

5 Lecture 5: Huber rings and continuous valuations

5.1 Introduction

The aim of this lecture is to introduce a special class of topological commutative rings, which we shall call Huber rings. These have associated spaces of valuations satisfying suitable continuity assumptions that will lead us to adic spaces later. We start by discussing some basic facts about Huber rings.

Notation: If S, S' are subsets of ring, we write $S \cdot S'$ to denote the set of finite sums of products ss' for $s \in S$ and $s' \in S'$. We define $S_1 \cdots S_n$ similarly for subsets S_1, \dots, S_n , and likewise for S^n (with $S_i = S$ for $1 \leq i \leq n$). For ideals in a ring this agrees with the usual notion of product among finitely many such.

5.2 Huber rings: preliminaries

We begin with a definition:

Definition 5.2.1 Let A be a topological ring. We say A is *non-archimedean* if it admits a base of neighbourhoods of 0 consisting of subgroups of the additive group $(A, +)$ underlying A .

Typical examples of non-archimedean rings are non-archimedean fields k and more generally k -affinoid algebras for such k .

Example 5.2.2 As another example, let $A := W(\mathcal{O}_F)$, where \mathcal{O}_F is the valuation ring of a non-archimedean field of characteristic $p > 0$. A base of open neighbourhoods of 0 required in Definition 5.2.1 is given by

$$p^n A = \{(0, \dots, 0, \mathcal{O}_F, \mathcal{O}_F, \dots), \text{ zeros in the first } n \text{ slots}\};$$

the p -adic topology on A is given by equipping $A = \prod_{n \geq 0} \mathcal{O}_F$ with the product topology in which each factor is discrete.

However, \mathcal{O}_F naturally comes with its *valuation* topology, and using this for the product topology yields a topology on A that can be described as follows. Let $\varpi \in \mathcal{O}_F$ be a *pseudo-uniformizer*; i.e., an element $\varpi \in F$ such that

$$0 < |\varpi| < 1,$$

(so \mathcal{O}_F has the ϖ -adic topology). Such an element exists because the valuation on F is non-trivial. For the Teichmüller lift

$$[\varpi] = (\varpi, 0, \dots, 0) \in A = \prod_{n=1}^{\infty} \mathcal{O}_F$$

it is an exercise to check that the $(p, [\varpi])$ -adic topology on A is the product topology for the valuation topology on \mathcal{O}_F .

For work with adic and perfectoid spaces we will need to topologize several algebraic constructions one can perform on A , such as localizations like $A[1/p] = W(\mathcal{O}_F)[1/p]$, or $A[1/[\varpi]]$. We will come to localizations next time.

The next examples give us motivation to define Huber rings.

Example 5.2.3 Let k be a non-archimedean field, and

$$A := k\langle X_1, \dots, X_n \rangle / I$$

be an affinoid Tate k -algebra. This has a unique k -Banach topology (defined via the residue norm via any such presentation, keeping in mind that all ideals in a Tate algebra are closed), though a k -Banach algebra norm is not canonical.

Consider the subring

$$\mathcal{O}_k\{X_1, \dots, X_n\}$$

of $k\langle X_1 \cdots X_n \rangle$ consisting of power series $\sum a_I X^I$ with coefficients in the valuation ring \mathcal{O}_k of k such that $a_I \rightarrow 0$ as $\|I\| \rightarrow \infty$. Note that for a pseudo-uniformizer ϖ , this subring is the ϖ -adic completion of $\mathcal{O}_k[X_1, \dots, X_n]$ (even when k is not discretely-valued, so \mathcal{O}_k is not noetherian).

Let A_0 be the image in A of $\mathcal{O}_k\{X_1, \dots, X_n\}$; this is the quotient of $\mathcal{O}_k\{X_1, \dots, X_n\}$ by $I \cap \mathcal{O}_k\{X_1, \dots, X_n\}$. Note that A_0 is an open subring of A with respect to the subspace topology (quotient maps of Banach spaces are open).

The topology on A_0 is the ϖ -adic topology, and the topology on A is “controlled” by A_0 , in the sense that a basis of neighbourhoods of 0 in A is given by $\varpi^n A_0$, for $n \geq 1$.

We are now ready to give the following:

Definition 5.2.4 A *Huber ring* A is a topological ring admitting an open subring $A_0 \subset A$ on which the subspace topology is the I -adic topology, for some finitely generated ideal I of A_0 . We call any such A_0 a *ring of definition* for the topology on A , or more briefly a *ring of definition*, and a couple (A_0, I) will be called a *couple of definition*.

Although Definition 5.2.4 may make it seem that the topology on such rings A is tightly linked to the assignment of a specific ring of definition A_0 , and of a finitely generated ideal in this latter, we will see that this is not really a problem. Let us first give a number of preliminary remarks:

Remark 5.2.5

- (1) Let $a \in A$. A base of open neighbourhoods of $\{a\}$ in A is given by

$$\{a + I^n\}_{n \geq 0}$$

where I^n is viewed as an ideal of A_0 , for each $n \geq 0$.

- (2) We fix a couple (A_0, I) as prescribed in Definition 5.2.4. Assigning separatedness and completeness assumption on A amounts to assigning them to the I -adic topology on A_0 . We *do not* assume the topology on A is separated neither complete! It is useful to avoid this assumption in order to handle algebraic constructions on A .
- (3) In [H1], Huber gives the following definition: a ring A is a Huber ring (f -adic ring, in the terminology adopted in [H1]) if there exists a subset U of A and a finite subset $T \subseteq U$ such that $\{U^n \mid n \geq 1\}$ is a fundamental system of neighborhoods of 0 in A and such that $T \cdot U = U^2 \subseteq U$. The advantage of such definition is that it does not mention a couple (A_0, I) as done in Definition 5.2.4, which gives us initial motivation to believe that the assignment of a *specific* ring of definition $A_0 \subset A$ and the I -adic topology on A_0 for some *specific* finitely generated ideal I of A_0 , is not so tightly related to the assignment of a Huber ring. See [Wed, Prop. 6.1] for a proof of the equivalence of the above definition with Definition 5.2.4.

5.3 Digression: continuity of valuations

Definition 5.3.1 Consider a valuation

$$v : A \rightarrow \Gamma \cup \{0\}$$

on a non-archimedean topological ring, and assume Γ is generated by $v(A - \text{supp}(v))$ (i.e., Γ is the “minimal” value group for v , so Γ is intrinsic to v as a point of $\text{Spv}(A)$). We say v is *continuous* if for all elements $a \in A$ and $\gamma \in \Gamma$ the subsets of A

$$\{x \in A \mid v(x - a) < \gamma\}$$

are open. (It is equivalent to consider such conditions with just $a = 0$.)

Note that a trivial valuation is continuous if and only if its support is an open prime ideal, since $\{v < 1\}$ is the support of v when $\Gamma_v = 1$.

Remark 5.3.2 Provided that $\Gamma \neq 1$, it is equivalent to require that the subsets

$$A_{\leq \gamma} := \{x \in A \mid v(x) \leq \gamma\}$$

are open for all $\gamma \in \Gamma$. Indeed, defining $A_{< \gamma}$: analogously, if $A_{< \gamma}$ is open then the additive subgroup $A_{\leq \gamma}$ is open, and conversely if Γ contains elements $\delta < \gamma$ then

$$A_{< \gamma} = \bigcup_{\delta < \gamma} A_{\leq \delta}.$$

Such δ exist precisely because we assumed $\Gamma \neq 1$.

On the other hand, when $\Gamma = 1$ then $A_{\leq 1} = A$ and $A_{< 1} = \text{supp}(v)$, and the former is always open whereas the latter may not be so.

In Lecture 7 we shall study the “continuous valuation spectrum” of a commutative non-archimedean topological ring A :

$$\text{Cont}(A) := \{v \in \text{Spv}(A) \mid v \text{ continuous}\} \subset \text{Spv}(A).$$

This is functorial in A in the sense that for a continuous map $f : A' \rightarrow A$ between non-archimedean rings and $v \in \text{Cont}(A)$ the valuation $v' := v \circ f \in \text{Spv}(A')$ is continuous. Indeed, the value group Γ' of v' is a subgroup of the value group Γ of v and for any $\gamma' \in \Gamma'$ clearly $A'_{< \gamma'} = f^{-1}(A_{< \gamma'})$ is open. Huber rings are non-archimedean rings, and for such rings A we will see later that $\text{Cont}(A)$ endowed with the subspace topology is a spectral space (hence qcqs). The quasi-compactness properties of the “adic spectrum” $\text{Spa}(A)$ of Huber rings A (to be defined later, using some additional structure on A that will have to be incorporated into the notation) will rest on this fact.

5.4 Boundedness

We recall that given a non-archimedean field k , and a k -affinoid algebra A , the set

$$A^0 := \{a \in A \mid \|a\|_{\text{sup}} \leq 1\} \subset A$$

consists of precisely the power-bounded elements of A (that is, those elements the set of whose powers is bounded in A in terms of the Banach topology on A). Provided that A is *reduced*, the sup-norm is a k -Banach algebra norm and thus A^0 is its unit ball. (For reduced A , whether or not this norm arises as the residue norm for a presentation of A as a quotient of a Tate algebra is a subtle question; it amounts to asking if A^0 is topologically of finite type over \mathcal{O}_k . One obvious obstruction is the possibility that there may exist $a \in A$ such that $\|a\|_{\text{sup}} \notin |k|$. This is the only obstruction when k is “stable” [BGR, 6.4.3/1]. See [BGR, 3.6.1] for examples of non-stable k and sufficient criteria for stability; stability holds in the discretely-valued case.)

Let $A_0 \subset A$ be a ring of definition. Then from a quotient presentation of A as in Example 5.2.3:

$$A \simeq k\langle T \rangle / I;$$

here, T is a finite set of indeterminates. Note that boundedness for a subset of A is equivalent to being contained in some $\varpi^{-n}A_0$ for some sufficiently large positive integer n .

Remark 5.4.1 Any ring of definition A_0 for a k -affinoid algebra A is contained in A^0 . We will give an analogue of this for Huber rings, but first we have to define boundedness without the crutch of a Banach space structure (since it is *important* to consider Huber rings for which I might not be able to be taken as principal; e.g., $\mathbf{Z}_p[[x]]$ with the (p, x) -adic topology).

Definition 5.4.2 Let A be a Huber ring. A subset $\Sigma \subset A$ is *bounded* if for every open neighbourhood U of 0 in A there exists an open neighbourhood V of 0 in A such that

$$V \cdot \Sigma \subset U.$$

(Here $V \cdot \Sigma$ denotes the additive span of the set of products vs for $v \in V$ and $s \in \Sigma$, and since we can shrink U to be an additive subgroup of A it is equivalent to demand that for every U there is some V such that $vs \in U$ for all $v \in V$ and $s \in \Sigma$.)

We say that Σ is *power-bounded* if the set of products

$$\{s_1 \cdots s_n \mid s_i \in \Sigma, i = 1, \dots, n, n \geq 1\}$$

is bounded. For example, $\{a\}$ is power-bounded with $a \in A$ precisely when the set $\{a, a^2, a^3, \dots\}$ is bounded.

If $\Sigma, \Sigma' \subset A$ are bounded subsets then so is the additive span $\Sigma \cdot \Sigma'$ of the set of products ss' for $s \in \Sigma$ and $s' \in \Sigma'$. Indeed, pick an open $U \subset A$, so there exists an open $V \subset A$ around 0 such that $V \cdot \Sigma \subset U$. There is also an open $V' \subset A$ around 0 such that $V' \cdot \Sigma' \subset V$, so $V' \cdot (\Sigma \cdot \Sigma) \subset (V' \cdot \Sigma') \cdot \Sigma \subset V \cdot \Sigma \subset U$. We will often use this fact without comment. By the same reasoning, if $\Sigma_1, \dots, \Sigma_n$ is a finite collection of bounded subsets of A then the additive span $\Sigma_1 \cdots \Sigma_n$ of the set of products $\prod s_i$ for $s_i \in \Sigma_i$ is also bounded. In particular, Σ^n is bounded for each $n \geq 1$ when Σ is bounded; power-boundedness is of course a much stronger condition on Σ .

Boundedness and continuous maps of Huber rings

In contrast with the case of morphisms of k -affinoid algebras in classical rigid geometry (where all maps are bounded between Banach spaces), not all continuous homomorphisms between Huber rings respect topological properties such as boundedness!

The issue is that if $f : A \rightarrow B$ is a continuous map between Huber rings, it is not clear how to control open neighbourhoods of 0 in B using just $f(A)$. This makes it hard to control boundedness of $f(\Sigma)$ in B for a bounded subset Σ in A . (One case where all works well is when f is surjective and B has the quotient topology from A .) Here is an example:

Example 5.4.3 Let $A = \mathbf{Q}_p$ with the (0) -adic topology, and let $B = A$ with the p -adic topology. So A has the discrete topology, and it is Huber because it is trivially adic, with ring of definition $A_0 = A$. Any ring homomorphism $A \rightarrow B$ is therefore continuous, but it cannot be adic. Call f the set-theoretic identity map $A \rightarrow B$. Then $f(A) = B$, which is not bounded in B while A is bounded in A .

Adic homomorphisms of Huber rings

Here is a particularly nice class of continuous homomorphisms between Huber rings, called *adic*, which avoids the problem in the preceding example (and is naturally motivated by experience with formal schemes as well).

Definition 5.4.4 Let A and B be Huber rings. A ring homomorphism $f : A \rightarrow B$ is called *adic* if there exist rings of definitions A_0 of A and B_0 of B such that $f(A_0) \subset B_0$ and for some ideal of definition $I \subset A_0$ the ideal $f(I)B_0$ is an ideal of definition in B_0 .

Any adic map of Huber rings is trivially continuous. It is also easy to check (exercise!) that if $f : A \rightarrow B$ is adic and A_0 and B_0 are rings of definition such that $f(A_0) \subset B_0$ then $f(I)B_0$ is an ideal of definition of B_0 for *any* ideal of definition I of A_0 ; this is very important in practice.

For any continuous map of Huber rings it is immediate from the definitions that there exist rings of definition $A_0 \subset A$ and $B_0 \subset B$, and ideals of definition $I \subset A_0$ and $J \subset B_0$, such that $f(A_0) \subset B_0$ and $f(I) \subset J$. However, it is not true in general that $f(I)B_0$ is still an ideal of definition. For a homomorphism between two Huber rings, being adic is a strictly stronger condition than just being continuous. By design, an adic map between Huber rings carries bounded sets to bounded sets.

Some examples and non-examples of adic maps:

Example 5.4.5

- (1) Consider the inclusion $f : \mathbf{Z}_p \rightarrow \mathbf{Z}_p[[x]]$, with the p -adic and (p, x) -adic topologies respectively. The map f is manifestly continuous, but not adic.
- (2) On the other hand, if we consider, analogously, f to be the inclusion $\mathbf{Z}_p \rightarrow \mathbf{Z}_p\{x\}$, then f is adic, as the topology on $\mathbf{Z}_p\{x\}$ is the p -adic topology.

Remark 5.4.6 Next time we shall study the following special class of Huber rings: a Huber ring A is *Tate* if there exists a topologically nilpotent unit $u \in A$ (beware that if u lies in a ring of definition A_0 then u cannot be a unit of A_0 when $A \neq 0$, as there are no topologically nilpotent units in A_0 when $A_0 \neq 0$; why not?). For example, if A is an affinoid algebra over some non-archimedean field k , then any pseudo-uniformizer ϖ is a topologically nilpotent unit. Tate rings are the class of Huber rings which exhibit behavior akin to Banach algebras.

If $f : A \rightarrow B$ is a continuous homomorphism of Tate rings then it is in fact adic! We prove a little more.

Proposition 5.4.7 *Let $f : A \rightarrow B$ be a continuous homomorphism of Huber rings. If A is Tate then so is B , and f is adic.*

Proof. The proof is immediate from the definitions! If we let A_0 and B_0 be rings of definitions such that $f(A_0) \subset B_0$ (given by the continuity assumption on f) and if $u \in A$ is a topologically nilpotent unit, then so is $f(u)$, thus proving that B is Tate.

Since A_0 is open, we may replace u by some power and assume $u \in A_0$, so that $f(u) \in B_0$. Multiplication by the unit u on A is a homeomorphism, so the ideal uA_0 in A_0 is *open* yet its powers tend to 0 (since u is topologically nilpotent and A_0 has its topology defined by powers of an ideal). Thus, uA_0 is an ideal of definition of A_0 . By the same reasoning, $f(u)B_0$ is an ideal of definition of B_0 . But this latter ideal is $f(uA_0)B_0$, so f is indeed adic. \square

Proposition 5.4.8 *Let A be a Huber ring. An open subring $A_0 \subset A$ is a ring of definition if and only if $A_0 \subset A$ is bounded.*

Proof. See [H1, Prop. 1] or [Wed, Lemma 6.2] for proofs. The proof makes use of the equivalent definition of Huber ring introduced in Remark 5.2.5. \square

Remark 5.4.9 Let A be a Huber ring, and (A_0, I) be given as usual. Then for any *open* subring $B \subset A$ we have $B \supset I^n$ for sufficiently large positive integers n . Such containment does *not* necessarily mean that I^n is an ideal of B : it will certainly be an ideal of A_0 , and the containment is purely set-theoretic.

As a corollary of Proposition 5.4.8, if A_0 and A_1 are two distinct rings of definition for the Huber ring A then we can produce more rings of definition from them. More precisely, we have the following Corollary whose proof is left as an exercise to the reader:

Corollary 5.4.10 *If A_0 and $A_1 \subset A$ are rings of definition for A , then so are $A_0 \cap A_1$ and $A_0 \cdot A_1$, where this latter “product ring” is the subring of A generated by the finite sums of products of elements of A_0 and A_1 respectively.*

(If $I_j \subset A_j$ is an ideal of definition then it isn’t clear if $I_0 \cap I_1$ is finitely generated, so we don’t know if it is an ideal of definition of $A_0 \cap A_1$; however, this open intersection is obviously bounded, so Proposition 5.4.8 applies. For the open subring $A_0 \cdot A_1$ that is visibly bounded, the finitely generated ideals $I_0 \cdot A_1$ and $A_0 \cdot I_1$ are easily seen to be open and in fact ideals of definition.)

Remark 5.4.11 Let A be a commutative ring endowed with the I -adic topology with respect to some nonzero proper finitely generated ideal I (so A is Huber). Then a somewhat bizarre ring of definition for A is $\mathbf{Z} + I$. Indeed, say A is a Huber ring, with ring of definition A_0 and ideal of definition $I \subset A_0$. Then $\mathbf{Z} + I \subset A_0$ is an open subring of A , and it is trivially bounded, so by Proposition 5.4.8 it is a ring of definition for A .

The property for a topological ring of “being a Huber ring” is inherited by open subrings, as explained in the following:

Proposition 5.4.12 *It A is a Huber ring, so is any open subring $B \subset A$.*

Proof. We pick a couple (A_0, I) , with I a finitely generated ideal of definition in A_0 . Since B is open, there is a sufficiently large integer $n > 0$ such that $I^n \subset B$. Consider the pair $(B \cap A_0, I^n)$. Since I^n is an ideal of A_0 contained in B , it is certainly an additive subgroup of $B \cap A_0$. Moreover, $BI^n \subseteq B$ and $(B \cap A_0)I^n \subseteq A_0$, so $(B \cap A_0)I^n \subseteq B \cap A_0$. Hence, I^n is an open ideal in $B \cap A_0$, and its powers go to 0 (as we can check in A_0 , so its adic topology is the one on $B \cap A_0$). But it is finitely generated *as an ideal of $B \cap A_0$* ? Not so clear! However, if f_1, \dots, f_r is a finite set of generators of I as an ideal of A_0 then every element of I^{2n} is a linear combination of n -fold products among the f_j ’s with coefficients in $I^n \subset B \cap A_0$, so the finite set of such n -fold products generates an ideal J of $B \cap A_0$ sandwiched between I^{2n} and I^n . This finitely generated ideal does the job. \square

We now prove a few important facts concerning Huber rings and subrings of definition. The second part of the following result provides an ample supply of rings of definition (in view of their characterization as the open bounded subrings).

Theorem 5.4.13 *Let A be a Huber ring, and let A^0 be the subset of power-bounded elements in A .*

- (1) A^0 is a subring of A , and it is the union of all the rings of definition of A .
- (2) Let $B \subset A$ be a bounded subring, and let B' be an open subring of A containing B . Then there exists a ring of definition A_0 of A which is contained in B' and contains B .

Proof. By Proposition 5.4.8, for (1) it suffices to show that A^0 is the union of all bounded subrings of A . Explicitly, we just have to check that if A_0 is a ring of definition and $a \in A$ is power-bounded then the open subring $A_0[a]$ is bounded (hence also is a ring of definition). The set of powers of a is bounded, so we just need to check that if Σ and Σ' are bounded subsets of A then the set $\Sigma \cdot \Sigma'$ of

sums of pairwise products is bounded. There is a base of open additive subgroups around 0, so the boundedness of $\Sigma \cdot \Sigma'$ is clear.

The proof of (2) is left to the reader as an exercise! \square

A consequence of Theorem 5.4.13 is that as soon as A^0 is a bounded subring of A , then it is a ring of definition (as it is obviously open). It may well happen, however, that A^0 is not bounded. For example, when A is not reduced: let $A = \mathbf{Q}_p[\varepsilon]$, with $\varepsilon^2 = 0$. Then

$$A^0 = \mathbf{Z}_p \oplus \mathbf{Q}_p \cdot \varepsilon,$$

which is not bounded (this will be the reason why perfectoid algebras over a perfectoid field will be reduced).

The case in which A^0 is bounded is quite important (it will indeed include the case of perfectoid algebras) and it deserves a name:

Definition 5.4.14 A Huber ring A is called *uniform* if A^0 is bounded in A .

Uniformity turns out to be a quite handy condition. In [BV], K. Buzzard and A. Verberkmoes show that a “localizable” version of uniformity can be used to prove sheaf-like properties of (pre-)adic affinoid spaces over uniform Tate rings without Noetherian conditions! The reader may refer to [BV, Cor. 5 and Thm. 7]; we will address this later.

5.5 Robustness of the Huber condition

The Huber condition (we shall often name this way the condition on a commutative ring A of “being Huber”) behaves very well under many operations on topological rings. We shall now discuss completions and tensor products; localizations and “polynomial rings” will be treated next time.

Completions of Huber rings

We consider a Huber ring A , with a ring of definition A_0 and a finitely generated ideal of definition $I \subset A_0$. Note that A_0 is an open subring of A , and I is an ideal of A_0 but not necessarily of A , so usually it does not make sense to take the I -completion of the ring A in the sense of commutative algebra.

On the other hand, we *can* do so for A_0 , though such completion may be quite wild, due to lack of Noetherianity. We want to formulate a reasonable sense of I -adic completion of A , in which A_0^\wedge should be a natural candidate for a subring of definition. This is handled by:

Definition 5.5.1 Let A be a Huber ring, A_0 a subring of definition with the I -adic topology, for I a finitely generated ideal of A_0 . We let

$$A^\wedge := \varprojlim A/I^n$$

be the I -adic completion of A in the sense of *topological groups*. This has a natural structure of topological ring (exercise!), and we define the *completion* of the Huber ring A to be A^\wedge .

Remark 5.5.2 It is not clear from Definition 5.5.1 what a ring of definition of A^\wedge might actually be, and especially what topology this may have. Consider A_0^\wedge , the commutative algebra I -adic completion of A_0 . This is visibly an open subring of \widehat{A} but A_0 may not be Noetherian, so it is not clear if the topology on A_0^\wedge is the I -adic topology.

Thanks to the finite generation of I , all works out well:

Theorem 5.5.3 *Let A be a Huber ring, and (A_0, I) be a couple of definition. Then, with notation as in Definition 5.5.1:*

- (1) *The open subring $A_0^\wedge \hookrightarrow A^\wedge$ has the $I(A_0^\wedge)$ -adic topology and for all $n \geq 1$ the natural map*

$$A_0/I^n \xrightarrow{\cong} A_0^\wedge/I^n(A_0^\wedge)$$

is an isomorphism; in particular, \widehat{A}_0 is I -adically separated and complete.

- (2) *The natural map of rings*

$$A \otimes_{A_0} A_0^\wedge \rightarrow A^\wedge$$

is an isomorphism.

Proof. The hard part of the proof is point (1), for which we refer the reader to [SP, Lemma 10.91.7]. (which also treats arbitrary modules over A , and uses a method entirely different from the Artin–Rees technique in commutative algebra). The proof of (2) is a little bit tricky, and is explained in [H1, Lemma 1.6]. \square

Tensor products of Huber rings

The setting is now the following: let A, B , and C be Huber rings, with *adic* morphisms $f : B \rightarrow A$ and $g : B \rightarrow C$. We may choose a couple of definition (B_0, J) for B and subrings of definition $A_0 \subset A$ and $C_0 \subset C$ such that $f(B_0) \subset A_0$ and $g(B_0) \subset C_0$. Note that $f(J)A_0$ and $g(J)C_0$ are automatically ideals of definition in A and C respectively, since f and g are *adic*.

We have:

Theorem 5.5.4 *Let $J \subset B_0 \subset B$, $A_0 \subset A$ and $C_0 \subset C$ be as above. Then equipping the subring*

$$\text{image}(A_0 \otimes_{B_0} C_0 \rightarrow A \otimes_B C)$$

of $A \otimes_B C$ with the J -adic topology and declaring it to be open makes $A \otimes_B C$ into a Huber ring. This Huber ring satisfies the expected universal property in the category of topological rings with continuous homomorphisms.

By means of the universal property one sees that the construction of this topology on $A \otimes_B C$ is independent of the choices of compatible rings of definition.

The next example shows that assuming in Theorem 5.5.4 that f and g are *adic* maps is necessary in order that the construction makes sense with appropriate topological properties.

Example 5.5.5 Let $A = \mathbf{Z}_p[[x]]$ be given the (p, x) -adic topology and define $C = \mathbf{Q}_p$ and $B = \mathbf{Z}_p$ with their usual topologies. Set $A_0 = A$, $B_0 = C_0 = \mathbf{Z}_p$, and $J = (p)$. Take f and g to be the natural inclusions, so $f : B \rightarrow A$ is not *adic* (as remarked in Example 5.4.5). Let's make the tensor product $A \otimes_B C = \mathbf{Z}_p[[x]][1/p]$ into a topological ring in which $\mathbf{Z}_p[[x]] = \text{image}(A_0 \otimes_{B_0} C_0 \rightarrow A \otimes_B C)$ is given the J -adic topology. Then the natural map $A \rightarrow A \otimes_B C$ isn't even continuous (e.g., x is topologically nilpotent in A but no power lies in the image of $J(A_0 \otimes_{B_0} C_0)$).

Suppose instead we try to make $A \otimes_B C$ into a topological ring so that $A_0 \otimes_{B_0} C_0 = A_0 = \mathbf{Z}_p[[x]]$ is an open subring in which (inspired by the case of completed tensor product topologies used to build fiber products of formal schemes) the topology is defined by powers of $I \otimes_{B_0} C_0 + A_0 \otimes_{B_0} J = I = (p, x)$. That is, we want $\mathbf{Z}_p[[x]][1/p]$ to be a topological ring in which $\mathbf{Z}_p[[x]]$ with its (p, x) -adic topology is an open subring. There is no such topological ring structure on $A \otimes_B C$! Indeed, were there such a structure then multiplication by the unit p would be an automorphism, so $p\mathbf{Z}_p[[x]]$ would have to be open in $\mathbf{Z}_p[[x]]$ with its (p, x) -adic topology. But that openness is false.

From a geometric point of view, the problem is that $\mathrm{Spa}(\mathbf{Z}_p[[x]]) \times_{\mathrm{Spa}(\mathbf{Z}_p)} \mathrm{Spa}(\mathbf{Q}_p)$ should correspond to the Berthelot “generic fiber” $\{|x| < 1\}$ of the formal unit disc over \mathbf{Z}_p , but this open disc is not affinoid and hence its structure should not be controlled by a ring. This is a prelude to the fact that fiber products among adic spaces do not exist in the same degree of generality that they do for schemes (or for formal schemes or rigid-analytic spaces), and when they do exist their construction entails some subtleties.

Localizations of Huber rings

We will discuss localizations of Huber rings in detail next time, providing in the process an abundant supply of new Huber rings from old ones (much like passing to rational domains in rigid-analytic geometry produces many new affinoid algebras from old ones).

Let us end by setting up the context via motivation from the case A is an affinoid algebra over some non-archimedean field k . We recall that a rational domain in $X := \mathrm{Sp}(A)$ is given as the locus U of those points of X over which a given collection of $f_1, \dots, f_n \in A$ satisfies

$$|f_1| \leq g, \dots, |f_n| \leq g$$

where $g \in A$, and f_1, \dots, f_n, g have no common zero. (Among other things, the absence of a common zero avoids any “pinching” in how this locus lies in X ; e.g., it underlies why such a locus is actually an open subset of X for the naive totally disconnected topology on classical geometric points, and also underlies why mapping into such loci can be classified by an affinoid algebra, due to a “minimum modulus principle” for a finite collection of analytic functions without a common zero on an affinoid.)

More precisely, calling

$$(*) \quad A \left\langle \frac{f_i}{g} \right\rangle := \frac{A \langle X_1, \dots, X_n \rangle}{\langle gX_i - f_i \rangle}$$

we have

$$U = \mathrm{Sp} \, A \left\langle \frac{f_i}{g} \right\rangle,$$

and the Banach A -algebra $A \langle f_i/g \rangle$ is intrinsically attached to the subset $U \subset X$ in the sense that for any k -affinoid B equipped with a (necessarily continuous) k -algebra homomorphism $f : A \rightarrow B$, the map $\mathrm{Sp}(f)$ factors through $U \subset X$ set-theoretically if and only if f factors through $A \langle f_i/g \rangle$ as an A -algebra (in which case such a factorization is unique).

The condition on f_1, \dots, f_n, g not having common zeros being rephrased in the condition:

$$1 \in Ag + \sum_{i=1}^n Af_i.$$

Given a Huber ring A , and $f_1, \dots, f_n, g \in A$, the construction of a suitable analogue of $(*)$ will be in the same spirit but we will impose the condition that the ideal generated by g and the f_i 's is merely *open*. (In a k -affinoid algebra, the only open ideal is the entire ring; this follows from precisely the “Tate ring” property!) The significance of the condition “generates an open ideal” will be addressed next time, when we carry out appropriate definitions for Huber rings and even general non-archimedean rings without the crutch of Banach topologies (or even a Tate-ring hypothesis).

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