6 Lecture 6: More constructions with Huber rings

6.1 Introduction

Recall from Definition 5.2.4 that a *Huber ring* is a commutative topological ring A equipped with an open subring A_0, such that the subspace topology on A_0 is I-adic for some *finitely generated* ideal I of A_0. This means that $\{I^n\}_{n\geq 0}$ is a base of open neighborhoods of $0 \in A$, and implies that A is a non-archimedean ring (see Definition 5.2.1). We stress that the topology on a Huber ring A need not be separated (i.e., Hausdorff) nor complete.

Recall from Lecture 5 that, given a Huber ring A with a subring of definition A_0 which is I-adic with respect to a finitely generated ideal I of A_0, its completion was defined to be

$$A^\wedge := \varprojlim A/I^n$$

where the limit is in the sense of (additive) topological groups and is seen to carry a natural structure of topological ring whose topology is in fact separated and complete. A subring of definition of $A^\wedge$ is given by $A_0^\wedge$, the usual I-adic completion of A_0 in the sense of commutative algebra, and we saw (by a non-trivial theorem when A_0 is not noetherian) that the inverse limit (same as subspace!) topology on $A_0^\wedge$ is its $I(A_0)^\wedge$-adic topology that is moreover separated and complete.

This lecture is devoted to introduce a special class of Huber rings which will play the role of Tate algebras (with "radii") in rigid geometry. We will make use of the constructions discussed here to define a good analogue of "rational domain" for adic spaces in the next lecture.

6.2 Topologically nilpotent units

We start off with an instructive class of Huber rings that play an essential technical role in later developments with adic spaces. Let K be a nontrivially valued field, which is to say a field equipped with a nontrivial valuation $v : K \to \Gamma \cup \{0\}$ (perhaps with rank > 1; see Remark 2.2.4). Let $R \subset K$ be its valuation ring.

We view K as a topological ring A with base of open neighbourhoods of 0 given by

$$A_\gamma := \{x \in K \mid v(x) < \gamma\}, \quad \gamma \in \Gamma,$$

so K is thereby a non-archimedean ring. Note that K is even a topological field, which is to say that $K^\times = K - \{0\}$ with its subspace topology is a topological group. Indeed, since multiplication is continuous (as K is a topological ring) we just have to check that inversion is continuous near 1, or in other words that for any $\gamma \in \Gamma$ there exists $\delta \in \Gamma$ such that if $v(x-1) < \delta$ then $v(x^{-1}-1) < \gamma$. Without loss of generality $\gamma < 1$, so $v(x) = 1$ and hence $v(x^{-1}-1) = v(1-x)/v(x) = v(x-1)$. Thus, we may use $\delta = \gamma$.

The question we wish to address is: when is K a Huber ring? By definition, we seek an open subring $A_0 \subset K$ such that the topology on A_0 is I-adic,for some finitely generated ideal $I \subset A_0$. Such A_0 may well not be the valuation ring R, and therefore I may well not be principal! The easy case is when v is a rank-1 valuation, and hence any pseudo-uniformizer ϖ (that is, $\varpi \in R$ with $0 < v(\varpi) < 1$) will generate an ideal $I = \varpi R$ making R into an open subring of K with the I-adic topology.

Suppose we are given $A_0 \subset K =: A$ and a nonzero finitely generated ideal $I \subset A_0$ with the desired properties. Since we are assuming K is non-trivially valued, I must be nonzero since $\{0\}$ is not an open subset of A. Therefore, we may pick a nonzero element $\varpi \in I$. But $\{I^n\}_{n\geq 0}$ is a base of open neighbourhoods of 0 in A, and they are open ideals of A_0, so ϖ is topologically nilpotent in A. Moreover, $A = K$ is a field and ϖ is nonzero, so ϖ is a topologically nilpotent *unit*.

How do we recognize topologically nilpotent units in $A = K$? As a first attempt we might guess that such elements are those nonzero elements $\varpi \in A$ with $v(\varpi) < 1$. This is the case when the valuation v is rank-1. Indeed, the "archimedean property" of rank-1 value groups gives that if $0 < v(\varpi) < 1$ then (via an inclusion $\Gamma \hookrightarrow \mathbf{R}^{\times}_{>0}$) we have $v(\varpi^n) = v(\varpi)^n \to 0$ as $n \to \infty$, and the principal ideal $I = \varpi R$ is moreover open since multiplication by ϖ is an automorphism of the topological space K (as multiplication by $1/\varpi$ is a continuous inverse).

But if v is a higher-rank valuation then the property of ϖ of being a pseudo-uniformizer generally does *not* imply that ϖ is topologically nilpotent. This is illustrated by the rank-2 valuation ring considered in Example 2.3.4 and Example 4.3.3:

Example 6.2.1 Let k be a field, and $K = k(\!(u)\!)(\!(t)\!)$. Let w be the rank two valuation produced endowing the residue field $k(\!(u)\!)$ of $k(\!(u)\!)[\![t]\!]$ with the u-adic valuation, as in Example 2.3.4. Its valuation ring is

$$R := k[\![u]\!] + tk(\!(u)\!)[\![t]\!],$$

with value group $\Gamma := a^{\mathbf{Z}} \times b^{\mathbf{Z}}$ for real numbers $0 < a, b < 1$ with $a := w(t)$ and $b := w(u)$. The total ordering on Γ is lexicographical, where the "t-factor" comes first. Such ordering arises naturally from the construction explained in §2.3 and detailed in Example 4.3.3.

Briefly, recall that Γ is a split extension of the value group of the t-adic valuation on K, which is $a^{\mathbf{Z}}$, and of the u-adic valuation on $k(\!(u)\!)$, which is $b^{\mathbf{Z}}$. Geometrically, the t-adic valuation on K (with valuation ring $k(\!(u)\!)[\![t]\!]$) seen as a point in $\mathrm{Spv}(K)$ is a vertical generization of w; i.e., it is obtained as $w_{/H}$ for the convex subgroup $H := 1 \times b^{\mathbf{Z}}$ of Γ, as explained in Example 4.3.3. Let us define $v := w_{/H}$, and recall that the picture is as follows, with η the trivial valuation on K:

$$\begin{tikzcd} \mathrm{Spv}(K) \arrow[dddd] & \mathrm{Spv}(K) \\ & \eta \arrow[d, rightsquigarrow] \\ & v \arrow[d, rightsquigarrow] \\ & w \\ \mathrm{Spec}(K) & (0) \end{tikzcd}$$

That w is a specialization of v means in particular that every open subset of $\mathrm{Spv}(K)$ containing w already contains v. We now interpret this in terms of the valuation topologies on K.

We claim that the valuation topologies on K arising from v and w *coincide*. Observe that the unit $t \in K^{\times}$ is topologically nilpotent for both topologies. Indeed, for v this is clear from the definition, and for w we just have to check that for any $\gamma = (a^i, b^j)$ there exists a large n so that $(a, 1)^n \leq \gamma$. That is, we want $(a^{n-i}, 1/b^j) \leq (1, 1)$. Taking $n > i$ ensures $a^{n-i} < 1$, so the lexicographical ordering implies $(a^{n-i}, 1/b^j) < (1, 1)$ regardless of the value of j. Hence, for the valuation rings R and $R_v = k(\!(u)\!)[\![t]\!]$ of w and v respectively, the collections $\{t^n R\}$ and $\{t^n R_v\}$ for $n \geq 0$ are respective bases of opens around 0 for the two topologies. But $t^{n+1} R_v \subset t^n R \subset t^n R_v$, so the topologies coincide.

Informally, the w-adic topology on K coincides with the "t-adic topology" even though w is a higher-rank valuation. In particular, although $w(u) < (1, 1) = 1_{\Gamma_w}$, u is *not* topologically nilpotent (a purely topological property) since we can see this in terms of the v-adic valuation. More concretely, $t/u^n \in R$ for all $n \geq 0$, so $w(t) \leq w(u)^n$ for all $n \geq 0$. Similarly we see that $1/u$ is power-bounded in the w-adic topology of K (a purely topological notion) yet this lies *outside* the w-adic valuation ring R.

Example 6.2.1 shows that given a nontrivially-valued field K with valuation $v : K \to \Gamma \cup \{0\}$, it is not true that $\varpi \in K^\times$ satisfying $0 < v(\varpi) < 1$ is topologically nilpotent. Nevertheless, for our favourite choice $K = k((u))((t))$ we can find a topologically nilpotent element t in K and this establishes the Huber condition for K.

Remark 6.2.2 We remark that if K is a nontrivially-valued field then the trivial valuation is *not* continuous in the sense of Definition 5.3.1 since $\{0\}$ is not open. We also note that in Example 6.2.1 *both* w and its vertical generization v are continuous, and the only possible further vertical generization of v is the trivial valuation η on K, which was not continuous.

In a later lecture we shall discuss the notion of *analytic points* in the "continuous" valuation spectrum of a Huber ring A: these are the $v \in \mathrm{Cont}(A)$ having support $\mathfrak{p}_v \in \mathrm{Spec}(A)$ *not* open (so w and v in Example 6.2.1 are both analytic, while η is not, since it is not even in $\mathrm{Cont}(A)$). We will see that within the subspace of analytic points of $\mathrm{Cont}(A)$ there are no horizontal specializations, and every analytic point is of rank ≥ 1, admitting a unique rank-1 vertical generization. For this reason, rank-1 points will wind up playing an important role in the general theory of adic spaces, despite the abundance of higher-rank points.

Here is a converse to the preceding considerations.

Proposition 6.2.3 *Let K be a nontrivially-valued field with valuation ring R, and suppose there exists a topologically nilpotent unit $\varpi \in K^\times$. Then K is Huber, with ring of definition $A_0 := R$ and ideal of definition $I = \varpi^e A_0$ for e large enough so that ϖ^e lies in the open subring R.*

Proof. Replace ϖ by ϖ^e for large enough e such that $\varpi \in R$. Since ϖ is a unit in K, multiplication by ϖ is invertible and continuous, and hence $I^n = \varpi^n R$ is an open ideal of R for all nonnegative integers n. We claim $\{\varpi^n R\}_{n \geq 0}$ is a base of neighbourhoods of 0 in K.

Choose γ in the nontrivial value group Γ of the valuation on K associated to R, so we want to show that for large enough $n \geq 0$, we have

$$v(\varpi^n x) < \gamma$$

for all $x \in R$. We have $v(x) \leq 1$, so $v(\varpi^n x) \leq v(\varpi^n)$, and since ϖ is topologically nilpotent we have that $v(\varpi^n) < \gamma$ for sufficiently large $n \geq 0$. This proves the claim. ☐

Remark 6.2.4 In the situation of Proposition 6.2.3, by inverting ϖ we recover K:

$$R[1/\varpi] = K.$$

Indeed, choose $x \in K$ so we seek some $n \geq 0$ such that $\varpi^n x \in R$. We may assume $x \neq 0$. Then we are reduced to showing that

$$v(\varpi)^n \leq 1/v(x) = v(1/x)$$

for some large n, where the right side is an element of Γ. But this is ensured by the fact that ϖ is topologically nilpotent.

We have seen that a nontrivially-valued field with higher-rank value group may have its valuation topology defined by powers of a single element. However, generally that topology is *not* the adic one defined by powers of the (nonzero) maximal ideal $\mathfrak{m}_R$. Here are two examples (one of which is our favourite usual one):

Example 6.2.5 (1) Let K be a non-archimedean field (i.e., complete for a nontrivial non-archimedean absolute value) with valuation ring R such that $\mathfrak{m}_R = \mathfrak{m}_R^e$ for some $e > 1$, as happens for $\mathbf{C}_p$ or (the completion of) $\mathbf{Q}_p^{\mathrm{cyc}}$. The $\mathfrak{m}_R$-adic topology on R is not even Hausdorff.

(2) Consider $R = k[\![u]\!] + tk(\!(u)\!)[\![t]\!]$ as in Example 6.2.1, with lexicographically ordered value group $a^{\mathbf{Z}} \times b^{\mathbf{Z}}$ for a, b real numbers $0 < a, b < 1$. Note that in contrast with the previous example, this one does not have any infinitely divisible elements in the value group. We have seen that u lies in the maximal ideal $\mathfrak{m}_R$ of R and is *not* topologically nilpotent (and for the same reason $1/u$ is power-bounded in the topology of the valued field $\mathrm{Frac}(R) = R[1/t]$), so the topology cannot be $\mathfrak{m}_R$-adic.

By the same reasoning, the same thing happens for *any* valuation ring whose maximal ideal contains *some* element that is not topologically nilpotent. Likewise, in all such cases there are power-bounded elements outside the valuation ring and inside the valued fraction field, namely the reciprocal of any element of $\mathfrak{m}_R$ that is not topologically nilpotent.

In the preceding considerations, we studied a nontrivially-valued field K with valuation ring R and found a topologically nilpotent unit $\varpi \in K^\times$ which is sufficient to cover the topology on $K = R[1/\varpi]$. This fits into a more general situation related to topologizing certain localizations of Huber rings. To explain this, we first prove the following general fact:

Proposition 6.2.6 *Let R be a ring, and $\varpi \in R$ an element that is not a zero-divisor. We endow R with the ϖ-adic topology. Then $R[1/\varpi]$ has a unique structure of topological ring making the topological ring R an* open *subring.*

Proof. Since ϖ is not a zero-divisor, multiplication by ϖ^n is injective on R. If there is such a topological ring structure then a neighborhood basis of 0 is given by $\varpi^n R$ for all $n \geq 0$, so likewise for all $x \in R[1/\varpi]$ a neighborhood basis is given by $x + \varpi^n R$ for $n \geq 0$.

Conversely, let's define a subset $U \subset R[1/\varpi]$ to be *open* if for every $u \in U$ we have $u + \varpi^n R \subset U$ for sufficiently large n (depending on u). This is clearly a topology on $R[1/\varpi]$ makes R an open subspace with its ϖ-adic topology and that makes $R[1/\varpi]$ a topological group under addition, and the only issue is to check that multiplication is continuous. For $x, y \in R[1/\varpi]$ we can pick $m \geq 0$ such that $x, y \in \varpi^{-m} R$, so for $n \geq m + N$ with $N > 0$ we have

$$(x + \varpi^n r)(y + \varpi^n r') \in xy + \varpi^N R,$$

giving the desired continuity. $\square$

We give a non-example which we have already encountered.

Example 6.2.7 Let $R := \mathbf{Z}_p[\![x]\!]$, with the (p, x)-adic topology. We claim there does not exist any extension of the (p, x)-adic topology on R to a topological ring structure on $R[1/p]$ making R into an open subring with the given topology, or even to a topological group structure making R an open subgroup. Indeed, if there were then the openness of the inclusion of R into $(1/p)R$ would imply that pR is open in R for the (p, x)-adic topology, which is not true. (The elements x^n converge to 0 for the (p, x)-adic topology but $x^n \notin pR$ for all $n \geq 0$.)

Remark 6.2.8 We recall Proposition 5.4.13, saying that given a Huber ring A, the subring of power-bounded elements A^0 is the filtered direct limit of the bounded subrings of A (that is, the subrings of definition of A, by Proposition 5.4.8). Example 6.2.1 gives an example of Huber ring A (the field

K) such that our ring of definition A_0 (that is, R in the example) is *strictly* contained in A^0. In fact, since $w(u^n) > w(t)$, $1/u$ is power-bounded, but it is not contained in R.

6.3 Topologizing localizations

In rigid-analytic geometry over a non-archimedean field k, for a k-affinoid algebra A we define the relative Tate algebra

$$A\langle X_1, \cdots, X_n\rangle = \{\sum_I a_I X^I \mid a_I \to 0 \text{ as } |I| \to \infty\}$$

that is equal to $A\widehat{\otimes}_k k\langle X_1 \cdots, X_n\rangle$ (where by X we mean $(X_1, \cdots, X_n)$). Note that $A\langle X_1, \cdots, X_n\rangle$ is the universal example of a "complete" non-archimedean topological A-algebra B equipped with an ordered n-tuple of power-bounded elements $b_1, \cdots, b_n$. By "complete" we mean that all Cauchy nets converge, and we implicitly include the separatedness assumption.

Rigid-analytic rational subdomains

Letting again A be a k-affinoid algebra, let $t_1, \cdots, t_n$ be units in A. Recall that A has an intrinsic Banach topology, though the "sup-norm" is only such a norm when A is reduced (and in the non-reduced case it is not a norm at all, since it vanishes on nonzero nilpotent elements). The Banach A-algebra

$$(*) \quad A\left\langle \frac{X_1}{t_1}, \cdots, \frac{X_n}{t_n} \right\rangle$$

corresponds to a rational subdomain of Sp $A\langle X_1, \cdots, X_n\rangle$, namely

$$\{(s, x) \in \mathrm{Sp}(A) \times \mathbb{A}_k^{n,\mathrm{an}} \mid |x_i| \leq |t_i(s)| \text{ for all } i\}$$

where by x we mean $(x_1, \cdots, x_n)$, and by $\mathbb{A}_k^{n,\mathrm{an}}$ we mean the *analytic* affine n-space over k. Informally, this is a family of closed polyndiscs parameterized by $\mathrm{Sp}(A)$, with fiber over $s \in \mathrm{Sp}(A)$ having polyradii $\{|t_i(s)|\}$.

Explicitly, $(*)$ is given by

$$\left\{ \sum_I a_I X^I \mid a_I t^I \to 0 \text{ as } |I| \to \infty \right\}.$$

This can be characterized in purely topological terms: it is the complete non-archimedean A-algebra $f : A \to B$ equipped with an ordered n-tuple of elements $b_1, \cdots, b_n \in B$ such that $b_i/f(t_i)$ is power-bounded for all i.

A slightly different variant

Let now $f_1, \cdots, f_n$ and g be elements of A with *no common zero*. Then we let

$$A\left\langle \frac{f_1}{g}, \cdots, \frac{f_n}{g} \right\rangle := \frac{A\langle X_1, \cdots, X_n\rangle}{\langle gX_i - f_i\rangle}.$$

Notice that since the f_i's and g have no common zero, g is a unit in $A\left\langle \frac{f_1}{g}, \cdots, \frac{f_n}{g} \right\rangle$ (as a zero of g in the MaxSpec of this k-affinoid algebra would project to a point in $\mathrm{Sp}(A)$ that is a common zero of the

f_i's, a contradiction). We can interpret $A\left\langle \frac{f_1}{g}, \cdots, \frac{f_n}{g} \right\rangle$ as the universal complete non-archimedean A-algebra in which g is a unit and f_i/g is power-bounded for all i.

Finally, back to Huber rings

Let A be a non-archimedean ring with *no* Hausdorff or completeness assumptions. We now address the problem of finding suitable universal mapping properties for analogous constructions as for the ones recalled above, but for "Tate algebras" over Huber rings. (The case of relative rational domains will be addressed next time.) For technical reasons, it really simplifies matters to first carry out constructions with universal topologies in the absence of completions, and then to bring in completions only afterwards.

This is essentially a game of appropriately topologizing the polynomial ring $A[X_1, \cdots, X_n]$, which we shall denote as $A[X]$ for brevity. In the sequel we shall indeed always write X for $(X_1, \cdots, X_n)$ when this will not cause confusion.

The idea is to describe the topology on $A[X]$ in terms of a base of neighbourhoods around 0 in A. We define:

$$U[X] := \left\{ \sum_{\text{finite}} u_I X^I \mid u_I \in U \right\} \subset A[X]$$

as *additive subgroups* of $A[X]$, where U runs through elements of a base of open neighbourhoods of 0 in A. We therefore wonder if we can make $U[X]$ into a neighbourhood base around $0 \in A[X]$ in order to generate a topological ring structure on $A[X]$.

We first observe that $\{f + U[X]\}_U$ is indeed a neighbourhood base around $\{f\}$ in $A[X]$, for all $f \in A[X]$, U running through a neighbourhood base of additive open subgroups of A. The only issue is to ensure continuity of the multiplicative structure on $A[X]$. More precisely:

Lemma 6.3.1 *Let R be a commutative ring, and $\Sigma := \{H_\alpha\}$ be a collection of additive subgroups of R. These are a base of neighbourhoods for a (necessarily unique) non-archimedean ring structure on R if and only if:*

(1) *for all $H, H' \in \Sigma$, $H \cap H'$ contains some H'' in Σ,*

(2) *for all $H \in \Sigma$, there exists $H' \in \Sigma$ such that $H \cdot H' \subseteq H$,*

(3) *for all $r \in R$ and all $H \in \Sigma$, there exists $H' \in \Sigma$ such that $rH' \subset H$.*

We remark that point (1) in the Lemma just encodes the fact that the H_α's are a base of neighbourhoods of 0 for an additive topological group structure on R. Point (2) just ensures that the multiplication of R is continuous at 0, and point (3) handles products of translates of the form

$$(r + x) \cdot (r' + x') = rr' + rx' + r'x + xx',$$

where rr' is already "small" by point (2) but $rx' + r'x + xx'$ must be handled.

Application: (uncompleted!) relative Tate algebras

Let A be a non-archimedean topological ring, and let $T_1, \cdots, T_n \subset A$ be finitely many finite nonempty subsets. For example T_i may be a singleton $\{t_i\}$ for all i.

We define, for every multi-index $I \in \mathbf{N}^n$:

$$T^I := T_1^{i_1} \cdot \cdots \cdot T_n^{i_n} := \{\xi_1 \cdot \cdots \cdot \xi_n \mid \xi_j \in T_j^{i_j},\ j = 1, \cdots, n\}$$

where $T_j^{i_j}$ is the set of all i_j-fold products (allowing repetitions) among elements in T_j.

Definition 6.3.2 We set

$$A[X]_T := A[X_1, \cdots, X_n]$$

as an A-algebra, equipped with a neighbourhood base of 0 given by

$$U[TX] := \left\{ \sum_{\text{finite}} a_I X^I \mid a_I \in T^I \cdot U \right\}$$

for open additive subgroups $U \subset A$, where by $T^I \cdot U$ we mean the set of all finite sums of monomials $\xi \cdot u$, for $\xi \in T^I$ and $u \in U$.

Remark 6.3.3 In Definition 6.3.2, one should think of the classical rigid setting, where (under mild hypotheses) the *completion* $A[X]_T$ is going to correspond to the adic space of points v with the property that $v(x_i t_{ij}) \leq 1$ for all $t_{ij} \in T_i$ and all i.

Remark 6.3.4 We warn the reader that the notation $U[TX]$ used to denote

$$\left\{ \sum_{\text{finite}} a_I X^I \mid a_I \in T^I U \text{ for all } I \right\}$$

is different from that in [Wed], where the same set is denoted by $U_{[X]}$ (suppressing the mention of T). Thus, when referring to [Wed] the reader should first read [Wed, Rem. 5.4.7] and keep in mind this Remark.

We now determine precisely when Definition 6.3.2 gives a topological ring structure on $A[X]_T$, which is to say when $A[X]_T$ satisfies the conditions in Lemma 6.3.1 with $\Sigma := \{U[TX]\}_U$ for U running over additive subgroups of A forming a neighbourhood base around 0 for the topology on A.

Property (1) always holds. Indeed, for all U, V open additive subgroups of A we have

$$U[TX] \cap V[TX] \supset (U \cap V)[TX].$$

Likewise, (2) always holds. This requires showing that if U is an open additive subgroup of A then there exists an open additive subgroup $V \subset A$ such that

$$V[TX] \cdot V[TX] \subset U[TX].$$

Consider V for which $V^2 \subset U$. Since $T^I \cdot T^J = T^{I+J}$ (with clear meaning of notation), for such V we just need to check that $vX^{I_0} \cdot V[TX] \subset U[TX]$ for any I_0 and $v \in V$, which is obviously true.

We now check exactly when (3) holds. Fix $f \in A[X]$ and $U \subset A$. We seek a $V \subset A$ such that

$$fV[TX] \subset U[TX].$$

Additivity is not an issue, so without loss of generality we can assume $f = aX^I$ for some $a \in A$ and some multi-index $I \in \mathbf{N}^n$. (This is why it is technically convenient that we focus now on topologizing merely a polynomial ring, postponing consideration of completions until later.)

Since $A[X]$ is commutative and A is a topological ring, we can actually absorb a in V and we are reduced to the case $f(X) = X^{I_0}$ for some fixed multi-index $I_0 \in \mathbf{N}^n$. By iteration, we may even assume $f(X) = X_{i_0}$ for some fixed $i_0 \in \{1, \cdots, n\}$. For every open additive subgroup $U \subset A$ we therefore seek a $V \subset A$ with the property that

$$X_{i_0} V[TX] \subset U[TX].$$

The typical element of $V[TX]$ will be, by definition, of the form

$$\sum_I a_I X^I,$$

where the sun runs over finitely many multi-indexes, and $a_I \in T^I V$ for such I's. We therefore seek V such that for all such a_I's we get

$$\sum_I a_I X^{e_{i_0}+I} \in U[TX]$$

where e_{i_0} is the standard vector with coordinates 0 at every slot but the i_0th, which is 1.

Since additivity is handled by definition of the $V[TX]$'s, we are reduced to analyzing each $a_I X^{e_{i_0}+I}$ separately. It is therefore necessary and sufficient to find $V \subset A$ such that whenever $a_I \in T^I V$ we have

$$a_I X^{e_{i_0}+I} \in U[TX].$$

This is equivalent to saying that for all I

$$T^I V \subset T_{i_0} T^I U,$$

which is equivalent to requiring that for every additive open subgroup $U \subset A$ there is an additive open subgroup $V \subset A$ such that

$$V \subset T_{i_0} U,$$

which is to say $T_{i_0} U$ is *open* for all such U and all i_0. If we iterate we can translate the condition of $A[X]_T$ being a (non-archimedean) topological ring to exactly the openness of $T^I \cdot U$ for all I and all U (or equivalently the openness of $T_{i_0} U$ for all i_0 and all U).

We now specialize to our case of interest: A a Huber ring. In this case we will show that $T \cdot U$ is open under a natural hypothesis that is reminscent of the condition "no common zeros" used to define rational domains in the rigid-analytic case (since an open ideal in an affinoid algebra is necessary the entire ring):

Proposition 6.3.5 *Let A be a Huber ring, and T a finite subset of A generating an open ideal. Then $T \cdot U$ is open in A for all open subgroups $U \subset A$.*

Proof. The ideal generated by T is $T \cdot A$, and this is open by hypothesis. But if S and S' are open subsets of A around 0 then the additive subgroup $S \cdot S'$ of finite sums of products ss' for $s \in A$ and $s' \in S'$ is open because A is Huber. (Indeed, if A_0 is a ring of definition and I is an ideal of definition in A_0 then by openness of S and S' there are large $n, n' > 0$ such that $I^n \subset S$ and $I^{n'} \subset S'$. Hence, the additive subgroup $S \cdot S'$ contains $I^n \cdot I^{n'} = I^{n+n'}$ and therefore is a neighborhood of 0 in A, so it is open.) Thus, $(T \cdot A)^n = T^n \cdot A$ is open for all $n \geq 1$ (where T^n means the set of n-fold products among elements of T).

Now fix $n > 0$ and an open subgroup U of A, and we aim to show that the additive subgroup $T^n \cdot U$ of A is open, or equivalently that it contains an open set around 0. Choose a ring of definition A_0, and choose an ideal of definition I of A_0 that is contained in the open subset $T^n \cdot A$ of A around 0. (Namely, pick some ideal of definition of A_0 and then replace it with a very large power.) We seek a large e so that $I^e \subset T^n \cdot A$. By definition of "ideal of definition", I has a *finite* set G of generators. Thus, $G \subset T^n \cdot A$ and hence we can find a *finite* subset $F \subset A$ such that $G \subset T^n \cdot F$ (the set of products of elements of F against n-fold products among elements of T). By finiteness of F and openness of U around 0, we can choose $m > 0$ large such that $F \cdot I^m \subset U$. Thus,

$$T^n \cdot U \supseteq T^n \cdot F \cdot I^m \supseteq G \cdot I^m = I^{m+1}$$

(the final equality by definition of G as a generating set for I as an ideal of the ring A_0). □

This concludes the proof that the assignement of $\Sigma := \{U[TX]\}_U$, for U running over the open additive subgroups of A, defines a base of neighbourhoods for a non-archimedean topological ring structure on $A[X]_T$, for A a Huber ring.

We have shown the following:

Proposition 6.3.6 *Let A be a Huber ring, and $T = \{T_1, \dots, T_n\}$ a collection of non-empty finite subsets of A such that $T_{i_0}U$ is open for all open additive subgroups $U \subset A$ and all i_0. Then the non-archimedean A-algebra $A[X]_T$ is initial among non-archimedean A-algebras $f : A \to B$ equipped with an ordered n-tuple of $b_1, \cdots, b_n \in B$ such that the elements*

$$f(t_{ij})b_i \in B$$

are power-bounded for all i and all $t_{ij} \in T_i$.

Proof. The essential point is to check that $A[X]_T$ satisfies the asserted power-boundedness property; the rest is then easy to check (and left to the reader). That is, for all i and all $t_{ij} \in T_i$ we claim that the element

$$t_{ij}X_i \in A[X]_T$$

is power-bounded. For all $m \geq 1$ we have

$$(t_{ij}X_i)^m \in T^{me_i}X^{me_i}$$

for any i and any $t_{ij} \in T_i$. On the other hand,

$$T^{I_0}X^{I_0}U[TX] \subset U[TX]$$

for all I_0 and all open additive subgroup $U \subset A$. We apply this to $I_0 = me_i$ for $m \geq 0$ to conclude. □

A non-example:

Example 6.3.7 Let $A := \mathbf{Z}_p[\![x]\!]$, equipped with the (p, x)-adic topology, and let $T := \{p\}$. Then T does not generate an open ideal, and calling $I := (p, x)$, none of the additive subgroups $pI^nA \subset A$, $n \geq 0$, is I-adically open.

Next time we will pass to completions, record the universal property as such (relative to maps from A to *complete* non-archimedean rings), and then especially carry out analogous constructions for certain rings of fractions (recovering a Huber-ring variant of the formation of rational domains in rigid-analytic geometry, this being the key reason we consider $T_1, \dots, T_n$ that are finite non-empty sets and not merely singletons inside A).

References

[Wed] T. Wedhorn, *Adic Spaces*, unpublished lecture notes.