Math 210C. Homework 6

1. (i) Let G be a *unimodular* Lie group, and K and B closed subgroups such that K is unimodular and multiplication $K \times B \to G$ is a diffeomorphism. (HW1 and HW2 give some such G for which unimodularity admits a unified proof via algebraic-group techniques.) Let $\mathrm{d}k$ be a Haar measure on K and $\mathrm{d}b$ be a *right* Haar measure on B. Let $\mathrm{d}g$ be a Haar measure on G. Prove the product measure $\mathrm{d}k\,\mathrm{d}b$ is equal to $f(g)\mathrm{d}g$ for some $f \in C^\infty(G)$, and then use invariance properties of the measures to deduce f is constant!

(ii) Let $U \subset G := \mathrm{SL}_n(\mathbf{R})$ be the subgroup of upper-triangular unipotents, $A \subset G$ the "positive" diagonal subgroup, and $K = \mathrm{SO}(n)$. Prove $\mathrm{d}u := \prod_{i<j} \mathrm{d}u_{ij}$ is a right Haar measure on U, $\mathrm{d}a := \prod_{i<n} \mathrm{d}a_i/a_i$ is a Haar measure on A, and $\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}u$ is a Haar measure on G for a Haar measure $\mathrm{d}k$ on K. (Proving invariance of $\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}u$ by bare hands is a mess!)

(iii) Make (ii) explicit for $\mathrm{SL}_2(\mathbf{R})$ via the equality $\mathrm{SO}(2) = S^1$.

2. By IV, 3.13 (whose proof you should read!), for $G = \mathrm{U}(n)$ and the diagonal maximal torus $T = (S^1)^n$, the injection $R(G) \hookrightarrow R(T)^W$ is an equality. The idea is to compute $R(T)^W$ as a ring and show that characters of specific G-representations (in fact, the first n exterior powers of the standard representation of $G = \mathrm{U}(n)$ and the dual of the nth) generate $R(T)^W$.

Explicitly, $R(T) = \mathbf{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] = \mathbf{Z}[z_1, \dots, z_n, 1/\prod z_i]$ on which $W = S_n$ acts by usual permutations, so $R(T)^W = \mathbf{Z}[z_1, \dots, z_n]^W[1/\prod z_i]$. Letting $\sigma_j = \sigma_j(z_1, \dots, z_n)$ be the jth symmetric polynomial ($1 \le j \le n$), so $\prod z_i = \sigma_n$, (ii) below computes $R(T)^W$.

(i) Using basic properties of transcendence degree, sketch the Galois-theoretic proof that $\sigma_1, \dots, \sigma_n$ are algebraically independent in $k[z_1, \dots, z_n]$ and $k(\sigma_1, \dots, \sigma_n) = k(z_1, \dots, z_n)^{S_n}$ for any field k. Using integrality considerations, deduce $k[z_1, \dots, z_n]^{S_n} = k[\sigma_1, \dots, \sigma_n]$.

(ii) Prove $A[z_1, \dots, z_n]^{S_n} = A[\sigma_1, \dots, \sigma_n]$ for $A = \mathbf{Z}, \mathbf{Z}/m\mathbf{Z}$ ($m > 0$).

(iii) For any commutative ring A, prove $\sigma_1, \dots, \sigma_n$ are algebraically independent over A inside $A[z_1, \dots, z_n]$ and that $A[\sigma_1, \dots, \sigma_n] = A[z_1, \dots, z_n]^{S_n}$. (Hint: the ring structure of A is irrelevant (!), so reduce to considering finitely generated $\mathbf{Z}$-modules A and apply (ii).)

3. IV, 3.14, Exercise 3.

4. (i) IV, 3.14, Exercise 4.

(ii) Using $R(\mathrm{SU}(2)) \simeq \mathbf{Z}[\sigma_1, \sigma_2]/(\sigma_2 - 1) = \mathbf{Z}[\sigma_1]$, describe the character χ_m of V_m as a polynomial in σ_1 for $m \le 5$. (For example, $t^2 + 1 + t^{-2} = (t + t^{-1})^2 - 1$, so $\chi_2 = \sigma_1^2 - 1$.)

5. For $\mathbf{C}$-valued continuous class functions f on $\mathrm{SU}(2)$ and the volume-1 measure $\mathrm{d}g$, prove

$$\int_{\mathrm{SU}(2)} f(g)\mathrm{d}g = 2\int_0^1 f(t(\theta))\sin^2(2\pi\theta)\mathrm{d}\theta$$

where $t(\theta) = \mathrm{diag}(e^{2\pi i\theta}, e^{-2\pi i\theta})$. (This agrees with II, 5.2 via change of variable. As a check on the normalizations, you can handle $f = 1$ by hand.) Also find an analogue for $\mathrm{SO}(3)$.

6. Let the group $\mathrm{SU}(2)$ of norm-1 quaternions act on $\mathbf{H} = \mathbf{R}^4$ in two commuting ways: left-multiplication ($v \mapsto uv$ for $v \in \mathbf{H}$) and right-multiplication through inversion ($v \mapsto vu^{-1}$ for $v \in \mathbf{H}$). These actions define a Lie group homomorphism $f : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{GL}_4(\mathbf{R})$.

(i) Check $\ker f$ is the diagonally embedded $\mu_2 = \{\pm 1\}$, and that f lands inside $\mathrm{SO}(4)$.

(ii) Prove that the map $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2 \to \mathrm{SO}(4)$ is a Lie group isomorphism.