Math 210C. Homework 5

1. Read the handout on Weyl groups, and give the justification (left for the reader to do) in the proof of Proposition 4.1 that the action of $n \in N_G(S)$ on $\mathrm{X}_*(S)_{\mathbf{R}}$ via f_S is carried by the canonical $\mathrm{X}_*(S)_{\mathbf{R}} \simeq \mathrm{Lie}(S)$ over to $\mathrm{Ad}_G(n)$ restricted to the subspace $\mathrm{Lie}(S) \subset \mathfrak{g}$.

2. (i) For a short exact sequence $0 \to V' \to V \to V'' \to 0$ of finite-dimensional vector spaces, construct a *canonical* isomorphism $\det(V') \otimes \det(V'') \simeq \det(V)$ with top exterior powers.

(ii) For a submersion $f : X \to Y$ and $y = f(x)$, prove short exactness of $0 \to \mathrm{T}_x(f^{-1}(y)) \to \mathrm{T}_x(X) \to \mathrm{T}_y(Y) \to 0$ and deduce $\det(\mathrm{T}_x^*(X)) \simeq \det(\mathrm{T}_x^*(f^{-1}(y))) \otimes \det(\mathrm{T}_y^*(Y))$ naturally.

(iii) For a Lie group G and closed subgroup H, left translations ℓ_h on G/H ($h \in H$) fix $\overline{e} = H$. Show $\mathrm{Ad}_{G/H}^* : H \to \mathrm{GL}(\mathrm{T}_{\overline{e}}^*(G/H))$ via $h \mapsto \mathrm{d}(\ell_{h^{-1}})(\overline{e})^*$ is a C^∞ homomorphism.

(iv) Show there is a left G-invariant top-degree $\overline{\omega} \neq 0$ on G/H if and only if $\det \mathrm{Ad}_{G/H}^* : H \to \mathbf{R}^\times$ is trivial, and that such $\overline{\omega}$ is C^∞. In such cases, for left-invariant top-degree nonzero ω on G and η on H choose the $\overline{\omega}$ so $\omega(e) = \eta(e) \otimes \overline{\omega}(\overline{e})$ via (ii), and for $f \in C_c(G)$ and the associated left-invariant measures $\mathrm{d}g$, $\mathrm{d}h$, and $\mathrm{d}\overline{g}$, show $g \mapsto \int_H f(gh)\mathrm{d}h$ arises from $C_c(G/H)$ and that $\int_{G/H}(\int_H f(gh)\mathrm{d}h)\mathrm{d}\overline{g} = \int_G f(g)\mathrm{d}g$ (so for compact G, $\int_{G/H} \mathrm{d}\overline{g} = 1$ if $\int_G \mathrm{d}g, \int_H \mathrm{d}h = 1$). This generalizes and clarifies 5.15 in Ch. I (and avoids 5.13 in Ch. I).

3. For topological spaces B and F ($\neq \emptyset$ to avoid silliness), a continuous $\pi : E \to B$ is an F-*fiber bundle* if there is an open cover $\{U_\alpha\}$ of B and homeomorphisms $f_\alpha : \pi^{-1}(U_\alpha) \simeq F \times U_\alpha$ "over U_α" (i.e., $\mathrm{pr}_2 \circ f_\alpha = \pi|_{\pi^{-1}(U_\alpha)}$). We say π is *trivial* if $E \simeq F \times B$ over B. A *section* to π is a continuous $s : B \to E$ with $\pi \circ s = \mathrm{id}_B$; note that π is not trivial if it has no section.

(i) Show covering spaces of connected B are the fiber bundles over B with discrete fibers.

(ii) Show the Möbius strip $\pi : M \to S^1$ (quotient of $[0,1] \times \mathbf{R}$ modulo $(0,x) \sim (1,-x)$, with evident π via $[0,1]/(0 \sim 1) \simeq S^1$) is an $\mathbf{R}$-fiber bundle. Using the Intermediate Value Theorem, show the (continuous) "zero section" e induced by $t \mapsto (t,0)$ on $[0,1]$ meets *every* section s (i.e., $s(b) = e(b)$ for some $b \in S^1$). Deduce π is not trivial.

(iii) Fix $n \geq 2$. Let $S = \partial_{\mathbf{R}^n}([0,1]^n) - \{1\} \times (0,1)^{n-1}$ (box with missing face). For a fiber bundle $\pi : E \to B$ and continuous $f : [0,1]^n \to B$, show any lift $\widetilde{f_S} : S \to E$ of $f|_S$ extends to a lift $\widetilde{f} : [0,1]^n \to E$ of f. (For $\{U_\alpha\}$ trivializing π, partition $[0,1]^n$ into cubes landing in U_α's. Systematic use of cubes reduces to trivial π, and $[0,1]^n$ retracts onto S since $n \geq 2$.)

(iv) If $\pi : E \to B$ is a fiber bundle with F path-connected, for continuous $\gamma : S^1 \to B$ with $\gamma(1) = b$ and $e \in E_b := \pi^{-1}(b)$ construct a lift $\widetilde{\gamma} : S^1 \to E$ carrying 1 to e.

(v) If $\pi_0(B), \pi_0(F) = 1$ show $\pi_0(E) = 1$, $\pi_1(E_b, e) \to \pi_1(E, e) \to \pi_1(B, b) \to 1$ is exact.

4. (i) Let G be a Lie group, H a closed subgroup. Show $G \to G/H$ is an H-fiber bundle, trivial if it admits a section. Deduce $\mathrm{SO}(n+1) \to \mathrm{SO}(n+1)/\mathrm{SO}(n) \simeq S^n$ is trivial if and only if S^n is parallelizable. (Lie groups parallelize via left-invariant vector fields, so S^1 and $S^3 \simeq \mathrm{SU}(2)$ do, as does S^7 using octonions. By algebraic topology, no other S^n does.)

(ii) Show $\pi_1(\mathrm{SU}(n)), \pi_1(\mathrm{Sp}(n)) = 1$ for $n \geq 1$ and deduce $\pi_1(\mathrm{SL}_n(\mathbf{C})) = 1$.

(iii) For a connected Lie group G and discrete normal (hence central!) subgroup Γ make a surjective $\pi_1(G/\Gamma) \twoheadrightarrow \Gamma$ and show it is an isomorphism if $\pi_1(G) = 1$. Deduce $\pi_1(\mathrm{SO}(3)) = \mathbf{Z}/2\mathbf{Z}$ and $|\pi_1(\mathrm{SO}(n))| \leq 2$ for $n > 3$ (so $|\pi_1(\mathrm{SL}_n(\mathbf{R}))| \leq 2$ for $n \geq 3$, whereas $\pi_1(\mathrm{SL}_2(\mathbf{R})) = \pi_1(\mathrm{SO}(2)) = \mathbf{Z}$); later we'll use spin groups to prove $\pi_1(\mathrm{SO}(n)) = \mathbf{Z}/2\mathbf{Z}$ for $n \geq 3$.