Let K be a p-adic field, K_∞/K a *ramified* $\mathbf{Z}_p$-extension, and L/K_∞ a finite extension. We write K_n to denote the nth layer in the $\mathbf{Z}_p$-extension (so $[K_n : K] = p^n$, $K_0 = K$, and K_{n+1}/K_n is totally ramified for sufficiently large n), and write $\mathfrak{m}_n$ and $\mathfrak{m}_\infty$ to denote the maximal ideals of the valuation rings of K_n and K_∞ respectively. Our aim is to prove the following result, called the "almost étale lemma" in class.

Theorem 0.1. *The trace map* $\mathrm{Tr}_{L/K_\infty} : L \to K_\infty$ *satisfies* $\mathfrak{m}_\infty \subseteq \mathrm{Tr}_{L/K_\infty}(\mathscr{O}_L)$.

Since the trace map is transitive, it suffices to prove the result after replacing L with a finite extension. Hence, we may and do assume that L/K_∞ is Galois. The description of L as a primitive extension of K_∞ and of the finite group $G = \mathrm{Gal}(L/K_\infty)$ as automorphisms in terms of this presentation does not require all of K_∞: it makes sense over K_n for sufficiently large n. More precisely, at the expense of renaming some K_{n_0} as K (which we may certainly do) there is no loss of generality in assuming that K_∞/K is totally ramified and $L = L_0K_\infty$ for a finite Galois extension L_0/K that is linearly disjoint from K_∞ over K (so $L = L_0 \otimes_K K_\infty$). In particular, $\mathrm{Gal}(L_0/K) = G$.

For any $n \geq 0$, let $L_n = L_0K_n = L_0 \otimes_K K_n$, so L_n/K is finite Galois with $\mathrm{Gal}(L_n/K) = G \times \mathrm{Gal}(K_n/K)$. The key to the proof will be to show that $\mathrm{ord}_K(\mathfrak{d}_{L_n/K_n})$ is $O(p^{-n})$ for large n (equivalently, $\mathrm{ord}_{K_n}(\mathfrak{d}_{L_n/K_n})$ is bounded). The transitivitty formula for differents gives

$$\mathfrak{d}_{L_n/K} = \mathfrak{d}_{L_n/K_n}\mathfrak{d}_{K_n/K},$$

so

$$\mathrm{ord}_K(\mathfrak{d}_{L_n/K_n}) = \mathrm{ord}_K(\mathfrak{d}_{L_n/K}) - \mathrm{ord}_K(\mathfrak{d}_{K_n/K}) = \int_{-1}^{\infty} \left(\frac{1}{\#\,\mathrm{Gal}(K_n/K)^y} - \frac{1}{\#\,\mathrm{Gal}(L_n/K)^y} \right) \, \mathrm{d}y;$$

here we have used our integration formula (at the end of Step 2 of our proof of the Trace Estimate) for the valuation of the relative different, applied to the two extensions K_n/K and L_n/K of the common ground field K. We will now give a useful upper bound by identifying a specific value h *independent of n* so that the integrand vanishes for $y \geq h$.

The compatibility of the upper numbering with quotients gives

$$\mathrm{Gal}(L_0/K)^y = \mathrm{Gal}(L_n/K)^y\mathrm{Gal}(K_n/K)/\mathrm{Gal}(K_n/K),$$

and this is trivial for sufficiently large y since L_0/K is finite. This formula also shows that for each y, $\mathrm{Gal}(L_0/K)^y = 1$ if and only if $\mathrm{Gal}(L_n/K)^y \subseteq \mathrm{Gal}(K_n/K)$, in which case the product decomposition of $\mathrm{Gal}(L_n/K)$ gives

$$\mathrm{Gal}(K_n/K)^y = \mathrm{Gal}(L_n/K)^y\mathrm{Gal}(L_0/K)/\mathrm{Gal}(L_0/K) = \mathrm{Gal}(L_n/K)^y$$

inside of $\mathrm{Gal}(K_n/K)$. We conclude that if $h \geq -1$ is chosen so that $\mathrm{Gal}(L_0/K)^y = 1$ for $y > h$ (note this h has nothing to do with n) then the integrand in the above integral formula for $\mathrm{ord}_K(\mathfrak{d}_{L_n/K_n})$ vanishes for $y > h$. Hence, for all n we have

$$\mathrm{ord}_K(\mathfrak{d}_{L_n/K_n}) \leq \int_{-1}^{h} \frac{1}{\#\,\mathrm{Gal}(K_n/K)^y} \, \mathrm{d}y.$$

Since this integral is over a bounded interval $[-1, h]$ rather than over $[-1, \infty)$, if we use the same kind of step-function calculation as in Step 3 of the proof of the Trace Estimate (using crucially the Filtration Lemma that relates the upper numbering and p-adic filtrations on $\mathrm{Gal}(K_\infty/K)$, with K_∞/K a totally ramified $\mathbf{Z}_p$-extension), we get $\mathrm{ord}_K(\mathfrak{d}_{L_n/K_n}) \leq a/p^n$ for some $a > 0$ and all n. We can take a to be an integer. (If we integrated out to ∞ we would only get the weaker upper bound $c + a/p^n$ for some $a, c > 0$, which is insufficient for what follows.)

Now we are finally in position to analyze the image of the trace map from L to K_∞. Choose $\alpha \in \mathfrak{m}_\infty$, so for sufficiently large n we have $\alpha \in K_n$ and hence $\alpha \in \mathfrak{m}_n$. Since α is fixed but K_∞/K is totally ramified, $i_n := \mathrm{ord}_{K_n}(\alpha) \to \infty$. Letting $e_n := e(L_n/K_n)$, for any $i \geq 0$ we have

$$\mathrm{Tr}_{L_n/K_n}(\mathfrak{m}_{L_n}^i) = \mathfrak{m}_n^j$$

where $j = [(i + \mathrm{ord}_{L_n}(\mathfrak{d}_{L_n/K_n}))/e_n]$. Taking $i = 0$,

$$\mathrm{Tr}_{L_n/K_n}(\mathscr{O}_{L_n}) = \mathfrak{m}_n^{j_n}$$

where $j_n \leq \mathrm{ord}_{L_n}(\mathfrak{d}_{L_n/K_n})/e_n = \mathrm{ord}_{K_n}(\mathfrak{d}_{L_n/K_n})$. But K_n/K is totally ramified of degree p^n, so this upper bound on j_n is equal to $p^n \, \mathrm{ord}_K(\mathfrak{d}_{L_n/K_n}) \leq a$. This proves $\mathfrak{m}_n^a \subseteq \mathrm{Tr}_{L_n/K_n}(\mathscr{O}_{L_n})$ for all n. Taking n large enough so that $i_n \geq a$, we get

$$\alpha \in \mathfrak{m}_n^{i_n} \subseteq \mathrm{Tr}_{L_n/K_n}(\mathscr{O}_{L_n}).$$

Hence, for such a large n there is some $\beta \in \mathscr{O}_{L_n}$ with $\mathrm{Tr}_{L_n/K_n}(\beta) = \alpha$. Since $L = L_n \otimes_{K_n} K_\infty$, we get $\mathrm{Tr}_{L/K_\infty}(\beta) = \mathrm{Tr}_{L_n/K_n}(\beta) = \alpha$.