11 Lecture 11: Affinoid adic spaces II

11.1 Introduction

The aim of this lecture is to take a detailed look at the adic unit disc, and subsequently address some important properties of the Spa functor (e.g., interaction with completion and localization). The first couple of lectures in the winter term will delve into the sheaf aspects (which we will not discuss today) and the relationship between global adic spaces and classical rigid geometry.

Before we get started, we address the relationship between $\mathbf{R}$-valued absolute values (the viewpoint of Berkovich spaces) and abstract valuations (with abstract value groups, unrelated from point to point, the viewpoint of adic spaces) in the setting of classical rigid geometry over a non-archimedean field k. Let us begin with the case of a point:

Example 11.1.1 The space $\mathrm{Spa}(k, k^0)$ consists of a single point, corresponding to the given rank-1 valuation ring k^0. Indeed, suppose $v \in \mathrm{Cont}(k)$ is a continuous valuation whose valuation ring R contains k^0. Then R is a k^0-subalgebra of k. But if $a \in k - k^0$ (i.e., $|a| > 1$) then for any $c \in k$ we have $|c/a^n| \leq 1$ for large n, so $c = (c/a^n)a^n \in k^0[a]$. In other words, any such a generates k as a k^0-algebra, so either $R = k^0$ or $R = k$. In the latter case v would be the trivial valuation on k, but that is not continuous because $\{0\}$ is not open in k. If instead $R = k^0$ then v coincides with the given valuation on k.

Beware that we do *not* claim that $\mathrm{Cont}(k)$ consists of just a single point, as this is false! Indeed, rank-1 valuation topologies on a field are often also defined by higher-rank valuations on the same field (even when the topology is complete). So typically $\mathrm{Cont}(k)$ contains many additional points v whose valuation rings simply do not contain k^0 (rather, k^0 is a localization of the valuation ring of v at a height-1 prime ideal).

A consequence of the preceding example is that for any k-affinoid algebra A, we have a natural continuous map $\mathrm{Spa}(A, A^0) \to \mathrm{Spa}(k, k^0)$ (more interesting once structure sheaves are introduced), and the target here is a point. A wrinkle in the theory of adic spaces, even those coming from rigid-analytic geometry, is that fibers of morphisms are adic spaces over a base whose underlying space is not a point! Indeed, the definition of adic spaces (as locally ringed spaces equipped with additional valuation-theoretic structure) will equip the residue field $\kappa(y)$ of the local ring at any point y of an adic space Y with a structure of valued field whose valuation ring $\kappa(y)^+$ is often of higher rank even though $\kappa(y)$ has the topology of a rank-1 valuation whenever y is an "analytic point" of Y (as occurs for every $y \in Y$ when Y is associated to a rigid-analytic space), and for any (reasonable) morphism $f : X \to Y$ the "adic-space fiber" of f over y will be an adic space over the base $\mathrm{Spa}(\kappa(y), \kappa(y)^+)$ whose underlying space is not a point whenever $\kappa(y)^+ \neq \kappa(y)^0$ (i.e., whenever $\kappa(y)^+$ is a higher-rank valuation ring). This is a huge deviation from experience with schemes, formal schemes, complex-analytic spaces, rigid-analytic spaces, and Berkovich spaces. Yet the higher-rank points give the theory its power (much as non-closed points are crucial in the theory of schemes), so these points cannot be ignored.

To explain the link with $\mathbf{R}$-valued valuations, we first recall that for a k-affinoid algebra A, the Tate property of A implies that all points in $\mathrm{Cont}(A)$ are analytic (Corollary 8.3.3), so every $x \in X$ has a unique rank-1 generization v_x that moreover admits no proper generizations inside X (or even inside $\mathrm{Cont}(A)$) due to Proposition 9.1.5 and Theorem 10.3.6. For a general ring A, at a rank-1 point of $\mathrm{Spv}(A)$ the value group embeds into $\mathbf{R}_{>0}$ as an ordered group but there is no preferred embedding (though any two such embeddings are related through $t \mapsto t^e$ for a unique $e > 0$, as is explained in the proof of the proposition below). However, the situation is better when A is k-affinoid, as the absolute value on k singles out a *preferred* embedding:

Proposition 11.1.2 *Let A be a k-affinoid algebra. Every rank-1 point $v \in X = \mathrm{Spa}(A, A^0)$ arises from a* unique *continuous multiplicative semi-norm $|\cdot|_v : A \to \mathbf{R}_{\geq 0}$ bounded on A^0 that extends the given absolute value $|\cdot|$ on k.*

Assigning to each $x \in \mathrm{Spa}(A, A^0)$ the semi-norm $|\cdot|_{\eta_x}$ thereby associated to the unique rank-1 generization $\eta_x \in X$ of x defines a continuous surjection $q : X \to M(A)$ onto the (compact Hausdorff) Berkovich spectrum, and this restricts to a bijection on the subset of rank-1 points in X. This map is initial among continuous maps from X to Hausdorff topological spaces.

We will never use the Berkovich aspects of this result in what follows. Also, beware that the evident section $\sigma : M(A) \to X$ assigning to each $z \in M(A)$ the unique rank-1 point in its fiber is essentially *never* continuous; this fails even for the closed unit disc (see Example 11.3.18).

Proof. For the first claim, our task is to show that Γ_v admits a unique order-preserving inclusion into $\mathbf{R}_{>0}$ whose restriction to $v(k^{\times})$ recovers the given absolute value on k. The rank-1 condition ensures that there is *some* order-preserving homomorphism $j : \Gamma_v \hookrightarrow \mathbf{R}_{>0}$. Choose nonzero $c \in k^{00}$, so $|c| < 1$ and $j(v(c)) < j(1) = 1$. Hence, there is a unique $e > 0$ such that $|c| = j(v(c))^e$ Replacing j with j^e then brings us to the case that the two $\mathbf{R}$-valued valuations $j \circ v|_k$ and $|\cdot|$ on k coincide at c. It therefore suffices to show that if Γ is a rank-1 totally ordered commutative group then any two order-preserving homomorphic inclusions $j, j' : \Gamma \to \mathbf{R}_{>0}$ satisfy $j' = j^e$ for a unique $e > 0$ (as then $j' = j$ when $j'(\gamma) = j(\gamma)$ for some $\gamma \in \Gamma - \{1\}$).

We can, as above, choose a unique e that "works" at one $\gamma \in \Gamma - \{1\}$ and then replace j with j^e to reduce to checking that if $j'(\gamma_0) = j(\gamma_0)$ for some $\gamma_0 \in \Gamma - \{1\}$ then $j' = j$. We may assume $\gamma_0 < 1$ (invert if necessary) and it suffices to show $j'(\gamma) = j(\gamma)$ for all $\gamma < 1$.

The rank-1 condition ensures that we can approximate γ by fractional powers of γ_0, or more formally for integers $m > 0$ the power γ^m lies between γ_0^{n+1} and γ_0^n for some integer $n > 0$ (depending on m) due to the absence of proper nontrivial convex subgroups of Γ (thanks to the rank-1 condition). Hence, $j(\gamma)^m$ lies between $j(\gamma_0)^{n+1}$ and $j(\gamma_0)^n$, so $j(\gamma)$ lies between $j(\gamma_0)^{(n+1)/m}$ and $j(\gamma_0)^{n/m}$. The same goes for j', yet $j'(\gamma_0) = j(\gamma_0)$, so the logarithms of $j(\gamma)$ and $j'(\gamma)$ to the common base $j'(\gamma_0) = j(\gamma_0)$ sit between the same consecutive elements n/m and $(n+1)/m$ of $(1/m)\mathbf{Z}$ for each $m > 0$ (where n depends on m). Taking $m \to \infty$ then forces equality of the logarithms in $\mathbf{R}$ and hence $j'(\gamma) = j(\gamma)$ as desired.

Now we address the second part of the proposition, concerning the map $q : X \to M(A)$ that is surjective by design. Since all generization relations in X are vertical and every point in X has a unique rank-1 generization in X, the fibers of $q : X \to M(A)$ have unique generic points that in turn are precisely the rank-1 points. But any continuous map to a Hausdorff space has to carry a pair of points related through specialization (or generization) to the same point, so any continuous map from X to a Hausdorff space factors uniquely through q on the set-theoretic level. It remains to show that q is continuous and a topological quotient map (so it will then have the initial mapping property in the sense of continuous maps to Hausdorff spaces). If q is continuous then it is automatically closed because any closed subset of X inherits compactness and thus has image in $M(A)$ that is compact and therefore closed (as $M(A)$ is Hausdorff!). In other words, q is automatically a topological quotient map once we know it is continuous.

From the definition of the topology on the Berkovich spectrum it follows (with some work) that a base of open sets in $M(A)$ is given by finite intersections of loci $V = \{z \in M(A) \,|\, |f|_z < |g|_z\}$ for $f, g \in A$, so it is enough to prove $q^{-1}(V)$ is open in X for such V. This preimage consists of $x \in X$

such that $\eta_x(f) < \eta_x(g)$ for the unique rank-1 generization η_x of x in X (the preimage is *not* given by the condition $x(f) < x(g)$!). In particular, $\eta_x(g) \neq 0$. Since η_x is a vertical generization of x, so their supports in A coincide, we can view η_x and x as valuations on a common field $\kappa(x)$. More specifically, the method of construction of η_x showed that the x-topology on $\kappa(x)$ is a rank-1 topology which is the same as that defined by η_x. Let $f(x), g(x) \in \kappa(x)$ denote the images of f and g, so $g(x) \neq 0$ since $\eta_x(g) \neq 0$. The hypothesis $|f|_{\eta_x} < |g|_{\eta_x}$ says exactly that $|f(x)/g(x)|_{\eta_x} < 1$. But η_x is a rank-1 valuation, so the element $f(x)/g(x) \in \kappa(x)$ is topologically nilpotent for the η_x-topology on $\kappa(x)$. This topology is the same as the one from x, whence $f(x)/g(x)$ is topologically nilpotent relative to the x-topology. This implies (but not conversely!) that $x(f)/x(g) < 1$ in the value group of x, which is to say $x(f) < x(g)$.

In contrast with rigid-analytic spaces and Berkovich spaces, it is generally *false* that loci of the form $\{x \in X \mid x(f) < x(g)\}$ are open. This fails even for the closed unit disc! The cause of such behavior is the presence of higher-rank points; see Example 11.3.17. To bypass this issue, we have to make effective use of k. Pick $c \in k$ with $0 < |c| < 1$. For $x_0 \in X$ and the associated rank-1 point $\eta_0 = \eta_{x_0}$, suppose $|f|_{\eta_0} < |g|_{\eta_0}$ in $\mathbf{R}$ (so $|g|_{\eta_0} \neq 0$). On the residue field $\kappa(x_0)$ the x_0-topology is the same as the η_0-topology, and the image $g(x_0)$ of g is nonzero with the fraction $f(x_0)/g(x_0)$ topologically nilpotent (as can be checked using η_0). Thus, for large enough n the element $(1/c)(f(x_0)/g(x_0))^n$ is in the valuation ring for x_0, so $x_0(f)^n \leq x_0(c)x_0(g)^n$ in the value group for x_0. So consider the subset

$$\Omega = \{x \in X \mid x(f^n) \leq x(cg^n) \neq 0\}$$

of X. This is open in X (as X has the subspace topology from $\mathrm{Spv}(A)$), and by design it contains x_0. We will show that $\Omega \subset q^{-1}(V)$, so since x_0 was arbitrary in $q^{-1}(V)$ the opennes of $q^{-1}(V)$ in X will be established.

For $x \in \Omega$, the image $g(x)$ of g in the residue field $\kappa(x)$ at x is nonzero. Thus, $f(x)^n/cg(x)^n$ makes sense in $\kappa(x)$ and lies in the valuation ring associated to x, so it is power-bounded for the valuation topology. But the rank-1 point η_x defines the same topology on that field, with its own valuation ring as the entire set of power-bounded elements (since η_x has rank 1). Hence, $\eta_x(f^n) \leq \eta_x(cg^n) \neq 0$. But $\eta_x(c) < 1$ since c is topologically nilpotent, so $\eta_x(cg^n) < \eta_x(g)^n$ since $\eta_x(g) \neq 0$. Thus, $\eta_x(f) < \eta_x(g)$. In other words $\Omega \subset q^{-1}(V)$ $\square$

11.2 The adic unit disc $\mathbf{D}_k = \mathrm{Spa}(k\langle t\rangle, k^0\langle t\rangle)$: preliminary constructions

We now discuss the topological space underlying the most basic nontrivial example of an adic space over a non-archimedean field k. We let $A := k\langle t\rangle$, and $A^+ = A^0 = k^0\langle t\rangle$. The associated topological space is denoted

$$\mathbf{D}_k := \mathrm{Spa}(A, A^+).$$

We describe $\mathbf{D}_k$ step by step, showing in particular that in the case, for example, when $k = \mathbf{C}_p$, it has five different types of points, one of which has no simple interpretation in terms of rigid-analytic geometry (whereas the other four types will have rigid-analytic meaning, and are encountered in Berkovich's approach to non-archimedean geometry).

We denote by $|\cdot|$ the non-archimedean absolute value on k, and $\mathfrak{m}$ the maximal ideal of k^0. Denote by κ the residue field $k^0/\mathfrak{m}$.

Consider $k^0[t]$ as a ring with the ϖ-adic topology for any pseudo-uniformizer $\varpi \in k^0$, and make this a ring of definition of $k[t] = k^0[t][1/\varpi]$. This makes $k[t]$ into a Huber ring (even Tate), and the completion of the pair $(k[t], k^0[t])$ is $(k\langle t\rangle, k^0\langle t\rangle)$. It is easy to check that $k^0[t]$ is the ring of power-bounded elements of the Tate ring $k[t]$. As we will review in §11.5, for any Huber ring A and ring of integral elements A^+ (i.e., open and integrally closed subring of A^0) the natural map $\mathrm{Spa}(\widehat{A}, \widehat{A^+}) \to \mathrm{Spa}(A, A^+)$ is a homeomorphism. As a special case, we obtain:

Proposition 11.2.1 *The natural continuous map*

$$\mathrm{Spa}(k\langle t\rangle, k^0\langle t\rangle) \to \mathrm{Spa}(k[t], k^0[t])$$

is a homeomorphism.

As a consequence, in order to study the points of $\mathbf{D}_k$ we are reduced to study those in $\mathrm{Spa}(k[t], k^0[t])$. This has the effect of making the algebra easier (e.g., $\mathrm{Cont}(k[t])$ is acted upon by $t \mapsto at + b$ for *any* $a \in k^\times$ and $b \in k$, whereas for $\mathrm{Cont}(k\langle t\rangle)$ that only works when $|a| = 1$ and $|b| \leq 1$ (a range of cases that is too limited to turn many small closed discs into the unit disc centered at the origin under suitable translation and rescaling).

The classical points: type 1. For each maximal ideal $\mathfrak{n}$ of $k[t]$, the quotient $k[t]/\mathfrak{n}$ is a field of finite degree over k, so it has a unique absolute value (also denoted $|\cdot|$) extending the one on k. This defines a valuation

$$v_{\mathfrak{n}} : A := k[t] \to \mathbf{R}_{\geq 0}$$

with support $\mathfrak{n}$ by sending f to $|f \bmod \mathfrak{n}|$. Since $k^0[t]$ is a ring of definition with ideal of definition generated by a pseudo-uniformizer ϖ of k and $v_{\mathfrak{n}}(\varpi) = |\varpi|$ is certainly cofinal in the value group (it is even cofinal in $\mathbf{R}_{>0}$), by the initial discussion in Example 10.2.3 we see that $v_{\mathfrak{n}}$ is continuous if and only if $v_{\mathfrak{n}}(a) < 1/|\varpi|$ for all $a \in k^0[t]$. By the non-archimedean property, it is equivalent that $v_{\mathfrak{n}}(t^n) < 1/|\varpi|$ for all $n \geq 1$, which is to say $v_{\mathfrak{n}}(t) < 1/|\varpi|^{1/n}$ for all $n \geq 1$, or in other words $v_{\mathfrak{n}}(t) \leq 1$, in which case $v_{\mathfrak{n}}(a) \leq 1$ for all $a \in k^0[t]$.

By comparing supports we see that distinct $\mathfrak{n}$'s give rise to distinct points in $\mathrm{Spv}(k[t])$ with non-generic support in $\mathrm{Spec}(k[t])$. Since a general $\mathfrak{n}$ has a unique monic generator h with positive degree, and all roots of h in $\overline{k}$ have the same absolute value, it makes sense to speak of $\mathfrak{n}$ having "integral" support (i.e., all roots lie in the valuation ring of $\overline{k}$), so $v_{\mathfrak{n}}$ is continuous if and only if $\mathfrak{n}$ has integral support. (It might sound paradoxical that continuity on $k[t]$ is characterized by such an integrality condition, as that integrality is not preserved by the k-algebra automorphisms of $k[t]$ given by $t \mapsto ct$ for general $c \in k^\times$, but there is no inconsistency since such k-algebra automorphisms are *not* continuous when $|c| > 1$! Indeed, continuity demands that $\varpi^n k^0[t]$ is carried into $k^0[t]$ for sufficiently large n, but $\varpi^n t^m \mapsto \varpi^n c^m t^m$ and if $|c| > 1$ then for any $n > 0$ (no matter how large) we have $\varpi^n c^m \notin k^0$ for large enough m, depending on n. In particular, the only $c \in k^\times$ for which $t \mapsto ct$ is continuous with continuous inverse are those that satisfy $|c| = 1$.)

We conclude that $v_{\mathfrak{n}} \in \mathrm{Spa}(k[t], k^0[t]) =: \mathbf{D}_k$ if and only if $\mathfrak{n}$ has integral support, which is to say $\mathbf{D}_k = \mathrm{Cont}(k[t])$. The set of such points corresponds bijectively to the underlying set of the maximal ideal space $\mathrm{Sp}(A)$, so these points are called "classical".

Definition 11.2.2 The points $v_{\mathfrak{n}}$ in $\mathrm{Cont}(k[t])$ are called *type* 1; if $\mathfrak{n} = (t - x)$ for $x \in k^0$ then it is denoted α_x.

When $k = \overline{k}$, points of type 1 have the same value group and residue field as does k.

Points on the limbs: types 2 and 3. Let us now consider x in k as before, and choose $r \in \mathbf{R}_{\geq 0}$. Define

$$v_{x,r} : k[t] \to \mathbf{R}_{\geq 0}$$

by sending $f = \sum a_m(t-x)^m$ (a finite sum centered at x) to

$$\sup_m \{|a_m| r^m\}.$$

This supremum is a maximum since $|a_m| \neq 0$ only for finitely many m, and we could have defined $v_{x,r}$ the very same way on the completion $k\langle t\rangle$, where the supremum would have still been a maximum. (Here we see concretely the implicit role of the homeomorphism $\mathrm{Spa}(k\langle t\rangle, k^0\langle t\rangle) \simeq \mathrm{Spa}(k[t], k^0[t])$.)

If $r = 0$ then we recover the k-rational type-1 points when moreover $|x| \leq 1$, so our main interest now is the case $r > 0$. Informally, if $r > 0$ then $v_{x,r}$ should be thought of as a "generic point" for a closed disc of radius r centered at x, except for two issues: in classical rigid-analytic geometry such a disc only has meaning when $r \in \sqrt{|k^\times|}$, and the points $v_{x,r}$ will turn out to be non-closed when $r \in \sqrt{|k^\times|}$ but closed for all other $r > 0$.

Lemma 11.2.3 *The function $v_{x,r}$ is a valuation on $k[t]$ for any $x \in k$ and $r > 0$.*

Proof. Without loss of generality, $x = 0$. The ultrametric inequality for $v_{x,r}$ follows from that of $|\cdot|$. Multiplicativity needs a small argument: for nonzero $f = \sum_m a_m t^m$ and $g = \sum_m b_m t^m$ in $k[t]$ we want to show that $v_{0,r}(fg) = v_{0,r}(f)v_{0,r}(g)$. Write $f \cdot g$ as $\sum_m c_m t^m$ and let i_0 be the smallest index such that

$$|a_{i_0}| r^{i_0} = v_{0,r}(f);$$

likewise define j_0 minimal such that

$$|b_{j_0}| r^{j_0} = v_{0,r}(g).$$

Let $m_0 := i_0 + j_0$. Then

$$c_{m_0} = \sum_{i+j=m_0} a_i b_j$$

and

$$\max_{i+j=m_0} (|a_i| r^i) \cdot (|b_j| r^j) = (|a_{i_0}| r^{i_0}) \cdot (|b_{j_0}| r^{j_0})$$

because $a_{i_0} b_{j_0}$ is a summand in c_{m_0}. It follows that

$$v_{0,r}(fg) = \max_m |c_m| r^m \geq (|a_{i_0}| r^{i_0}) \cdot (|b_{j_0}| r^{j_0}) = v_{0,r}(f) v_{0,r}(g),$$

and since the inequality $v_{0,r}(fg) \leq v_{0,r}(f) \cdot v_{0,r}(g)$ is already implied by the ultrametric inequality, we obtain the desired multiplicative equality. $\square$

Since $v_{x,r} = \max(|x|, r)$ via the expansion $t = (t-x) + x$, arguing similarly to the type-1 case shows that $v_{x,r}$ is continuous if and only if $x \in k^0$ and $r \leq 1$. Thus, $v_{x,r}(f) \leq 1$ for all $f \in A^+$ if and only if $x \in k^0$ and $r \leq 1$ (the "if" direction being immediate from the ultrametric inequality and consideration of $t \mapsto t - x$ as an automorphism of the pair $(k[t], k^0[t])$ when $x \in k^0$). Thus, $v_{x,r}$ is a point in $\mathbf{D}_k$ precisely when $x \in k^0$ and $r \leq 1$, which is to say precisely when it is continuous.

We shall distinguish two cases: $r \in \sqrt{|k^\times|}$ or not.

Points of type 2.

Definition 11.2.4 Given $r \in \sqrt{|k^\times|}$ with $r \leq 1$ and $x \in k^0$, the points $v_{x,r} \in \mathrm{Cont}(k[t])$ are callled *type* 2, and are denoted $\beta_{x,r}$. The point $v_{0,1}$ is the *Gauss point*, and denoted η.

Note that $\eta = v_{x,1}$ for all $x \in k^0$ via the automorphism $t \mapsto t - x$ on $(k[t], k^0[t])$).. The notion of "type 2" can be defined a bit more generally (roughly speaking, to allow for the possibility of a disc of radius $r \in \sqrt{|k^\times|}$ with no k-points), but for k algebraically closed (the eventual case of interest) this will be sufficient.

We now investigate a few properties of type-2 points. We denote by $D(x, r) \subset \overline{k}$ the closed disc in $\overline{k}$ of center $x \in k$ and radius $r > 0$ (so this determines r since the divisible subgroup $|\overline{k}^\times| \subset \mathbf{R}_{>0}$ is dense). If $k \neq \overline{k}$ then $\overline{k}$ is not complete, but that will not be a problem.

Proposition 11.2.5 *For $x \in k$ and $r \in \sqrt{|k^\times|}$,*

$$v_{x,r}(f) = \sup_{y \in D(x,r)} |f(y)|$$

for all $f \in k[t]$. In particular, for such r the point $v_{x,r} \in \mathrm{Spv}(k[t])$ depends only on $D(x, r)$, and $v_{x,r} = v_{x',r}$ if $|x - x'| \le r$.

Proof. Since $r^n \in |k^\times|$ for some integer n, we may increase k a finite amount so that $r = |\rho|$ for some $\rho \in k^\times$. The automorphism $t \mapsto (t - x)/\rho$ of the ring $k[t]$ allows us to replace x with 0, r with 1, and $D(x, r)$ with $D(0, 1)$. But the case $x = 0$ and $r = 1$ is classical from the study of affinoid Tate algebras. $\square$

Remark 11.2.6 When k is algebraically closed, points $v_{x,r} \in \mathrm{Spv}(k[t])$ for $x \in k$ and $r \in |k^\times|$ have the same value group as k but their residue field is $\kappa(t)$ for the residue field κ of k. Indeed, by recentering the disc and scaling (generally not a continuous automorphism, hence our preference to think in terms of $\mathrm{Spv}(k[t])$ rather than $\mathrm{Cont}(k[t])$ at the moment) we reduce to the easy case $x = 0$ and $r = 1$ (i.e., the Gauss point of $\mathbf{D}_k$). The properties $\sqrt{\Gamma_v} = \sqrt{|k^\times|}$ and non-algebraic residual extension define the appropriate notion of "type 2" for general k.

Points of type 3. Observe that for $x \in k$ and $r \in \mathbf{R}_{>0} - \sqrt{|k^\times|}$, we can exhaust $D(x, r)$ by closed discs $D(x, r')$ with $r' \in \sqrt{|k^\times|}$ satisfying $0 < r' < r$.

Proposition 11.2.7 *For $x \in k$ and $r \in \mathbf{R}_{>0} - \sqrt{k^\times|}$,*

$$v_{x,r}(f) = \sup_{y \in D(x,r)} |f(y)|$$

for all $f \in k[t]$.

Proof. Since f involves only finitely many monomial terms, it is clear that $v_{x,r}(f)$ is the supremum of the values $v_{x,r'}(f)$ for $0 < r' < r$ with $r' \in \sqrt{|k^\times|}$. But by Proposition 11.2.5, $v_{x,r'}(f)$ is the supremum of $|f(y)|$ over all $y \in D(x, r')$. Passing to the limit over such $r' \to r^-$, the supremum of these latter suprema recovers exactly the right side of the desired identity precisely because $D(x, r)$ is covered by the discs $D(x, r')$ for such r'. $\square$

Now we record the precise extent to which $v_{x,r}$ is unaffected by certain changes in $x \in k$ and $r \ge 0$:

Corollary 11.2.8 *For $x, x' \in k$ and $r, r' \ge 0$, $v_{x,r} = v_{x',r'}$ as rank-1 points of $\mathrm{Spv}(k[t])$ if and only if $r = r'$ and $|x - x'| \le r$, or equivalently if and only if $D(x, r) = D(x', r')$.*

Proof. By distinguishing whether the support is a closed point or generic point in $\mathrm{Spec}(k[t])$ it is enough to separately treat the two cases that either $r, r' = 0$ or $r, r' > 0$. The former is easy by analyzing supports (since $x, x' \in k$), so now assume $r, r' > 0$.

By Propositions 11.2.5 and 11.2.7, the equality of the discs implies the equality of the valuations. Conversely, assuming $v_{x,r} = v_{x',r'}$ as points of $\mathrm{Spv}(k[t])$ we get equality as $\mathbf{R}_{\geq 0}$-valued valuations by Proposition 11.1.2. Hence,

$$r = v_{x,r}(t - x) = v_{x',r'}(t - x) = \max(r', |x' - x|),$$

so $|x - x'| \leq r$ and $r' \leq r$. By symmetry $r \leq r'$, so we are done. $\square$

Exactly as in the type-2 case, for $r \notin \sqrt{|k^\times|}$ the valuation $v_{x,r} \in \mathrm{Spv}(k[t])$ is continuous if and only if $x \in k^0$ and $r \leq 1$. In view of the preceding corollary, the following class of points is disjoint from those of type 2.

Definition 11.2.9 Given $r \in (0, 1)$ lying outside $\sqrt{|k^\times|}$ and $x \in k^0$, the points $v_{x,r} \in \mathrm{Cont}(k[t])$ are called *type* 3.

This definition could be made more generally (allowing for discs with no k-point), but when k is algebraically closed the above definition is appropriate (and sufficient). An interesting feature of points of types 2 and 3 is their topological dependence on the real-valued parameter r:

Proposition 11.2.10 *For $x \in k$, the map $\gamma : \mathbf{R}_{\geq 0} \to \mathrm{Spv}(k[t])$ defined by $r \mapsto v_{x,r}$ is continuous precisely at points in $\mathbf{R}_{\geq 0} - \sqrt{|k^\times|}$.*

In contrast, the analogous map $[0, 1] \to M(k\langle t\rangle)$ into the Berkovich spectrum is continuous everywhere (and such continuity is very useful when analyzing the topology of Berkovich spaces).

Proof. We may assume $x = 0$. In view of how the topology on $\mathrm{Spv}(k[t])$ is defined, continuity at r amounts to checking for $f, g \in k[t]$ satisfying $v_{0,r}(f) \leq v_{0,r}(g) \neq 0$ that the same holds for nearby r'. The case $r = 0$ is obvious (as then $|f(0)| = |g(0)| \neq 0$).

Now consider $r > 0$. Writing $f = \sum a_m t^m$ and $g = \sum b_m t^m$, let I and J be the non-empty finite sets of indices such that $v_{0,r}(f) = |a_i|r^i$ for $i \in I$ and $v_{0,r}(g) = |b_j|r^j$ for $j \in J$. For r' near r certainly $|a_m|r'^m < v_{0,r'}(f)$ for $m \notin I$ and $|b_m|r'^m < v_{0,r'}(g)$ for $m \notin J$. Thus, for considering r' near r we only need to focus on the indices from I and J respectively when computing $v_{0,r'}(f)$ and $v_{0,r'}(g)$. Likewise, it suffices to separately consider the cases $0 < r' \leq r$ and $r' \geq r$.

First consider $0 < r' \leq r$. Let $i_0 \in I$ and $j_0 \in J$ be maximal. Thus, for any $i \in I$ the equality $|a_i|r^i = |a_{i_0}|r^{i_0} \neq 0$ can be rewritten as $|a_i/a_{i_0}| = r^{i_0 - i}$ with $i_0 - i > 0$, so for $r' \geq r$ we have $|a_i/a_{i_0}| < r'^{i_0 - i}$, and hence $|a_i|r'^i < |a_{i_0}|r'^{i_0}$. That is,

$$v_{0,r'}(f) = |a_{i_0}|r'^{i_0}$$

for r' slightly larger than r. Likewise, if $j_0 \in J$ is maximal then

$$v_{0,r'}(g) = |b_{j_0}|r'^{j_0}$$

for r' slightly larger than r.

The equality $v_{0,r}(f) = v_{0,r}(g)$ says $|a_{i_0}/b_{j_0}| = r^{j_0 - i_0}$. Hence, if $i_0 \neq j_0$ then $r \in \sqrt{|k^\times|}$. Thus, if $r \notin \sqrt{|k^\times|}$ then necessarily $i_0 = j_0$, so $v_{0,r'}(f) = v_{0,r'}(g)$ for r' slightly larger than r.

If we instead work with the smallest indices in I and J then we similarly conclude for $r \notin \sqrt{|k^\times|}$ that $v_{0,r'}(f) = v_{0,r'}(g)$ for r' slighty smaller than r. This establishes continuity at any $r \notin \sqrt{|k^\times|}$.

Now consider $r \in \sqrt{|k^\times|}$, say $r^n = |c|$ with $c \in k^\times$. The inequality $v_{0,r'}(1) \leq v_{0,r'}(t^n/c)$ holds for $r' = r$ but this fails for all $r' < r$, and likewise the inequality $v_{0,r'}(t^n/c) \leq v_{0,r'}(1)$ holds for $r' = r$

but fails for all $r' > r$. Thus, both open loci $v(1) \leq v(t^n/c)$ and $v(t^n/c) \leq v(1)$ in $\mathrm{Spv}(k[t])$ contain $v_{0,r}$ but there is no interval of values r' around r for which all points $v_{0,r'}$ lie in either such open set. This proves discontinuity at r. $\square$

Remark 11.2.11 If k is algebraically closed then points $v = v_{x,r}$ of type 3 (so $x \in k^0$ and $r \leq 1$) have the same residue field as k but rank-1 value group $|k^\times| \times r^{\mathbf{Z}}$ for some $r \in \mathbf{R}_{>0} - |k^\times|$. To prove this we may recenter the disc so that $x = 0$, and the value group is then clearly as claimed. To compute the residue field at such a valuation we have to do a bit more work, as follows.

First note that if $f = \sum a_m t^m$ then $v_{0,r}(f) \leq 1$ if and only if $|a_m| \leq r^{-m}$ for all m, but equality cannot ever hold when $m > 0$ since $r \notin |k^\times| = \sqrt{|k^\times|}$. For similar reasons, the numbers $|a_m| r^m$ for varying m are pairwise distinct when nonzero. Hence, $v_{0,r}(f) \leq 1$ if and only if $a_0 \in k^0$ and $|a_m| < r^{-m}$ for all $m > 0$, and $v_{0,r}(f) < 1$ if and only if $|a_0| < r^{-m}$ for all $m \geq 0$. It follows that $f \mapsto f(0) \bmod \mathfrak{m} \in \kappa$ identifies

$$\{f \in k[t] \,|\, v_{0,r}(f) \leq 1\}/\{f \in k[t] \,|\, v_{0,r}(f) < 1\}$$

with the residue field of k. Likwise, for nonzero $f/g \in k(t)$ with $f = \sum a_m t^m$ and $g = \sum b_m t^m$, if $v_{0,r}(f/g) = 1$ then the *same* index i must uniquely maximize $|a_m| r^m$ and $|b_n| r^n$ since $r \notin |k^\times|$. Hence, we can adapt the preceding calculation to work on $k(t)$ rather than $k[t]$ to identify the residue field of $v_{0,r}$ on $k(t)$ with that of κ as well.

Beyond the case of algebraically closed k, the correct notion of "type 3" is defined by the conditions that $\kappa(v)$ is algebraic over κ and $\sqrt{\Gamma_v} \neq \sqrt{|k^\times|}$. If k is algebraically closed, types 2 and 3 account for all increase in residue field or value group for rank-1 points with generic support. To make this precise, we first require:

Lemma 11.2.12 *For any $v \in \mathrm{Cont}(k[t])$ of rank 1 with generic support, the valuation ring R_v satisfies $R_v \cap k = k^0$. In particular, the residue field $\kappa(v)$ of R_v is naturally an extension of the residue field κ of k and the value group Γ_v naturally contains the value group of $(k, |\cdot|)$ as an ordered subgroup.*

Proof. Consider the valuation ring $k \cap R_v$ with fraction field k. This is not equal to k, which is to say that $v|_k$ is a nontrivial valuation. Indeed, a nonzero element ϖ of the maximal ideal of k^0 is topologically nilpotent in k and hence is topologically nilpotent in R_v (by *continuity* of v on $k[t]$), forcing ϖ to lie in the maximal ideal of R_v. Thus, $1/\varpi \notin R_v$, so $1/\varpi$ lies in k but not in $k \cap R_v$. The nontrivial valuation ring $k \cap R_v$ of k has rank 1 (as its value group is a subgroup of the rank-1 value group Γ_v), so there are no rings lying strictly between $k \cap R_v$ and k. To prove that $k \cap R_v = k^0$ it therefore suffices to show that $k \cap R_v \subset k^0$, or equivalently that $k - k^0$ is disjoint from R_v.

Choose $x \in k - k^0$, so $1/x$ is topologically nilpotent in k since k^0 is a rank-1 valuation. By *continuity* of the rank-1 valuation v on $k[t]$ it follows that $1/x$ is topologically nilpotent relative to v on $k(t)$, so $1/x$ lies in the maximal ideal of R_v. Thus, $x \notin R_v$ as desired. $\square$

Proposition 11.2.13 *Assume k is algebraically closed. Let $v \in \mathrm{Cont}(k[t])$ be a rank-1 point with generic support (so its valuation ring is a local extension of k^0, by Lemma 11.2.12). If Γ_v strictly contains $v(k^\times)$ then v is type 3 (so $\kappa(v) = \kappa$), and if $\Gamma_v = v(k^\times)$ but $\kappa(v)/\kappa$ is a non-trivial extension then v is of type 2.*

Proof. Since k is algebraically closed, $k(t)^\times$ is generated multiplicatively by elements $t - x$ for $x_0 \in k$. Consider the case when $\Gamma_v \neq v(k^\times)$, so some $v(t - x)$ must lie outside $v(k^\times)$. As we saw in the proof of Proposition 11.1.2, there exists a unique order-preserving injective homomorphism $\Gamma_v \hookrightarrow \mathbf{R}_{>0}$ such that $v|_k$ coincides with the given absolute value on k, so we can say that Γ_v strictly contains $|k^\times|$ inside $\mathbf{R}_{>0}$. Hence, as in our study of type 3, for any $f = \sum a_m(t-x_0)^m \in k[t]$ we have

$$v(f) = \max_m |a_m|\rho^m$$

where $\rho = v(t-x_0) \in \Gamma_v \subset \mathbf{R}_{>0}$. In other words, $v = v_{x_0,\rho}$, a type-3 point (as $\rho \notin |k^\times|$) upon recalling our earlier verification that $v_{x_0,\rho}$ is continuous if and only if $x_0 \in k^0$ and $\rho \leq 1$.

Next, assume instead that $\Gamma_v = v(k^\times)$ but $\kappa(v)$ is larger than κ. Since κ is algebraically closed, $\kappa(v)$ has to contain an element that is trancendental over κ. To get a handle on what such an element can look like, we need to do some work with the condition $\Gamma_v = v(k^\times)$. This equality implies that for all $x \in k$ we have $v(t-x) = v(c)$ for some $c \in k^\times$, which is to say that $v((t-x)/c) = 1$. Since $k(t)^\times$ is generated multiplicatively by elements of the form $t-x$, we conclude that every $f \in k(t)^\times$ can be written as a product

$$f = b \cdot \prod_j ((t-x_j)/c_j)^{m_j}$$

for some $b \in k^\times$, pairwise distinct $x_j \in k$ with $c_j \in k^\times$ such that $v((t-x_j)/c_j) = 1$, and integers $m_j \neq 0$. Thus, $v(f) = v(b)$, so f lies in the valuation ring of v if and only if $b \in k \cap R_v = k^0$, in which case the image of f in $\kappa(v)$ vanishes precisely when $|b| < 1$ and othewise this image is the analogous product of reductions in $\kappa(v)^\times$.

We conclude that $\kappa(v)^\times$ is generated by $\kappa^\times$ and the classes of elements $(t-x)/c$ with $v((t-x)/c) = 1$. In particular, since $\kappa(v)$ contains transcendental elements over κ, the reduction of some such $(t-x)/c$ has to be of this form. Dropping the assumption that v is continuous for a moment (so we work in $\mathrm{Spv}(k[t])$ rather than $\mathrm{Cont}(k[t])$, thereby permitting the use of a general affine-linear but typically discontinuous k-algebra automorphism to $k[t]$), we may rename such $(t-x)/c$ as t, so $t \in R_v^\times$ and the reduction $\bar{t} \in \kappa(v)$ lies outside κ. In particular, $k^0[t] \subset R_v$. In this setting, any $f = \sum a_m t^m \in k^0[t] - k^{00}[t]$ has reduction $\overline{f} = \sum \overline{a}_m \overline{t}^m$ with some $\overline{a}_m \neq 0$, so $\overline{f} \neq 0$ since $\bar{t}$ is transcendental over κ. Thus, $v(f) = 1$ for any such f, so combining with $k^\times$-scaling implies that v coincides with the type-2 Gauss norm. Undoing the affine-linear change of variable, the original v has the form $v_{x,r}$ for some $x \in k$ and $r \in |k^\times|$. Continuity of v then forces $x \in k^0$ and $r \leq 1$, so v is of type 2. $\square$

11.3 The adic unit disc $\mathbf{D}_k = \mathrm{Spa}(k\langle t\rangle, k^0\langle t\rangle)$: advanced constructions

Immediate extensions and spherical completness. In view of Proposition 11.2.13 and the characterization of type-1 points in terms of non-generic support, if k is algebraically closed then the rank-1 points of $\mathrm{Cont}(k[t])$ not of types 1, 2, or 3 have to be precisely rank-1 points v with generic support whose value group and residue field *coincide* with those of k. In other words, the valued field $(k(t), v)$ as an extension of the valued field k (see Lemma 11.2.12) satisfies the following condition:

Definition 11.3.1 A local extension $R \to R'$ of valuation rings (or equivalently of valued fields) is *immediate* if the induced maps between value groups and residue fields are isomorphisms. A valued field (F, w) is *maximally complete* if it admits no nontrivial immediate extensions.

The completion $R \to \widehat{R}$ is an immediate extension, and it is an exercise to check that any trivially-valued or complete and discretely-valued field is maximally complete. Note that any maximally complete valued field has to be complete since its completion is an immediate extension! Likewise, if (F, w) is a maximally complete valued field whose residue field is algebraically closed and whose value group is divisible then an extension $\overline{w}$ of w to an algebraic closure $\overline{F}$ (such $\overline{w}$ exists by Zorn's Lemma) is necessarily an immediate extension (this is classical in the rank-1 case, and is a consequence of the behavior of value groups and residue fields under finite extensions of valued fields in general, as discussed in Bourbaki), whence $F = \overline{F}$; i.e., such a maximally complete F is automatically algebraically closed too! This might seem like a vacuous assertion, as it isn't at all obvious how to build maximally complete fields without assuming the value group to be discrete of rank ≤ 1. Yet such fields exist in great abundance:

Theorem 11.3.2 (Kaplansky/Poonen) *For any valued field (F, w) there exists a maximally complete valued extension field. If w has divisible value group and algebraically closed residue field then (F, w) has a maximally complete immediate extension (F', w') and moreover it is unique up to isomorphism (so it is called the* maximal completion*). In this latter case, (F', w') (which we have seen above is complete and algebraically closed) satisfies two additional properties:*

(1) *Every immediate extension of F admits an inclusion into F' as valued fields.*

(2) *The valued field F' is* spherically complete*: for closed discs $D_i = \{x \in F' \mid w'(x - a_i') \leq \gamma_i\}$ with $a_i' \in F'$ and $\gamma_i \in \Gamma_{w'}$ such that $D_i \cap D_{i'} \neq \emptyset$ for all $i \neq i'$, necessarily $\cap_{i \in I} D_i \neq \emptyset$.*

It is elementary to check that a disc in F with radius in Γ uniquely determines the radius, so we can speak of the "radius" of such a disc without ambiguity.

Proof. The existence and uniqueness, as well as the amazing property (1), are originally due to Kaplansky, but his proof did not provide a concrete description of maximal completions. Poonen revisited the topic for his undergraduate thesis and discovered a very concrete description of maximal completions as a vast generalization of the description of the completed algebraic closure of $F(\!(t)\!)$ for fields F of characteristic 0 via Pusieux series, by means of which he gave a very slick proof Kaplansky's results and obtained spherical completeness rather vividly; see [Po, §1–§5] (which also provides references to the relevant background references on valuation theory in Bourbaki, though we only require the case of rank-1 value groups, for which the input needed from valuation theory is more familiar). The method of proof shows as a remarkable consequence of (1) that a valued field with specified value group and residue field has bounded cardinality. □

The condition of spherical completness has a nice visualization, upon noting that if two closed discs in a valued field meet then one must be contained inside the other (as we see upon using a common point as the "center" and then comparing nonzero "radiii" in the totally ordered value group): a collection $\{D_i\}$ of closed discs with radii in Γ_w satisfies $D_i \cap D_{i'}$ for all $i \neq i'$ if and only if $\{D_i\}$ is totally ordered by inclusion. Thus, spherical completeness is a substitute for the "finite intersection property" of compactness. (Any locally compact topological field is spherically complete, precise by the finite intersection property for compact spaces.)

Remark 11.3.3 In the special case of rank-1 valued fields with divisible value group (thereby dense in $\mathbf{R}_{>0}$), the same visualization applies when using closed discs with radii allowed to lie in $\mathbf{R}_{>0} - |k^\times|$ too, since each disc with radius in $\mathbf{R}_{>0} - |k^\times|$ is exhausted by subdiscs with radiii in $|k^\times|$ (approximating from below). This will be tacitly used in our considerations with spherical completeness in rank-1 settings below.

Beware that $\mathbf{C}_p$ is *not* spherically complete (hence not maximally complete). For example, if D_n is the closed disc of radius $p^{1/n}$ centered at $s_n = p^{-1} + p^{-1/2} + \cdots + p^{-1/n}$ for a compatible system of root extractions of $1/p$ then $s_{n+1} \in D_n \cap D_{n+1}$, so D_n is also centered at s_{n+1} and hence $D_{n+1} \subset D_n$ for all n. But we claim that $\cap_{n \geq 1} D_n = \emptyset$, so $\{D_n\}$ violates the "finite intersection property" for spherical completeness. (The maximal completion of $\overline{\mathbf{Q}}_p$ is actually a huge extension of $\mathbf{C}_p$, described concretely in [Po, §7].)

The proof that $\cap_{n\geq 1} D_n = \emptyset$ requires a genuine technique, not just formalism with non-archimedean fields, since it is specific to $\mathbf{C}_p$ and is not true for the much larger field given by its maximal completion! We sketch the idea now, using results from [Po], and in Remark 11.3.4 we give a more elementary proof of the failure of spherical completeness of $\mathbf{C}_p$ by using Krasner's Lemma with a slightly different nested sequence of discs. Suppose to the contrary that there exists $s \in \cap_{n\geq 1} D_n$, so the closed unit disc centered at s also lies in every D_n (as the radius of each D_n is larger than 1). We can move s slightly so that it lies inside $\overline{\mathbf{Q}}_p$. The maximal completion of $\overline{\mathbf{Q}}_p$ as explicitly constructed in [Po, §4–§5] is given in terms of formal infinite sums $s = \sum [a_q] p^q$ where $[a]$ is the Teichmüller lift of $a \in \overline{\mathbf{F}}_p$ and the set of q's for which $a_q \neq 0$ (called the *support* of s) is a well-ordered subset of $\mathbf{Q}$ (and p^q means $(p^{1/n})^m$ for the reduced-form expression m/n of q); the precise meaning of such a "sum" and its relation with the valuation structure is described explicitly in [Po, §4]. By [Po, §7, Cor. 8], algebraicity of s over $\mathbf{Q}_p$ forces the support of s to be contained in $(1/N)\mathbf{Z}[1/p]$ for some $N > 0$ not divisible by p; i.e., the prime-to-p parts of the denominators of those q in the support of s is bounded. The explicit description of the valuation on the maximal completion in [Po, §4, Lemma 3] implies that if $s = \sum [a_q] p^q$ (viewed in the maximal completion of $\overline{\mathbf{Q}}_p$) satisfies $|s - s_n| \leq p^{-1/(n+1)}$ then the only $q < -1/(n+1)$ for which $a_q \neq 0$ are $q = -1/j$ for $j \leq n$ and $a_q = 1$ for such q. But then taking $n \to \infty$ implies that the prime-to-p parts of the denominators of the q's in the support of s is unbounded, a contradiction!

Remark 11.3.4 Let $\{e_1, e_2, \dots\}$ be the ordered enumeration of the positive integers not divisible by p (so $e_{j+1} \geq e_j + 1$ for all j) and let $s'_n = \sum_{j=1}^{n} p^{-1/e_j}$. Let D'_n be the closed disc in $\mathbf{C}_p$ around s'_n of radius $p^{1/(e_n+1)}$ ($\geq |p^{-1/e_{n+1}}|$). Clearly this is a nested sequence of discs with radii > 1. The emptiness of $\cap_n D'_n$ can be proved by using just Krasner's Lemma, via the following argument suggested by Zev Rosengarten.

Since all radii are > 1, if the intersection is non-empty then we can wiggle around a common point to find $s \in \overline{\mathbf{Q}}_p$ that lies in every D'_n. The key claim is that if $t_n \in \overline{\mathbf{Q}}_p$ is any Galois conjugate of s_n over $\mathbf{Q}_p$ that is *distinct* from s_n then

$$|s_n - t_n| \geq p^{1/e_n}.$$

Granting this for a moment, since $|s - s_n| \leq p^{1/(e_n+1)} < p^{1/e_n} \leq |s_n - t_n|$ for all such t_n it follows from Krasner's Lemma that $\mathbf{Q}_p(s_n) \subset \mathbf{Q}_p(s)$ for all n, so $\mathbf{Q}_p(p^{-1/j}) \subset \mathbf{Q}_p(s)$ for all $j > 0$ not divisible by p. That is impossibe when j exceeds the ramification degree of the finite extension $\mathbf{Q}_p(s)$ over $\mathbf{Q}_p$.

To establish that s_n has distance at least p^{1/e_n} from each of its distinct Galois conjugates over $\mathbf{Q}_p$, we first note that any such conjugate t_n is a sum of terms $\zeta_j p^{-1/e_j}$ where $1 \leq j \leq n$ and ζ_j is an e_jth root of unity, and the hypothesis $t_n \neq s_n$ forces $\zeta_j \neq 1$ for some such j. But if ζ is a nontrivial e_jth root of unity then $|1 - \zeta| = 1$ since $p \nmid e_j$, so $|p^{-1/e_j} - \zeta p^{-1/e_j}| = |1 - \zeta||p^{-1/e_j}| = p^{1/e_j}$. Such absolute values are pairwise distinct as j varies, so $|s_n - t_n| = p^{1/e_j}$ for some $1 \leq j \leq n$. Hence, this distance is at least as large as p^{1/e_n}.

Remark 11.3.5 If (F, w) is a valued field whose value group is divisible and residue field is algebraically closed then its (algebraically closed and complete) maximal completion (F', w') is sometimes also called its *spherical completion* in view of property (2) above, so it may also be denoted $(F^{\mathrm{sc}}, w^{\mathrm{sc}})$. To justify this name, we claim that there is no spherically complete proper subfield of F' containing F (so a valued field with divisible value group and algebraically closed residue field is spherically complete *if and only if* it is maximally complete; the viewpoint of spherical completeness is more visual but the algebraic perspective of maximal completeness is better-suited to proving existence and the basic properties). Any intermediate valued field is immediate under F' since F'/F is immediate, so it suffices to prove rather generally that there is no nontrivial *immediate* extension L/K between spherically complete fields.

Suppose there exists $x \in L - K$, so for every $a \in K$ we have $x - a \neq 0$. Define $\gamma_a = v(x - a) \in \Gamma$, where Γ is the common value group of the valuation v on L and its restriction on K. The family of closed discs $D_a = D(a, \gamma_a)$ in L with radii in Γ is totally ordered by inclusion because there is a common center x and we can compare γ_a's inside Γ. Thus, for any $a, a' \in K$ we have $D_a \subset D_{a'}$ or $D_{a'} \subset D_a$, so either $a \in D_{a'}$ or $a' \in D_a$. Hence, $\{D_a \cap K\}$ is a family of closed discs in K with radii in Γ and totally ordered by inclusion. Spherical completness of K then produces $a_0 \in K$ lying in every $D_a \cap K$, so $v(x - a_0) \leq \gamma_a = v(x - a)$ for all $a \in K$. In other words, a_0 is a "best approximation" in K to $x \in L - K$. We shall use immediacy of L/K to show that no such best approximation can exist.

Letting $w = v|_K$ denote the given valuation on K, for any $b \in K$ by immediacy we have $\gamma_b = w(c)$ for some $c \in K^\times$, so $v((x - b)/c) = 1$. Again using immediacy, the image of $(x - b)/c \in R_v^\times$ in the common residue field κ of v and w is represented by some $u \in R_w^\times$, so

$$v\left(\frac{x - b}{c} - u\right) < 1;$$

i.e., $x = b + c(u + t)$ with $t \in \mathfrak{m}_v$. Thus, $b + cu$ is an element of K such that

$$v(x - (b + cu)) = v(ct) < v(c) = w(c) = \gamma_b = v(x - b).$$

This says that the element $b + cu \in K$ is a better approximation to x than b is, and $b \in K$ was arbitrary, so there is no best approximation to x in K.

Dead ends: points of type 4. Now we return to the non-archimedean field k, and aim to construct some new rank-1 points in $\mathbf{D}_k$ when $k = \overline{k}$ but $k \neq k^{\text{sc}}$. By Proposition 11.2.13 (and Lemma 11.2.12), the rank-1 points v of $\text{Cont}(k[t])$ not of types 1, 2, or 3 are exactly the valuations v on $k(t)$ local over k^0 and continuous on $k[t]$ such that the valued field $(k(t), v)$ is an immediate extension of k. The maximality of k^{sc} among immediate extensions of k thereby implies that every such v must occur via a k-algebra injection of $k(t)$ into k^{sc}. Such a map is precisely given by sending t to some $x \in k^{\text{sc}} - k$, which is to say

$$v(f) = |f(x)|$$

upon using the unique injection $\Gamma_v \hookrightarrow \mathbf{R}_{>0}$ of ordered groups extending the absolute value on k. The continuity of v on $k[t]$ amounts to the requirement $v(t) \leq 1$ (exactly as in our study of the rank-1 points of types 1, 2, and 3), or equivalently $x \in (k^{\text{sc}})^0$.

Beware however that such v *never* uniquely determines x. Indeed, by completeness k is closed in k^{sc}, so the infimum $\rho = \inf_{a \in k} |x - a|$ in $\mathbf{R}_{>0}$ is positive, and hence $|x - a| = |x' - a|$ for any $x' \in k^{\text{sc}}$ satisfying $|x - x'| < \rho$. For any such x' (which we can arrange to satisfy $|x'| \leq 1$ as well) we have $|f(x')| = |f(x)| = v(f)$ for all $f \in k(t)$. But this method of building such v is rather exotic-looking relative to k, and it involves the choice of x that is not unique, so we seek a concrete intrinsic description of such v in terms of k alone without reference to k^{sc} (but k^{sc} is theoretically useful in our analysis of the situation).

For conceptual clarity, we shall carry out the necessary calculations initially assuming just that $|k^\times|$ is divisible and the residue field κ is algebraically closed rather than the stronger assumption that $k = \overline{k}$. (Note that $\overline{k}/k$ is an immediate extension for such k.) Under these hypotheses on the value group and residue field, we will be able to carry out certain "geometric" supremum computations over discs in $\overline{k}$ by using only k-points. Note also that divisibility ensures $|k^\times|$ is dense in $\mathbf{R}_{>0}$, so for any $r \in \mathbf{R}_{\geq 0}$ (perhaps not in $|k^\times|$!), the disc $D_k(x, r) = \{y \in k \mid |y - x| \leq r\}$ inside k determines r.

Lemma 11.3.6 *If $|k^\times|$ is divisible and κ is algebraically closed then for $r > 0$, $f \in k[t]$, and $x \in k$ we have*

$$\sup_{y \in D_k(x,r)} |f(y)| = \sup_{y \in D(x,r)} |f(y)|$$

(*right side using* $\overline{k}$*-points*). *In particular, to each closed disc* D *in* k *there is a canonically associated rank-*1 *point* $v_D \in \mathrm{Spv}(k[t])$, *so it is continuous* (*hence of type* 2 *or type* 3) *if and only if* $x \in k^0$ *and* $r \leq 1$.

Proof. We may assume $x = 0$, and the left side is bounded above by the right side, which in turn we know is $\max_m |a_m| r^m$ where $f = \sum a_m t^m$. The left side is bounded from below by analogous suprema with $r' \in |k^\times| \cap (0, r)$ due to density of $|k^\times|$, so it suffices to prove the asserted equality for such r' in place of r; i.e., we may assume $r \in |k^\times|$. But then we can scale to the case $r = 1$ that is classical on k-points precisely because the residue field is algebraically closed. □

For k as in Lemma 11.3.6, assume k is *not* spherically complete, so there exists a non-empty collection $\mathscr{D} = \{D_i\}$ of closed discs in k such that $\mathscr{D}$ is totally ordered by inclusion (writing $i \geq j$ when $D_i \supseteq D_j$) but $\cap_i D_i$ is *empty.* Consider the function $k[t] \to \mathbf{R}_{\geq 0}$ defined by

$$v_{\mathscr{D}}(f) := \inf_{i \in I} \sup_{y \in D_i} |f(y)| = \inf_{i \in I} v_{D_i}(f).$$

This is unaffected by enlarging the collection $\mathscr{D}$ to contain any closed disc of k which contains a member of the initial choice of $\mathscr{D}$, so the collection of radii of the discs in $\mathscr{D}$ now sweeps out an interval I in $\mathbf{R}_{>0}$ unbounded above, where the indexing of the discs is now given by the radius. This "maximaizing" of the collection $\mathscr{D}$ is designed to to make our construction canonical in the end.

It is easy to verify direclty that $v_{\mathscr{D}}$ is a valuation, as follows. The inequality

$$v_{\mathscr{D}}(fg) \geq v_{\mathscr{D}}(f) \cdot v_{\mathscr{D}}(g)$$

follows from properties of inf since each v_{D_i} is a rank-1 valuation on $k[t]$. The reverse inequality is formal from properties of inf since $\{D_i\}$ is totally ordered. Finally, the ultrametric inequality is a consequence of the same for the points v_{D_i}.

Since $k[t]$ is Tate and $v_{\mathscr{D}}$ is of rank 1, the continuity of $v_{\mathscr{D}}$ is precisely the condition $v_{\mathscr{D}}(t) \leq 1$. Each closed disc D in k has the form $D^{\mathrm{sc}} \cap k$ for a unique closed disc D^{sc} in k^{sc} (namely, take the center to be any point of D and the radius in $\mathbf{R}_{>0}$ to be the same as that of D), and $D \subseteq D'$ forces $D^{\mathrm{sc}} \subseteq D'^{\mathrm{sc}}$ (by comparing radii in $\mathbf{R}_{>0}$ relative to a common center in D). Thus, the collection $\{D_i^{\mathrm{sc}}\}$ of closed discs in k^{sc} with positive radius (sweeping out an interval unbounded above) are totally ordered under inclusion. Hence, by spherical completness we can choose $x \in \cap_i D_i^{\mathrm{sc}}$, so $x \in k^{\mathrm{sc}} - k$ since $\cap_i D_i = \emptyset$ by design.

Clearly $\mathscr{C} := \{D_i^{\mathrm{sc}}\}$ is the set of *all* closed discs of k^{sc} that contain x and meet k. Any two such discs coincide if they have the same radius since they can be centered at x. Therefore, $\mathscr{C}^{\mathrm{sc}}$ is a totally ordered set:

$$\mathscr{C}^{\mathrm{sc}} = \{D_i^{\mathrm{sc}}, i \in I\}$$

with I an interval in $\mathbf{R}_{>0}$ unbounded above.

In Remark 11.3.5 we saw that the interval I does not contain its infimum; i.e., there is no "best approximation" to x in k. Let $\rho \geq 0$ be the infimum of the radii in I, so $I = (\rho, \infty)$. Concretely, ρ is the infimum of the distances from x to points of k, so $\rho > 0$ since $x \notin k$ and k is closed in k^{sc} (due to completness of k). This procedure can be run in reverse:

Lemma 11.3.7 *For any* $x \in k^{\mathrm{sc}} - k$ *and* $\rho > 0$ *its distance to* k, *let* $I = (\rho, \infty)$ *and* $D_i = D^{\mathrm{sc}}(x, i) \cap k$ *for each* $i \in I$. *Each* D_i *is a closed disc in* k *of radius* i *and the intersection* $\cap_{i \in I} D_i$ *is empty whereas* $\cap_{i \in J} D_i$ *is non-empty for any proper subinterval* $J \subset I$

Proof. The definition of ρ as an infimum implies that each D_i is non-empty, so we can recenter $D^{\mathrm{sc}}(x,i)$ at a point of D_i to see that D_i is a closed disc in k with radius i.) If there exists a lying in each D_i then $|a-x|=\rho$, so a is a "best approximation" in k to $x \in k^{\mathrm{sc}} - k$. But we have seen that no such approximation exists. If J is a proper subinterval of I then it omits the subinterval $(\rho, \rho+\varepsilon)$ of I for some $\varepsilon > 0$ (as I does not contain its infimum ρ), so for i_0 in this interval we have $D_{i_0} \subset D_i$ for all $i \in J$. $\square$

For any $f \in k[t]$ we define

$$v^{\mathrm{sc}}_{x,\rho}(f) := \sup_{y \in D^{\mathrm{sc}}(x,\rho)} |f(y)|_{k^{\mathrm{sc}}} \in \mathbf{R}_{\geq 0}$$

and this is clearly a valuation on $k[t]$ (and when $|x|, \rho \leq 1$ it is the restriction of a type-2 or type-3 point in $\mathrm{Cont}(k^{\mathrm{sc}}[t])$, depending on whether $\rho \in |k^\times| = |(k^{\mathrm{sc}})^\times|$ or not). By expanding $f \in k[t]$ as a polynomial in $t-x$ with coefficients in k^{sc}, so only finitely many powers $(t-x)^e$ appear, it is clear via consideration of limits (in $\mathbf{R}_{>0}$) of fixed powers of the radius around x that

$$v^{\mathrm{sc}}_{x,\rho}(f) = \inf_{i \in I} v^{\mathrm{sc}}_{x,i}(f).$$

The supremum of $|f|$ over any disc in k with a given center and radius in $\mathbf{R}_{>0}$ is given by an explicit formula in terms of the radius and center and coefficients (as in the proof of Lemma 11.3.6), so the supremum remains *unchanged* by passing to the analogous supremum for the disc with the same radius and center over an extension of the ground field. Applying this to equate suprema over k-points or k^{sc}-points in a disc, we obtain

$$v^{\mathrm{sc}}_{x,\rho}(f) = \inf_{i \in I} \sup_{y \in D_i} |f(y)| = v_{\mathscr{D}}(f)$$

where y is varying through the disc D_i in k for each $i \in I$.

Lemma 11.3.8 *If k is algebraically closed then $v_{\mathscr{D}}(f) = |f(x)|$ for all $f \in k[t]$. In particular, $v_{\mathscr{D}}$ on $k(t)$ is an immediate extension of the valuation on k, and $v_{\mathscr{D}}$ is continuous on $k[t]$ if and only if $D_i \subset D(0,1)$ for some i.*

Proof. For any $a \in k$, $v_{\mathscr{D}}(t-a) = \max(\rho, |x-a|) = |x-a|$. Since $k(t)^\times$ is multiplicatively generated by $k^\times$ and such monomials $t-a$, the formula for $v_{\mathscr{D}}$ follows. Since $v_{\mathscr{D}}$ is a rank-1 valuation local over the given one on k, continuity amounts to the requirement $v_{\mathscr{D}}(t) \leq 1$. This says exactly $|x| \leq 1$. We have $0 < |x-1| \leq 1$ (recall that $x \notin k$, so $x \neq 1$), so the disc $D(x, |x-1|)$ centered at x meets k at the point 1 and is contained in $D(0,1)$. This disc must then lie in the collection $\mathscr{D}$, so a necessary consequence of continuity is that $D_i \subset D(0,1)$ for some i. Conversely, if such containment holds then certainly $|x| \leq 1$, so $v_{\mathscr{D}}$ is continuous on $k[t]$. $\square$

Keeping in mind Lemma 11.2.12, we have shown:

Proposition 11.3.9 *If k is algebraically closed then the valuations on $k(t)$ which are an immediate extension of the valuation on k and are continuous on $k[t]$ are in bijective correspondence with such (maximal) collections $\mathscr{D}$ of discs in k such that $D_i \subset D(0,1)$ for some i, where to such $\mathscr{D}$ we associate the valuation $v_{\mathscr{D}}$.*

Definition 11.3.10 For algebraically closed k, to each maximal family $\mathscr{D} := \{D_i\}_{i \in I}$ of nested closed balls of k with positive radii and empty intersection (which exists if and only if k is not spherically complete) satisfying $D_i \subset D(0,1)$ for some i, the associated point $v_{\mathscr{D}}$ in $\mathrm{Cont}(k[t])$ is called *type* 4 if it is continuous.

Each point of type 4 has an associated "radius" $\rho \in \mathbf{R}_{>0}$ as in the discussion above, and a center $x \in k^{\mathrm{sc}} - k$ that is well-defined up to translation by a distance $\leq \rho$ (which automatically preserves the property of lying outside k since such x admit no "best approximation" in k), and necessarily $\rho \leq 1$ and $|x| \leq 1$.

Beyond the case of algebraically closed k, the definition of "type 4" is as the rank-1 points $v \in \mathrm{Cont}(k[t])$ with generic support such that $\sqrt{\Gamma_v} = \sqrt{|k^\times|}$ and $\kappa(v)/\kappa$ is algebraic.

Higher rank: points of type 5. Now it is time to build some higher-rank points. These will be clustered around points of type 2, and for this discussion we *assume k is algebraically closed.*

Define the abelian group $\Gamma := \mathbf{R}_{>0} \times (1^-)^{\mathbf{Z}}$ endowed with the unique total order such that $s < 1^- < 1$ for all $s < 1$ in $\mathbf{R}_{>0}$; informally, 1^- is infinitesimally less than 1. This says that for $t \in \mathbf{R}_{>0}$, $t(1^-)^m \leq 1$ if and only if $t < 1$ or $t = 1$ and $m \geq 0$, so more rigorously

$$\Gamma = \mathbf{R}_{>0} \times \mathbf{Z}$$

with the lexicographical order (according to which $(t, m) \leq (1, 0)$ precisely when $t < 1$ or $t = 1$ with $m \leq 0$, so $1^- := (1, -1)$).

For $x \in k$ and $r > 0$, define $r^- = r \cdot 1^- \in \Gamma$ and $v_{x,r^-} : k[t] \to \Gamma \cup \{0\}$ by

$$f = \sum a_m (t - x)^m \mapsto \max_m \{|a_m|(r^-)^m\}.$$

Since the nonzero terms in this maximum are pairwise distinct, when $x \in k^0$ this valuation only depends on the reduction of x in the residue field κ of k (as we can check when $r \in |k^\times|$ by passage to the case $r = 1$, and in other cases via approximation to the case $r \in |k^\times|$). It is easy to check (by Corollary 9.3.3, as usual) that v_{x,r^-} is continuous on $k[t]$ if and only if $x \in k^0$ and $r^- \leq |\varpi^{1/n}|$ in Γ for all $n \geq 1$, which is to say $x \in k^0$ and $r \leq 1$.

Analogously, define $r^+ = r/1^- \in \Gamma$ and

$$v_{x,r^+} : f = \sum a_m (t - x)^m \mapsto \max_m \{|a_m|(r^+)^m\}.$$

This is also readily checked to be continuous on $k[t]$ if and only if $x \in k^0$ and $r \leq 1$ (since $1^+ < |\varpi^{1/n}|$ for all $n > 0$!). Note that $v_{0,1^+}$ does *not* lie in $\mathbf{D}_k$ because $v_{0,1^+}(t) = 1^+ > 1$.

If $r \notin |k^\times|$ then since $|a|r^i \neq |b|r^j$ whenever $a, b \in k^\times$ and $i \neq j$, one can build order-preserving isomorphisms of value groups to obtain an identification

$$v_{x,r^+} = v_{x,r^-} = v_{x,r}$$

as points in $\mathrm{Spv}(k[t])$ (so this point determines r and x up to translation by $D(0, r)$).

Thus, we are mainly interested in the case $r \in |k^\times|$, for which we shall see below that $v_{x,r^\pm}$ determines r, the sign, and x up to translation by $D(0, r)$. Assume $r \in |k^\times|$. For every type-2 point $v_{x,r}$ of $\mathbf{D}_k$ other than the Gauss point we have just built an associated collection of higher-rank points in $\mathbf{D}_k$ parameterized by $\mathbf{P}^1(\kappa)$ (with ∞ corresponding to v_{x,r^+}) and associated to the Gauss point we have built a collection of higher-rank points in $\mathbf{D}_k$ parameterized by $\mathbf{A}^1(\kappa)$.

Definition 11.3.11 For $x \in k^0$ and $r \in |k^0| - \{0\}$, the points $v_{x,r^\pm} \in \mathrm{Cont}(k[t])$ are called *type* 5.

Remark 11.3.12 These points do not have an interpretation via rigid-analytic geometry, as they correspond to rank-2 valuations. (They do however have an interpretation in terms of generic points of special fibers of Raynaud's formal models of qcqs rigid-analytic spaces. This is explained systematically in [vdPS].) If $x \in k^0$ and $r < 1$ then $v_{x,r^\pm} \in \mathbf{D}_k$ and $v_{x,1^-} \in \mathbf{D}_k$, but $v_{x,1^+}$ is *outside* $\mathbf{D}_k$ as noted above. For $x \in k^0$ and $0 < r < 1$, we visualize v_{x,r^+} as a first infinitesimal point on the ray from $v_{x,r}$ heading towards the Gauss point, whereas for $x \in k^0$ and $0 < r \leq 1$ the point v_{x,r^-} is a first infinitesimal point on the ray from $v_{x,r}$ heading towards the classical point x and depending on x only through its reduction. This visualization will be justified in our discussion of specialization relations among points of types 1, 2, 3, 4, 5 below.

Here is a partial picture.

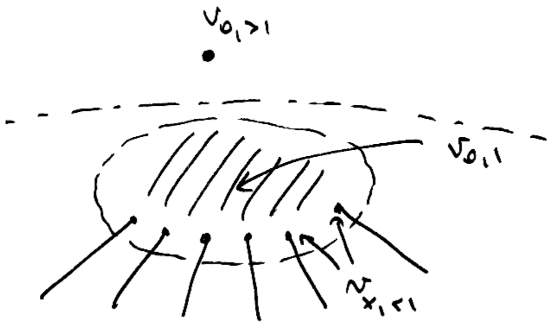

Figure 1. Around the Gauss point.

Theorem 11.3.13 *Assume k is algebraically closed. The points in* $\mathrm{Cont}(k[t])$ *whose valuation ring contains k^0 are precisely the points of type* $1, 2, 3, 4, 5$. *In particular, all points in $\mathbf{D}_k$ arises in this way.*

Proof. Let v be a point in $\mathrm{Cont}(k[t])$ whose valuation ring contains k^0. In our discussion of type-1 points we showed that those give precisely such v with non-generic support. Hence, for the rest of the argument we only consider v with generic support. The points of types 2, 3, and 4 are exactly the rank-1 possibilities for v due to Proposition 11.2.13 (and Lemma 11.2.12) and our discussion of the relationship between k^{sc} and the construction $v_{\mathscr{D}}$.

It remains to consider higher-rank possibilities for v with generic support. We will suggestively write $|a|$ instead of $v(a)$ for $a \in k^\times$ (i.e., we use the unique order-compatible isomorphism between $v(k^\times)$ and $|k^\times|$). Since $k(t)^\times$ is generated multiplicatively by $k^\times$ and elements $t - a$ for $a \in k$, assuming Γ_v is larger than its subgroup $|k^\times|$ forces the existence of some $c \in k$ such that $\gamma := v(t-c) \notin |k^\times|$. Since $|k^\times|$ is divisible, it follows that $\gamma^m \notin |k^\times|$ for all nonzero integers m. We claim that v has to be one of the type-5 points associated to the "center" c. One of the tricky points in the proof will be that a-priori elements of $\mathbf{R}_{>0} - |k^\times|$ have nothing to do with Γ whereas the type-5 construction was informed by thinking about infinitesimal relations with $\mathbf{R}_{>0}$.

Writing $f \in k[t]$ in the form $\sum a_m (t-c)^m$ implies

$$v(f) = \max_m |a_m| \gamma^m$$

inside Γ_v since the valuations $v(a_m(t-c)^m) = |a_m|\gamma^m$ for nonzero a_m are pairwise distinct. By continuity of v on $k[t]$, for a nonzero topologically nilpotent $\varpi \in k$ we see that $v(\varpi^n(t-c)) < 1$ for sufficiently large integers n, which is to say $\gamma = v(t-c) < |\varpi^{-n}|$ in Γ. If $v(t-c)$ were infinitesimal relative to $|k^\times|$, which is to say that $v(t-c) < |a|$ for all $a \in k^\times$, then the formula for $v(f)$ would be $v(f) = |a_0| = |f(c)|$, forcing v to be of rank 1 (even type-1), a contradiction. Hence, inside Γ the value γ is bounded both above and below by elements of $|k^\times|$; such bounds have to be strict since $\gamma \notin |k^\times|$.

Let ρ^- be the supremum in $\mathbf{R}_{>0}$ of the set of elements in $|k^\times|$ bounded above by γ inside Γ, and let ρ^+ be the infimum in $\mathbf{R}_{>0}$ of the set of elements in $|k^\times|$ bounded below by γ inside Γ, so $\rho^- \leq \rho^+$ inside $\mathbf{R}_{>0}$. Necessarily $\rho- = \rho^+$, since otherwise by the density the $|k^\times|$ in $\mathbf{R}_{>0}$ there would exist $a \in k^\times$ such that $\rho^- < |a| < \rho^+$, so $|a|$ would be incomparable with γ, contradicting that Γ is ordered. Let ρ denote this common supremum and infimum. A priori this is just an element of $\mathbf{R}_{>0}$, so it has nothing to do with Γ (in particular, it doesn't a-priori make sense to compare ρ and γ inside Γ).

The key fact to be shown, resting on the hypothesis that Γ is higher-rank, is that $\rho \in |k^\times|$, so actually it *is* meaningful to ask if $\rho < \gamma$ or $\rho > \gamma$ (certainly $\rho \neq \gamma$!). Assume to the contrary that $\rho \notin |k^\times|$, so ρ lies outside the two subsets of $|k^\times|$ for which it is a sup and inf inside $\mathbf{R}_{>0}$. For all $a \in k^\times$, we claim that $|a|\rho^n < 1$ in $\mathbf{R}_{>0}$ if and only if $|a|\gamma^n < 1$ in Γ. To prove this, we may pick $b_n \in k^\times$ satisfying $|b_n|^n = |a|$ since $|k^\times|$ is divisible, so $|a|\rho^n < 1$ if and only if $\rho < |1/b_n|$ in $\mathbf{R}_{>0}$, which in turn (by the equality $\rho = \rho^+$) is equivalent to $\gamma < |1/b_n|$ in Γ, or in other words $|a|\gamma^n < 1$. By similar reasoning using that $\rho = \rho^-$, we see that for all $a \in k^\times$, $|a|\rho^n > 1$ in $\mathbf{R}_{>0}$ if and only if $|a|\gamma^n > 1$ in Γ. Of course, for any nonzero n we know that $|a|\rho^n \neq 1$ in $\mathbf{R}_{>0}$ and $|a|\gamma^n \neq 1$ in Γ since $|k^\times|$ is a divisible subgroup of the torsion-free groups $\mathbf{R}_{>0}$ and Γ. Hence, it follows that the unique injective (!) group homomorphism

$$\Gamma = |k^\times| \times \gamma^{\mathbf{Z}} \to \mathbf{R}_{>0}$$

defined by the identity on $|k^\times|$ and $\gamma \mapsto \rho$ is *order-preserving*, contradicting that Γ is higher-rank. Thus, indeed $\rho \in |k^\times|$.

Disregarding the continuity condition on v for a moment, we may find an affine-linear change of parameter over k so that $a = 0$ and $\rho = 1$. Either $v(t) < 1$ or $v(t) > 1$. If the former then (in view of our formula for v on $k[t]$ in terms of γ) we recover exactly the type-5 point based on 1^-, and if the latter then we recover the type-5 point based on 1^+. Undoing the change of parameter gives that $v = v_{x,r^\pm}$ for some $x \in k$ and $r \in |k^\times|$, for which we have seen that continuity forces $|x|, r \leq 1$; i.e., v is a point of type 5. $\square$

Example 11.3.14 Here is an interesting example of the role of type-5 points in illustrating the effect on $\mathrm{Spa}(B, B^+)$ when we change B^+. Let $A = k[t]$ for algebraically closed k, and recall that $A^0 = k^0[t]$. Inside $\mathrm{Cont}(A)$, consider the "affinoid" subspace $\mathrm{Spa}(A, A^+)$ where

$$A^+ = k^0 + t\mathfrak{m}[t] = k^0 + A^{00}$$

where A^{00} is the ideal of topological nilpotents (i.e., the set of $\sum a_m t^m$ where $|a_m| < 1$ for all m). This is clearly an open subring of A (recall that $k^0[t]$ is a ring of definition, and a base of open neighborhoods of 1 is given by $\varpi^n \cdot k^0[t]$ for $n \geq 0$ and a fixed pseudo-uniformizer ϖ), and we claim that it *is* integrally closed. To prove this, since A^0 is integrally closed (as in any Huber ring) we just have to check that A^+ is integrally closed in A^0. That is, if $f \in A^0$ is integral over A^+ then we claim that $f \in A^+$. The image of f in $A^0/A^{00} = \kappa[t]$ is integral over the image of A^+. This latter image is κ, which is clearly integrally closed in $\kappa[t]$, so the image of f in A^0/A^{00} lies in κ; i.e., $f \in A^+$.

We claim that the subset $\mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(k[t])$ consists of *exactly* the union of $\mathbf{D}_k$ and the additional type-5 point $v_{0,1^+}$. In particular, this subset of $\mathrm{Cont}(k[t])$ is "affinoid". The condition on $v \in \mathrm{Cont}(k[t])$ that $v(f) \leq 1$ for all $f \in A^+$ are that $v(c) \leq 1$ for all $c \in k^0$ and that $v(ct) \leq 1$ for all $c \in \mathfrak{m}$. The condition that $v(c) \leq 1$ for all $c \in k^0$ says that k^0 lies in the valuation ring of $v \in \mathrm{Cont}(k[t])$, and by Theorem 11.3.13 these are exactly the points of types 1, 2, 3, 4, and 5. But if v is a rank-1 point then uniquely $\Gamma_v \subset \mathbf{R}_{>0}$ extending the absolute value on k, so the condition $v(ct) \leq 1$ for all $c \in \mathfrak{m}$ says $v(t) \leq 1/|c|$ for all $c \in k$ with $0 < |c| < 1$, and then the density of $|k^\times|$ in

$\mathbf{R}_{>0}$ then implies that it is the same to say $v(t) \leq 1$. In other words, for rank-1 points such v already lie in $\mathrm{Spa}(A, A^0) = \mathbf{D}_k$, so we get nothing new. Hence, we just have to consider type-5 points. All type-5 points lie in $\mathbf{D}_k$ except for $v_{0,1^+}$, and it is clear by inspection that $v_{0,1^+}$ takes values ≤ 1 on all elements of A^+, so this remaining point does lie in $\mathrm{Spa}(A, A^+)$.

Specialization relations. Assume k is algebraically closed. All specialization relations in $\mathbf{D}_k$ are concentrated in the fibers over $\mathrm{Spec}(k[t])$, so type-1 points are closed and have no proper generizations either. So for the study of specialization and generization among other points we can ignore type 1; i.e., we can work inside $\mathrm{Spv}(k(t))$.

Lemma 11.3.15 *Points in $\mathbf{D}_k$ of type $2, 3, 4$ have no nontrivial generization, and points of type $3, 4, 5$ have no nontrivial specialization. In particular, types $3, 4, 5$ are closed.*

Proof. Since all specialization relations are vertical and hence correspond to forming proper convex subgroups in the value group of a valuation, the absence of points of rank more than 2 (due to Theorem 11.3.13) forces points of type 5 to have no nontrivial specialization.

We know that every point has a unique rank-1 generization that itself has no proper generizations inside $\mathbf{D}_k$, so types 2, 3, and 4 have no nontrivial generization. It remains to show that types 3 and 4 have no nontrivial specialization. Since types 3 and 4 have residue field agreeing with that of k (unlike type 2), any specialization of them inside $\mathrm{Spv}(k[t])$ lying in $\mathrm{Spv}(k(t))$ and compatible with the unique point in $\mathrm{Spa}(k, k^0)$ has to arise from a valuation on κ which is forced to be *trivial* (by compatibility with $\mathrm{Spa}(k, k^0)$). Thus, such points have no such proper specialization at all. □

The only remaining possibilities for nontrivial specialization relations in $\mathbf{D}_k$ is that points of type 2 might have some type-5 specializations. We shall show that this happens in abundance, with a concrete description of all such specializations (and a modest wrinkle at the Gauss point compared with all other type-2 points). In particular, our arguments will justify the picture in Figure 2 below.

Proposition 11.3.16 *Points of type 2 are not closed. For any such point $v = \beta_D = \beta_{x,r}$ distinct from the Gauss point, the closure of v in $\mathbf{D}_k$ is topologically identified with $\mathbf{P}^1_\kappa$ with v as the generic point and all closed points of type 5 (with v_{x,r^+} as ∞). If v is the Gauss point η then the same holds except with $\mathbf{A}^1_\kappa$ as the closure.*

The special behavior at the Gauss point is related to the fact that its closure in $\mathrm{Cont}(k[t])$ also contains the point $v_{0,1^+}$ whose valuation ring dominates k^0 (as we see from the proof below, or by applying $k^\times$-scaling to move the Gauss point to a type-2 point with $r < 1$). In particular, by using $\mathrm{Spa}(A, A^+)$ instead of $\mathbf{D}_k = \mathrm{Spa}(A, A^0)$ as in Example 11.3.14 the description of the closures of the type-2 points in the same would treat the Gauss point on equal footing with the rest.

Proof. Let $D \subset \mathbf{D}_k$ be a disc centered at $x \in k^0$ with radius $r = |c|$ for nonzero $c \in k^0$. The residue field of the type-2 point β_D is naturally identified with $\kappa(t_{x,c})$ where $t_{x,c}$ is the residue class of $(t - x)/c$. Note that changing x or c (for a fixed D) has the effect of changing $t_{x,c}$ by an affine-linear transformation over κ.

By the same method as used for types 3 and 4, proper specializations in $\mathbf{D}_k$ of points β_D of type 2 have to arise from *non-trivial* valuations on the residue field $\kappa(t_{x,c})$ that extend the *trivial* valuation on κ (due to compatibility with $\mathrm{Spa}(k, k^0)$); we are not claiming that all points in $\mathrm{Spv}(k(t))$ arising from such a residual construction actually lie in $\mathbf{D}_k$, and that will actually be false at the Gauss point (but true everywhere else). Since κ is algebraically closed, the Riemann-Zariski space $\mathrm{RZ}(\kappa(t_{x,c}), \kappa) \subset \mathrm{Spv}(\kappa(t_{x,c}))$ is parameterized by $\mathbf{P}^1(\kappa)$ (with ∞ corresponding to $1/t_{x,c}$).

By working in $\mathrm{Spv}(k[t])$ rather than in $\mathrm{Cont}(k[t])$ (so we may avail ourselves of discontinuous affine-linear change of variables on t over k), using recentering and $k^\times$-scaling to pass to $x = 0$ and $r = 1$ makes it easy to verify by direct computation that the points $v_{a,r^-} \in \mathrm{Spv}(k[t])$ for $a \in D$ making $(a-x)/c$ vary through all possible residue classes in κ account for exactly the points in $\mathrm{RZ}(\kappa(t_{x,c}), \kappa)$ labelled by $\mathbf{A}^1(\kappa) \subset \mathbf{P}^1(\kappa)$, and that the point $v_{x,r^+} \in \mathrm{Spv}(k[t])$ corresponds to exactly the point $\infty \in \mathbf{P}^1(\kappa)$. Thus, when $D \neq D(0,1)$ (i.e., $r < 1$) we have identified an additional type-5 point $v_{x,r^+} \in \mathbf{D}_k$ with closure containing β_D and these are *all* of the generaizations of β_D in $\mathrm{RZ}(\kappa(t_{x,c}), \kappa)$. So in such cases, the closure of β_D is identified with the topological space of $\mathbf{P}^1_\kappa$ (with β_D as the generic point, and all other points of type 5).

On the other hand, if $r = 1$ (i.e., β_D is the Gauss point) then we know that $v_{x,1^+} \notin \mathbf{D}_k$ inside $\mathrm{Cont}(k[t])$, so similarly we conclude that the closure of $\beta_{D(0,1)}$ is the topological space $\mathbf{A}^1_\kappa$ (with $\beta_{D(0,1)}$ as the generic point, and all other points of type 5). $\square$

Example 11.3.17 The preceding shows that generally loci of the form $\{v \in X \mid v(f) < v(g)\}$ in $X = \mathrm{Spa}(A, A^+)$ is *not* open, precisely due to the intervention of higher-rank points. Indeed, consider $X = \mathbf{D}_k$ (with $A = k[t]$ and $A^+ = k^0[t]$) with k algebraically closed and choose $c \in k$ with $0 < |c| < 1$. We claim that the condition $v(c) < v(t)$ is not open, or equivalently $Y = \{v \in X \mid v(t) \leq v(c)\}$ is not closed. Indeed, the type-2 point $v_{c,|c|}$ has closure which contains the type-5 point $w = v_{c,|c|^+}$ but $w(t) > w(c)$, so $w \notin Y$ and hence Y is not closed.

A better visualization of $\mathbf{D}_k$. We now explain how to use the specialization relations to partially visualize $\mathbf{D}_k$. For every $x \in k^0$ and the type-1 point α_x, there is a "path" emanating from α_x and terminating at the Gauss point η, namely the collection of points $v_{x,r}$ for $r \in [0,1]$. These points are of type 2 or 3 for $0 < r \leq 1$, though we saw in Proposition 11.2.10 that $r \mapsto v_{x,r}$ is discontinous at precisely the type-2 points (i.e., $r \in |k^\times|$), so the "path" terminology should not be taken too seriously.

Let $\alpha_{x'}$ be a distinct type-1 point in $\mathbf{D}_k$ (so $x' \in k^0$ and $x' \neq x$). For $r, r' \leq 1$ we know that $v_{x,r} = v_{x',r'}$ if and only if $D(x,r) = D(x',r')$, which is in turn equivalent to having $r = r'$ and $|x - x'| \leq r$. Letting $s := |x - x'| \in |k^\times|$, the two "paths" $r \mapsto v_{x,r}$ and $r' \mapsto v_{x',r'}$ coincide for $r, r' \geq s$ but are disjoint for $r, r' < s$. This point of first intersection is the type-2 point $\beta_{x,s} = \beta_{x',s}$.

Consider all such "paths" that pass through the type-2 point $\beta_D = \beta_{x,s}$ corresponds to a disc $D = D(x,s)$ with $0 < s \leq 1$ and $x \in k^0$. Define the map

$$\psi : D \to k^0/\mathfrak{m} = \kappa$$

by

$$\psi(a) := \frac{x-a}{c} \bmod \mathfrak{m}$$

where $c \in k$ is fixed with $|c| = s$ (so ψ depends on $x \in D$ and the auxiliary c up to an overall affine transformation on κ). This is certainly surjective, as $b \bmod \mathfrak{m}$ is the image of $a + bc \in D$. Two points $y, y' \in D$ are sent to the same residue class $\psi(y) = \psi(y')$ in κ if and only if

$$|y - y'| < s,$$

which amounts to having $v_{y,r} = v_{y',r}$ for some positive $r < s$. Thus, ψ establishes a bijection between κ and the set of equivalence classes of paths "converging" to β_D, where equivalence is taken in the sense of paths that meet strictly before β_D.

If β_D is actually the Gauss point η (i.e., $D = D(0,1)$) then this bijection recovers the homeomorphism $\overline{\{\eta\}} \simeq \mathbf{A}^1_\kappa$ with a description now in terms of "paths". For any other D there is also the path from β_D

to η, and by including that we likewise get a path-based description of the earlier homeomorphism

$$\overline{\{\beta_{x,s}\}} \simeq \{\infty\} \cup \mathbf{A}^1_\kappa = \mathbf{P}^1_\kappa.$$

We can draw the classical points as in the picture below, and from each one of them we can go further inside the disc, until we reach $v_{x,1}$, which is the same "target" for all limbs starting from classical points, since it does not depend on x. This "central" point is the *Gauss point* η.

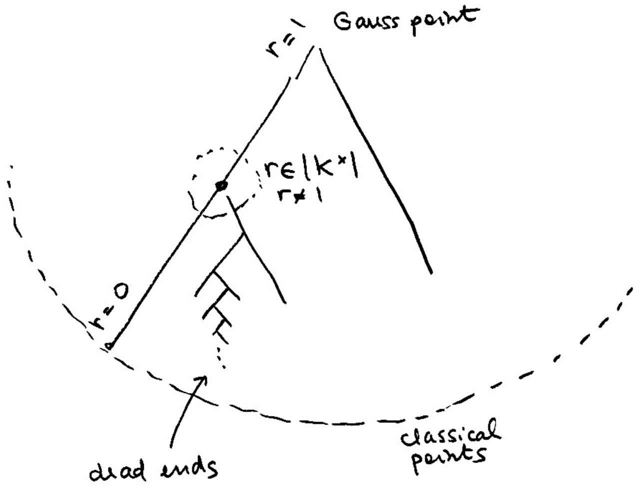

Figure 2. The adic unit disc.

The above description simply says that, starting from the classical points, we find rays tending to the Gauss point, where the various points merge, depending on their distance. Since k is algebraically closed, so the residue field κ is infinite, we have infinitely many branches at each type-2 point! The above picture is just a psychological device and should not be taken too seriously for our purposes, especially in view of the discontinuities in Proposition 11.2.10 (e.g., there is no sense in which $\mathbf{D}_k$ is a very branched tree, though the Berkovich analogue does have tree-like structure which is useful). We will never use this picture.

Example 11.3.18 We conclude our discussion of the unit disc over algebraically closed k by proving that the natural section σ to the surjection $q : \mathbf{D}_k \to M := M(k\langle t\rangle)$ is *not* continuous. In the theory of Berkovich spaces, one shows early on that the 1-parameter paths analogous to Proposition 11.2.10 are continuous. (This underlies the path-connectedness properties of Berkovich spaces.) For example, the map $\gamma : [0,1] \to \mathbf{D}_k$ defined by $r \mapsto v_{0,r}$ has composition $q \circ \gamma : [0,1] \to M$ that is continuous. If σ were continuous then $\sigma \circ (q \circ \gamma)$ would be continuous. But $\sigma \circ q$ is the identity map on rank-1 points, so it would follow that γ is continuous. However, by Proposition 11.2.10 we know that γ is discontinuous at every $r \in |k^\times| \cap (0,1)$! This argument really shows that σ is discontinuous at every type-2 point of the Berkovich unit disc.

In more geometric terms, if X_1 denotes the topological subspace of rank-1 points inside $X = \mathbf{D}_k$ then σ becomes the inclusion $j : X_1 \hookrightarrow X$ (with change of topology on the source!) and q becomes a retraction $r : X \to X_1$ (with change of topology on the target). We have now rigged that j is continuous, but the retraction r is *not* continuous. Indeed, if r were continuous then X_1 would inherit quasi-compactness from X, but X_1 with its subspace topology from X is not quasi-compact. To prove this we just need to exhibit an open cover with no finite subcover, and this can be done with the help of higher-rank points. For each $r \in |k^\times|$ with $0 < r < 1$ the locus $\{|t| \le r\}$ is open in X, and $\{|t| = 1\}$ is also open in X. Their union certainly contains X_1 (since at a type-3 point v we can choose a nonzero element a of the maximal ideal of k such that $v(t) < |a| < 1$), but no finite subcollection covers X_1 since we can build type-2 points v satisfying $v(t) = r$ for $r \in |k^\times| \cap (0,1)$ arbitrarily close to 1. As a sanity-check, note that this construction doesn't contradict the quasi-compactness of X because this collection of open subsets of X fails to cover X, as it misses the type-5 points around the Gauss point (and it doesn't lead to inconsistencies in Berkovich's theory that since in the compact Hausdorff topology of the Berkovich closed unit disc on the set X_1 these loci are not open)!

11.4 Rational domains

Returning to the general theory, we have an analogoue of Proposition 9.2.5 for adic affinoid spaces (in place of continuous valuation spectra):

Theorem 11.4.1 *Let (A, A^+) be a Huber pair, and define $X := \mathrm{Spa}(A, A^+)$.*

1. *The space X has a base of qc open subsets given by*

$$X(T/s) = \{v \in X \mid v(t_i) \le v(s) \ne 0, \text{ for all } t_i \in T\}$$

where $s \in A$ and $T \subset A$ is a finite nonempty subset such that $T \cdot A$ is open in A. This base is stable under finite intersection:

$$X(T/s) \cap X(T'/s') = X(TT'/ss').$$

2. *Let $A^+[T/s] \subset A(T/s) =: A_s$ be the open A^+-subalgebra generated by fractions t/s for $t \in T$. The natural map*

$$\mathrm{Spa}\left(A\left(\frac{T}{s}\right), A^+\left[\frac{T}{s}\right]\right) \to X$$

is a homeomorphism onto $X(T/s)$ respecting rational domains in both directions.

The open sets $X(T/s)$ as in (1) are called *rational domains*. Before we discuss the proof of Theorem 11.4.1, we make some comments about parts of the result.

Remark 11.4.2 By Proposition 6.3.5, the Huber hypothesis on A ensures that if $T \cdot A$ is open in A then $T \cdot U$ is open in A for every additive open subgroup U of A. This latter property was used to build the topological ring structure on $A(T/s)$ (under which it is even a Huber ring when A is Huber, by Proposition 7.4.2). This ensures that the statement of part (2) at least “makes sense”.

Remark 11.4.3 In part (2) we can replace $A^+[T/s]$ with its integral closure inside $A(T/s)$ (and we may suggestively denote that integral closure as $A(T/s)^+$) without changing the adic spectrum, but if $A^+ = A^0$ then there is no reason whatsoever to expect that even $A(T/s)^+$ actually coincides with $A(T/s)^0$. This illustrates the tremendous importance of allowing the case $A^+ \ne A^0$, and also of even allowing general subsets of A^0 to be used when defining the topological adic space (since in the setting of (2) is may be more convenient to work with $A^+[T/s]$ rather than its integral closure for some calculations).

Let us illustrate what the Theorem is saying in the important case when A is a k-affinoid algebra and $A^+ = A^0$; i.e., $X = \mathrm{Spa}(A, A^0)$. Firstly, $T \cdot A$ is open if and only if $T \cdot A = A$, as A is Tate (i.e., contains topologically nilpotent units). It follows that the (finitely many) elements of T have no common geometric zero, because otherwise $T \cdot A$ should be contained in a maximal ideal of A. Moreover, we know that:

$$A\left(\frac{T}{s}\right) = A\left(\frac{T \cup \{s\}}{s}\right)$$

so we lose nothing by assuming $s \in T$. Hence, in these cases

$$A(T/s) = A\left(\frac{f_1, \ldots, f_n}{f_1}\right)$$

where the f_i's have no common geometric zero. In other words, rational domains provide a base for the topology of X, much as they do in $\mathrm{Sp}(A)$ in rigid-analytic geometry.

Now we sketch the proof of part (1):

Proof. The formula for $X(T/s) \cap X(T'/s')$ was verified in the proof of Proposition 9.2.5(2). As for the other assertions, first recall that we have an inclusion

$$\mathrm{Cont}(A) \subset \mathrm{Spv}(A, A^{00} \cdot A) =: Y$$

realizing $\mathrm{Cont}(A)$ as a proconstructible subset of Y with a base of qc open subsets of the form:

$$\mathrm{Cont}(A)(T/s) = \{v \in \mathrm{Cont}(A) \mid v(t_i) \leq v(s) \neq 0 \ \text{ for all } t_i \in T\}$$

for $s \in A$ and finite nonempty subsets $T \subset A$ such that $A^{00} \cdot A \subset \mathrm{rad}(T \cdot A)$. We obtained $\mathrm{Cont}(A)(T/s)$ as $\mathrm{Cont}(A) \cap Y(T/s)$. Thus,

$$X(T/s) = X \cap \mathrm{Cont}(A)(T/s)$$

is a base of open subsets of X, and the issue is showing that the open subsets of this form are indeed quasi-compact.

We outline two reasons for this quasi-compactness to hold. First, one can argue as in the end of the proof of Lemma 10.2.6 and obtain that $\mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(A)$ is pro-constructible, so it meets any qc open subset of $\mathrm{Cont}(A)$ in a qc open subset of itself. Alternatively, one can check that $X(T/s)$ is the image of $\mathrm{Spa}(A(T/s), A^+[T/s])$ in $\mathrm{Spa}(A, A^+)$: it is clear that this latter image is contained in $X(T/s)$ (check!), but for exhaustion one has to check that for any $v \in X(T/s)$ the induced valuation on $A(T/s)$ really is continuous, and that in turn is seen by reviewing how the topology on $A(T/s)$ was defined. $\square$

The continuous map

$$\mathrm{Spa}((A(T/s), A^+[T/s]) \to X$$

is easily seen to be injective, and we remarked above on the verification that its image is $X(T/s)$. The main work in (2) is to prove that the bijection onto $X(T/s)$ respects rational domains in both directions (and so in particular is a homeomorphism).

Fix a Huber ring A. The natural map

$$A \to A\left(\frac{T}{s}\right)$$

is an adic map of Huber rings by design. (Recall that an *adic* map of Huber rings $f : B \to B'$ is a continuous ring homomorphism such that for some rings of definition B_0 and B'_0 satisfying $f(B_0) \subset B'$ we have that $f(I)B'_0$ is an ideal of definition of B'_0 for some, or equivalently any, ideal of definition I of B_0.) As discussed in Example 5.4.5, a natural example of non-adic (although continuous) ring homomorphism between Huber rings is the inclusion $\mathbf{Z}_p \hookrightarrow \mathbf{Z}_p[\![x]\!]$, this latter ring being endowed with the (p, x)-adic topology.

The fact that the map in (2) respects rational domains in one direction is now a special case of:

Lemma 11.4.4 *Let* $(B, B^+) \xrightarrow{h} (B', B'^+)$ *be an adic map of Huber pairs, and denote by* $\mathrm{Spa}(h)$ *the induced map on adic spectra:*

$$\mathrm{Spa}(h) : \mathrm{Spa}(B', B'^+) =: Y' \to Y := \mathrm{Spa}(B, B^+).$$

Then the inverse image under $\mathrm{Spa}(h)$ *of a rational domain in* Y *is a rational domain in* Y'*.*

Proof. Define $\varphi := \mathrm{Spa}(h)$, and observe that set-theoretically

$$\varphi^{-1}(Y(T/s)) = Y'(h(T)/h(s))$$

where the right side "makes sense" without any knowledge of openness for $h(T) \cdot B'$ in B'. However, to affirm that this is a rational domain we have to check that $h(T) \cdot B'$ is indeed open in B'! This openness holds precisely because $T \cdot B$ is open in B and h is adic by assumption. $\square$

The remaining direction for the proof of part (2) of Theorem 11.4.1 is much harder: we have to show that for the natural injective map

$$\varphi : \mathrm{Spa}(B, B^+) = Y := X(T/s) \subset X$$

and a rational subset $Y(T'/s')$ of Y there exists a suitable element $s_1 \in A$ and finite nonempty subset $T_1 \subset A$ such that

$$Y(T'/s') = X(T_1/s_1).$$

The proof is surprisingly nontrivial (in contrast with the analogue for schemes that is very easy), and it relies on the adic version of the "minimum modulus principle" [H1, Lemma 3.11]) saying that given a qc subset $K \subset \mathrm{Spa}(D, D^+)$ for a Huber pair (D, D^+) and $f \in D$ such that $v(f) \neq 0$ for all $v \in K$ then there exists an open subset $U \subset D$ around 0 such that $v(f) > v(u)$ for all $v \in K$ and *all* $u \in U$. The proof of this Minimum Modulus Principle is quite short and crucially uses quasi-compactness arguments in adic spaces; it is totally different from the proof in the rigid-analytic case.

To apply this Minimum Modulus Principle to the quasi-compact subset $K = \varphi(Y(T'/s')) \subset X$, we first note that by some denominator-chasing in $A(T/s)$ we can arrange the choices $s' \in A(T/s)$ and $T' \subset A(T/s)$ to come from A without affecting $Y(T'/s')$ (using that $A(T/s) = A_s$ as A-algebras). Now we take f in the Minimum Modulus Principle to be $s' \in A$, and the open set $U \subset A$ can be shrunken to be an ideal of definition I in a ring of definition $A_0 \subset A$. Thus, $v(u) < v(s')$ for all $v \in \varphi(Y(T'/s'))$ and all $u \in I$. Letting T'' be a finite generating set of I as an ideal of A_0, we see that $T'' \cdot A$ is *open*. Hence, $X((T' \cup T'')/s')$ makes sense as a rational domain in X and it meets $X(T/s) = Y$ in exactly $Y(T'/s')$ due to how T'' was chosen. In other words, $\varphi^{-1}(X((T' \cup T'')/s')) = Y(T'/s')$ inside Y, as desired. This completes the proof of Theorem 11.4.1.

11.5 Invariance under completion

The formalism of structure sheaves on adic spaces will rest on completions, and generally perfectoid spaces are built as completions of certain direct limits, so it is technically very very useful in approximation arguments that the underlying topological space of a Huber pair can be correctly computed *without* passing to completions yet *retaining* knowledhe of which subsets are rational domains. More precisely:

Theorem 11.5.1 *Let* (A, A^+) *be a Huber pair. The natural continuous map*

$$\mathrm{Spa}(A^\wedge, (A^+)^\wedge) \to \mathrm{Spa}(A, A^+)$$

is a homeomorphism preserving analyticity and rational subsets in both directions.

Remark 11.5.2 The open subring $(A^+)^\wedge$ is integrally closed in $A^\wedge$. Indeed, suppose $x \in A^\wedge$ satisfies $x^n + a'_{n-1}x^{n-1} + \cdots + a'_0 = 0$ for $a'_j \in (A^+)^\wedge$. For $a_j \in A^+$ near a'_j and $y \in A$ near x we have that the element $y^n + a_{n-1}y^{n-1} + \cdots + a_0 \in A$ is very close to 0 and so in particular lies in A^+ by openness. But A^+ is integrally closed and hence $y \in A^+$. We could have taken such y to be as close as we wish to x, so $x \in (A^+)^\wedge$ as we claimed. By openness of the ring of power-bounded elements in a Huber ring, it is also not difficult to check that if $A^+ = A^0$ then $(A^+)^\wedge = (A^\wedge)^0$.

Theorem 11.5.1 is [H1, Prop. 3.9]. The proof of preservation of analyticity in both directions is very easy, as is bijectivity (as we are working inside spaces of *continuous* valuations, after all). For preservation of rationality in the "preimage" direction one uses Lemma 11.4.4 since $A \to \widehat{A}$ is adic. The more interesting reverse direction rests on a much more serious ingredient given in [H1, Lemma 3.10]: if $X = \mathrm{Spa}(B, B^+)$ where B is complete then the subset $X(T/s) \subset X$ is insensitive to small perturbations of T and s inside B (applied to $B = \widehat{A}$).

The proof of this latter fact uses both the adic Minimum Modulus Principle as well as a nontrivial property of rings B complete for the topology of a finitely generated ideal J: if we give the dual module $\mathrm{Hom}(B^n, B)$ the J-adic topology then the locus of linear forms ℓ with open image in B is itself an open subset of the dual module. This result is cited from Bourbaki's *Commutative Algebra* but unfortunately appears to have two typos in the citation: it should cite Corollary 2 in III.2.8 from Bourbaki. This idea is that if $\ell : B^n \to B$ has open image then all nearby linear forms ℓ' have image contained inside $\ell(B^n)$, so the aim is to show that if ℓ' is close enough to ℓ then $\ell'(B^n) = \ell(B^n)$.

This latter property for ℓ' near ℓ is proved by successive approximation via the *completeness* of B, but the details are delicate since there are no noetherian hypotheses on B (for many good reasons). For example, one has to show that the subspace topology on $\ell(B^n) \subset B$ coincides with the J-adic topology on the B-module $\ell(B^n)$, a tricky matter without noetherian conditions.

11.6 A non-emptiness criterion

As with spectra of rings, it is important to know that affinoid adic spectra of Huber pairs (A, A^+) are non-empty in reasonable situations. It is easy to check as an exercise that the natural map

$$\mathrm{Spa}(A/\overline{\{0\}}, A^+/\overline{\{0\}}) \to \mathrm{Spa}(A, A^+)$$

is a homeomorphism respecting rational domains and analyticity in both directions. (Every open subring of A contains the closure of (0), since it contains 0 and is closed, as are open subgroups of all topological groups.) Hence, to investigate what Huber pairs (A, A^+) yeld nonempty adic spectra or even have non-empty loci of analytic points we may assume A is Hausdorff.

Proposition 11.6.1 *Let A be a Hausdorff Huber ring, and $A^+ \subset A^0$ any open subring. Then* $\mathrm{Spa}(A, A^+)$ *is empty if and only if $A = (0)$, and* $\mathrm{Spa}(A, A^+)_{\mathrm{an}}$ *is empty if and only if A is discrete.*

Proof. The "only if" directions are obvious. Granting the converse for existence of analytic points, let us first handle the other converse. Assuming emptiness of $\mathrm{Spa}(A, A^+)$ then forces A to be discrete, so the trivial valuation on the residue field at any prime ideal of A would be a point in $\mathrm{Spa}(A, A^+)$. Consequently, there could not be any primes, so $A = 0$ as desired.

Now comes the real work: assuming $\mathrm{Spa}(A, A^+)$ has no analytic points, we need to prove that the topology of A is discrete. Let (A_0, I) be a pair of definition. We aim show that $I^{n+1} = I^n$ for some $n > 0$, so by the Hausdorff property of A it would follow that $I^n = 0$ for large n; i.e., A is discrete as desired. (That was the only place where the Hausdorff property of A is going to be used.) The proof that $I^{n+1} = I^n$ for large n is the content of [H1, Prop. 3.6(ii)] and is a rather nontrivial argument in commutative algebra. It follows into two Steps as follows.

As a first step, Huber employ a fluent command of valuation theory (using vertical generization and horizontal specialization is clever ways) to prove that in the absence of any analytic points, the subset $\mathrm{Spec}(A_0/I) \subset \mathrm{Spec}(A_0)$ contains every irreducible component of $\mathrm{Spec}(A)$ that it touches. Here, by "irreducible component" we mean "maximal irreducible closed subset", but the collection of these need not even be locally finite for a non-noetherian ring. (If A_0 were noetherian we could then conclude right now that $\mathrm{Spec}(A_0/I)$ is clopen and hence $\mathrm{rad}(I)$ is idempotent, so $I^{n+1} = I^n$ for large n. But of course we are absolutely forbidden to make any noetherian assumptions on A_0.)

For the second step, consider the auxiliary A-algebra $A_1 = (1+I)^{-1}A_0$. Any maximal ideal $\mathfrak{m}$ of this ring must lie over IA_1 since all elements of I lie in the Jacobson radical of A_1 by design of A_1 (due to I being an ideal of A_0). Viewing $\mathrm{Spec}(A_1)$ as a subset of $\mathrm{Spec}(A_0)$, we have just noted that its closed points all lie in $\mathrm{Spec}(A_0/I)$. But every irreducible component of $\mathrm{Spec}(A_1)$ contains such a closed point and in turn is contained in an irreducible component of $\mathrm{Spec}(A_0)$. Hence, by the first step, every irreducible component of $\mathrm{Spec}(A_1)$ is contained in $\mathrm{Spec}(A_0/I)$, so $\mathrm{Spec}(A_1) \subset \mathrm{Spec}(A_0/I)$. In other words, all elements of I are nilpotent in the A_0-algebra A_1. But I is a finitely generated ideal of A_0, so for some large integer m we have $I^m \cdot A_1 = 0$.

Since I^m is a finitely generated A_0-module, the definition of A_1 as a localization of A_0 gives that for some $i \in I$ we have $(1+i)I^m = 0$ in A_0. This implies $I^m \subset I^{m+1}$, so $I^m = I^{m+1}$. $\square$

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