square$

Example 7.3.5 Let $A = \mathbf{Z}_p$ with its p-adic topology, and $T := T_1 = \{p\}$, so that we have

$$\mathbf{Z}_p\langle X\rangle_T \supset \mathbf{Z}_p\langle pX\rangle$$

where the latter is an *open* subring whose subspace topology is its own p-adic topology (the p-adic completion of $\mathbf{Z}_p[pX]$). Note that $\mathbf{Z}_p\langle X\rangle_T$ is *not* adic, as p *is* topologically nilpotent but pX is not. Hence, the condition that A be adic is generally not preserved under passage to $A\langle X\rangle_T$. By contrast, the Huber property as preserved, as we next record.

Proposition 7.3.6 *Let A be Huber, and let (A_0, J) be a pair of definition. Let $T_1, \ldots, T_n \subset A$ be finite nonempty subsets such that $T_i A$ is open for $i = 1, \ldots, n$. Then we have:*

(1) *$A[X]_T$ is Huber and a pair of definition will be given by the open subring $A_0[TX]$ and the ideal $J[TX]$ that is moreover equal to $J \cdot A_0[TX]$.*

(2) *Assume now A is complete. Then $A\langle X\rangle_T$ is Huber and a pair of definition is given by the open subring $A_0\langle TX\rangle$ and the ideal $J\langle TX\rangle$ that is moreover equal to $J \cdot A_0\langle TX\rangle$.*

Proof. We refer the reader to [Wed, Prop. 6.21(1),(2)] (the main point being that the idicated ideals of definition really are generated by the images of J as indicated and hence are finitely generated). $\qquad\square$

7.4 Topologized rings of fractions

We now want to discuss a suitable analogue for non-archimedean rings of the construction

$$A\left\langle \frac{f_1}{g}, \cdots, \frac{f_n}{g} \right\rangle := A\langle X_1, \cdots, X_n\rangle / \langle gX_i - f_i\rangle,$$

for k-affinoid algebra A and elements $f_1, \ldots, f_n, g$ generating the unit ideal. As with the analogue of relative Tate algebras, it will simplify matters to first build a topological ring structure on an algebraic localization, and only afterwards to pass to completions thereof.

Let A be a non-archimedean ring and $T_1, \ldots, T_n \subset A$ finite nonempty subsets such that for all open additive subgroups $U \subset A$,

$$T_i U \quad \text{is open for all } i = 1, \ldots, n.$$

We choose $s_1, \ldots, s_n$ in A. For complete A we aim to construct a ring of fractions

$$A \left\langle \frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n} \right\rangle$$

as the completion of a topological A-algebra

$$A \left(\frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n} \right)$$

that as an A-algebra is just $S^{-1}A$, for $S = s_1^{\mathbf{Z}} \cdots \cdot s_n^{\mathbf{Z}}$, the key property of the topology being that t_{ij}/s_i will be power-bounded for all $t_{ij} \in T_i$, and all $i = 1, \ldots, n$.

Remark 7.4.1 Since

$$S^{-1}A = A[X_1, \cdots, X_n]/(1 - X_i s_i),$$

this suggests to consider the non-archimedean A-algebra

$$S^{-1}A = A[X_1, \cdots, X_n]_T/(1 - X_i s_i)$$

and then form its completion (when A is complete). As an example, if $n = 1$, A is k-affinoid, $T := T_1 = \{f_1, \cdots, f_n\}$ and $s_1 := g$, this construction should recover the usual k-affinoid algebra

$$A \left\langle \frac{f_1}{g}, \cdots, \frac{f_n}{g} \right\rangle$$

(as will follow from consideration of universal properties once the construction is given below).

Proposition 7.4.2 *Let A be a non-archimedean ring. Then*

$$A \left(\frac{T}{s} \right) := A[X]_T/(1 - X_i s_i)$$

is initial among non-archimedean A-algebras

$$f : A \to B$$

such that $f(s_i) \in B^\times$ for all $i = 1, \ldots, n$ and $f(t_{ij})/f(s_i)$ is power-bounded for all $t_{ij} \in T_i$, and all $i = 1, \ldots, n$. If A is Huber then so is $A(T/s)$.

Proof. As remarked in §5.4, quotient maps of non-archimedean rings preserve boundedness (not true for general continuous homomorphisms between such rings), whence mapping property of $A(T/s)$ follows easily. If A is Huber then so is $A[X]_T$ (by Proposition 7.3.6(1)), so its quotient $A(T/s)$ is also Huber. □

Remark 7.4.3 We have the equality of topologies

$$A \left(\frac{T}{s} \right) = A \left(\frac{T'}{s} \right)$$

for $T_i' := T_i \cup \{s_i\}$ for all $i = 1, \ldots, n$.

Definition 7.4.4 Let A be a complete non-archimedean ring. We define

$$A \left\langle \frac{T}{s} \right\rangle := A \left(\frac{T}{s} \right)^{\wedge}.$$

By the universal property of completion, $A\langle T/s \rangle$ is initial among complete non-archimedean A-algebras in which each s_j is a unit and t_{ij}/s_i is power-bounded for each $t_{ij} \in T_i$ for all i. This immediatelty yields the consistency with the analogous construction in rigid geometry (which is characterized by the same universal property within the category of Banach algebras over the non-archimedean ground field).

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7.5 Tate rings

Recall that a Huber ring A is *Tate* if there exists a topologically nilpotent unit $u \in A$ (Remark 5.4.6). For example, if A is an affinoid algebra over some non-archimedean field k, then any pseudo-uniformizer $u := \varpi$ of k is a topologically nilpotent unit. This example is in fact enlightening: Tate rings will be the non-archimedean rings that behave most like Banach algebras.

Proposition 7.5.1 *Let $f : A \to B$ be a continuous homomorphism of Huber rings. If A is Tate, then so is B, and f is adic.*

This was Proposition 5.4.7, and the reader is encouraged to review its proof.

Example 7.5.2 Another example which we shall encounter and study in detail in the context of perfectoid algebras later is the completion of the inductive limit of $\mathbf{Q}_p \langle T^{p^{-n}} \rangle$ along the obvious transition maps. (The precise topology on this will be discussed later, but briefly it is the "p-adic" topology.) This is an important example of a *perfectoid* $\mathbf{Q}_p$-algebra, and is Tate.

In Lecture 6, we discussed the fact that when $A = k$ a non-archimedean field, then A is Huber if and only if it is Tate (see §6.2, particularly Example 6.2.1).

Now let us consider A a Tate ring, with pair of definition (A_0, I) and a topologically nilpotent unit u.

Proposition 7.5.3 *The natural inclusion*

$$A_0[1/u] \hookrightarrow A$$

is an equality, and the subspace topology on A_0 is the u-adic topology.

Proof. We may replace u with u^e for sufficiently large e to ensure $u \in A_0$. Let a be an element of A. Then

$$u^n a \to 0$$

as $n \to \infty$. Hence, $u^n a \in A_0$ for large n, so $a \in A_0[1/u]$. Fix such a choice of n. Since multiplication by u is a homeomorphism, $u^n A_0 \subset A$ is open since A_0 is open. On the other hand, u is topologically nilpotent, and therefore for any m we have $u^{n'} \in I^m$ for all large n' (depending on m), giving $u^{n'+n} A_0 \subset I^m$ for all large m. $\square$

Again boundedness

Essential fact to keep in mind: let A be a Huber ring, and $A^\wedge$ its completion. The natural map $f : A \to A^\wedge$ preserves boundedness (see Example 5.4.5). Quotient maps have the same property (but general continuous homomorphisms of non-archimedean rings do not!), and the Huber condition for non-archimedean rings is therefore inherited by completions and quotients by ideals (with the quotient topology). For a finite subset $s_1, \ldots, s_n \in A$ and the multiplicative set S that they generate, recall that $A(T/s)$ is $S^{-1}A$ equipped with a quotient topology from $A[X]_T$, making it a topological A-algebra. Being a quotient topology, we obtain:

Corollary 7.5.4 *The ring $A(T/s)$ is Huber. If A is complete then $A\langle T/s \rangle$ is likewise Huber.*

Let us discuss a couple of examples.

Example 7.5.5

(1) Let $T := \{1, t\}$, for some $t \in A$ with A complete. Then we have:

$$A\langle t/1 \rangle = (A\langle X \rangle_T/(X - t))^\wedge \quad \text{and} \quad A\langle 1/t \rangle = (A\langle X \rangle_T/(Xt - 1))^\wedge.$$

Later we will see that in the setting of adic spaces, the adic spaces attached to these A-algebras will correspond to loci "$X = t$" and "$Xt = 1$" in a relative closed unit ball over the adic space for A (when A admits a reasonable adic space).

(2) Consider $A = A_0 = \mathbf{Z}_p[\![x]\!]$ equipped with the (p, x)-adic topology. Choose $T := T_1 = \{1\}$, and $s = p \in A$ (or x). Then

$$A(T/s) = \mathbf{Z}_p[\![x]\!][1/p]$$

(or $A(T/s) = \mathbf{Z}_p[\![x]\!][1/x]$) which is its own ring of definition, with ideal of definition equal to the unit ideal. Therefore, $A\langle T/s \rangle = 0$. Informally, this is reasonable from a geometric point of view (think about what its "points" should mean, at least for the case $s = p$).

7.6 An example to keep in mind

Let A be a Huber ring, and $T = \{T_1, \cdots, T_n\}$ a collection of non-empty finite subsets of A that each generate an open ideal. Fix $s_1, \ldots, s_n \in A$ and let S be the multiplicative set in A that they generate. Recall that by construction

$$A(T/s) = A[X]_T / (1 - X_1 s_1, \ldots., 1 - X_n s_n)$$

with the quotient topology from $A[X]_T$. As a ring (ignoring its topology), this is the localization $S^{-1}A$, and by definition of quotient topology a base of opens around 0 in $A(T/s)$ is given by the images of a base of opens in $A[X]_T$ around 0.

Let J be an ideal of definition of A_0 (so J is finitely generated). From our study of the topology on $A[X]_T$ in terms of A_0 and J, by the *Huber property* of A the ring $A_0[TX]$ is a ring of definition of $A[X]_T$ and it has the $J \cdot A_0[TX]$-adic topology. The image of X_i in $A(T/s)$ is $1/s_i$, so a ring of definition of $A(T/s)$ (ignoring the topology) is given by the A_0-subalgebra of $S^{-1}A$ generated by the fractions t_{ij}/s_i for $t_{ij} in T_i$.

It might be tempting to denote this ring of definition suggestively as $A_0(T/s)$, but that would be a very bad idea because if $A = A_0$ then this ring of definition need *not* coincide with $A(T/s)$! (Remember that $A(T/s)$ is just suggestive notation for the ring $S^{-1}A$ with a topology determined by T, so don't be misled by notation.) The point is that by design the s_i's are units in $A(T/s)$ whereas the s_i's can all fail to be units in this ring of definition (as we will see below in an example). Thus, we shall instead denote this ring of definition as $A_0[T/s]$, and it is equipped with the adic topology associated to the ideal generated by the image of J; this ideal will be denoted $J[T/s]$.

Remark 7.6.1 One has to be extra-careful with the suggestive notation $A_0[T/s]$ if some s_i does not belong to A_0 or if some T_i is not contained in A_0. If $T_i \subset A_0$ for all i and if $s_i \in A_0$ for all i then the localization $S^{-1}A_0$ makes sense and is a subring of $S^{-1}A = A(T/s)$ with the ring of definition $A_0[T/s]$ equal to the A_0-subalgebra of $S^{-1}A_0$ generated by the fractions t_{ij}/s_i for $t_{ij} \in T_i \subset A_0$, and $J(T/s) = J \cdot A_0(T/s)$. But more generally it doesn't make sense to try to describe $A_0[T/s]$ without reference to the ambient A, even though the role of A is hidden from the notation.

Let's now see in an example that the s_i's can all fail to be units in $A_0[T/s]$, even when $s_i \in A_0$ and $T_i \subset A_0$ for all i. For example, consider the case $A = A_0 = \mathbf{Z}_p[\![x]\!]$ with the J-adic topology for $J = (p, x)$, with $T = T_1 = \{p, x\}$ (which indeed generates an open ideal) and $s = s_1$ equal to either p or x. In this case T and s are contained in $A = A_0$ and $A_0[T/s]$ is the A-subalgebra of $A[1/s]$ generated by p/s and x/s, equipped with the topology for the ideal generated by the image of J. That is, for $s = p$ we get $A[x/p]$ with $J \cdot A[x/p]$-adic topology and for $s = x$ we get $A[p/x]$ with the $J \cdot A[x/p]$-adic topology. In both cases we claim that s is *not* a unit in $A_0(T/s)$.

More generally if R is a UFD and $a, b \in R$ are nonzero elements not generating the unit ideal and a is irreducible and does not divide b then b is *not* a unit in the subring $R[a/b]$ of $R[1/b]$. (We apply this with $R = \mathbf{Z}_p[\![x]\!]$ and either $a = p$ and $b = x$ or instead $a = x$ and $b = p$.) To see this, we just have to rule out an identity of the form

$$b(r_n(a/b)^n + \cdots + r_1(a/b) + r_0) = 1$$

for $r_j \in R$. If $n = 1$ then this contradicts that $1 \notin (a, b)$, so $n \geq 2$. Scaling through by b^{n-1} gives that $b^{n-1} \in a(a, b)^{n-2}$, so $a|b^{n-1}$. But R is a UFD and we are assuming a is irreducible and does not divide b, so this is impossible.

Now assume the Huber ring A is complete, so we may form the completion $A\langle T/s \rangle$ of the Huber ring $A(T/s)$ and it is Huber with ring of definition given by the $J \cdot A_0[T/s]$-adic completion of $A_0[T/s]$ (as we saw in our general discussion of completions of Huber rings); this latter completion will be denoted $A_0[T/s]^{\wedge}$ (we cannot denote it as $A_0\langle T/s \rangle$ because even when $A_0 = A$ it can fail to be equal to $A\langle T/s \rangle$); this completion generally depends on the ambient A, as for $A_0[T/s]$. The topology on $A_0[T/s]^{\wedge}$ is the adic topology for the ideal generated by J (as we saw in our general discussion of completions of Huber rings).

In the special case that $s_i \in A_0$ for all i and $T_i \subset A_0$ for all i then $A_0[T/s]^{\wedge}$ is the adic completion of the A_0-subalgebra $A_0[T/s] \subset S^{-1}A_0$ for the ideal generated by the image of J. Let's see how this all works out for the example $\mathbf{Z}_p[\![x]\!]$ that was considered above.

Example 7.6.2 Let $A = A_0 = \mathbf{Z}_p[\![x]\!]$ with $J = (p, x)$, $T = T_1 = \{p, x\}$ and $s = s_1$ equal to p or x. Then the ring of definition $A_0[T/s]^{\wedge}$ of $A\langle T/s \rangle$ is equal to the adic completion of $A[x/p]$ for the topology defined by powers of the ideal $J \cdot A[x/p]$. Now comes the key point: although p and x are not units in $A[x/p]$, in this ring we can write $x = (x/p)p$, so $J \cdot A[x/p] = p \cdot A[x/p]$. Thus, for $s = p$ a ring of definition is $A[x/p] = \mathbf{Z}_p[\![x]\!][x/p]$ with the p-adic topology, and similarly for $s = x$ a ring of definition is $A[p/x]$ with the x-adic topology. In particular, s is topologically nilpotent in $A_0[T/s]$.

These rings of definition are *Hausdorff* for their topologies, and so they inject into their completions. Let's prove the Hausdorff property for the case $s = p$, as the case $s = x$ goes similarly. A general non-zero element of $A[x/p]$ has the form $(x/p)^e f$ for some $f \in A$ and $e \geq 0$, and we can assume p doesn't divide f because of the non-zero hypothesis. Thus, if this element lies in $p^n \cdot A[x/p]$ then $(x/p)^e f = p^n(x/p)^{e_n} g_n$ for some $e_n \geq 0$ and $g_n \in A$, so comparison of ord_p's (recall that we arranged p doesn't divide f) implies that $e_n \geq e + n$, whence the right side has $\mathrm{ord}_x \geq e_n \geq e + n$. Going back to the left side, $\mathrm{ord}_x(f) \geq n$. But $f \neq 0$, so this puts an upper bound on the possibilities for such n, thereby establishing the Hausdorff property.

Since the *unit* s in $A\langle T/s \rangle$ is topologically nilpotent, $A\langle T/s \rangle$ is Tate and so is equal to the algebraic localization $A_0[T/s]^{\wedge}[1/s]$ (by the general relationship between Tate rings and rings of definition containing a given topologically nilpotent unit). Thus, $A\langle T/s \rangle$ is the p-localization of the p-adic completion of $\mathbf{Z}_p[\![x]\!][x/p]$ when $s = p$ and it is the x-localization of the x-adic completion of $\mathbf{Z}_p[\![x]\!][p/x]$ when $s = x$. We will write $A\langle x/p \rangle$ to denote $A\langle T/s \rangle$ for $s = p$ (dropping the mention of the fraction $p/p = 1$), and likewise write $A\langle p/x \rangle$ to denote $A\langle T/s \rangle$ for $s = x$, but this is strictly speaking abusive since neither x nor p individually generates an open ideal in A.

Even though A is an adic topological ring (i.e., its topology is defined by powers of an ideal), $A\langle T/s \rangle$ is Tate and hence is *not* an adic topological ring (since an adic topological ring with a topologically nilpotent *unit* must have 1 topologically nilpotent and thus is the zero ring). So we see (as we did earlier for relative Tate algebras) that the Huber property is more robust than the adic property for the constructions on non-archimedean rings that we are considering. Observe also that the infinite

sum

$$\sum_{n\geq 0} p^n (x/p)^{2n} = \sum_{n\geq 0} p^{-n} x^{2n}$$

converges in the Huber ring $A\langle x/p\rangle$ in which p is a (topologically nilpotent) unit, and it lies in the ring of definition $A_0[x/p]^\wedge$, but the second expression for the sum only "makes sense" from the perspective of the ambient ring $A\langle x/p\rangle$ because only in here is p a unit (the topologically nilpotent p cannot be a unit in the (nonzero) adic topological ring $A_0[x/p]^\wedge$).

For $s = p$, the ring $A\langle T/s\rangle = A[x/p]^\wedge[1/p]$ is a p-adic Banach algebra which admits an explicit description as follows. Consider the $\mathbf{Q}_p\langle x\rangle$-affinoid algebra $\mathbf{Q}_p\langle x/p\rangle$ corresponding to the disc $|x/p| \leq 1$ inside the closed unit disc. This contains A as a subring via restriction of analytic functions from the closed unit disc to the subdisc $|x/p| \leq 1$ and it contains the element x/p, so it contains the subring $A[x/p]$, and all elements of $A[x/p]$ has sup-norm at most 1 when viewed on the disc $|x/p| \leq 1$, so since the sup-norm on $\mathbf{Q}_p\langle x/p\rangle$ defines its topology, we see that the natural inclusion $A[x/p] \to \mathbf{Q}_p\langle x/p\rangle$ is continuous for the p-adic topology on the source (an adic topology in the sense of commutative algebra) and the Banach topology on the target.

Thus, by the universal property of completion, we get a unique continuous A-algebra map

$$A\langle T/s\rangle \to \mathbf{Q}_p\langle x/p\rangle.$$

But the right side is the p-localization of the p-adic completion of $\mathbf{Z}_p[x/p] \subset A[x/p]$, so since $A[x/p]$ is equipped with the p-adic topology we likewise get a unique continuous A-algebra map in the opposite direction. These two maps restrict to the identity on the subring $A[1/p]$ that has dense image in each side, so these are inverse to each other. In other words, $A\langle T/s\rangle$ as topological algebra over $\mathbf{Z}_p[\![x]\!]$ is the coordinate ring of the $\mathbf{Q}_p$-affinoid disc centered at 0 with radius $|p|$.

On the other hand, for $s = x$ the ring $A\langle T/s\rangle = A\langle p/x\rangle = A[p/x]^\wedge[1/x]$ with the "x-adic" topology does *not* contain $\mathbf{Q}_p$, or in other words p is *not* a unit in this Tate ring. Indeed, since $A[Y]/(xY-p) \simeq A[p/x]$ as A-algebras (check!), the quotient of $A[p/x]^\wedge[1/x]$ modulo p is the x-localization of

$$A[p/x]/(p) = A[Y]/(p, xY - p) = \mathbf{F}_p[\![x]\!][Y]/(Yx),$$

so this x-localization is the ring $\mathbf{F}_p(\!(x)\!) \neq 0$ (affirming that p isn't a unit). It follows that $A\langle p/x\rangle$ has no interpretation in terms of rigid-analytic geometry (much as arithmetic curves over a mixed-characteristic discrete valuation ring do not have a literal interpretation in terms of algebraic geometry over a field).

However, much as we can make fruitful analogies between arithmetic surfaces and surfaces over a field by considering arithmetic curves over the base ring $k[t]_{(t)}$, if we replace $\mathbf{Z}_p$ with $k[\![t]\!]$ for a field k and replace p with t in all of the preceding calculations then we would find that $A\langle T/s\rangle$ *does* have geometric meaning: it is the coordinate ring of the disc with radius $|x|$ in the parameter t over the Laurent series field $k(\!(x)\!)$ (equipped with an absolute value making $0 < |x| < 1$). In this respect, the Tate ring $A\langle p/x\rangle$ should be "seen" as a unit disc in the parameter p with radius $|x|$ (whatever that might mean) Informally, we want to regard $\mathbf{Z}_p$ as a power series ring in the parameter p with "coefficients" in $\mathbf{F}_p$, but this is strictly speaking meaningless. Nonetheless, the theory of adic spaces makes it possible to assign precise "analytic geometry" meaning to such arithmetic constructions, and this turns out to be incredibly fruitful via Scholze's geometrization of the period ring formalism of p-adic Hodge theory by means of his theory of diamonds (a generalization of perfectoid spaces much as algebraic spaces are a generalization of schemes).

Remark 7.6.3 Here is an easy trap one can fall into, due to the abusive use of the phrase "s-adic" to refer to both adic topology in the sense of commutative algebra and in a more topological sense

akin to the "p-adic" topology on $\mathbf{Q}_p$. Consider the ring $A[1/s]$ topologized via the s-adic topology on $A = \mathbf{Z}_p[\![x]\!]$. This is complete since A is s-adically separated and complete, and we generally then speak of the "s-adic" topology on $A[1/s]$ even though it is not literally an adic topology on the ring $A[1/s]$ in which s is a unit (much as we speak of the "p-adic" topology on $\mathbf{Q}_p$). This ring $A[1/s]$ contains $A_0[T/s]$ as a subring in an evident manner.

Since $A_0[T/s]$ is equipped with its own s-adic topology (in the commutative algebra sense), it is easy to make the mistake of thinking that the inclusion of $A_0[T/s]$ into $A[1/s]$ is continuous (s-adic topologies have to be compatible, right?) and hence by the universal property of completion factors through a continuous A-algebra map $A_0[T/s]^\wedge \to A[1/s]$, and thus a continuous A-algebra map

$$A\langle T/s \rangle = A_0[T/s]^\wedge[1/s] \to A[1/s].$$

But this is totally wrong!

The point is that the "s-adic" topology on $A[1/s]$ is *not* a topology defined by powers of the ideal generated by s in the ring $A[1/s]$ (where that ideal is the unit ideal), but rather it is extended from the commutative-algebra s-adic topology on the subring A which does *not* contain the subring $A_0[T/s]$ of $A[1/s]$. And this shows quite directly that the natural map $A_0[T/s] \to A[1/s]$ really is not continuous relative to the commutative-algebra s-adic topology on the source and the more topological "s-adic" topology on the target: a base of opens around 0 in $A_0[T/s]$ consists of the ideals $s^n \cdot A_0[T/s]$ for $n \geq 0$, but via the inclusion into $A[1/s]$ *none* of these lands inside the open subring A around 0 in $A[1/s]$ (e.g., $s^n \cdot A_0[T/s]$ contains $s^n(t/s)^{n+1} = t^{n+1}/s$ for the unique $t \in T - \{s\}$; i.e., x^{n+1}/p or p^{n+1}/x).

7.7 Introduction to analytic points in $\mathrm{Cont}(A)$

Definition of analytic points in the continuous valuation spectrum of a Huber rings.

Definition 7.7.1 Let A be a Huber ring. A point $v \in \mathrm{Cont}(A)$ is called an *analytic point* if its support $\mathfrak{p}_v \in \mathrm{Spec}(A)$ is *not* an open ideal of A.

We shall discuss next time that the subspace of analytic points in $\mathrm{Cont}(A)$, denoted by $\mathrm{Cont}(A)_{\mathrm{an}}$, is in fact a quasi-compact open subset of $\mathrm{Cont}(A)$, and that there are no horizontal specializations among analytic points. The reader may also wish to look back at Examples 6.2.1 and 6.2.2. In the general theory of adic spaces, we shall see next time that analytic points will be precisely those admitting an open neighborhood that is associated to a Tate ring.

References

[SP] A. J. de Jong, *Stacks Project*.

[Wed] T. Wedhorn, *Adic Spaces*, unpublished lecture notes.