12 Lecture 12: Structure Presheaves and Rational Domains

12.1 Introduction

Let (A, A^+) be a Huber pair, so A^+ is an open and integrally closed subring of A. Let $X = \mathrm{Spa}(A, A^+)$. The aim of this lecture is to revisit the significance of the choice of A^+ with focus on the rigid-analytic case and to begin defining the structure presheaf on X.

A familiar situation is when A is k-affinoid for a non-archimedean field k and $A^+ = A^0$. We know that all points x are analytic (recall the meaning of the natural map $\mathrm{Spa}(A, A^0) \to \mathrm{Spa}(k, k^0)$ with the latter a single point), so each such x has a unique rank-1 generization $\eta_x \in X$ (this is a general fact for analytic points). The η_x-topology coincides with the v_x-topology on the associated field $\kappa(x) = \mathrm{Frac}(A/\operatorname{supp} x)$, and the subring $\kappa(x)^0$ (defined in terms of the valuation topology) is therefore the valuation ring for η_x since η_x is a rank-1 valuation.

Example 12.1.1 Consider the closed unit disc, which is the case $A = k[t]$ (or $\widehat{A} = k\langle t\rangle$) and $A^+ = A^0[t]$ (or $\widehat{A}^+ = \widehat{A^0} = k^0\langle t\rangle$). Assume as well that k is algebraically closed.

We saw last time that there are five types of points (and type-4 occurs if and only if k is not spherically complete, as can occur for some algebraically closed fields such as $\mathbf{C}_p$ but not for others such as its spherical completion). This naturally maps continuously onto the Berkovich closed unit disk as its maximal Hausdorff quotient by sending x to the rank-1 generization η_x upon recalling that for each rank-1 point v there is a unique embedding of ordered groups $\Gamma_v \hookrightarrow \mathbf{R}_{>0}$ extending the given absolute value on k.

We also saw that type-2 points are not closed but the other types are closed, and that for $x \in k^0$ the (radius-parametrized) paths $[0, 1] \to \mathbf{D}_k$ defined by $r \mapsto v_{x,r}$ (type-1 for $r = 0$, type-2 for $r \in |k^\times|$, and type-3 otherwise) are *not* continuous at $r \in |k^\times|$ but their composition down to the Berkovich disc is continuous. Recall as well that the natural embedding of the Berkovich disk into the adic disc is also *not* continuous, nor is the retraction onto this topological subspace of rank-1 points via $x \mapsto \eta_x$ (though the retraction is continuous when this subspace is equipped with its compact Hausdorff Berkovich topology).

12.2 Changing A^+

In this section, we investigate the effect of the choice of A^+. The connection with rigid-analytic geometry will come from taking $A^+ = A^0$, but it is important to carry along an A^+ more generally (not to only take $A^+ = A^0$ as in the rigid-analytic setting) because the formalism of A^+ behaves better than that of A^0 under basic operations on Huber pairs.

Example 12.2.1 Here is a modification of Example 12.1.1 that gives an example of working with an $A^+ \neq A^0$. Let $A = k[t]$ and $A^+ = k^0 + t\mathfrak{m}[t]$. (This is the preimage of the residue field κ of k under $A^0 \to A^0/A^{00} = \kappa[t]$.) Then inside $\mathrm{Cont}(A)$ we have $\mathrm{Spa}(A, A^+) = \mathbf{D}_k \cup \{v_{0,1^+}\}$, where the one extra point in the closure of the Gauss point (corresponding to the point at infinity in the $\mathbf{P}^1_\kappa$ describing the topological closure of the Gauss point inside any strictly larger closed disc). This is explained in Example 11.3.13.

Example 12.2.2 For k-affinoid A, we know that $\mathrm{Spa}(A, A^0)$ has a base of rational domains, each of which has (many) classical points when non-empty (arising from the MaxSpec of A), so we naturally have $\mathrm{Sp}(A) \subset \mathrm{Spa}(A, A^0)$ as a dense subset. However, as we saw rather vividly for the closed unit disc, these are generally *not* the only closed points (*unlike* the situation in algebraic geometry, comparing classical varieties built with MaxSpec and the associated schemes built with Spec)!

For affinoid algebras, it turns out that working with A^0 as the choice of A^+ is "well-behaved" under basic operations. This is to be expected, since in the classical rigid-analytic case we have a good theory keeping track of A alone without the baggage of an extra A^+. This is formalized in the following result, where k is any non-archimedean field, A, B, C are k-affinoid algebras, and $f_1, \ldots, f_n \in A$ have no common zero in any finite extension of k (or equivalently they generate the unit ideal):

Proposition 12.2.3 $(A\widehat{\otimes}_C B)^0$ *is the integral closure of the image of* $A^0\widehat{\otimes}_{C^0} B^0 \to A \otimes_C B$ *and* $A\langle f/g\rangle^0$ *is the integral closure of the image of* $A^0\langle f/g\rangle \to A\langle f/g\rangle$*.*

Remark 12.2.4 Recall that $A^0\langle f/g\rangle$ is defined to be the ϖ-adic completion of the subring $A^0[f/g] \subset A[1/g]$, where $\varpi \in k$ is a pseudo-uniformizer.

The key fact for the proof of Proposition 12.2.3 is that for a (necessarily continuous) k-algebra map $f\colon A' \to A''$ of k-affinoid algebras, the induced map of rings $f^0\colon (A')^0 \to (A'')^0$ is integral if and only if f is integral. The forward direction is evident by inverting ϖ. The converse is rather non-obvious, especially when k is not discretely-valued (so methods of "noetherian" commutative algebra are not available over k^0); see [BGR, §6.3] for a comprehensive treatment of this foundational input (where it is also shown that f is finite when it is integral, another non-trivial fact [BGR, 6.3.5/1]). Before we address the proof, to appreciate the subtleties involved we first record:

Example 12.2.5 For $k \to L$ a finite extension of non-archimedean fields, $k^0 \to L^0$ need *not* be finite (but it is always integral). An example is given just after [BGR, 6.4.1/2] (based on a criterion in [BGR, 6.4.1/1]).

If k is either discretely-valued or algebraically closed then in fact f^0 is *finite* when it is integral (see [BGR, 6.4.1, Cor. 5, Cor. 6]) provided that the target is reduced. To see why such a reducedness hypothesis is necessary, consider the finite map $k \subset k[\epsilon]/(\epsilon^2)$: the inclusion of power-bounded elements is $k^0 \hookrightarrow k^0 + k\epsilon$, which is certainly not module-finite.

Since surjective maps are integral, we get the very important consequence that if f is surjective then f^0 is integral. Note however that in such cases f^0 need not be surjective:

Example 12.2.6 Consider $A' = k\langle t\rangle \twoheadrightarrow k\langle t\rangle/t(t-\pi) = k \times k$ where $0 < |\pi| < 1$. Then $(A')^0 = k^0\langle t\rangle$, but its image in $k^0 \times k^0$ is contained in (in fact, consists of precisely) the set of pairs (c_1, c_2) such that $c_1 \equiv c_2$ modulo π.

Since affinoid are quotients of Tate algebras, the idea for proving Proposition 12.2.3 will be to use the relationship between surjectivity at the affinoid level and integrality at the power-bounded level to transport things to the land of Tate algebras where everything can be computed explicitly:

Proof. The first step is to reduce to the case $C = k$ for the case of completed tensor products. By closedness of ideals in affinoid algebras, the natural map $A\widehat{\otimes}_k B \to A\widehat{\otimes}_C B$ is surjective, induced by the quotient map $C\widehat{\otimes}_k C \twoheadrightarrow C$. Thus, $(A\widehat{\otimes}_k B)^0 \twoheadrightarrow (A\widehat{\otimes}_C B)^0$ is integral. Hence, if we knew the desired result for $C = k$ (i.e., $(A\widehat{\otimes}_k B)^0$ is integral over $A^0\widehat{\otimes}_{k^0} B^0$) then this would give it for all C, so now we assume $C = k$.

Likewise, now with $C = k$, since A and B are quotients of Tate algebras we can further assume without loss of generality for the treatment of completed tensor products that A, B are Tate algebras. In this setting (with $C = k$) we can just calculate everything explicitly and find that both sides are equal on the nose.

For completed localization corresponding to a rational domain, recall that

$$A\langle f/g\rangle = A\langle t_1, \ldots, t_n\rangle/(gt_i - f_i),$$

a quotient of a relative Tate algebra $A\langle t_1, \ldots, t_n\rangle$. The power-bounded subring of this relative Tate algebra is $A^0\langle t_1, \ldots, t_n\rangle$, and the map from that power-bounded subring to $A\langle f/g\rangle^0$ has to be integral due to our preceding discussion with affinoid surjectivity and power-bounded integrality. But the integral map

$$A^0\langle t_1, \ldots, t_n\rangle \to A\langle f/g\rangle^0$$

clearly factors through $A^0\langle f/g\rangle := A^0[f/g]^\wedge$, so this latter ring is also integral under $A\langle f/g\rangle^0$ as desired. □

Example 12.2.7 In fact, $A^0\widehat{\otimes}_{k^0}B^0 \to (A\widehat{\otimes}_k B)^0$ is an isomorphism if $k = \overline{k}$: see [Bo, §6, Satz 5]. (By [Bo, §6, Satz 4], likewise $A^0\widehat{\otimes}_{k^0}k'^0 \to (A\widehat{\otimes}_k k')^0$ is an isomorphism for algebraically closed k and any non-archimedean extension field k'/k.) However, over any k whatsoever it is generally not true that $A^0 \to (A/I)^0$ is surjective for k-affinoid A and ideals $I \subset A$; see Example 12.2.6.

Example 12.2.8 The map $\varphi : A^0\langle f/g\rangle \to A\langle f/g\rangle^0$ generally fails to be an *isomorphism* in the rigid-analytic setting (so one cannot do better there than to say it is integral). To build an example for which this is not an isomorphism, inspired by Raynaud's theory of formal models (which interprets rational domains in terms of charts for blow-ups supported in the special fiber) and the relationship between power-bounded functions and normalization in Berthelot's theory of formal models as expressed in [dJ, §7] one is led to search for normal arithmetic schemes having a non-normal blow-up; e.g., an arithmetic surface with isolated singularity in the special fiber whose blow-up has a curve of singularities.

For example, considering discretely-valued k having uniformizer π, we observe that the domain $R = k^0[x, y]/(x^2 + \pi^4 + y^5)$ is regular away from (x, π, y) (so it is normal by Serre's criterion) and its blow-up there has y-chart

$$R' = R[\pi/y, x/y] = k^0[x', t, y]/(x'^2 + t^4y^2 + y^3, \pi - ty)$$

over which the fraction $x'/y = x/y^2$ is integral but $x'/y \notin R'$ (as x' doesn't vanish in $R'/yR' = \kappa[x', t]/(x'^2)$). Passing to π-adic completions, for

$$A := k\langle x, y\rangle/(x^2 + \pi^4 + y^5)$$

(so $\mathrm{Sp}(A)$ is a smooth affinoid curve) the π-adic completion $k^0\langle x, y\rangle/(x^2 + \pi^4 + y^5)$ of the normal R is normal by excellence considerations (or by using Serre's normality criterion). Hence, by [dJ, Prop. 7.3.6], this completion computes A^0. Thus,

$$A^0\langle \pi/y, x/y\rangle = k^0\langle x', t\rangle/(x^2 + \pi^4 + y^5),$$

and this is non-normal (again by using x'/y). But $A\langle \pi/y, x/y\rangle$ is normal by excellence considerations since A is normal, so its ring of power-bounded functions is also normal. It follows that the integral inclusion

$$A^0\langle \pi/y, x/y\rangle \hookrightarrow A\langle \pi/y, x/y\rangle^0$$

is not an equality (e.g., x/y^2 is a power-bounded analytic function on the rational domain but it does not arise from $A^0\langle \pi/y, x/y\rangle$).

Example 12.2.9 Fiber products will turn out not to exist among adic spaces in the complete generality, but rational domains play a fundamental role in the entire theory of adic spaces. Since $\mathrm{Spa}(A, S)$ depends on a given subset $S \subset A$ only through the open integrally closed subring generated by $S + A^{00}$, it is therefore natural to ask if the induced map

$$\varphi : A^0\langle f/g\rangle \to A\langle f/g\rangle^0$$

(for any Huber ring A and $f_1, \ldots, f_n \in A$ generating an open ideal) is integral, as we have shown always holds in the rigid-analytic setting (whereas by Example 12.2.8 it generally fails to be an *equality* in the rigid-analytic case). The failure of integrality of φ for Huber rings A, if it really occurs, would be a compelling illustration of the importance of permitting rather general open integrally closed subrings $A^+ \subset A^0$ when developing the theory of adic spaces. We have not yet found such an example, but Scholze is confident that they exist and he has suggested to try to adapt some of the constructions in [Mi, §2–§3] to do this. We have not yet succeeded in this task, so in lieu of that we offer a few thoughts on the task of finding examples of non-integral φ.

Boundedness of a subset Σ in a Huber ring B is equivalent to boundedness of the image of Σ in $\widehat{B}$ (as boundedness is equivalent to the property that for an ideal of definition I in a ring of definition B_0 and every $n > 0$ we have $I^{e(n)} \cdot \Sigma \subset I^n$ for some $e(n) > 0$, and I^n is the preimage under $B \to \widehat{B}$ of $\widehat{I^n}$), and the image of φ is open, so it is equivalent to ask that the image of $A(f/g)^0$ in $A\langle f/g\rangle^0$ is not integral over the completion $A^0\langle f/g\rangle$ of $A^0[f/g]$. Thus, a *necessary* condition for finding an example of non-integral φ is that the Huber ring $A(f/g) = A_g$ contains an element power-bounded relative to the topology on $A(f/g)$ (generally not the topology of A when $g = 1$!) yet not integral over the subring $A^0[f/g]$. Scaling by g^m is harmless for this purpose, so a warm-up to finding a counterexample would be to find $a \in A$ *not* integral over the subring $A^0[f]$ yet which is nonetheless power-bounded for the topology of $A(f/g)$.

Consider a Huber pair (A, A^+) with $A^+ = A^0$ such that A^0 is *bounded* in A. For $a \in A$ we have the rational localization $A\langle a\rangle := A\langle 1/1, a/1\rangle$. The open subring $A^0\langle a\rangle$ is a topological quotient of $A^0\langle U\rangle$, and more specifically it is the image of $A^0\langle U\rangle$ under $A\langle U\rangle \twoheadrightarrow A\langle a\rangle$, so $A^0\langle a\rangle$ is bounded in $A\langle a\rangle$ because $A^0\langle U\rangle$ is bounded in $A\langle U\rangle$. Hence, to prove for some such pair (A, a) that $A\langle a\rangle^0$ is not integral over $A^0\langle a\rangle$ it would suffice to show (for the same pair) that $A\langle a\rangle^0$ is not bounded whereas the integral closure in $A\langle a\rangle$ of $A^0\langle a\rangle$ is bounded. We do not know any such example (and [BV, §4.5] gives a Huber ring R and element $r \in R$ such that R^0 is bounded but the integral closure in $R\langle r\rangle$ of the bounded open subring $R^0\langle r\rangle$ is *not* bounded).

Example 12.2.10 Suppose $X = \mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+) and $x \in X$ is an analytic point for which the associated valuation v_x is higher-rank; such x exist in abundance even when $A^+ = A^0$ with affinoid algebras A over a non-archimedean field (even simply the closed unit disc over such a field). The valued field $\kappa(x)$ has the topology of a rank-1 valuation, namely corresponding to the unique rank-1 generization of such an x in X, so the subring of power-bounded elements in $\kappa(x)$ (which is determined by the topology on $\kappa(x)$ from v_x) is the valuation ring of that rank-1 valuation. Note that the valuation ring $\kappa(x)^+$ of v_x on this field is higher-rank, so it is a proper subring of $\kappa(x)^0$.

This is a very important situation because when we want to study fibers of morphisms of adic spaces $Y \to X$ (upon equipping our spaces with structure sheaves and constructing fiber products in sufficient generality), the "adic-space fiber" over $x \in X$ is really an adic space over the base $\mathrm{Spa}(\kappa(x), \kappa(x)^+)$ (which has the subspace topology from $\mathrm{Spv}(\kappa(x))$ with x as its unique closed point and the unique rank-1 generization as its unique generic point). So in addition to the fact that this base space for the fiber is generally not a single point, we see that on this affinoid base the Huber pair does not satisfy the condition $R^+ = R^0$. Since higher-rank points are pervasive and fundamental to the entire theory, this shows that demanding $R^+ = R^0$ is incompatible with the ability to make effective use of higher-rank points even when studying concrete adic spaces (such as fibers of morphisms between adic spaces associated to rigid-analytic spaces).

12.3 The Structure Presheaf and Rational Domains

Let (A, A^+) be a Huber pair and $X = \mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(A)$. This has a base of opens given by rational domains $U = X(T/s) = \{x \in X \mid x(t_i) \le x(s) \neq 0\}$ for $T = \{t_1, \dots, t_n\} \subset A$, $s \in A$, such that $T \cdot A$ is open in A.

We saw in the previous lecture that U can be described as follows. We set $A(T/s)$ to be the ring A_s with a topology coming from T, and $A^+[T/s]$ to be the A^+-subalgebra of $A(T/s) = A_s$ generated by the fractions t_i/s, and $A(T/s)^+$ denotes the integral closure of $A^+[T/s]$ in $A(T/s) = A_s$. There are also the Huber pairs $(\widehat{A}, \widehat{A}^+)$ and $(A\langle T/s\rangle, A\langle T/s\rangle^+)$ arising via completion (where the second member of each pair is really the integral closure of a completed open subring), and we saw that the natural continuous maps

$$\mathrm{Spa}(A\langle T/s\rangle, A\langle T/s\rangle^+) \to \mathrm{Spa}(A(T/s), A(T/s)^+) \to X(T/s)$$

are homeomorphisms respecting (in all directions!) the condition of being a rational domain.

In order to define a structure presheaf on rational domains, we first need to understand the precise sense in which the completed Huber pair $(A\langle T/s\rangle, A\langle T/s\rangle^+)$ over (A, A^+) is intrinsically determined by the subset $X(T/s) \subset X$. To do this we will write down a universal property inspired by the definition of an affinoid subdomain in classical rigid geometry, so we will first recall how the latter goes.

(Since there is no cohomological criterion to be affinoid, the notion of a general affinoid subdomain rather than just a rational domain is even more "accidental" than the notion of a general affine open subscheme of an affine scheme.)

Definition 12.3.1 A subset $U \subset \mathrm{Sp}(A)$ is an *affinoid subdomain* if there exists a k-affinoid map $f \colon A \to B$ such that

1. $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ lands in U,
2. f is initial among k-algebra maps $A \to C$ to k-affinoids such that $\mathrm{Sp}C \to \mathrm{Sp}A$ lands in U.

In [BGR, 7.2.1–7.2.6] it is shown that if such an f exists, then (i) the map $\mathrm{Sp}B \to U$ is bijective (and even a homeomorphism for the naive totally disconnected topology), (ii) $U \subset \mathrm{Sp}A$ is open for the naive topology, and (iii) the induced map on completed local rings $\widehat{A_u} \to \widehat{B_u}$ for $u \in U$ (which makes sense by (i)) is an isomorphism.

The most important instance of the preceding rigid-analytic notion is:

Example 12.3.2 If $U = \{|f_i| \leq |g| \text{ for all } i\}$, where the f_i have no common zero, then one can take $B = A\langle f_1/g, \ldots, f_n/g\rangle$ (justified by using the universal mapping property of completed localizations).

We now generalize to the adic setting:

Proposition 12.3.3 *For any morphism of Huber pairs $\varphi : (A, A^+) \to (B, B^+)$ with B complete, $\mathrm{Spa}(\varphi) \colon \mathrm{Spa}(B, B^+) \to X = \mathrm{Spa}(A, A^+)$ lands in $X(T/s) =: U$ if and only if there is a factorization as morphisms of Huber pairs*

$$\begin{array}{ccc} (A, A^+) & \longrightarrow & (B, B^+) \\ & \searrow \qquad \nearrow & \\ & (A\langle T/s\rangle, A\langle T/s\rangle^+) & \end{array}$$

in which case the factorization is unique. (*Keep in mind that maps of Huber pairs are required to be continuous on underlying topological rings*).

This result, to be proved below, implies that we may unambiguously define a Huber pair $(F_A(U), F_A^+(U))$ attached to a rational domain $U \subset X = \mathrm{Spa}(A, A^+)$ via

$$F_A(U) := A\langle T/s\rangle, \;\; F_A^+(U) = A\langle T/s\rangle^+$$

for any description of U as $X(T/s)$. Setting $U = X$, we see that for a Huber pair (A, A^+) not assumed to be complete, by *definition* $F_A(\mathrm{Spa}(A, A^+))$ is the completion $\widehat{A}$ (not A) in general. At the other extreme, the empty set is also rational (let $s = 0$ and $T = \{1\}$) and if U is empty then $F_A(U) = 0 = F_A^+(U)$ as one would expect (it does have the required universal property!).

Proof. It is equivalent to work with the uncompleted $A(T/s) = A_s$ and the open subring $A^+(T/s)$ (without passing to its integral closure inside A_s) since B is complete and B^+ is integrally closed (keep in mind how $A\langle T/s\rangle^+$ is defined in terms of a commutative-algebra completion and an integral closure). Thus, it suffices to show that $\mathrm{Spa}(\varphi)$ lands in U if and only if the following conditions all hold: $\varphi(s) \in B^\times$, all $\varphi(t_i)/\varphi(s)$ lies in B^+, and the unique A-algebra map $A_s \to B$ is continuous for the "T/s" topology on A_s.

By the universal property for that topology, the latter is equivalent (given that $\varphi(s) \in B^\times$) to the combined conditions that $A \to B$ is continuous and $\varphi(t_i)/\varphi(s) \in B^0$. But the former continuity is given to us at the outset and the latter condition is subsumed by the desired property $\varphi(t_i)/\varphi(s) \in B^+$ since $B^+ \subset B^0$ (by the definition of a Huber pair).

In summary, we need to check that $\mathrm{Spa}(\varphi)\colon \mathrm{Spa}(B, B^+) \to X$ lands in $X(T/s)$ if and only if $\varphi(s) \in B^\times$ and $\varphi(t_i)/\varphi(s) \in B^+$.

Given these latter two algebraic conditions, it is easy to see from the definition of $\mathrm{Spa}(B, B^+)$ and $\mathrm{Spa}(\varphi)$ that $\mathrm{Spa}(\varphi)$ lands inside $X(T/s) = U$. It is more subtle to conversely use the topological conditions to deduce the algebraic ones, as we now explain.

Assume $\mathrm{Spa}(\varphi)$ lands in $X(T/s)$, so the element $\varphi(s) \in B$ has no zeros on $\mathrm{Spa}(B, B^+)$. To deduce it must be a unit, it suffices to show that any element $b \in B$ with no zeros on $\mathrm{Spa}(B, B^+)$ is a unit. This will rest crucially on the completeness of B (see Remark 12.3.4 for an easy counterexample otherwise). Assuming no zeros, if we let J denote the closure of the ideal $\varphi(s)B$ then the Huber ring B/J is complete (check!) and the Huber pair $(B/J, B^+/(B^+ \cap J))$ has *empty* adic spectrum. But recall that a complete Huber pair has empty adic spectrum if and only if it vanishes, so we conclude that the closure of $\varphi(s)B$ is the ring B.

Again using the completeness of B, the unit group $B^\times$ is open as it contains the open set $1 + I$ around 1 where I is any ideal of definition in a ring of definition. (Here we use the geometric series for $1/(1 + x)$ with topologically nilpotent x in a complete Hausdorff topological ring; see Remark 12.3.4 for an easy counterexample to openness of the unit group without completeness.) Some multiple $\varphi(s)b$ has to lie in such an open $1 + I$ around 1 due to the density of $\varphi(s)B$ in B, so that multiple is a unit, and therefore $\varphi(s)$ is a unit too.

Now using that $\varphi(s) \in B^\times$, the fractions $\varphi(t_i)/\varphi(s) \in B$ make sense and $\mathrm{Spa}(\varphi)$ lands in $X(T/s)$ precisely when $v(\varphi(t_i)/\varphi(s)) \leq 1$ for all $v \in \mathrm{Spa}(B, B^+)$. By Theorem 10.3.6 (the "adic Nullstellensatz"), it follows that $\varphi(t_i)/\varphi(s) \in B^+$. □

Remark 12.3.4 Let's show by example that if we drop the completeness hypothesis on B then certain parts of the preceding proof break down. Firstly, the openness of $B^\times$ in B certainly fails: for non-archimedean k we have the Huber pair $(k[t], k^0[t])$ as usual (with completion $(k\langle t\rangle, k^0\langle t\rangle)$) but $k[t]^\times = k^\times$ is not open in $k[t]$ (whereas $1 + \varpi k^0\langle t\rangle \subset k\langle t\rangle^\times$ for any pseudo-uniformizer ϖ of k). Secondly, $1 + \varpi t$ has empty zero locus on $\mathrm{Spa}(k[t], k^0[t]) = \mathbf{D}_k$ but it is not a unit in $k[t]$ (though it is a unit in the completion $k\langle t\rangle$, as we know it must be).

Next we apply the universal property in Proposition 12.3.3 to equip $(F_A(U), F_A^+(U))$ with a functoriality in the pair (A, U); doing this with A fixed will then provide restriction maps for a presheaf structure (on rational domains).

Consider a map of Huber pairs

$$\varphi\colon (A', A'^{+}) \to (A, A^{+})$$

inducing a continuous map $f : X \to X'$, and suppose we are given rational domains $U \subset X$ and $U' \subset X'$ such that $f(U) \subset U'$. Then the composite map $(A', A'^{+}) \to (A, A^{+}) \to (F_A(U), F_A^{+}(U))$ satisfies exactly the topological condition to obtain from the universal property a unique factorization through some

$$F(\varphi) : (F_{A'}(U'), F_{A'}^{+}(U')) \to (F_A(U), F_A^{+}(U))$$

as Huber pairs. By uniqueness this is seen to be transitive in φ, so it is really a functor (obvious $F(\mathrm{id})$ is the identity map). Taking (A', A'^{+}) to be (A, A^{+}) then defines restriction maps

$$(F_A(W), F_A^{+}(W)) \to (F_A(W'), F_A^{+}(W'))$$

for inclusions of rational domains $W' \subset W$ inside X with the desired transitivity as in the presheaf axioms. (Note that taking W and/or W' to be empty behaves well on equal footing with everything and does not require ad hoc procedures.)

The following fact, from [H2, Lemma 1.5], is used all the time without comment (and prevents total confusion that would result if it were not true):

Proposition 12.3.5 *For rational $W \subset X = \mathrm{Spa}(A, A^{+})$, a subset $W' \subset W$ is rational with respect to (A, A^{+}) if and only if it is rational with respect to $(F_A(W), F_A^{+}(W))$.*

This result is non-trivial; the key difficulties in its proof are the same as those which underlie the proof of the result discussed last time concerning the invariance of Spa and its notion of rational domain relative to completion of a Huber pair.

We can now finally define the structure presheaf $\mathcal{O}_A$: for open $U \subset X$, we define

$$\mathcal{O}_A(U) = \varprojlim_{W \subset U} F_A(W) = \{(f_w) \in \prod F_A(W) \mid f_W|_{W\cap W'} = f_{W'}|_{W \cap W'} \text{ inside } F_A(W \cap W')\}$$

where W varies through rational domains contained in U. This is viewed with the subspace topology of $\prod_W F_A(W)$ (so $\mathcal{O}_A(U)$ really is a topological inverse limit, for those who are more categorically-inclined). In particular, if U is rational then this recovers $F_A(U)$ as a topological ring since the inverse system of such W's has U itself as a unique object dominating all others. Likewise define the subring

$$\mathcal{O}_A^{+}(U) = \varprojlim_{W \subset U} F_A^{+}(W) \subset \mathcal{O}_A(U).$$

Beware that this is generally *not* open (though it is open when U is quasi-compact, as we will address in Remark 12.3.10):

Example 12.3.6 Consider U that is a union of infinitely many disjoint "closed discs" D_i inside the closed unit disk $\mathbf{D}_k$. (These "closed subdiscs" are really open and *not* closed for the topology on $\mathbf{D}_k$!) Then it is easy to verify that $\mathcal{O}_A(U) = \prod \mathcal{O}_A(D_i)$ with the product topology (check!) and $\mathcal{O}_A^{+}(U)$ is the corresponding infinite product of $\mathcal{O}_A^{+}(D_i)$'s, so openness certainly fails in such cases.

Here is an important fact:

Proposition 12.3.7 *For $x \in X$, $\mathcal{O}_{A,x}$ is a local ring.*

Proof. Unraveling the definitions, $\mathcal{O}_{A,x} = \varinjlim_{W \ni x} F_A(W)$ where the limit is over rational domains only.

For $W' \subset W$ rational, consider the valuations

$$\begin{array}{ccc} F_A(W) & \xrightarrow{v_x} & \Gamma_x \cup \{0\} \\ \downarrow & & \downarrow \\ F_A(W') & \xrightarrow{v'_x} & \Gamma'_x \cup \{0\} \end{array}$$

As $F_A(W')$ is a completed localization of $F_A(W)$, both have the same value group. Therefore we can extend this to a valuation v_x on $\mathcal{O}_{A,x}$. (We do not try to make sense of continuity for v_x on $\mathcal{O}_{A,x}$, such as with direct limit topologies, as that is of no use to us.) It now suffices to show that $v_x(f) \neq 0$ implies $f \in \mathcal{O}_{A,x}^{\times}$.

Choose W so that f comes from $B = F_A(W)$, with associated $B^+ = F_A^+(W)$ (an open subring). Let $I \subset B$ be an ideal of definition of a ring of definition. Since elements of I are topologically nilpotent (so $v_x(t) \leq 1$ in Γ_x for all $t \in I$) and I is finitely generated, by replacing I with I^N for large enough N we can ensure that $v_x(t_i) \leq v_x(f)$ for $t_1, \ldots, t_n$ generating I over B_0. The domain

$$W' = W(T/f) := \{v \in W \mid v(t_i) \leq v(f) \neq 0 \text{ for all } i\}$$

is rational in W and hence rational in X by Proposition 12.3.5. (Note that such t_i's generate an *open* ideal in B (as they generate an open ideal in a ring of definition of B), so it is meaningful to use them as the "numerators" in the definition of a rational domain W' inside $W = \mathrm{Spa}(B, B^+)$.) But the restriction $f|_{W'} \in B' := F_A(W')$ has no zeros in $\mathrm{Spa}(B', B'^+) = W'$, hence $f \in B'^{\times}$ because B' is complete. Passing all the way back to the limit $\mathcal{O}_{A,x}$ thereby recovers f as a unit in here. □

The device of computing with an ideal of definition in the preceding proof is necessary precisely because the value groups Γ_x vary from point to point on X. Indeed, if $f \in B := F_A(W)$ has $v_x(f) \neq 0$ then in the "easy" case that $v_x(f) \geq 1$ we can use the condition $v(f) \geq 1$ to define a rational domain $W' \subset W$ containing x such that f is a unit in the *complete* Huber ring $\mathcal{O}_A(W')$. But if $v_x(f) < 1$ then we have nothing to grab onto for defining an appropriate W'! (Recall that a condition such as "$v(f) < 1$" generally does *not* define an open locus in an adic spectrum due to the presence of higher-rank points, and anyway it certainly doesn't enforce a unit condition on f.) The consideration of generators $\{t_1, \ldots, t_n\}$ of a sufficiently small ideal of definition inside a ring of definition of B is the device by which we can define a rational domain W' inside W around x for which f restricts to a unit in $F_A(W')$ (and thus f is a unit in the x-stalk).

Remark 12.3.8 Although $\mathcal{O}_A$ has stalks that are local, it is non-trivial to see noetherian properties even if A is very nice. Suppose A is an affinoid algebra and $x \in X = \mathrm{Spa}(A, A^+)$ is a classical point (i.e., comes from $\mathrm{Sp}(A)$). By consideration of rational domains we see that $\mathcal{O}_{A,x}$ is exactly the same stalk as computed at x in rigid-analytic geometry, so it is noetherian by classical rigid geometry. But for non-classical x it isn't evident if this stalk is noetherian, and one might be worried particularly if x is not a rank-1 point. (Surprisingly, none of the literature on adic spaces appears to address this basic question.) But it turns out that in the affinoid-algebra setting the stalks at all points *are* noetherian, though the proof is not at all elementary (it rests on excellence of affinoid algebras); we will address this in a later lecture on the connection between adic spaces and rigid geometry.

In the spirit of the adic Nullstellensatz for rational domains U, in general we have a valuation-theoretic formula:

Lemma 12.3.9 *For any open $U \subset X$, $\mathcal{O}_A^+(U) = \{f \in \mathcal{O}_A(U) \mid v_x(f) \leq 1$ for all $x \in U\}$. In particular, $\mathcal{O}_A^+(U)$ is integrally closed in $\mathcal{O}_A(U)$.*

Proof. For any rational domain $W \subset U$ and $x \in W$, the valuation $v_x : \mathcal{O}_A(U) \to \Gamma_x$ is built by composing the valuation $v_{x,W}$ on $F_A(W) = \mathcal{O}_A(W)$ associated to $x \in W \subset \mathrm{Spa}(F_A(W), F_A^+(W))$ with the restriction map $\mathcal{O}_A(U) \to \mathcal{O}_A(W)$, and the adic Nullstellensatz for the Huber pair attached to W says

$$\mathcal{O}_A^+(W) = \{f \in \mathcal{O}_A(W) \,|\, v_{x,W}(f) \leq 1 \text{ for all } x \in U\}.$$

Passing to the inverse limit over all $W \subset U$ then yields the desired formula for $\mathcal{O}_A^+(U)$ inside $\mathcal{O}_A(U)$. □

Remark 12.3.10 One might object that the counterexample to openness in Example 12.3.6 is cheap insofar as U is not quasi-compact, and indeed in general if U is quasi-compact then $\mathcal{O}_A^+(U)$ is open inside $\mathcal{O}_A(U)$ (so it is an open and integrally closed subring, just as for the case of rational U).

To see this, first note that if U is a rational domain then the equality $\mathcal{O}_A^+(W) = F_A^+(W)$ inside the topological ring $\mathcal{O}_A(W) = F_A(W)$ for rational W (adic Nullstellensatz!) does the job. In general, a quasi-compact open U is covered by a finite collection of rational domains $W_1, \ldots, W_n$. Now comes a problem: we do not have any sheaf-theoretic results and so we lack a "formula" for $\mathcal{O}_A(U)$ in terms of a finite amount of data (such as the $\mathcal{O}_A(W_i)$'s and $\mathcal{O}_A(W_i \cap W_j)$'s)! We circumvent this with the help of Lemma 12.3.9 (in effect, the adic Nullstellensatz).

For each $x \in U$, the valuation v_x on $\mathcal{O}_A(U)$ arises by choosing any one of the W_i's containing x and composing the continuous (!) restriction map $\mathcal{O}_A(U) \to \mathcal{O}_A(W_i)$ with the valuation on the latter associated to $x \in W_i$. Hence, $\mathcal{O}_A^+(U)$ is nothing but the preimage under the *continuous* map

$$\mathcal{O}_A(U) \to \prod_i \mathcal{O}_A(W_i)$$

of the subset $\prod \mathcal{O}_A^+(W_i)$ of the target. This subset is open precisely because we use a product over just a *finite* index set (or more concretely we are intersecting just finitely many open subsets of $\mathcal{O}_A(U)$).

12.4 Warm-up for sheaves of topological rings

To set up what it means to speak of the sheaf property on an adic space, we use:

Definition 12.4.1 : A presheaf G of topological groups (or topological rings, or topological spaces, etc.) on a topological space X (so continuous transition maps) is a *sheaf* as such when it is a sheaf of sets with the further topological property that for any open cover $\{U_i\}$ of an open subset U the natural injection

$$G(U) \to \prod G(U_i)$$

is a topological embedding.

Remark 12.4.2 For the more categorically-minded among you, this is equivalent to saying that

$$G(U) \to \prod G(U_i) \rightrightarrows \prod G(U_i \cap U_j)$$

is an equalizer diagram in the category of topological groups (or rings, spaces, etc.).

Example 12.4.3 Although rigid-analytic spaces are not topological spaces, the rings of sections over affinoid opens are naturally topologized (as k-affinoid algebras) and the left-exact sequence expressing the sheaf axiom for a finite affinoid open cover $\{U_i\}$ of an affinoid space $X = \mathrm{Sp}(A)$ does satisfy the topological embedding property: the evident continuous map

$$A \to \ker(\prod \mathcal{O}(U_i) \to \prod \mathcal{O}(U_i \cap U_j))$$

is not only a bijection but is a topological isomorphism. This is an absolutely essential fact (e.g., it underlies the good behavior of completed tensor products to build arbitrary ground field extension functors, and it pervades the foundation of Berkovich's approach to non-archimedean geometry), and it is an immediate consequence of the Banach Open Mapping Theorem over k since the kernel in question is a closed subspace.

Returning to the general development, recall that when we form the sheaf associated to a presheaf of sets, we do it by two iterations of the construction associating to any open U the inverse limit

$$G^+(U) = \varprojlim G(W)$$

as W varies through opens in U. (The traditional notation G^+ here of course has nothing whatsoever to do with $\mathcal{O}_A^+$ in relation to $\mathcal{O}_A$ in the adic space situations considered earlier in this lecture! Please don't get confused.)

Such G^+ itself inherits a structure of presheaf of topological groups (or rings, spaces, etc.). As usual, G^+ is a separated presheaf, meaning that maps such as displayed in the preceding definition are injective, but even better the topology on $G^+(U)$ is rigged so that this injection is a topological embedding (exercise!). Hence, G^{++} is not only a sheaf in the non-topological sense, but really is a sheaf of topological groups (or rings, spaces, etc.) in the sense defined above.

For a Huber pair (A, A^+) with its presheaf $\mathcal{O}_A$, suppose $\mathcal{O}_A$ satisfies the sheaf axiom in the purely algebraic sense when covering a rational domain U by finitely many rational subdomains U_i. Since every open cover of such U has a refinement by a finite cover of rational subdomains (due to the quasi-compactness of such U and the cofinality of rational domains among opens in the ambient space), elementary sheaf theory exactly as implicit in the construction of the structure sheaf on Spec of a ring implies that $\mathcal{O}_A$ as we defined it on *arbitrary* open sets is a sheaf as well (ignoring topological aspects). Likewise, if $\mathcal{O}_A$ satisfies the "topological sheaf" axiom for a covering of a rational domain by finitely many rational subdomains then it is an elementary exercise with product topologies to check that the $\mathcal{O}_A(U)$'s topologized as we have done are sheaves of topological rings.

Hence, for the problem of verifying whether or not $\mathcal{O}_A$ is a sheaf of topological rings, the problem is entirely about rational domains and finite covers by other rational domains. The main result of Huber's paper [H2] is that this sheaf property for $\mathcal{O}_A$ (and the vanishing of its higher cohomology!) holds in either of the following two favorable-looking situations:

(i) there exists a noetherian ring of definition $A_0 \subset A$,

(ii) A is "strongly noetherian", meaning that the relative Tate algebras $A\langle t_1, \dots, t_n\rangle$ are noetherian for all $n > 0$.

The hypothesis in (i) is satisfied for situations arising from noetherian formal schemes as well as from affinoid algebras over discretely-valued fields, and the hypothesis in (ii) is satisfied for all affinoid algebras over non-archimedean fields (even algebraically closed, for which (i) fails). In case (i) the proof by Huber uses methods inspired by the treatment in [EGA, III_1] of cohomology of coherent sheaves on noetherian formal schemes, and in case (ii) the proof by Huber uses ideas from the proof of Tate's Acyclicity Theorem for affinoid algebras. But neither is applicable to perfectoid algebras, which essentially never satisfy any noetherian-like properties at all!

In Scholze's paper on perfectoid spaces, his proof of the sheaf property for adic spaces associated to perfectoid algebras is deep (especially in characteristic 0), relying on the tilting equivalence (one of the main results in that paper). Later, Buzzard and Verberkmoes found a much simpler approach to the sheaf property for Huber rings satisfying the Tate property, using a hypothesis called "stably uniform" that can be verified without too much difficulty for perfectoid algebras. A convenient feature of the Tate condition is that handling the topological aspect of the sheaf axioms is reasonably easy, so one can largely focus on the traditional difficulty of the purely algebraic task of unique gluing of local sections (but it is necessary to pay attention to topological aspects). Next time we'll discuss the Buzzard–Verberkmoes paper [BV].

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