Math 210C. Homework 7

1. (i) Let C be a commutative compact Lie group. Prove $C = C^0 \times F$ as Lie groups for a finite F. Deduce $\mathrm{X}(C) := \mathrm{Hom}(C, S^1)$ is a finitely generated $\mathbf{Z}$-module with torsion subgroup $\mathrm{X}(C/C^0)$. Discuss the behavior of $\mathrm{X}(C)$ with respect to direct products in C.

(ii) Let M be a finitely generated $\mathbf{Z}$-module. For any surjection $q : \mathbf{Z}^n \twoheadrightarrow M$, prove $\mathrm{Hom}(M, S^1) \hookrightarrow \mathrm{Hom}(\mathbf{Z}^n, S^1) = (S^1)^n$ has closed image and the resulting compact Lie group structure on $\mathrm{Hom}(M, S^1)$ is independent of q. We call this Lie group the (Pontryagin) *dual* $D(M)$ of M. Discuss the behavior of $D(M)$ with respect to direct products in M.

(iii) Construct natural isomorphisms $M \to \mathrm{X}(D(M))$ and $C \to D(\mathrm{X}(C))$ (so the categories of such M and C are anti-equivalent via inverse functors $D(\cdot)$ and $\mathrm{X}(\cdot)$). Prove that a diagram $M' \to M \to M''$ is a short exact sequence if and only if the same holds for the dual diagram.

2. Let G be compact connected, Z_G its center, T a maximal torus, and $\Phi = \Phi(G, T)$.

(i) Let $Z' \subset Z_G$ be a closed subgroup. Establish a bijection between the sets of maximal tori of G and G/Z', and prove $Z_G/Z' = Z_{G/Z'}$ (so $Z_{G/Z_G} = 1$) and $\mathrm{X}(T/Z') \hookrightarrow \mathrm{X}(T)$ carries $\Phi(G/Z', T/Z')$ onto Φ with $\mathfrak{g}_{\mathbf{C}} \to \mathrm{Lie}(G/Z')_{\mathbf{C}}$ giving isomorphisms between root spaces.

(ii) For the upper unipotent subgroup U of $\mathrm{SL}_3(\mathbf{R})$ show $Z_U \neq 1$ and U/Z_U is commutative.

(iii) Prove $Z_G = \ker \mathrm{Ad}_G|_T = \bigcap_{a\in\Phi} \ker(a)$, and deduce $\mathrm{X}(T)/\mathbf{Z}\Phi \simeq \mathrm{Hom}(Z_G, S^1)$ and that $\# Z_G < \infty$ if and only if Φ spans $\mathrm{X}(T)_{\mathbf{Q}}$, in which case $[\mathrm{X}(T) : \mathbf{Z}\Phi] = \# Z_G$. Use this formula for Z_G to show $Z_{\mathrm{SU}(n)} = \mu_n$ (so $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$ has trivial center).

3. (i) Prove any $g \in \mathrm{SU}(2)$ is $xyx^{-1}y^{-1}$ for $x, y \in \mathrm{SU}(2)$. (The Conjugacy Theorem helps.)

(ii) Using conjugation of the diagonal on upper and lower unipotent subgroups, show $\mathrm{SL}_2(k) = \mathrm{SL}_2(k)'$ for any field k with $|k| > 3$. (Hint: for $u^+(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $u^-(x) = \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$, and $t(\lambda) = \mathrm{diag}(\lambda, 1/\lambda)$, check $t(\lambda) = u^+(1)u^-(\lambda - 1)u^+(-1/\lambda)u^-(\lambda(1-\lambda))$.)

(iii) Show $\mathrm{PGL}_2(k)$ has abelianization $k^\times/(k^\times)^2$ when $|k| > 3$.

4. (i) Let $f : X \to Y$ be a C^∞ map, and $q : Y' \to Y$ a C^∞ submersion. Show that $X' := \{(x, y') \in X \times Y' \mid f(x) = q(y')\}$ (also denoted $X \times_Y Y'$) is a C^∞ closed submanifold of $X \times Y'$, $q' = \mathrm{pr}_1 : X' \to X$ is a C^∞ submersion (called the *pullback* of q along f), and $f' = \mathrm{pr}_2 : X' \to Y'$ is a C^∞ map (called the *pullback* of f along q).

(ii) Let $f : G \to H$ be a map of Lie groups, and $q : H' \to H$ a surjective Lie group homomorphism with kernel K. Prove $G' := G \times_H H'$ with its group structure is a Lie group and $1 \to \{1\} \times K \to G' \to G \to 1$ is exact (called the *pullback extension* of G by K).

(iii) If $K \subseteq Z_{H'}$ (H' is a *central extension* of H by K) then show $\{1\} \times K \subseteq Z_{G'}$.

5. (i) For compact G, prove the complete reducibility of finite-dimensional continuous $\mathbf{R}$-linear G-representations.

(ii) Prove the irreducible $\mathbf{R}$-representations of S^1 are $\mathbf{R}$ and n-fold rotations of $\mathbf{R}^2$ ($n \geq 1$).

(iii) Let V be an irreducible $\mathbf{C}$-linear representation of compact G, $\chi = \chi_V$. Make a G-equivariant $\mathbf{C}$-linear $f : \overline{V} \simeq V^*$ with $\overline{f}^* = f$, and show $\chi(G) \subset \mathbf{R}$ if and only if there is a G-equivariant $L : V \simeq V^*$, with L *unique up to* $\mathbf{C}^\times$*-scaling* if it exists. In such cases, show V is defined over $\mathbf{R}$ if and only if $L = L^*$ (via $V = V^{**}$). Deduce V is defined over $\mathbf{R}$ if and only if $(\mathrm{Sym}^2(V)^*)^G = \mathrm{SymBil}_G(V \times V, \mathbf{C})$ is nonzero, and that then $(\wedge^2(V)^*)^G = 0$. Hint: $\mathrm{Hom}(V^*, V) = V^{\otimes 2} = \mathrm{Sym}^2(V) \oplus \wedge^2(V)$. (By the proof of Prop. 39 in §13.2 of Serre's book on finite group representations, it's the same as "$\int_G \chi(g^2)\mathrm{d}g = 1$", where $\int_G \mathrm{d}g = 1$.)