14 Lecture 14: Basic generallities on adic spaces

14.1 Introduction

The aim of this lecture and the next two is to address general adic spaces and their connection to rigid geometry.

14.2 Two open questions

We begin by pointing out two questions which naturally come as analogous to facts from classical rigid geometry, and are still open.

Around noetherianity

Huber shows in his paper [H1] that a strongly noetherian complete Huber ring yields a sheafy pre-adic space; i.e., its adic spectrum is an adic space in the sense that the natural presheaf of topological rings is a sheaf. It is not know if a noetherian complete Huber ring is always strongly noetherian.

Rational open immersions

In classical rigid geometry, given an affinoid algebra A and a rational domain $\mathrm{Sp}(A\langle f/g\rangle) \hookrightarrow \mathrm{Sp}(A)$, it is known that $A \to A\langle f/g\rangle$ is flat. The reader can find the argument in Proposition 1.2.13. Roughly, it is sufficient to show that the above natural ring map is an isomorphism of completed local rings at every maximal ideal.

For a Huber pair (A, A^+) rather than an affinoid algebra A, one is led to ask if the natural map

$$A \to A\langle T/s\rangle$$

is always flat when $T \cdot A \subset A$ is open. This is believed to be false in general.

It seems unlikely that one can set up a theory of coherent sheaves on adic spaces without some noetherian assumptions (but note that on formal models of rigid spaces we do have a good theory of coherent sheaves despite the coordinate rings being non-noetherian in general).

14.3 Complements on adic spaces

Recall from last time that for a Huber pair (A, A^+) and $X := \mathrm{Spa}(A, A^+)$, we defined the functor on the category of rational domains of X:

$$\mathscr{O}_A : W := X(T/s) \mapsto A\langle T/s\rangle$$

where $s \in A$ and $T = \{t_1, \ldots, t_n\}$ is a non-empty finite subset of A such that $T \cdot A \subset A$ is open (and we saw that $\mathscr{O}_A(W)$ is intrinsic to W and is uniquely functorial with respect to inclusions among such W's). We extend $\mathscr{O}_A$ to the whole category of open subsets $U \subset X$ by setting

$$\mathscr{O}_A(U) := \varprojlim_{W \subset U} \mathscr{O}_A(W)$$

with W running over all rational domains of X contained in U. We know from Lecture 13 that $\mathscr{O}_A$ is a sheaf if either A is stably uniform (i.e., $A\langle T/s\rangle^0$ is bounded in $A\langle T/s\rangle$ for all choices of T and

s) or if A has a noetherian ring of definition or if A is strongly noetherian. The first of these three conditions is what we will need for the study of perfectoid spaces (but the latter two are extremely useful for adic spaces associated to noetherian formal schemes and to rigid-analytic spaces, each an important source of examples; keep in mind that non-reduced affinoid algebras are never uniform whereas perfectoid rings are always reduced!).

Units in $\mathscr{O}_{A,x}$, $x \in X$
Let us recall the proof that for $x \in X$ the stalk $\mathscr{O}_{A,x}$ is a local ring (and we do *not* try to put a topology on it). For any rational domain $W := X(T/s)$ containing x we have a valuation $v_x : A \to \Gamma_x \cup \{0\}$ which factors uniquely continuously through a valuation $v'_x : {}_A(W) = A\langle T/s\rangle \to \Gamma_x \cup \{0\}$ (because it factors continuously through $A[1/s]$ with the topology of $A(T/s)$). Such a factorization is natural in W in the obvious sense and therefore yields a valuation on the filtered colimit

$$\mathscr{O}_{A,x} = \varinjlim_{x \in W} \mathscr{O}_A(W) \to \Gamma_x \cup \{0\}$$

with W running over all rational domains containing x; this latter map will also be denoted v_x without risk of confusion. Obviously $\ker v_x$ is a proper ideal in $\mathscr{O}_{A,x}$, and we will show that any $f \in \mathscr{O}_{A,x}$ outside this kernel is a unit (so the non-units form an ideal, making the stalk local).

Consider $f \in \mathscr{O}_{A,x}$ such that $v_x(f) \neq 0$. Certainly f comes from an element $f' \in \mathscr{O}_A(W) =: A'$ for some rational domain W, and if I' is an ideal of definition of some ring of definition then upon replacing I' with a large enough power if necessary we get for generators $t'_1, \dots, t'_m$ of I' in A' that

$$v_x(t'_i) \leq v_x(f') \neq 0$$

for all i. Define $W' := W(T'/f')$, with $T' := \{t'_1, \dots, t'_m\}$ (which obviously generates in A' an open ideal), so $W' \subset W$ is a rational domain around x with f' a unit in the *completion* ${}_A(W') = A\langle T'/f'\rangle$, so $f \in \mathscr{O}_{A,x}^{\times}$ as desired.

Caveat on residue fields
Given that $\mathscr{O}_{A,x}$ is a local ring for all $x \in X$, let us denoted by $\mathfrak{m}_x$ its maximal ideal, and define $\kappa(x) := \mathscr{O}_{A,x}/\mathfrak{m}_x$. Clearly v_x factors through a valuation $\kappa(x) \to \Gamma_x \cup \{0\}$, which we still call v_x. Let us denote by $\kappa(x)^+$ the valuation ring of v_x in $\kappa(x)$.

How do we concretely compute $\kappa(x)$? Geometrically, we consider rational domains $W = X(T/s)$ in $X = \mathrm{Spa}(A, A^+)$ containing x with $T \cdot A$ open in A. We know that $v_x : A \to \Gamma_x \cup \{0\}$ factors naturally through $A\langle T/s\rangle$ and the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{v_x} & \Gamma_x \cup \{0\} \\ \downarrow & \nearrow_{v'_x} & \\ A' := A\langle T/s\rangle & & \end{array}$$

Thus, the support $\mathfrak{p}$ of x in A is the contraction of the prime ideal support $\mathfrak{p}'$ of v'_x on A', yielding unique factorizations through the respective residue fields in the form of a commutative diagram of valued fields:

$$\begin{array}{ccc} \mathrm{Frac}(A/\mathfrak{p}) & \xrightarrow{v_x} & \Gamma_x \cup \{0\} \\ \downarrow & \nearrow_{v'_x} & \\ \mathrm{Frac}(A'/\mathfrak{p}') & & \end{array}$$

We conclude that

$$\kappa(x) = \varinjlim_{\mathrm{Spa}(A', A'^+) \ni x} \mathrm{Frac}(A'/\mathfrak{p}')$$

with $A' = \mathscr{O}_A(W)$ for rational $W \ni x$ as above. But in contrast with the case of schemes or of rigid-analytic spaces and classical points, the map of valued fields $\mathrm{Frac}(A/\mathfrak{p}_v) \to \mathrm{Frac}(A'/\mathfrak{p}'_v)$ will in general be a *huge* extension, so it is not clear how to "compute" the limit! (This is already seen for the closed unit ball with x the Gauss point.)

The way out is the observation that each such map of valued fields has *dense* image for the valuation topology because $A[1/s] = A(T/s) \to A\langle T/s\rangle =: A'$ has dense image for the topology defined by v'_x on A'. Indeed, v'_x is continuous on A', and $A[1/s] \to A'$ has dense image for the Huber ring topology on A'. The denseness of the images of such transition maps between valued fields implies that the completions all canonically coincide: the field

$$k(x) := \kappa(x)^\wedge$$

coincides with the completion of $\mathrm{Frac}(A'/\mathfrak{p}')$ for *every* (A', A'^+) as above. In other words, $k(x)$ can be "directly computed" using any choice of rational domain containing x whereas $\kappa(x)$ requires the consideration of arbitrarily small such rational domains. (The better behavior of the "completed residue field" over the actual residue field is also seen in the theory of Berkovich spaces.)

Local integral structures

We recall that we have the following commutative diagram of maps of rings:

$$\begin{array}{ccc} \mathscr{O}_{A,x} & \longrightarrow & \kappa(x) \\ \uparrow & & \uparrow \\ \mathscr{O}^+_{A,x} & \longrightarrow & \kappa(x)^+ \end{array}$$

For all rational domains $W = X(T/s) \subset X$ we have

$$\mathscr{O}^+_A(W) = A\langle T/s\rangle^+$$

inside $\mathscr{O}_A(W) = A\langle T/s\rangle$, where $A\langle T/s\rangle^+$ is the integral closure of the open image of $A^+\langle T/s\rangle$ in $A\langle T/s\rangle$. Finally, recall that we have a natural homeomorphism

$$\mathrm{Spa}(A\langle T/s\rangle, A\langle T/s\rangle^+) \simeq X(T/s)$$

and the adic Nullstellensatz gives

$$A\langle T/s\rangle^+ = \{f \in A(T/s) \mid v(f) \leq 1 \text{ for all } v \in X(T/s)\}.$$

We prove the following:

Proposition 14.3.1 *The commutative diagram*

$$\begin{array}{ccc} \mathscr{O}_{A,x} & \longrightarrow & \kappa(x) \\ \uparrow & & \uparrow \\ \mathscr{O}^+_{A,x} & \longrightarrow & \kappa(x)^+ \end{array}$$

is Cartesian; in other words, $\mathscr{O}^+_{A,x} = \{f \in \mathscr{O}_{A,x} \mid v_x(f) \leq 1\}$.

Proof. Consider $f \in \mathscr{O}_{A,x}$ such that $v_x(f) \leq 1$. We have to show that f comes from $\mathscr{O}^+_{A,x}$. There exists a rational domain $W = X(T/s) \subset X$ around x such that f comes from $f' \in A' := \mathscr{O}_A(W)$, and necessarily $v_x(f') \leq 1$. We seek a rational domain $U \subset W$ around x such that $f'|_U \in \mathscr{O}^+_A(U)$. Define

$$U := \{v \in W \mid v(f') \leq v(1) = 1 \neq 0\} = W(f'/1);$$

this contains x, is rational in $W = \mathrm{Spa}(A', A'+)$ (hence is rational in $X = \mathrm{Spa}(A, A^+)$, by a nontrivial argument of Huber discussed earlier), and clearly does the job. □

We observe that as a consequence, the natural map $\mathscr{O}^+_{A,x} \twoheadrightarrow \kappa(x)^+$ is surjective and the full preimage of the units in $\kappa(x)^+$ is exactly $(\mathscr{O}^+_{A,x})^\times$. This yields:

Corollary 14.3.2 *The stalk $\mathscr{O}^+_{A,x}$ is local with residue field equal to that of the valuation ring $\kappa(x)^+$.*

The adic Nullstellensatz applied to rational domains yields the following concrete formula for all open $U \subset \mathrm{Spa}(A, A^+)$:

$$\mathscr{O}^+_A(U) = \{f \in \mathscr{O}_A(U) \mid v_x(f) \leq 1 \text{ for all } x \in U\}.$$

(For rational U this literally is the adic Nullstellensatz.)

14.4 Global adic spaces

To define adic spaces, a choice needs to be made in the axioms: do we prioritize the stalkwise valuations or the subsheaf $^+$ of "integral elements"? It seems a bit simpler to use stalkwise valuations in the axioms, so (following Huber) we shall proceed that way. Define $\mathfrak{C}$ to be the category of triples $(X, \mathscr{O}, \{v_x\}_{x\in X})$ where $(X, \mathscr{O})$ is a topologically locally ringed space (i.e., $\mathscr{O}$ is a presheaf of topological rings with local stalks and for any open $U \subset X$ and open cover $\{U_i, i \in I\}$ of U the diagram

$$\mathscr{O}(U) \longrightarrow \textstyle\prod_{i\in I} \mathscr{O}(U_i) \rightrightarrows \prod_{i,j\in I} \mathscr{O}(U_i \cap U_j)$$

is a topological equalizer) and v_x is a valuation on the residue field $\kappa(x)$ of $\mathscr{O}_x$ for all $x \in X$. We now define morphisms between such triples.

A *morphism* of such triples

$$(X', \mathscr{O}', \{v'_x\}) \xrightarrow{f} (X, \mathscr{O}, \{v_x\}),$$

is a map $(X', \mathscr{O}') \xrightarrow{\varphi} (X, \mathscr{O})$ of locally ringed spaces such that

(1) $\mathscr{O}(U) \to \mathscr{O}'(\varphi^{-1}U)$ is continuous for all open subsets $U \subset X$,

(2) for all $x' \in X'$ and $x = \varphi(x') \in X$ the natural map $\kappa(x) \to \kappa(x')$ is valuation-compatible; i.e., it induces a local map of valuation rings $\kappa(x)^+ \to \kappa(x')^+$ or equivalently there exists a (necessarily unique) map of ordered abelian groups $\Gamma_x \to \Gamma_{x'}$ making the following diagram commute:

$$\begin{array}{ccc} \kappa(x) & \xrightarrow[v_x]{} & \kappa(x') \\ {\scriptstyle v_x}\big\downarrow & & \big\downarrow{\scriptstyle v_{x'}} \\ \Gamma_x \cup \{0\} & \xrightarrow[\exists]{} & \Gamma_{x'} \cup \{0\} \end{array}$$

Example 14.4.1 As an example, if (A, A^+) is a sheafy Huber pair, then

$$(\mathrm{Spa}(A, A^+), \mathscr{O}_A, \{v_x\})$$

is such a triple.

Given a continuous map of Huber pairs $f : (A, A^+) \to (B, B^+)$ we get a continuous map

$$Y := \mathrm{Spa}(B, B^+) \xrightarrow{\varphi} \mathrm{Spa}(A, A^+) =: X$$

of topological spaces. Let us explain how to naturally enhance this to a morphism in $\mathfrak{C}$. Given $W = X(T/s)$ a rational domain in X, its preimage under the continuous φ is an open set and so is covered by rational domains $U = Y(T'/s')$ in Y. (Beware that this preimage is generally *not* a rational domain since "$Y(\varphi(T)/\varphi(s))$" may not make sense as such: $\varphi(T) \cdot A$ might not be open! In general $\varphi^{-1}(Y'(T'/s'))$ can fail to be quasi-compact.) By the universal mapping property for the A-algebra $A\langle T/s\rangle$ and B-algebra $B\langle T'/s'\rangle$ we get a unique commutative diagram of topological rings

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A\langle T/s\rangle & \xrightarrow[\exists]{} & B\langle T'/s'\rangle \end{array}$$

Varying U in $\varphi^{-1}(W)$ thereby defines a *continuous* map $\mathscr{O}_A(W) \to \mathscr{O}_B(\varphi^{-1}(W))$ and by varying W this yields a map $\varphi^{\#} : \mathscr{O}_A \to \varphi_*(\mathscr{O}_B)$ which provides a map $Y \to X$ as topologically ringed spaces. This is readily verified to be local on stalks and valuation-compatible, so in the sheafy case it is a morphism $\mathrm{Spa}(f) : Y \to X$ in $\mathfrak{C}$ which is clearly functorial in f.

We are finally ready to globalize the notion of affinoid adic space:

Definition 14.4.2 A triple $(X, \mathscr{O}, \{v_x\})$ is called an *adic space* if it is locally isomorphic to $\mathrm{Spa}(A, A+)$ for some sheafy Huber pair (A, A^+). Given an adic space X, we define $\mathscr{O}^+ \subset \mathscr{O}$ by:

$$\mathscr{O}^+(U) := \{f \in \mathscr{O}(U) \mid v_x(f) \leq 1 \text{ for all } x \in U\}.$$

Remark 14.4.3 If $X = \mathrm{Spa}(A, A^+)$ for some sheafy Huber pair (A, A^+) then the above definition of $\mathscr{O}^+$ recovers $\mathscr{O}_A^+ \subset \mathscr{O}_A$.

By earlier considerations on $\mathscr{O}_A$ and $\mathscr{O}_A^+$, we deduce the following facts for general adic spaces:

(1) $\mathscr{O}_{X,x}^+$ is local for all points $x \in X$.

(2) $\mathscr{O}_{X,x}^+ = \{f \in \mathscr{O}_x \mid v_x(f) \leq 1\}$.

The following basic but important result is [H1, Prop. 2.1]:

Proposition 14.4.4

(1) *If X and $Y := \mathrm{Spa}(A, A^+)$ are adic spaces with A complete, then*

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}_{\mathrm{cont}}((A, A^+), (\mathscr{O}_X(U), \mathscr{O}_X^+(U)))$$

is a bijection.

(2) *For adic spaces X and Y, a map*

$$f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$$

of topologically locally ringed spaces respects valuations if and only if we can fill in the bottom of a commutative diagram

$$\begin{array}{ccc} \mathscr{O}_Y & \longrightarrow & f_*\mathscr{O}_X \\ \uparrow & & \uparrow \\ \mathscr{O}_Y^+ & \longrightarrow & f_*\mathscr{O}_X^+ \end{array}$$

and the induced natural map of ringed spaces $(X, \mathscr{O}_X^+) \to (Y, \mathscr{O}_Y^+)$ *is a map of* locally *ringed spaces.*

Part (2) provides an alternative viewpoint on how to think about morphisms of adic spaces, and it suggests that even the very definition of adic space can be given via axioms on triples $(X, \mathscr{O}, \mathscr{O}^+)$, but this does not seem to be as elegant as axiomatizing with triples $(X, \mathscr{O}, \{v_x\})$.

Comments on the proof. To prove (1), one reduces to the case $X = \mathrm{Spa}(B, B^+)$, and for

$$f : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$$

inducing a continuous map on Huber pairs $\varphi : A \to B$, one checks that $\mathrm{Spa}(\varphi) = f$ by adapting the arguments one runs for schemes, but using rational domains and some extra care with topological aspects.

The key ingredient in the proof of (2) is that valuation rings are maximal with respect to local domination among local domains with given fraction field. □

Example 14.4.5 In Proposition 14.4.4(1), if we choose $Y = \mathbf{D}_{\mathbf{Z}} := \mathrm{Spa}(\mathbf{Z}[t], \mathbf{Z}[t])$ (giving $\mathbf{Z}[t]$ the discrete topology) then

$$\mathrm{Hom}(X, \mathbf{D}_{\mathbf{Z}}) = \mathscr{O}_X^+(X);$$

that is, $\mathbf{D}_{\mathbf{Z}}$ represents the functor $X \mapsto \mathscr{O}_X^+(X)$ from the category of adic spaces (in which $\mathrm{Spa}(\mathbf{Z}, \mathbf{Z})$ is the terminal object!) to the category of topological rings. Instead, if we choose $Y = \mathbf{A}_{\mathbf{Z}}^{1,\mathrm{ad}} := \mathrm{Spa}(\mathbf{Z}[t], \mathbf{Z})$ (again giving $\mathbf{Z}[t]$ the discrete topology, so $\mathbf{Z}$ is an open subring) then

$$\mathrm{Hom}(X, Y) = \mathscr{O}_X(X);$$

that is, $\mathbf{A}_{\mathbf{Z}}^{1,\mathrm{ad}}$ represents the functor $X \mapsto \mathscr{O}_X(X)$ from the category of adic spaces to the category of topological rings.

14.5 Analytic points

Recall the following notion:

Definition 14.5.1 A point $x \in \mathrm{Spa}(A, A^+)$ is *analytic* if its support $\mathfrak{p}_x \in \mathrm{Spec}(A)$ is *not* open, or equivalently, for some (equivalently, any) ring of definition A_0 for A and ideal of definition $I \subset A_0$ we have $v_x(I) \neq 0$.

Note that if A is Tate then all points in $\mathrm{Spa}(A, A^+)$ is analytic, as the only open ideal is the unit ideal (due to the existence of a topolocially nilpotent unit); see Corollary 8.3.3. More generally, analyticity is of local nature in the affinoid case:

Lemma 14.5.2 *For* $x \in X := \mathrm{Spa}(A, A^+)$ *and affinoid open subspace* $W = \mathrm{Spa}(B, B^+)$ *around* x, *the point* x *is analytic in* X *if and only if it is analytic in* W.

Proof. First consider the case that W is a rational domain $X(T/s)$ for some $s \in A$ and non-empty finite subset $T \subset A$ such that $T \cdot A$ is open in A. Thus, $B = A\langle T/s \rangle$ and $B^+ = A\langle T/s \rangle^+$. Pick a pair (A_0, I) for A as usual, so $B = A(T/s)$ has ring of definition $B_0 = A_0[T/s] \subset A_0[1/s]$ with ideal of definition $I \cdot B_0$. Clearly $v_x(I) = 0$ if and only if $v_x(I \cdot B_0) = 0$, due to commutativity of

$$\begin{array}{ccc} A & \xrightarrow{v_x} & \Gamma_x \cup \{0\} \\ \downarrow & \nearrow_{v_x} & \\ B & & \end{array}$$

Now consider the general case, so W is not necessarily rational in X. There is a rational domain $U \subset X$ around x that is contained in W, and our earlier study of rational domains then implies that such U is also rational in W. Hence, the above special case establishes that the analyticity of x in either X or W is equivalent to the common condition of analyticity of x in U. □

The preceding lemma allows us to globalize the notion of analyticity:

Definition 14.5.3 For a general adic space X, a point $x \in X$ is *analytic* if and only if it is so in some (equivalently, any) affinoid open neighbourhood.

Clearly any point $x \in X$ admitting an affinoid open neighborhood that is Tate must be analytic. The converse is true:

Corollary 14.5.4 *For the subset* $X_{\mathrm{a}} := \{x \in X \mid x \text{ analytic}\}$*, a point* $x \in X_{\mathrm{a}}$ *admits an affinoid neighbourhood* $U := \mathrm{Spa}(A, A^+)$ *with* A *Tate. In particular,* X_{a} *is open in* X*.*

Proof. Since an affinoid subdomain of a Tate neighbourhood is still Tate, we reduce to the affinoid case, which was dealt with in Proposition 8.3.8 and Remark 8.3.9. □

Now we discuss how the non-analytic locus behaves under morphisms.

Lemma 14.5.5 *Let* $f : X \to Y$ *be a morphism of adic spaces. Defining* X_{na} *and* Y_{na} *to be the respective closed loci of non-analytic points in* X *and* Y*, we have*

$$f(X_{\mathrm{na}}) \subset Y_{\mathrm{na}}.$$

Proof. One reduces at once to the affinoid case, so $X = \mathrm{Spa}(B, B^+)$ and $Y = \mathrm{Spa}(A, A^+)$. Given that the induced map $\varphi : (A, A^+) \to (B, B^+)$ is a continuous map of Huber pairs, we can choose compatible pairs of definitions (A_0, I) and (B_0, J) for each, so for all $x \in X$, if $x(J) = 0$ then $(x \circ \varphi)(I) = 0$. (The converse is false since generally the finitely generated ideal $I \cdot B_0$ inside J might not be open.) □

Example 14.5.6 As a first example, if we consider *any* adic space X over $\mathrm{Spa}(k, k^0)$ for a non-archimedean field k then $X = X_{\mathrm{a}}$. Let us instead consider the open unit disc over $\mathbf{Z}_p$:

$$X := \mathrm{Spa}(\mathbf{Z}_p[\![t]\!], \mathbf{Z}_p[\![t]\!])$$

where we use the (p,t)-adic topology. Then X_{na} is the single point corresponding to the unique open prime (p,t) with (completed) residue field $\mathbf{F}_p$, and the analytic locus X_{a} contains the open subspace $\{p \neq 0\}$ that is the adic space associated to the open unit disc as well as an additional closed point in characteristic p with completed residue field $\mathbf{F}_p(\!(t)\!)$; this underlies Scholze's notion of the adic Fargues–Fontaine curve for the study of *integral* p-adic Hodge Theory

We recall that a map $f : (A, A^+) \to (B, B^+)$ of Huber pairs is *adic* if there exists a pair of definition (A_0, I) for A and a ring of definition B_0 of B such that $f(A_0) \subset B_0$ and $f(I) \cdot B_0$ is an ideal of definition. The typical non-example is the inclusion $\mathbf{Z}_p \to \mathbf{Z}_p[\![t]\!]$, the former being given the p-adic topology and the latter the (p,t)-adic topology.

In a positive direction, if A is Huber then the quotient B of $A\langle x_1, \ldots, x_n\rangle$ by a closed ideal makes $A \to B$ an adic morphism. A typical example is

$$B = A\langle T/s\rangle := A\langle x_1, \ldots, x_n\rangle/\overline{(sx_i - t_i)}.$$

Remark 14.5.7 It is easy to show that if $f : X := \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+) =: Y$ is $\mathrm{Spa}(\varphi)$ for an adic map $\varphi : (A, A^+) \to (B, B^+)$ then $f(X_{\mathrm{a}}) \subset Y_{\mathrm{a}}$ (as one sees by unravelling the definitions).

Proposition 14.5.8 ([H1, Prop. 3.8]) *Let $f : X := \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+) =: Y$ be a morphism of adic spaces.*

(1) *Assume B is complete, so $f = \mathrm{Spa}(\varphi)$ for a map of Huber pairs $\varphi : (A, A^+) \to (B, B^+)$. If $f(X_{\mathrm{a}}) \subset Y_{\mathrm{a}}$ then φ is adic.*

(2) *If $f = \mathrm{Spa}(\varphi)$ for $\varphi : (A, A^+) \to (B, B^+)$ adic then f is spectral (and in particular it is quasi-compact!).*

A comment on the proof: in the proof of (1), if φ is not adic then one must construct a point $x \in X_{\mathrm{na}}$ such that $f(x) \in Y_{\mathrm{a}}$ if φ; this involves non-trivial valuation-theoretic arguments (with vertical and horizontal specializations).

14.6 What is a point?

Let X be an adic space, and $x \in X$ a point. We want to understand what x actually is from various points of view, generalizing the identification of points of a scheme with morphisms from spectra of fields.

Any such x yields a pair $(\kappa(x), \kappa(x)^+)$, so the first question is whether or not this pair (or at least its completion $(k(x), k(x)^+)$) is Huber; if so then $\mathrm{Spa}(\kappa(x), \kappa(x)^+)$ makes sense and would be a good candidate to encode $x \in X$ via a morphism $\mathrm{Spa}(\kappa(x), \kappa(x)^+) \to X$.

However, we observe that if A is a noetherian ring with the discrete topology then for

$$X := \mathrm{Spa}(A, A) = \{v \in \mathrm{Spv}(A) \mid v(a) \leq 1 \text{ for all } a \in A\}$$

we have no control on the nonzero topologically nilpotent elements in $\kappa(x)^+$ for typical $x \in X$.

The situation is more promising when x lies in the analytic locus $X_{\mathrm{a}} \subset X$. In such cases $\kappa(x)$ is equipped with the topology induced by a rank-1 valuation and $\kappa(x)^+$ is an open valuation subring of the valuation ring $\kappa(x)^0$. Thus, the Huber property easily holds since the topology on $\kappa(x)^+$ is defined by powers of a pseudo-uniformizer chosen to lie inside $\kappa(x)^+$. For any affinoid naighbourhood $\mathrm{Spa}(A, A^+)$ of such an x we get a canonical composite morphism of adic spaces

$$j_x : \mathrm{Spa}(\kappa(x), \kappa(x)^+) \to \mathrm{Spa}(A, A^+) \to X$$

which is easily verified to be independent of the choice of affinoid neighborhood (compare through rational domains).

Next time we will look more closely at j_x, and in particular see that the source has a unique closed point s_x (and unique generic point η_x that is the unique rank-1 generization of x in X), and that $j_x(s_x) = x$. The fact that $\mathrm{Spa}(\kappa(x), \kappa(x)^+)$ is *not* a 1-point space when x is a higher-rank analytic point is a huge deviation from experience with schemes, formal schemes, complex-analytic spaces, and rigid-analytic spaces, yet pullback along j_x is essential when we wish to study fibers of morphisms between adic spaces. For this reason, even if we are ultimately most interested in adic spaces over non-archimedean fields it is inevitable that we have to work with adic spaces over complete fields equipped with a higher-rank valuation.

References

[H1] R. Huber, *Continuous valuations.*