Math 210C. Homework 9

1. (i) Read the handout on root system decomposition (including a strong uniqueness).

(ii) Exer. 4 (allowing any char. 0 field k) and Exer. 6 (over $\mathbf{Q}, \mathbf{R}$) in 5.14 of Ch. V. For irreducible (V, Φ), deduce the $W(\Phi)$-action on $V_{k'}$ is irreducible for *any* field extension k'/k.

2. Read the handout computing the root systems for classical groups (types A, B, C, D).

(i) *Using the root notation of that handout* (not the course text!), for Propositions 6.4 (for $n \geq 3$) and 6.6 in Ch. V verify that the indicated basis in each "part (iii)" is a basis, with associated positive system of roots as in each "part (ii)" and associated Weyl chamber whose closure is indicated in each "part (v)".

(ii) Verify the diagram in "part (iv)" for each, labeling nodes by corresponding basis roots.

3. (i) Let $f : E \to M$ be a covering space of a C^∞ manifold. Prove E admits a unique C^∞ structure making f a local diffeomorphism. (Prove general uniqueness, then existence for split covers, and then general existence by gluing locally over M via *uniqueness* on overlaps.)

(ii) Let M be a topological manifold, $m_0 \in M$, and $f : E \to M$ a covering space. For $[\gamma] \in \pi_1(M, m_0)$ and $x \in E_{m_0}$, define $x.\gamma := \widetilde{\gamma}_x(1) \in E_{m_0}$ where $\widetilde{\gamma}_x : [0,1] \to E$ is the unique lift of $\gamma : [0,1] \to M$ with $\widetilde{\gamma}_x(0) = x$. Prove $x.\gamma$ depends on γ through $[\gamma]$, and is a right action of $\pi_1(M, m_0)$ on E_{m_0}, *transitive* if E is *connected*. Deduce f splits if $\pi_1(M, m_0) = 1$.

(iii) Any connected topological manifold M has a connected covering space $\widetilde{M} \to M$ with $\pi_1(\widetilde{M}) = 1$ (e.g., $\widetilde{S}^1 = \mathbf{R}$). Prove *universality*: for covering spaces $E \to X$, continuous $h : (M, m_0) \to (X, x_0)$, and $e_0 \in E_{x_0}$, $\widetilde{m}_0 \in \widetilde{M}_{m_0}$, there is a unique continuous $\widetilde{h} : (\widetilde{M}, \widetilde{m}_0) \to (E, e_0)$ over h. (By (ii), $\widetilde{M} \times_X E \to \widetilde{M}$ splits; look at pr_2 on the component of $(\widetilde{m}_0, e_0)$.)

(iv) For a connected Lie group (G, e), show $(\widetilde{G}, \widetilde{e})$ has a unique Lie group structure making $\widetilde{G} \to G$ a Lie group homomorphism. (Hint: apply (iii) to $m : G \times G \to G$ as h, etc.) Show the discrete kernel is naturally isomorphic to $\pi_1(G, e)$ (so $\pi_1(G, e)$ is *commutative*).

(v) Accept $\pi_1(M)^{\mathrm{ab}}$ $(= \mathrm{H}_1(M, \mathbf{Z}))$ is finitely generated for connected compact manifolds. For a connected compact Lie group G with $\# Z_G < \infty$, prove $\deg(\widetilde{G} \to G) < \infty$, so $\widetilde{G}$ is *compact*. (Use (iv) above and 4(iv) in HW8 to uniformly bound finite quotients of $\pi_1(G, e)$.)

4. For compact connected G, let $H \to G/Z_G^0$ be the *compact* universal cover (see Exer. 3).

(i) Show $\mathfrak{h} = \mathfrak{g}/\mathfrak{z}_{\mathfrak{g}}$, $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, and $\mathrm{ad}_{\mathfrak{g}}$ factors through a representation $\mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$.

(ii) Prove semisimplicity of finite-dimensional $\mathbf{R}$-representations of $\mathfrak{h}$ ("unitary" trick, as $\pi_1(H) = 1$). Deduce there is a *unique* $\mathfrak{g}$-equivariant section $s : \mathfrak{h} \to \mathfrak{g}$. Using $H \to G$ lifting s, show G' is closed, $(G')' = G'$, and $Z_G^0 \times G' \to G$ is an isogeny (so $\# Z_G < \infty$ iff $G = G'$!).