DESCENDING PROPERTIES OF MORPHISMS

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Notation 0 If $\phi : S' \to S$ is a morphism of schemes, and X an S scheme with structural morphism $f : X \to S$, then X' will denote $X \times_S S'$, f' will denote the structural morphism $X' \to S'$ (projection onto the second factor) and ϕ' the morphism $X' \to X$ (projection onto the first factor):

$$\begin{array}{ccc} X' & \stackrel{\phi'}{\to} & X \\ \downarrow f' & & \downarrow f \\ S' & \stackrel{\phi}{\to} & S \end{array} .$$

Definition 1 If $\mathbf{P}_1$ and $\mathbf{P}_2$ are two properties of morphisms of schemes, we say that the property $\mathbf{P}_1$ descends along morphisms of type $\mathbf{P}_2$ if for any $f : X \to S$, if $\phi : S' \to S$ is a morphism of type $\mathbf{P}_2$ such that $f' : X' \to S'$ has property $\mathbf{P}_1$, then f has property $\mathbf{P}_1$.

Remark 2 Suppose that $\phi : S' \to S$ is a faithfully flat and quasi-compact morphism of schemes (abbreviated from now on by fpqc, for "fidelment plat, quasi-compact"). Let $U \subset S$ be affine open. Then $U' = \phi^{-1}(U)$ is a finite union of affine opens $U' = \bigcup_{i=1}^n U_i'$, and $U' = \coprod_{i=1}^n U_i'$ is affine. The natural morphism

$$\coprod_{i=1}^n U_i' \to \bigcup_{i=1}^n U_i' = \phi^{-1}(U) \to U$$

is surjective, since ϕ is surjective, and flat, because flatness is local on the source, and thus is faithfully flat.

Now suppose that $\mathbf{P}$ is a property of morphisms which is preserved under base change and can be checked locally on the target, i.e. $f : X \to S$ satisfies $\mathbf{P}$ iff there exists an open cover U_i' of S such that $f_{|f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ satisfies $\mathbf{P}$ for all i. Suppose also that $\mathbf{P}$ satisfies **affine fpqc descent**: if $\operatorname{Spec} A' \to \operatorname{Spec} A$ is faithfully flat and X is an A-scheme such that $X' \to A'$ satisfies $\mathbf{P}$, then $X \to A$ satisfies $\mathbf{P}$.

Then we claim that $\mathbf{P}$ satisfies **fpqc descent**: if $\phi : S' \to S$ is fpqc and we have a morphism $f : X \to S$ such that f' satisfies $\mathbf{P}$, then f satisfies $\mathbf{P}$.

To see this, let $U \subset S$ be affine open, and $\phi^{-1}(U) = \coprod_{i=1}^n U_i'$ with each U_i open affine in S'. By assumption $X' \to S'$ satisfies $\mathbf{P}$, and so the base change $X' \times_{S'} \coprod_{i=1}^n U_i' \to \coprod_{i=1}^n U_i'$ satisfies $\mathbf{P}$. Note that

$$X' \times_{S'} \coprod_{i=1}^n U_i' = (X \times_S S') \times_{S'} \coprod_{i=1}^n U_i' = X \times_S \coprod_{i=1}^n U_i' = f^{-1}(U) \times_U \coprod_{i=1}^n U_i',$$

since $\phi(\coprod_{i=1}^n U_i') = U$, an open subset of X. Thus $f^{-1}(U) \times_U \coprod_{i=1}^n U_i' \to \coprod_{i=1}^n U_i'$ satisfies **P**. By the assumption of affine descent, $f^{-1}(U) \to U$ satisfies **P**. Since **P** is local on the target, $X \to S$ satisfies **P**.

Example 3 Let **P** be the property $X = \emptyset$. Then this property is local on the target and preserved under base change. Let us show that it satisfies affine fpqc descent. Let $f : X \to \operatorname{Spec} A$. If $X \neq \emptyset$ there exists a non-empty affine open $\operatorname{Spec} B \subset X$, and then $\operatorname{Spec} B \otimes_A A'$ is affine open in X'. Now $\operatorname{Spec} B \neq \emptyset$ implies $B \neq 0$, and thus $B \otimes_A A' \neq 0$, since A' is faithfully flat over A. Thus $\operatorname{Spec} B \otimes_A A' \neq \emptyset$. Hence if $X' = \emptyset$ then $X = \emptyset$, so that **P** does indeed satisfy affine fpqc descent. So by Remark 1, **P** satisfies fpqc descent, i.e. if X is an S-scheme and $X' = \emptyset$ after base change by an fpqc morphism $S' \to S$ then $X = \emptyset$.

Remark 4 Surjectivity of $f : X \to S$ is preserved under arbitrary base change $\phi : S' \to S$.

For f is surjective iff $f^{-1}(s) \neq \emptyset$ for all $s \in S$ iff $X_s \neq \emptyset$ for all $s \in S$. Now let $s' \in S'$ with $f(s') = s \in S$. Then

$$\begin{aligned} X'_{s'} &= X' \times_{S'} \operatorname{Spec} k(s') = (X \times_S S') \times_{S'} \operatorname{Spec} k(s') = X \times_S \operatorname{Spec} k(s') \\ &= X \times_S \operatorname{Spec} k(s) \times_{\operatorname{Spec} k(s)} \operatorname{Spec} k(s') = X_s \times_{\operatorname{Spec} k(s)} \operatorname{Spec} k(s'). \end{aligned}$$

Since $k(s')$ is faithfully flat over $k(s)$ (being an extension of fields) it follows by Example 2 that if $X_s \neq \emptyset$ then $X_s \times_{\operatorname{Spec} k(s)} \operatorname{Spec} k(s') \neq \emptyset$. Thus $X'_{s'} \neq \emptyset$ for every $s' \in S'$, implying that f' is surjective.

More generally, whether or not f is surjective, the same remark shows that $\phi^{-1}(\operatorname{image} f) = \operatorname{image} f'$: if $\phi(s') = s = f(x)$, then $X_s \neq \emptyset$ implies that $X'_{s'} \neq \emptyset$.

Remark 5 Let $f : X \to S$, $\phi : S' \to S$, with ϕ surjective. If f' is injective, the same is true for f, that is, the property of being injective descends with respect to surjective maps.

To see this, let $s \in S$, and choose $s' \in S'$ such that $\phi(s') = s$ (possible since ϕ is surjective). We wish to show that X_s has at most one element. Now the argument of Remark 3 shows that $X'_{s'} = X_s \times_{\operatorname{Spec} k(s)} \operatorname{Spec} k(s') \to X_s$ is the base change of the surjective morphism $\operatorname{Spec} k(s') \to \operatorname{Spec} k(s)$ via the morphism $X_s \to \operatorname{Spec} k(s)$. Thus by the argument of Remark 4, the morphism $X'_{s'} \to X_s$ is surjective. Now by assumption f' is injective, so that $X'_{s'}$ has at most one element. The same is therefore true of X_s.

Lemma 6 The properties of being surjective and universally injective are preserved under arbitrary base change, and descend along surjective morphisms.

Note that Remark 4 shows that surjectivity is preserved under arbitrary base change, while universal injectivity is preserved under arbitrary base change essentially by definition.

Now suppose that we have a map $f : X \to S$ and that $\phi : S' \to S$ is surjective, yielding the diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\phi'} & X \\ \downarrow f' & & \downarrow f \\ S' & \xrightarrow{\phi} & S \end{array} .$$

If the base change f' is surjective, the composite $f \circ \phi' = \phi \circ f$ is a composite of surjective maps and so is surjective. Thus f is surjective.

On the other hand, suppose that f' is universally injective. Let T be an S-scheme, and T' the base change of T to an S'-scheme. Then by assumption $f'_{T'} : X' \times_{S'} T' \to T'$ is injective. We compute

$$X' \times_{S'} T' = (X \times_S S') \times_{S'} T' = X \times_S T' = (X \times_T) \times_T T',$$

and thus rephrase our previous statement: $(X \times_S T) \times_T T' \to T'$ is injecitve.

This morphism is the base change via $T' \to T$ of $f_T : X \times_S T \to T$. Since $S' \to S$ is surjective, Remark 4 shows that $T' \to T$ is surjective. Remark 5 now shows that f_T is injective. Thus f is indeed universally injective.

Remark 7 Suppose that a particular property of morphisms $\mathbf{P}_2$ is preserved under arbitrary base change, and that another property of morphisms $\mathbf{P}_1$ descends along morphisms of type $\mathbf{P}_2$. Let $f : X \to Y$ be a morphism of S-schemes, let $\phi : S' \to S$ have property $\mathbf{P}_2$, and suppose that the base change $f' : X' \to Y'$ has property $\mathbf{P}_1$. Then the same is true of f.

To see this, note that

$$X' = X \times_S S' = X \times_Y Y \times_S S' = X \times_Y Y'.$$

Thus $f' : X' \to Y'$ is the base change of $f : X \to Y$ via $Y' \to Y$. Since $\phi : S' \to S$ has property $\mathbf{P}_2$, the same is true of the base change $Y' \to Y$ by assumption. The conclusion now follows from the assumption that $\mathbf{P}_1$ descends along $\mathbf{P}_2$.

Lemma 8 Each of the properties of being quasi-compact and quasi-separated is preserved under arbitrary base-change and descends along surjective quasi-compact morphisms.

To see this, deal with the case of quasi-compact first. Let $f : X \to S$ be quasi-compact and let $\phi : S' \to S$ be an arbitrary morphism. If $U \subset S$ is an affine open, let $f^{-1}(U) = \bigcup_{i=1}^n V_i$ with each $V_i \subset X$ an affine open. Let $U' \subset \phi^{-1}(U)$ be an affine open. Then

$$f'^{-1}(U') = f^{-1}(U) \times_U U' = (\bigcup_{i=1}^n V_i) \times_U U' = \bigcup_{i=1}^n V_i \times_U U',$$

and each $V_i \times_U U'$ is an affine open subset of X'. Thus f' is quasi-compact.

Now suppose that ϕ is surjective and quasi-compact, and that f' is quasi-compact. We wish to conclude that f is quasi-compact. The morphism $\phi' : X' \to X$ is the base-change via $X \to S$ of the surjective morphism ϕ, and thus is surjective by Lemma 7. Now since $f \circ \phi' = \phi \circ f'$ is the composite of quasi-compact maps, it is quasi-compact. Thus the composite of f with the surjective map ϕ is quasi-compact, implying that f is quasi-compact. (If U is any quasi-compact open in S, then since ϕ is surjective, $f^{-1}(U) = \phi((f \circ \phi)^{-1}(U))$ is the image of a quasi-compact set and so is quasi-compact.)

Now we deal with the case of quasi-separated. By definition $f : X \to S$ is quasi-separated iff $\Delta : X \to X \times_S X$ is quasi-compact. Thus the claims concerning quasi-separatedness follows from those concerning quasi-compactness via Remark

7, together with the base change properties of diagonal maps.

Remark 9 If $f' : X' \to S'$ is the base change of $f : X \to S$ via the surjective morphism $\phi : S' \to S$, then if f' has finite fibres the same is true of f.

The same argument as used in Remark 5 shows that if $s \in S$ and we write $s = \phi(s')$ for some $s' \in S$ (which is possible since ϕ is surjective), then $X'_{s'} \to X_s$ is surjective. Thus if $X'_{s'}$ is finite, the same is true of X_s.

Lemma 10 The property of having universally finite fibres is preserved under arbitrary base change, and descends along surjective morphisms.

This follows by the same arguments that were used in proving Lemma 6 for universally injecive morphisms, using Remark 9 in place of Remark 5.

Example 11 Flatness is preserved under arbitrary base change, and can be checked locally on the target (in fact locally on the source). We will show that flatness satisfies fpqc descent. From Remark 2 it suffices to show that flatness satisfies affine fpqc descent.

Thus suppose that $A \to A'$ is a faithfully flat morphism and that $X \to \operatorname{Spec} A$ is an A-scheme, such that the base change $X' \to \operatorname{Spec} A'$ is flat. Let $\operatorname{Spec} B$ be an open affine subset of X. We must show that B is a flat A-algebra.

By assumption, $B' = B \otimes_A A'$ is a flat A'-algebra, since $\operatorname{Spec} B' = \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} A'$ is affine open in X' which is flat over $\operatorname{Spec} A'$.

We must show that $- \otimes_A B$ is an exact functor on A-modules. But $- \otimes_A A'$ is an exact and faithful functor on A-modules, so $- \otimes_A B$ is exact iff $(- \otimes_A B) \otimes_A A'$ is exact. But

$$(- \otimes_A B) \otimes_A A' = - \otimes_A (B \otimes_A A') = - \otimes_A B' = (- \otimes_A A') \otimes_{A'} B'$$

is the composition of the exact functor $- \otimes_{A'} B'$ with the exact functor $- \otimes_A A'$, and so is indeed an exact functor. Thus B is flat over A, and we see that flatness does satisfy affine fpqc descent, and thus satisfies fpqc descent.

Since faithful flatness equals flatness together with surjectivity, we see by combining what we have just proved for flatness with Lemma 6 that faithful flatness also satisfies fpqc descent.

Proposition 12 The following properties are preserved by arbitrary base change, and may be checked locally on the target: quasi-compact, quasi-separated, locally finite type, finite type, locally finite presentation, finite presentation, universally finite fibres, quasi-finite, universally injective, surjective, flat, faithfully flat, affine, finite, isomorphism, open immersion, closed immersion, separated, quasi-affine, universally closed, proper.

We have already observed that base change preserves the properties of being quasi-compact, quasi-separated, universally injective, surjective, flat, universally finite fibres, and faithfully flat. Since base change is functorial, it preserves the property of being an isomorphism. Base change preserves universally closed essentially by definition. This leaves the following properties still to be checked: locally finite type (from which finite type also follows, since finite type equals locally finite type plus quasi-compact), locally finite presentation (from which finite presentation follows also, since finite presentation equals locally finite presentation plus quasi-compact and quasi-separated), quasi-finite, affine, finite, open immersion, closed immersion, and separated (properness then follows, as the combination of separated, finite type and universally closed).

We will consider a morphism $f : X \to S$, assumed to possess the property being checked, and shows that the base change $f' : X' \to S'$ via any morphism $\phi : S' \to S$ has the same property.

(a) locally finite type and locally finite presentation: suppose that f is locally finite type (respectively locally finite presentation), and let $U = \operatorname{Spec} A \subset S$ and $U' = \operatorname{Spec} A' \subset \phi^{-1}(U) \subset S'$ be affine opens. Write $f^{-1}(U) = \bigcup \operatorname{Spec} B_i$, with B_i finite type (respectively finite presentation) over A. Then

$$f'^{-1}(U') = f^{-1}(U) \times_U U' = \left(\bigcup \operatorname{Spec} B_i\right) \times_{\operatorname{Spec} A} \operatorname{Spec} A' = \bigcup \operatorname{Spec} B_i \otimes_A A'.$$

Since B_i has finite type (respectively finite presentation) over A, B_i' has finite type (respectively finite presentation) over A'. Thus f' is locally finite type (respectively finite presentation), so locally finite type (respectively locally finite presentation) is preserved by all base changes. As already noted, combining this with the known result for quasi-compact and quasi-separated shows that finite type and finite presentation are both preserved by all base changes.

(b) quasi-finite: quasi-finite equals finite type together with finite fibres. Suppose that f is quasi-finite. Let $s' \in S'$ and let $s = \phi(s')$. Then $X'_{s'} = X_s \times_{\operatorname{Spec} k(s)} \operatorname{Spec} k(s')$. Now by (a) X_s is finite type over $k(s)$. Since it has only finitely many points, it is equal to the spectrum of a finite-dimensional $k(s)$-algebra. Thus $X'_{s'}$ is equal to the spectrum of a finite-dimensional $k(s')$-algebra, and so also has only finitely many points. By (a), f' is finite type, and so (b) is proved. We also see that quasi-finite equals finite type together with universally finite fibres.

(c) affine: suppose that f is affine. Let $U = \operatorname{Spec} A \subset S$ be affine open, then $f^{-1}(U) = \operatorname{Spec} B$ is an affine open subset of X. Let $U' = \operatorname{Spec} A' \subset \phi^{-1}(U) \subset S'$ be affine open. Then $f'^{-1}(U') = f^{-1}(U) \times_U U' = \operatorname{Spec} B \otimes_A A'$ is affine open in X', showing that f' is affine.

(d) finite: Let us continue the notation argument of (c). If f is finite as well as affine, then B is finite over A, and so $B \otimes_A A'$ is finite over A', showing that f' is finite.

(e) open immersion: it is a basic property of fibre products that base change preserves open immersion, which we have been using freely: if U is open in S, then the projection of $U \times_S S'$ onto the second factor is an open immersion identifying $U \times_S S'$ with the open subscheme $\phi^{-1}(U)$ of S.

(f) closed immersion: f is a closed immersion iff it is affine and for all $U = \operatorname{Spec} A$ affine open in S the natural map $\Gamma(U, \mathcal{O}_U) \to \Gamma(f^{-1}(U), \mathcal{O}_{f^{-1}(U)})$ is surjective. Also, it suffices to check this condition with respect to a particular open affine cover of S.

Suppose that f is a closed immersion. Let $U = \operatorname{Spec} A$ be affine open in S and let $f^{-1}(U) = \operatorname{Spec} B$ be its preimage, an affine open in X. By assumption the natural map $A \to B$ is surjective. Let $U' = \operatorname{Spec} A' \subset \phi^{-1}(U) \subset S'$ be affine open. Then $f'^{-1}(U') = f^{-1}(U) \times_U U' = \operatorname{Spec} A' \otimes_A B$ is affine, and the natural map $A' \otimes_A B \to A'$ is surjective, since it is the base change of $A \to B$ via $A \to A'$, and tensor product preserves surjections. Thus f' is a closed immersion.

(g) separated: $f : X \to S$ is separated iff the diagonal map $X \to X \times_S X$ is a closed immersion. The diagonal map $X' \to X' \times_{S'} X'$ is the base change of this, map, and so by (f) above we see that this property is preserved under arbitrary base change. As observed, the case of properness follows from that of separatedness together with that of universally closed and finite type.

(h) quasi-affine: $f : X \to S$ is quasi-affine if it is quasi-compact and S is covered by open affines $U = \operatorname{Spec} A$ such that the canonical map $f^{-1}(U) \to \operatorname{Spec} \Gamma(f^{-1}(U), \mathcal{O}_{f^{-1}(U)})$ is an open immersion. Moreover, this is equivalent to $f^{-1}(U)$ admitting an open immersion into some affine scheme. (We will use this formulation when showing that f' is quasi-affine.) For ease of notation, write $B = \Gamma(f^{-1}(U)$. Now consider the base change of f via $\phi : S' \to S$. From what we have already proved, we know that f' is quasi-compact. Let $U' = \operatorname{Spec} A'$ be an open affine subset of $\phi^{-1}(U)$. Then $f^{-1}(U') = f^{-1}(U) \times_U U'$, which has an open immersion into the affine scheme $\operatorname{Spec} B \times_U U' = \operatorname{Spec} B \otimes_A A'$. Since S' is covered by such open affine subsets U', we conclude that f' is quasi-affine.

This completes the proof that all the properties listed are preserved by base change. Essentially by definition, the property of a morphism being quasi-compact, quasi-separated, locally finite type, finite type, locally finite presentation, finite presentation, affine, flat, finite, an isomorphism, an open immersion, or quasi-affine can be checked locally on the base. Since being quasi-finite, universally injective, surjective, or having universally finite fibres refers to properties of the fibres, these properties can be checked locally on the base. Since the property of being a closed topological embedding can be checked locally on the target, as can surjectivity of a morphism of sheaves, the property of being a closed immersion can be checked locally on the base. Thus separatedness is local on the base. The property of being a closed subset of a space can be checked locally, so that the property of being universally closed can be checked locally on the base. Thus properness is local on the base.

Lemma 13 If $f : X \to S$ is dominant and quasi-compact, the same is true for the base change $f' : X' \to S'$ via any flat morphism $\phi : S' \to S$.

Let $U = \operatorname{Spec} A \subset S$ be an affine open subset, and write $f^{-1}(U) = \bigcup_{i=1}^n V_i$, with each $V_i = \operatorname{Spec} B_i \subset X$ affine open in X. Now X dominates S, and this property is local on the base (since $f : X \to S$ is dominant iff $f^{-1}(U) \neq \emptyset$ for every non-empty open U in S). Thus $\bigcup_{i=1}^n V_i$ dominates U, and so the same is true of $\coprod_{i=1}^n V_i \to U$. But now $\coprod_{i=1}^n V_i = \operatorname{Spec} B$, with $B = \prod_{i=1}^n B_i$. Thus $\operatorname{Spec} B \to \operatorname{Spec} A$ is dominant, which is equivalent to saying that $A/\operatorname{nil} A \to B/\operatorname{nil} B$ is injective.

Now choose $U' = \operatorname{Spec} A' \subset \phi^{-1}(U) \subset S'$ affine open. By assumption, A' is flat over A, and thus $(A/\operatorname{nil} A) \otimes_A A' \to (B/\operatorname{nil} B) \otimes_A A'$ is injecitve, which implies (writing $B' = B \otimes_A A'$) that $A'/\operatorname{nil} A' \to B'/\operatorname{nil} B'$ is injective. (For suppose that $a' \in A'$ has nilpotent image in B'. Then for some n, a'^n has zero image in B, and hence in $B'/(\operatorname{nil} B \otimes_A A') = (B/\operatorname{nil} B) \otimes_A A'$. So by what we have just proved, $a'^n \in \operatorname{nil} A \otimes_A A'$, and so a'^n is nilpotent in A', showing that a' is nilpotent in A.) Thus $\operatorname{Spec} B' \to \operatorname{Spec} A'$ is dominant, i.e. (writing $V_i' = V_i \times_U U'$) $\coprod_{i=1}^n V_i' \to U'$ is dominant. Thus $f'^{-1}(U') = \bigcup_{i=1}^n V_i' \to U'$ is dominant. Since dominance can be checked locally on the base, and we can cover S' by such open sets U', we see that $f' : X' \to S'$ is dominant. Together with Lemma 8, which shows that f' is quasi-compact, this proves the Lemma.

Theorem 14 Let $\phi : S' \to S$ be fpqc. Then the topology of S is the quotient via

ϕ of the topology of S', i.e. $U \subset S$ is open iff $\phi^{-1}(U) \subset S'$ is open.

Equivalently, by taking complements, the claim may be phrased as $F \subset S$ is closed iff $\phi^{-1}(F) \subset S'$ is closed. So suppose that $\phi^{-1}(F)$ is closed in S'. Let $\bar{F}$ be the closure of F in S. We wish to show that $F = \bar{F}$. Give $\bar{F}$ its reduced induced scheme structure. Base change $S' \to S$ over the closed immersion $\bar{F} \to S$, to get the diagram

$$\begin{array}{ccccc} \phi^{-1}(F) & \subset & \phi^{-1}(\bar{F}) & \hookrightarrow & S' \\ \downarrow & & \downarrow & & \downarrow \\ F & \subset & \bar{F} & \hookrightarrow & S. \end{array}$$

Then $\phi^{-1}(\bar{F})$ is fpqc over $\bar{F}$ (by Proposition 12), and $\phi^{-1}(F)$ is a closed subset of $\phi^{-1}(\bar{F})$ which dominates $\bar{F}$ (since $F = \phi(\phi^{-1}(F))$ is dense in $\bar{F}$). Thus suppose we are in this situation: $S' \to S$ is fpqc and X is a closed subset of S' which dominates S and which is the preimage of its image. (We have relabelled $\bar{F}$ as S, $\phi^{-1}(\bar{F})$ as S', and $\phi^{-1}(F)$ as X.) We wish to show that the image of X is equal to S, or equivalently (since X is the preimage of its image) that $X = S'$. If we can do this, we are done.

There are (at least) two possible arguments to show this. The first is as follows: give X its reduced induced scheme structure. The inclusion $X \to S'$ is a closed embedding and so is a quasi-compact map. Thus the composition $X \to S' \to S$ is quasi-compact, so by Lemma 4.5 in Chapter II of Hartshorne, the image of X in S is closed iff it is closed under specialization. If we can show this, then since we have assumed that the image of X is dense in S, we see that this image is equal to S, completing the argument.

To see that the image of X is closed under specialization, we argue as follows: let P be a point in the image of X, and let P' be a specialization of P. Let Q' be a point lifting P' in S'. Then since flat maps satisfy the Going Down Theorem, we may find a generization Q of Q' mapping onto P. (To be more precise, let $U = \operatorname{Spec} A$ be an open affine neighbourhood of P', and let $V = \operatorname{Spec} B$ be an open affine neighbourhood of Q' contained in $\phi^{-1}(U)$. Then $P \in U$, because P' is a specialization of P. Now since ϕ is flat, B is a flat A-algebra, $P \subset P'$ are prime ideals of A, and Q' is a prime ideal of B which contracts to P' in A. We apply the Going Down Theorem in this situation to find a prime ideal $Q \in \operatorname{Spec} B$ contracting to P.) Since X is the preimage of its image, we see that Q lies in X. Since X is closed, it is stable under specialization, and so Q' lies in X. Thus P' lies in the image of X, and so we see that the image of X is indeed closed under specialization.

The second argument illustrates an important technique, that of base changing a faithfully flat map over itself. Assume as before that $S' \to S$ is fpqc, and that X is a closed subset of S' which is the preimage of its image and which dominates S. We will show that $X = S'$, and then we will be done. Give X its reduced induced scheme structure: we have the diagram

$$X \overset{i}{\hookrightarrow} S' \overset{\phi}{\to} S.$$

Now base change this diagram with respect to $S' \to S$. Write $S'' = S' \times_S S'$. We have

$$\begin{array}{ccccc} X' & \overset{i'}{\hookrightarrow} & S'' & \overset{\phi'}{\to} & S' \\ \downarrow & & \downarrow \phi' & & \downarrow \phi \\ X & \overset{i}{\hookrightarrow} & S' & \overset{\phi}{\to} & S\ . \end{array}$$

By Remark 4, $\operatorname{image}\phi' \circ i' = \phi^{-1}(\operatorname{image}\phi \circ i)$, i.e. $\phi'(X') = \phi^{-1}(\phi(X)) = X$ (since X is assumed to be the preimage of its image via ϕ), which is closed. But $\phi \circ i$ is quasi-compact and dominant, so by Lemma 13, $X' \to S'$ is dominant. Thus we see that $X = S'$.

Lemma 15 Let $f : X \to \operatorname{Spec} A$ be quasi-compact and quasi-separated, let A' be flat over A, and consider the base change $f' : X' \to \operatorname{Spec} A'$. Then the natural map $\Gamma(X, \mathcal{O}_X) \otimes_A A' \to \Gamma(X', \mathcal{O}_{X'})$ is an isomorphism.

First note that since X is an A-scheme, $\Gamma(X, \mathcal{O}_X)$ is naturally an A-algebra. Similarly $\Gamma(X', \mathcal{O}_{X'})$ is an A'-algebra. There is a natural map of A algebras $\Gamma(X, \mathcal{O}_X) \to \Gamma(X', O_{X'})$ obtained by pulling back global sections of the structure sheaf via the morphism of locally ringed spaces $X' \to X$. Thus there is an induced A'-algebra map $\Gamma(X, \mathcal{O}_X) \otimes_A A' \to \Gamma(X', \mathcal{O}_{X'})$. This is the map which we claim is an isomorphism.

Let us first prove this in the case when $X = \operatorname{Spec} B$ is affine. Then $X' = \operatorname{Spec} B \otimes_A A'$, and the natural map $\Gamma(X, \mathcal{O}_X) \otimes_A A' \to \Gamma(X', \mathcal{O}_{X'})$ is simply the identity map $B \otimes_A A' = B \otimes_A A'$, which is certainly an isomorphism.

Now in general, if X is not affine, then we may cover X with finitely many affine opens V_i, and cover the intersection $V_i \cap V_j$ with finitely many affine opens W_{ijk}. (This is possible since we are assuming that X is quasi-compact and quasi-separated.) Then $\Gamma(X, \mathcal{O}_X)$ is the kernel of the diagram

$$\prod_i \mathcal{O}(V_i) \rightrightarrows \prod_{ijk} \mathcal{O}(W_{ijk}).$$

Since A' is flat over A, we see that $\Gamma(X, \mathcal{O}_X) \otimes_A A'$ is the kernel of

$$\prod_i \mathcal{O}(V_i) \otimes_A A' \rightrightarrows \prod_{ijk} \mathcal{O}(W_{ijk}) \otimes_A A'.$$

(Here we are using the fact that tensor products commute with finite products.) Since V_i is affine, our above discussion shows that $V_i' = V_i \times_{\operatorname{Spec} A} \operatorname{Spec} A'$ is also affine, and pullback of global sections from V_i to V_i' induces the identity map $\mathcal{O}(V_i') = \mathcal{O}(V_i) \otimes_A A'$. Similarly, $W_{ijk}' = W_{ijk} \otimes_{\operatorname{Spec} A} \operatorname{Spec} A'$ is affine, and the pullback of global sections from W_{ijk} to W_{ijk}' induces the identity map $\mathcal{O}(W_{ijk}') = \mathcal{O}(W_{ijk}) \otimes_A A'$. Thus pullback of global sections on X identifies $\Gamma(X, \mathcal{O}_X) \otimes_A A'$ with the kernel of

$$\prod_i \mathcal{O}(V_i') \rightrightarrows \prod_{ijk} \mathcal{O}(W_{ijk}').$$

But the V_i' provide an open cover of X' whose intersections are covered by the W_{ijk}'. Thus this kernel is canonically isomorphic to $\Gamma(X', \mathcal{O}_{X'})$. This proves the Lemma.

Remark 16 Lemma 15 gives a quick proof of Exercise 2.16 of Chapter II of Hartshorne: suppose that X is a quasi-compact, quasi-separated scheme, and write $A = \Gamma(X, \mathcal{O}_X)$. Then there is a natural map $X \to \operatorname{Spec} A$. Let $f \in A$. Then A_f is flat over A, so the hypotheses of Lemma 15 are satisfied. Note that the fibre product $X \times_{\operatorname{Spec} A} \operatorname{Spec} A_f$ is simply X_f (the non-vanishing locus of f in X), and so we see from Lemma 15 that we obtain a natural isomorphism $A_f \xrightarrow{\sim} \Gamma(X_f, \mathcal{O}_X)$. (This map is the tautological map arising from the fact that $\Gamma(X_f, \mathcal{O}_X)$ is an A-algebra in which f is invertible.)

Theorem 17 All the properties listed in Proposition 12 satisfy fpqc descent.

Any fpqc map is surjective and quasi-compact, so by Lemmas 6 and 8 we are done for surjective, universally injective, quasi-compact and quasi-separated. By Example 11 we are done for flat and faithfully flat. By Lemma 10 we are done for universally finite fibres. By Remark 2 together with Proposition 12, we see that in proving the remainder of the Theorem, it suffices to consider the affine case in which $S = \operatorname{Spec} A$ and $S' = \operatorname{Spec} A'$, with A' faithfully flat over A.

(a) locally finite type and locally finite presentation: let $f : X \to \operatorname{Spec} A$ and suppose that $f' : X \to \operatorname{Spec} A'$ is locally finite type (respectively locally finite presentation). Let $\operatorname{Spec} B \subset X$ be affine open, then if $B' = B \otimes_A A'$, $\operatorname{Spec} B'$ is an open affine subset of X', and thus is finite type over A'. We wish to show that B is finite type (respecitvely finite presentation) over A.

Write $B = \varinjlim B_i$ as the direct limit of its finite type A-subalgebras. Then $B' = \varinjlim B_i \otimes_A A'$. By assumption $B' = B_i \otimes_A A'$ for some B_i, and thus $B = B_i$ since A' is faithfully flat over A. Thus B is finite type over A, showing that f is indeed locally finite type.

Suppose furthermore that B' is of finite presentation over A: let

$$0 \to I \to A[T_1, \dots, T_n] \to B \to 0$$

be an exact sequence showing that B is finite type over A, and write $I = \varinjlim I_i$ as the limit of its finitely generated subideals. Since A' is flat over A,

$$0 \to I \otimes_A A' \to A'[T_1, \dots, T_n] \to B \to 0$$

is exact, and $I \otimes_A A' = \varinjlim I_i \otimes_A A'$. Since B' is finitely presented, $I \otimes_A A' = I_i \otimes_A A'$ for some I_i, and thus $I = I_i$, since A' is faithfully flat over A. This shows that B is finitely presented over A. Thus the property of being locally of finite presentation descends.

Since finite type equals locally finite type together with quasi-compact, finite presentation equals locally finite presentation together with quasi-compact and quasi-separated, and quasi-finite equals finite type together with universally finite fibres, these are also dealt with via (a) and our previous remarks.

(b) isomorphism: Supppose that $f : X \to \operatorname{Spec} A$ is such that $f' : X' \to A'$ is an isomorphism. We wish to show that f is an isomorphism.

Since f' is an isomorphism, it is in particular universally injective and faithfully flat. So by what we have already proven, the same properties are true of f. By Theorem 14, f induces the quotient topology on $\operatorname{Spec} A$, and since f is a bijection, we see that it is a homeomorphism $X \to S$. Now cover X by open affines $U_i = \operatorname{Spec} B_i$. Then each $f(U_i)$ is open in $\operatorname{Spec} A$, so we may cover $f(U_i)$ by open affines of the form $\operatorname{Spec} A_{a_j}$ ($a_j \in A$). Then $X_{a_j} = f^{-1}(\operatorname{Spec} A_{a_j}) = \operatorname{Spec}(B_i)_{a_j}$ is an open affine in X. Since the $\operatorname{Spec} A_{a_j}$ cover $\operatorname{Spec} A$, the elements a_j generate the unit ideal of A. Thus they generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$, and so by Exercise 2.17 of Hartshorne, X is affine, say $X = \operatorname{Spec} B$. Now we have a morpism $A \to B$ such that $A' \to B \otimes_A A'$ is an isomorphism. Since A' is faithfully flat over A, we see that $A \to B$ is an isomorphism, and we are done.

(c) open immersion: Suppose that $f : X \to S$ is such that the base change $f' : X' \to S'$ via the fpqc morphism $\phi : S' \to S$ is an open immersion. By Remark 4, $\phi^{-1}(\operatorname{image} f) = \operatorname{image} f'$. The image of f' is open by assumption. Theorem 14 now shows that the image of f is open. Write $U = \operatorname{image} f$, $U' = \phi^{-1}(U) = \operatorname{image} f'$. Then $U' = S' \times_S U$ is fpqc over U, and by assumption $X' \to U'$ is an isomorphism. By (b) we see that $X \to U$ is an isomorphism, and so $X \to S$ is an open immersion.

(d) affine: Suppose that we have $f : X \to \operatorname{Spec} A$ such that the base change $X' \to \operatorname{Spec} A'$ over the faithfully flat A-algebra A' is affine. We must show that f is affine. Since affine maps are quasi-compact and quasi-separated, it follows from what we have already done that f has these properties. Write $B = \Gamma(X, \mathcal{O}_X)$. By Lemma 15 it follows that $B' = B \otimes_A A' \xrightarrow{\sim} \Gamma(X', \mathcal{O}_{X'})$. Denote this ring by B'.

Since $\operatorname{Spec} A'$ is affine, our assumption implies that the natural map $X' \to \operatorname{Spec} B'$ is an isomorphism, and we wish to show that the same is true of the natural map $X \to \operatorname{Spec} B$. But the first of these maps is the base change of second via the faithfully flat map of rings $A \to A'$. Thus (b) above together with Remark 7 shows that indeed $X \to \operatorname{Spec} B$ is an isomorphism, and we are done.

(e) finite: By (d) we may assume that we have a morphism of rings $A \to B$ such that after making a faithfully flat base change $A' \to B' = B \otimes_A A'$, B' is a finite A'-module. We wish to show that B is a finite A-module. But this follows by writing $B = \varinjlim B_i$ as the limit of its finite submodules and arguing as in the proof of (a).

(f) closed immersion: By (d) we may assume that we have a morphism of rings $A \to B$ such that the faithfully flat base change $A' \to B' = B \otimes_A A'$ is a surjection. By definition of faithfully flat, $A \to B$ must be surjective, and we are done.

(g) separated: Suppose that $f : X \to S$ such that the base change $f' : X' \to S'$ via the fpqc map $S' \to S$ is separated. This means that the diagonal map $X' \to X' \times_{S'} X'$ is a closed immersion. But this is the base change of the diagonal map $X \to X \times_S X$. Remark 7 together with (f) above shows that f is separated.

(h) quasi-affine: Suppose that $f : X \to \operatorname{Spec} A$ such that the base change $f' : X' \to \operatorname{Spec} A'$ via the fpqc map $A \to A'$ is quasi-affine. In particular f' is quasi-compact and separated, so by what has already been proved, the same is true of f. Write $B = \Gamma(X, \mathcal{O}_X)$. Write $B' = B \otimes_A A'$. Then Lemma 15 shows that $B' = \Gamma(X', \mathcal{O}_{X'})$. Since f' is quasi-affine, $X' \to \operatorname{Spec} B'$ is an open immersion. But this is the base change of the natural map $X \to \operatorname{Spec} B$ via $A \to A'$, so Remark 7 together with (c) above, shows that this map is an open immersion, so that f is quasi-affine.

(i) universally closed: suppose that $f : X \to S$ is such that the fpqc base change $f' : X' \to S'$ is universally closed. We wish to show that f is universally closed. Let T be any S-scheme. We must show that $f_T : X \times_S T \to T$ is closed. If we base change this via the map $T' = T \times_S S' \to T$ (which by Proposition 12 is fpqc, since $S' \to S$ is fpqc) we obtain $f'_{T'} : X' \times_{S'} T' \to T'$ (compare the manipulation of fibre products in the proof of Lemma 6) which is closed, because we have assumed that f' is universally closed. Therefore it is enough to prove that if $f : X \to S$ yields a closed map $f' : X' \to S'$ after fpqc base change, then f is itself closed.

So let F be closed in X; we must show that $f(F)$ is closed in S. Give F its reduced induced structure, and base change $F \hookrightarrow X$ by $S' \to S$ to obtain $F' \hookrightarrow X'$ a closed immersion. Then the composition $F' \hookrightarrow X' \to S'$ is a composition of closed maps and so is closed. Thus, after replacing X by F, it is enough to show that X has closed image under f.

Remark 4 shows that $\phi^{-1}(\text{image } f) = \text{image } f'$, which is closed by our assumption on f'. Now Theorem 14 shows that the image of f is closed, and we are done. The case of properness (equals universally closed, separated and finite type) now follows.

Remark 18 Recall that if $f : X \to S$ is a quasi-compact, quasi-separated morphism of schemes, we can define the scheme-theoretic image of f as follows: $f_*\mathcal{O}_X$ is a quasi-coherent sheaf on S, and so the kernel $\mathcal{I}$ of the map $\mathcal{O}_S \to f_*\mathcal{O}_X$ is a quasi-coherent ideal sheaf on S, which cuts out a closed subscheme of S, called the scheme-theoretic image of f. Denote this closed subscheme by Z. Then f factors as $X \to Z \to S$.

Suppose that $\phi : S' \to S$ is a flat morphism. Then we may base change the above factorization of f via ϕ to obtain the factorization $X' \to Z' \to S'$ of the base changed morphism $f' : X' \to S'$. Since $Z \to S$ is a closed immersion, it follows from Proposition 12 that $Z' \to S'$ is a closed immersion. We claim that Z' is the scheme-theoretic image of f'.

To check this, it suffices to work locally on S'. Thus we may assume that $S = \operatorname{Spec} A$ and that $S' = \operatorname{Spec} A'$, with A' a flat A-algebra. Then Z is the closed subscheme of S cut out by the kernel of the natural morphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus Z' (being the base change of Z via $A \to A'$) is the closed subscheme of S' cut out by the kernel of the morphism $A' \to \Gamma(X, \mathcal{O}_X) \otimes A'$. But now Lemma 15 shows that this is precisely the kernel of the morphism $A' \to \Gamma(X', \mathcal{O}_{X'})$, and so Z' is indeed the scheme-theoretic image of f'.

Remark 19 Recall that a morphism $f : X \to S$ is called an immersion if it can be factored as a closed immersion followed by an open immersion. Since both closed and open immersions are preserved by base change, the base change of an immersion is again an immersion. Since closed immersions and open immersions are both separated, immersions are separated. Suppose that a morphism f is a quasi-compact and quasi-separated morphism, so that it has a scheme-theoretic image. Then f is an immersion iff it is an open immersion into its scheme-theoretic image. (In this case we see that we can factor f as an open immersion followed by a closed immersion.)

Now suppose given $f : X \to S$ such that the base change $f' : X' \to S'$ via an fpqc map $\phi : S' \to S$ is a quasi-compact immersion. Then by Theorem 17, f must be quasi-compact and separated, since the same is true of f'. In particular, f has a scheme-theoretic image, which we will denote by Z. Using Remark 18 we see that the scheme-theoretic image of f' is the base change via ϕ of Z, which we will denote by Z'. Thus we have the following diagram in which the left-hand side is the base change via ϕ of the right hand side:

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ Z' & \to & Z \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\phi} & S. \end{array}$$

By assumption f' is an immersion, and so the morphism $X' \to Z'$ is an open immersion. But this is the base change of the morphism $X \to Z$, so Theorem 17 shows that this latter morphism is also an open immersion. Thus f induces an open immersion into its scheme-theoretic image, and so f is an immersion.