3 Lecture 3: Spectral spaces and constructible sets

3.1 Introduction

We want to analyze quasi-compactness properties of the valuation spectrum of a commutative ring, and to do so a digression on constructible sets is needed, especially to define the notion of constructibility in the absence of noetherian hypotheses. (This is crucial, since perfectoid spaces will not satisfy any kind of noetherian condition in general.) The reason that this generality is introduced in [EGA] is for the purpose of proving openness and closedness results on the locus of fibers satisfying reasonable properties, without imposing noetherian assumptions on the base. One first proves constructibility results on the base, often by deducing it from constructibility on the source and applying Chevalley's theorem on images of constructible sets (which is valid for finitely presented morphisms), and then uses specialization criteria for constructible sets to be open. For our purposes, the role of constructibility will be quite different, resting on the interesting "constructible topology" that is introduced in [EGA, IV_1, 1.9.11, 1.9.12] but not actually used later in [EGA].

This lecture is organized as follows. We first deal with the constructible topology on topological spaces. We discuss useful characterizations of constructibility in the case of spectral spaces, aiming for a criterion of Hochster (see Theorem 3.3.9) which will be our tool to show that $\mathrm{Spv}(A)$ is spectral, our ultimate goal.

Notational convention. From now on, we shall write everywhere (except in some definitions) "qc" for "quasi-compact", "qs" for "quasi-separated", and "qcqs" for "quasi-compact and quasi-separated". (We wlll review the meaning of quasi-separatedness below.)

3.2 Constructible sets in topological spaces

Recall the following basic fact:

Proposition 3.2.1 *Let A be a commutative ring. An open subset $U \subset \mathrm{Spec}(A)$ is qc if and only if $\mathrm{Spec}(A) - U$ has the form $\mathrm{Spec}(A/I)$, for some finitely generated ideal I of A.*

Proof. If I is finitely generated, then we can cover U with the non-vanishing loci of each of a finite set of generators of I. Conversely, if U is qc then if we consider the open affine cover $U = \cup_i D(f_i)$ by basic affine opens, from which we can extract a finite subcover $D(f_1) \cup \cdots \cup D(f_N)$, so the complement of U is the zero scheme of the ideal $I = (f_1, \ldots, f_N)$. □

The above remark should immediately remind one that if A is non-Noetherian, it may well happen that some open subset of $\mathrm{Spec}(A)$ is not qc (although $\mathrm{Spec}(A)$ is). As a typical example, one can consider the following non-Noetherian topological space:

Example 3.2.2 Let A be the polynomial ring over a field k in infinitely many indeterminates:

$$A = k[x_1, x_2, \ldots]$$

We write $\mathbb{A}_k^\infty := \mathrm{Spec}(A)$. The open complement U of the closed point $(0, 0, \ldots)$ is not qc, as no finite open subcover of the affine open cover $\{D(x_i)\}_{i \geq 1}$ can cover U.

Definition 3.2.3 A continuous map between topological spaces $f : X' \to X$ is *quasi-compact* if for every quasi-compact open subset $U \subset X$, $f^{-1}(U)$ is quasi-compact.

Concretely, assuming X has a base of qc open subsets, then checking quasi-compactness according to Definition 3.2.3 is equivalent to checking the stated condition for U running over the elements of that base.

Example 3.2.4 If we glue two copies U_1 and U_2 of $\mathbb{A}_k^\infty$ along the identity morphism on U, we obtain a scheme X over k whose diagonal map $X \to X \times_k X$ is not qc (as the two copies $U_i \subset X$ of $\mathbb{A}_k^\infty$ have overlap U that is not qc, so $\Delta_{X/k}^{-1}(U_1 \times U_2)$ is not quasi-compact).

To define the notion of "constructible" subset in a general topological space (going beyond the familiar noetherian case), we want to allow not arbitrary closed sets but only those whose open complement is quasi-compact (as the idea behind constructibility is that it encodes what can be described with "a finite amount of information"; cf. Proposition 3.2.1). As a first step towards making such a definition, we introduce:

Definition 3.2.5 A subspace Y of X is *retrocompact* if the inclusion map $Y \hookrightarrow X$ is quasi-compact.

Remark 3.2.6 If X admits a base of qc open subsets then to check retrocompactness of a subspace it is enough to do against the members of that base. Note also that all qc open subsets of X are retrocompact in X if and only if $U \cap V$ is qc for every pair of qc open subsets U and V in X. If X is a *separated* scheme then the inclusion of any affine open subscheme $U \to X$ is qc (as any two open affines in X have affine overlap), and so U is retrocompact in X.

Example 3.2.7 A first example of a qc but not retrocompact open affine in a scheme X is given by Example 3.2.4 (for which X is of course not separated). In that example we glued two copies of $\mathbb{A}_k^\infty$ along the identity on $U = \mathbb{A}_k^\infty - \{0\}$ to obtain a scheme X over k. The embedding of either of the two copies of $\mathbb{A}_k^\infty \hookrightarrow X$ is not retrocompact.

Definition 3.2.8 A topological space X is *quasi-separated* if every quasi-compact open subset is retrocompact.

If X has a base of qc open subsets, then we can check quasi-separatedness using members of the base. Provided such a base exists, Definition 3.2.8 is equivalent to saying that the diagonal map $\Delta : X \to X \times X$ is qc. (If X is a scheme over an affine base S then it is equivalent to work with the diagonal morphism $X \to X \times_S X$.)

If X is qc then a retrocompact subset is precisely a qc subspace Y such that $Y \cap U$ is qc for all qc open subsets U of X. In particular, if X is qcqs then an open subset is retrocompact if and only if it is qc. We regard non-qs spaces as pathological (for the purposes of algebraic geometry), so as long as we work with qcqs spaces we do not need to speak of retrocompactness for open sets: it is then just qc by another name (and so the word "retrocompact" is not often seen in practice).

Finally, we arrive at:

Definition 3.2.9 A subset Y of a topological space X is *constructible* if it is a finite Boolean combination of retrocompact open sets.

Remark 3.2.10 Let X be a topological space.

(1) In concrete terms, a subspace $Y \subset X$ is constructible if and only if Y is of the form

$$Y = \bigcup_{i=1}^{r} (U_i \cap (X - V_i))$$

for retrocompact open subsets $U_i, V_i \subset X$ (see [EGA, IV_3, Prop. 9.1.3]).

(2) If X is qc and $Y \subset X$ is constructible then Y is qc.

(3) If X is qcqs (so every qc open in X is retrocompact), then a closed $Z \subset X$ is constructible if and only if the open set $X - Z$ is qc (due to (2)).

(4) If X is qcqs then the constructible subsets of X are exactly the finite Boolean combinations of *quasi-compact* open subsets of X (so one doesn't need to speak of retrocompactness in such cases). This is the typical situation of interest. (Recall the exercise from Hartshorne's textbook on algebraic geometry that a topological space is noetherian precisely when all subspaces are quasi-compact, so in the noetherian setting we recover the usual notion of constructibility.)

As an example of Remark 3.2.10(3), if $X = \operatorname{Spec}(A)$ then X is qcqs and the constructible closed subsets of X are exactly those of the form

$$\operatorname{Spec}\left(A/(f_1, \dots, f_n)\right),$$

as these are exactly the closed sets having qc open complement (by Proposition 3.2.1).

Definition 3.2.11 The *constructible topology* on a topological space X is the topology on X generated by the constructible sets in X (i.e., a base of opens is given by the constructible subsets). We denote by X_{cons} the set X endowed with the constructible topology.

As before, we note that such topology is interesting only in the case X has a basis of qc open subsets! (so that we can check constructibility using a single open cover by qc open subsets, and in general we can use them to probe the topology on X).

Observe that the identity map $X_{\text{cons}} \to X$ is continuous if and only if every open subset of X is a union of constructible sets; this holds when X is qs with a base of qc open subsets (as then every qc open subset is constructible and there are "enough" such open subsets), as will be the case for all X we will care about.

Some examples: constructible topology on schemes

In this paragraph we discuss the details of two examples, in which we make explicit the constructible topology on the spectrum of a Noetherian domain of dimension 1, and on $\mathbb{A}^2_k$. These are qcqs, as is any noetherian scheme.

Example 3.2.12 Let $X = \operatorname{Spec}(A)$, for A a Noetherian domain of dimension 1. Then we have just the generic point η corresponding to the prime ideal (0), and the closed points

$$x \in X_{\text{closed}} =: X^0,$$

corresponding to the maximal ideals $\mathfrak{m}_x$ of A.

Since X is noetherian, the constructible subsets are exactly the finite Boolean combinations of open subsets, so all open subsets and all closed subsets are constructible. It follows at once that X^0 is discrete in X_{cons} (since every closed point of X is both open and closed in X_{cons}).

Let us deal with the generic point η. If we pick any closed point $x \in X^0$, then $\{x\}$ is clopen, and its (constructible!) open complement U in X contains η set-theoretically. Hence, we can separate η from any other point of X via a pair of disjoint opens in the constructible topology, thus implying that X_{cons} is Hausdorff. Since every constructible set containing η must contain a dense open of X (see Remark 3.2.10), the Hausdorff X_{cons} is seen to be the 1-point compactification of the discrete space X^0. In particular, X_{cons} is Hausdorff and qc.

In the above example, we have seen in concrete a general fact (which we will record with a reference for general proof later): the constructible topology on a spectral space is always Hausdorff qc. Here is another illustration:

Example 3.2.13 Let k be a field, and $X = \mathbb{A}^2_k$ the affine plane over k (hence qcqs, like any noetherian topological space). Exactly as before, all the closed points X^0 of X are clopen in the constructible topology on X. The remaining points of X are the generic points η_C of the irreducible closed curves $C \subset X$ and the generic point η of X.

Let us choose an irreducible curve $C \subset X$. We claim that we can separate η_C from any other point of X_{cons} (i.e., using a pair of disjoint open sets in X_{cons}). Obviously we can separate it from the points $x \in X^0$ (as they are clopen in X_{cons}). Likewise, if $C' \neq C$ then $C' \cap C$ is a finite set of closed points and hence $C - (C \cap C')$ and $C' - (C \cap C')$ are disjoint constructible sets (open in X_{cons}!) that separate η_C from $\eta_{C'}$. Finally, $\{C, X - C\}$ is a pair of disjoint constructible sets separating η_C and η.

This also implies that η can be separated in X_{cons} from all the other points, since it can be separated from the closed points and from the generic points of the irreducible curves. We conclude that X_{cons} is Hausdorff.

Let's show that X_{cons} is quasi-compact. For any open cover $\{U_i\}$, to make a finite subcover we can first refine the cover to consist of constructible sets. Any constructible set U_{i_0} containing the generic point must contain a dense Zariski-open U, and $X - U$ has Zariski closure that is a union of finitely many irreducible closed curves $C_1, \ldots, C_n$ and closed points. But for any closed subset of X, the subspace topology on it from X_{cons} is clearly its own constructible topology, so we are reduced to prove the quasi-compactness of C_{cons} for irreducible closed curves in X. These are instances of Example 3.2.12.

[pictures coming soon!]

3.3 Spectral spaces: Hochster's criterion

We recall the following notion from last time (motivated by properties of the spectrum of a ring):

Definition 3.3.1 A topological space X is *spectral* if it is qcqs, sober, and has a base of quasi-compact open subsets.

Note in particular that the retrocompact open subsets in such an X are exactly the qc open subsets. (Hochster proved that every spectral space is in fact the spectrum of a ring. The rings built in this way are rather bizarre in general; e.g., odd as it may sound, since every qcqs scheme is spectral it follows that the underlying space of such a scheme coincides with that of an affine scheme too.)

As suggested by Examples 3.2.12 and 3.2.13, one has:

Theorem 3.3.2 *Let X be a spectral space. The constructible topology on X is Hausdorff and qc.*

For a proof, see [SP, Lemma 5.22.2] (Tag 08YF).

Limits of qc topological spaces

Theorem 3.3.2 gives nontrivial control on the topology of an inverse limit of spectral spaces X_α, α under suitable assuptions on the transition maps. We will not address that here, but just want to point out that away from the Hausdorff setting, an inverse limit of qc spaces may not be qc (basically because the conditions defining the inverse limit inside the direct product may not be closed conditions, so the inverse limit may not be closed in the direct product).

Example 3.3.3 Let $X_i := \mathbf{N}$ as a *set* for $i \geq 1$, and let us define on X_i the topology making the points $1, \ldots, i$ all clopen and giving $\{j > i\}$ the indiscrete topology (only open sets in it are the whole space and the empty set). Each X_i is qc by design. We define the transition maps $\varphi_{ji} : X_j \to X_i$, for $j \geq i$ to be the identity map, which is obviously continuous with respect to the respective topologies. The limit $X = \varprojlim X_i$ is $\mathbf{N}$ endowed with the discrete topology, which is not qc! Note that the X_i are not spectral.

The above phenomenon turns out not to occur for (usually non-Hausdorff!) spectral spaces when the continuous transition maps are "spectral" in a sense given in Definition 3.3.8 (and see item (4) following that definition).

Pro-constructibility, and a motto to keep in mind

Proposition 3.3.4 *Let X be a spectral topological space. A subset $Z \subset X_{\text{cons}}$ is closed if and only if*

$$Z = \bigcap_{i \in I} C_i,$$

where I is an index set, and C_i is a constructible subset of X for all $i \in I$.

Proof. $Z \subset X_{\text{cons}}$ is closed if and only if $X - Z \subset X_{\text{cons}}$ is open, which is equivalent to saying that

$$X - Z = \bigcup_{i \in I} C_i'$$

where each one of the C_i' is constructible. This is equivalent to saying that Z is $\cap_i (X - C_i')$, where each one of the $X - C_i'$ is constructible. □

Definition 3.3.5 We call a subset Z of the topological space X *pro-constructible* if Z is the intersection of a collection of constructible subsets of X.

Remark 3.3.6 To get a feeling of what pro-constructible sets are, it is worthwhile saying that the pro-constructible subsets of a qcqs scheme X are exactly the images of morphisms from affine schemes to X, as discussed in [EGA, IV_3, 1.9.5(ix)]. More in general, the motto to keep in mind is that several "finiteness questions" can find an answer dealing with the constructible topology. The reader may think, for example, to the well known Chevalley's Theorem, saying that given a finitely presented morphism $f : X \to Y$ of schemes, Y qcqs, then $f(X)$ is constructible.

Here is a useful result:

Proposition 3.3.7 *Let X be a spectral topological space. Then a subset $Y \subset X$ is constructible if and only if it is clopen in X_{cons}.*

Proof. The "only if" part is clear by definition of the constructible topology. Conversely, if Y is open in the constructible topology, then it can be covered with constructible subsets of X:

$$Y = \bigcup_{i \in I} C_i.$$

On the other hand, if Y is also closed in X_{cons} then by *compactness* of X_{cons} (as X is spectral) it follows that the open cover $\{C_i\}$ of Y can be refined to a finite subcover, so Y is a finite union of constructible sets in X and hence is constructible. □

Generalities on spectral spaces

Definition 3.3.8 A continuous map $f : X' \to X$ between spectral spaces is called *spectral* if $f_{\text{cons}} : X'_{\text{cons}} \to X_{\text{cons}}$ is continuous.

It is equivalent that for every pro-constructible set Y of X, $f^{-1}(Y)$ is pro-constructible in X' (as pro-constructible sets are precisely the closed sets for the constructible topology). This is in turn equivalent to saying that for every constructible subset C of X, $f^{-1}(C)$ is constructible in X', due to Proposition 3.3.7. Finally, since an constructible open set in a qc space is qc, it is also equivalent to say that for every qc open subset U of X, the open subset $f^{-1}(U)$ in X' is qc.

In other words, a spectral map between spectral spaces is just a quasi-compact continuous map by another name.

We collect here some nontrivial facts about spectral spaces. The setup is that of X being a spectral space, and $Z \subset X$ a pro-constructible set (that is, a closed subset of X_{cons}, by Proposition 3.3.4).

(1) Z is constructible if and only if $X - Z$ is pro-constructible. Indeed, by Proposition 3.3.7 we see that Z is constructible if and only if Z is open in X_{cons} (being closed already), which is to say if and only if $X - Z$ is closed in X_{cons}; i.e., pro-constructible.

(2) The closure $\overline{Z} \subset X$ of the pro-constructible Z in the given topology of the spectral X is

$$\{x \in X \mid x \in \overline{\{z\}} \text{ for some } z \in Z\}.$$

This is shown by applying Lemma 5.22.5 in [SP] to Z (which is closed in X_{cons}).

(3) With its subspace topology from X, Z is spectral. This is shown in the *proof* of Lemma 5.22.4 in [SP] (which also gives the useful properties that every constructible subspace Y of Z is of the form $Z \cap C$ for some constructible subset C of X, and likewise for "constructible open").

(4) If $\{X_i\}$ is an inverse system of spectral spaces with spectral transition maps $f_{ij} : X_j \to X_i$ for $j \geq i$ then the topological inverse limit L is spectral (and in particular, is quasi-compact; cf. Example 3.3.3).

As a first step, we note that $\prod X_i$ (with product topology) is spectral: the case of a product of two spectral spaces is Lemma 5.22.9 in [SP], and the general case is a minor variant on the same method (using the finiteness conditions built into the definition of the product topology, as well as Tychonoff's theorem).

It now suffices to show that L is pro-constructible inside $\prod X_i$, by item (3). By definition, L is an intersection of the conditions "$f_{ji}(x_j) = x_i$" over all pairs $j \geq i$, so it suffices to show that each such condition is pro-constructible. More specifically, we just need to check that the graph of a spectral map $f : X \to Y$ between spectral spaces is pro-constructible inside $X \times Y$. Since the graph is the preimage of $\Delta(Y)$ under the spectral map $f \times 1 : X \times Y \to Y \times Y$, it suffices to show that the subset $\Delta(Y) \subset Y \times Y$ is pro-constructible in the spectral space $Y \times Y$. But it is easy to check that $(Y \times Y)_{\text{cons}} = Y_{\text{cons}} \times Y_{\text{cons}}$ (using the base of qc opens), so the Hausdorff property of Y_{cons} expresses exactly the closedness of $\Delta(Y)$ in $(Y \times Y)_{\text{cons}}$, which in turn is the desired pro-constructibility of the diagonal in $Y \times Y$.

We add one more item to our list: a criterion which enables us to reconstruct the given topology on a spectral space X from the constructible topology, but formulated more widely as a *criterion* for a topology to be spectral. Such criterion will be at the heart of our proof of the fact that $\text{Spv}(A)$ is a spectral space.

Theorem 3.3.9 (Hochster) *Let X' be a qc topological space, and let Σ be a collection of clopen sets of X'. Endow the set underlying X' with the topology generated by Σ, calling the resulting topological space X. If X is T_0 then it is spectral with Σ as a base of qc open subsets, and $X_{\text{cons}} = X'$.*

The reader may refer to [Wed, §3.4] for further details. We conclude this section with two comments on the above theorem. We are not assuming at all that Σ is stable under complement. This will be the reason why X need not be Hausdorff even when X' is Hausdorff (as will be the case in situations of interest). Moreover, the hypothesis that the space X built from X' is T_0 is a very strong condition on Σ. (Intuitively, if $X' = T_{\text{cons}}$ for a spectral space T that we do not yet have, taking Σ to be the collection of quasi-compact open subsets of T recovers T as X in Hochster's criterion.)

3.4 $\mathrm{Spv}(A)$ is spectral

We are ready to prove the following:

Theorem 3.4.1 [H1, Prop. 2.2] *Let A be a commutative ring. Then $X = \mathrm{Spv}(A)$ is spectral, and all the open subsets*

$$X\left(\frac{f}{s}\right) := \{v \in X \mid v(f) \leq v(s) \neq 0\}, \quad f, s \in A$$

are qc.

Remark 3.4.2 A warning. The subset $X(f/s)$ is *not* the image of Spv $A[f/s]$ in general, nor does it usually *contain* that image, since often $s \notin A[f/s]^\times$ (in which case $A[f/s]$ has valuations that kill s, in contrast with any valuation on A coming from $X(f/s)$). For example, take A to be a UFD and s and f irreducible elements that are not $A^\times$-multiples of each other. Hence, in the theorem it is crucial that quasi-compactness for the subsets $X(f/s)$ is recorded as well; it is not an instance of the spectrality of valuation spectra of rings in general.

Proof of Theorem 3.4.1. We divide the proof in a few steps. Let $X := \mathrm{Spv}(A)$.

Step 1 We show that the collection of open subsets

$$\Sigma := \{X(f/s) \mid f, s \in A\}$$

satisfies a good "separatedness property", which will later verify the T_0 aspect of the criterion in Theorem 3.3.9. Let $v \neq w$ in X. We prove there exists $U \in \Sigma$ with the property that either $v \notin U$ and $w \in U$, or $v \in U$ and $w \notin U$. Assume, first, that $\mathrm{supp}(v) \neq \mathrm{supp}(w)$. Then there exists some $s \in A$ such that $v(s) = 0$ and $w(s) \neq 0$, or $v(s) \neq 0$ and $w(s) = 0$. Then $U = X(s/s)$ does the job. (This subset is not X when $s \notin A^\times$!)

Suppose instead that the supports coincide, and let

$$\mathfrak{p} := \mathrm{supp}(v) = \mathrm{supp}(w).$$

Then v and w are distinct valuations on $\kappa(\mathfrak{p})$, implying that $R_v \not\subseteq R_w$ or $R_w \not\subseteq R_v$, R_v and R_w being the respective valuation rings in $\kappa(\mathfrak{p})$. Suppose the first case holds true, and let $\overline{h} \in R_v - R_w$. Then $v(\overline{h}) \leq 1$, but $w(\overline{h}) > 1$. In particular, $\overline{h} \neq 0$. Pick $s, f \in A - \mathfrak{p}$ such that the fraction $f/s \in \kappa(\mathfrak{p})^\times = \mathrm{Frac}(A/\mathfrak{p})$ is $\overline{h}$. Then $X(f/s)$ does the job.

Step 2 We realize X as the underlying set of a closed subspace X' of the power-set $\wp(A \times A) = \{0, 1\}^{A\times A}$ of $A \times A$, endowed with the product topology (which is compact Hausdorff, by Tychonoff). We define a map

$$j : X \to \wp(A \times A)$$

by sending $v \in X$ to its "set of v-divisibilities"

$$|_v := \{(f, s) \mid v(f) \leq v(s)\}.$$

One can roughly think of $|_v$ as encoding the set of pairs $(f, s) \in A \times A$ such that "$s \mid_v f$" in the sense of the valuation v. Notice that in the definition of $|_v$, we do not require $v(s) \neq 0$!

Define

$$U(a, b) := \{v \in X \mid v(a) \leq v(b)\} = \{v \in X \mid (a, b) \in |_v\}.$$

Then $X(f/s) = U(f, s) \cap (X - U(s, 0))$. Since distinct w and w' in X are distinguished by some $X(f/s)$ (which contains one of w or w' but not the other), it follows that the collection of pairs $(a, b) \in A \times A$ such that $w \in U(a, b)$ cannot be the same as the collection of such pairs for which $w' \in U(a, b)$. But $U(a, b)$ is the preimage of $(a, b) \in A \times A$ under the map $j : v \mapsto |_v$, so $|_w \neq |_{w'}$. In other words, the map $j : X \mapsto \wp(A \times A)$ is *injective*. What is its image?

As suggestive notation, let us write "$|$" to denote an element of $\wp(A \times A)$, and write $b|a$ to mean that $(a, b) \in |$; we speak of elements of $\wp(A \times A)$ as binary relations on A. We claim that the image of j consists of those binary relations $|$ on A satisfying the following axioms. For all $a, b, c \in A$:

(1) either $a \mid b$ or $b \mid a$ (so $a|a$). Moreover, $0 \nmid 1$.

(2) $a \mid b$, $b \mid c$ implies $a \mid c$.

(3) $a \mid b$ implies $ad \mid bd$ for all $d \in A$.

(4) $a \mid b$ and $a \mid c$ imply $a \mid b + c$.

(5) $ac \mid bc$ and $0 \nmid c$ imply $a \mid b$.

These axioms (which are obviously satisfied by $|_v$ for any valuation v on A) don't directly encode the entire totally ordered abelian group of the desired valuation, but rather the totally ordered cancellative abelian monoid

$$v(A - \mathfrak{p}_v)$$

for $\mathfrak{p}_v = \mathrm{supp}(v)$, which *generates* the totally ordered abelian group $v(\kappa(A/\mathfrak{p}_v))$.

Step 3 Now we verify that the above axioms encode exactly the image of j. Fix such a binary relation $|$. We shall use this to build a cancellative commutative monoid M with identity, from which we will get a totally ordered commutative group Γ "generated" by M (via "fractions") and a valuation v on A valued in Γ for which $|_v = |$.

Declare $a \sim a'$ if $a \mid a'$ *and* $a' \mid a$; let $[a]$ denote the equivalence class of a. It is well-posed to define multiplication among equivalence classes via multiplication of representatives in A. If a, a' are not equivalent to 0 then neither is aa'. Indeed, if $0 \mid aa'$ then since $0 = a \cdot 0$, yet $0 \nmid a$, we can "cancel" a by axiom (5) to get that $0 \mid a'$, a contradiction.

Now it is well-posed to consider the set of $\sim$ equivalence classes *away from that of* 0, made into a commutative monoid M with identity (the class of 1, which is distinct from that of 0 by axiom (1)) via multiplication as defined. This is cancellative by axiom (5). (That is, if $mm' = mm''$ for $m, m', m'' \in M$ then $m' = m''$.)

For non-zero equivalence classes $[a]$ and $[a']$, whether or not $a \mid a'$ is independent of the representative (by the "transitivity" axiom (2)), and when that holds we declare $[a'] \leq [a]$ (note the order of the inequality). From the axioms it follows that this makes M a totally ordered cancellative commutative monoid (e.g., axiom (1) makes M totally ordered).

Now we imitate the formation of rings of fractions, in the setting of cancellative commutative monoids with identity: declare two pairs (m_1, m_1') (think: m_1/m_1') and (m_2, m_2') to be *equivalent* if $m_1 m_2' = m_2 m_1'$ in M. This really is an equivalence relation: its transitivity rests on axiom (3) (similar to forming the ring of fractions). We define Γ to be the set of equivalance classes, considered as "fractions" (i.e., m/m' denotes the class of (m, m')), and we multiply two such "fractions" in the usual manner that is easily seen to be well-defined due to the cancellative property. This makes Γ an abelian group (with identity $1/1$ and $(m/m')^{-1} = m'/m$). Intrinsically, $M \to \Gamma$ is initial among homomorphisms from M into abelian groups, and it is *injective* since M is cancellative.

We extend the total ordering from M to Γ in the usual way (namely, $m/m' \leq n/n'$ when $mn' \leq m'n$), and this is a well-defined and transitive relation because of axiom (3) and the cancellative property. Note that if $[b] \leq [a]$ and $[c] \leq [a]$ then $[b + c] \leq [a]$ by axiom (4). The mapping $v : A \to \Gamma \cup \{0\}$ sending a to $[a]$ when $a \not\sim 0$ and sending a to 0 when $a \sim 0$ is then seen to be a valuation (e.g., axiom (4) ensures that $v(b + c) \leq \max(v(b), v(c))$). It is easy to check that $|_v = |$. Hence, the axioms listed in Step 2 really do catch exactly the image of $\mathrm{Spv}(A)$ in $\wp(A \times A)$.

Step 4 The axioms listed in Step 2 are closed conditions on $\wp(A \times A)$ (check!), so the subspace $X' := j(X)$ is a closed subspace $X' \subset \wp(A \times A)$ and hence is *compact*. The projection $\wp(A \times A) \to \{0, 1\}$ to the (a, b)-factor is a continuous map to a discrete space, and the clopen preimage of $\{1\}$ meets X' in exactly $U(a, b)$, so these $U(a, b)$'s are clopen in X'. As a consequence, the $X(f/s)$'s are all clopen in X'. We want to apply Hochster's criterion (Theorem 3.3.9) with $\Sigma := \{X(f/s) \mid f, s \in A\}$. The resulting topology on X' via Σ is precisely the topological space $X = \mathrm{Spv}(A)$, and Step 1 verifies the required T_0 property. Hence, we are done. □

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