Math 210C. Homework 8

1. Let $\{H_i\}$ be a locally finite set of *affine* hyperplanes in $V = \mathbf{R}^n$,, so $H_i = \ell_i^{-1}(c_i)$ for $c_i \in \mathbf{R}$ and $\ell_i \in V^* - \{0\}$. Its *chambers* are connected components of $V - (\cup H_i)$. Pick a chamber K, $x_0 \in K$, and let $\varepsilon_i = \mathrm{sgn}(\ell_i(x_0) - c_i) = \pm 1$, so $x_0 \in H_i^{>0} := \{\varepsilon_i(\ell_i - c_i) > 0\}$.

(i) Prove $K = \bigcap_i H_i^{>0}$ (convex) and such intersection with any signs ε_i is a chamber if $\neq \emptyset$. Use closed segments to show $\overline{K} = \bigcap_i H_i^{\geq 0}$ (evident definition for $H_i^{\geq 0}$). Deduce $\mathrm{int}_V \overline{K} = K$.

(ii) Call H_i a *wall* of K if $\mathrm{int}_{H_i}(H_i \cap \overline{K}) \neq \emptyset$ (so H_i is the only affine hyperplane of V containing $H_i \cap \overline{K}$!). Show the union of the walls of K contains $\partial_V \overline{K}$. (Hint: For $x \in \partial_V \overline{K}$, arrange $x \in \cap H_j$. Find a closed ray L based at $x_0 \in K$ missing all $H_j \cap H_{j'}$ when $j' \neq j$ but meeting some H_j; study $L \cap \overline{K}$.) Deduce $K = \bigcap_{H \in \mathrm{walls}(K)} H^{>0}$ and $\overline{K} = \bigcap_{H \in \mathrm{walls}(K)} H^{\geq 0}$.

2. (i) Read the handout on dual root systems.

(ii) Let G be a connected compact Lie group with $\#Z_G < \infty$ and choose a maximal torus T, so $(\mathrm{X}(T)_{\mathbf{Q}}, \Phi)$ is a root system. Show $[\mathrm{X}_*(T) : \mathbf{Z}\Phi^\vee] < \infty$. Deduce that for $G_a := Z_G(T_a)'$ (SU(2) or SO(3)), we have $\sum_{a \in \Phi} \mathrm{Lie}(G_a) = \mathfrak{g}$ (the key is that $\mathfrak{t}$ is contained in this span).

(iii) For G in (ii), prove $G' := [G, G]$ is a neighborhood of e. Deduce $G = G'$!

3. Let (V, Φ) be a root system over a field k of characteristic 0, and define the *canonical* symmetric bilinear form $(v|v') = \sum_{a \in \Phi} \langle v, a^\vee \rangle \langle v', a^\vee \rangle$ on V. Read dual root system handout.

(i) Prove $(\cdot|\cdot)$ is $W(\Phi)$-invariant, and moreover positive-definite if $k = \mathbf{Q}$.

(ii) Let $V_0 = \mathbf{Q}\Phi$ (so $k \otimes_{\mathbf{Q}} V_0 = V$; see the handout on dual root systems). Show $(\cdot|\cdot)$ is the scalar extension of the analogue $(\cdot|\cdot)_0$ on V_0, so V admits a (canonical) non-degenerate $W(\Phi)$-invariant symmetric bilinear form (so $(V^*, \Phi^\vee) \simeq (V, \Phi')$ via $a^\vee \mapsto a' := 2a/(a|a)$).

(iii) For $\Psi \subset \Phi$ and $V' = k\Psi$, show $(V', \Phi \cap \mathbf{Z}\Psi)$ is a root system with coroots $a^\vee|_{V'}$. Also show $V^{W(\Phi)} = 0$ (hint: consider $W(\Phi)$-equivariant *quotients* of V with trivial $W(\Phi)$-action).

4. Let G be a connected compact Lie group, and T a maximal torus of G; we make no finiteness hypotheses on Z_G. Define $V = \mathrm{X}(T)_{\mathbf{Q}}$.

(i) For $a \in \Phi$ let $a^\vee \in \mathrm{X}_*(T) \subset V^*$ be the coroot arising from $(Z_G(T_a)', T'_a, a|_{T'_a})$ as in the handout on coroots. Show $\mathrm{X}_*(T)_{\mathbf{Q}} \twoheadrightarrow \mathrm{X}_*(T/Z_G)_{\mathbf{Q}} = \mathrm{X}(T/Z_G)^*_{\mathbf{Q}}$ carries $a^\vee$ to the coroot associated to a for $(G/Z_G, T/Z_G)$, and for $b \in \Phi$ prove $b^\vee = a^\vee$ if and only if $b = a$.

(ii) Let $\Phi^\vee$ be the set of coroots for (G, T). Prove $\mathbf{Q}\Phi^\vee$ is a complement to $\mathrm{X}_*(Z_G^0)_{\mathbf{Q}} = \mathrm{X}_*(Z_G)_{\mathbf{Q}}$ in $\mathrm{X}_*(T)_{\mathbf{Q}}$ (equivalently, it maps isomorphically onto $\mathrm{X}_*(T/Z_G)_{\mathbf{Q}} = \mathrm{X}(T/Z_G)^*_{\mathbf{Q}}$) inducing an isomorphism $W(G, T) \simeq W(\Phi(G/Z_G, T/Z_G)) = W(\Phi(G, T))$. Hint: make a $W(G, T)$-invariant positive-definite quadratic form $q : \mathrm{X}(T)_{\mathbf{Q}} \to \mathbf{Q}$.

(iii) In (ii) you showed $\mathrm{X}_*(Z_G)_{\mathbf{Q}}$ has a *canonical* $W(G, T)$-equivariant complement in $\mathrm{X}_*(T)_{\mathbf{Q}}$ (so likewise for $\mathrm{X}(Z_G^0)_{\mathbf{Q}}$ inside $\mathrm{X}(T)_{\mathbf{Q}}$). Prove $\mathrm{X}_*(Z_G)_{\mathbf{Q}} = \mathrm{X}_*(T)_{\mathbf{Q}}^{W(G,T)}$ (see Exercise 3(iii)) and verify this directly for $G = \mathrm{U}(n)$ and T the diagonal torus. Deduce $\mathbf{Z}\Phi^\vee$ has finite index in $\mathrm{X}_*(T)$ if and only if Z_G is finite; see Exercise 2(ii). (Later: index is $\#\pi_1(G)$.)

(iv) Assume Z_G is finite. For any lattice L in V^* (e.g., $\mathbf{Z}\Phi^\vee$), let L' denote the $\mathbf{Z}$-dual $\mathrm{Hom}(L, \mathbf{Z}) \subset \mathrm{Hom}(L, \mathbf{Z})_{\mathbf{Q}} = V^{**} = V$ consisting of $v \in V$ such that $\ell(v) \in \mathbf{Z}$ for all $\ell \in L$. For any isogeny $\pi : \mathcal{G} \to G$ from a connected compact Lie group $\mathcal{G}$, show its maximal torus $\mathcal{T} = \pi^{-1}(T)$ satisfies $\mathbf{Z}\Phi \subset \mathrm{X}(T) \subset \mathrm{X}(\mathcal{T}) \subset (\mathbf{Z}\Phi^\vee)'$ inside V and prove $\#\ker \pi = [\mathrm{X}(\mathcal{T}) : \mathrm{X}(T)]$. Deduce an upper bound on $\#\ker \pi$ determined entirely by (V, Φ). (This will underlie our later proof that $\pi_1(G)$ is *finite* when Z_G is finite.)