Math 210C. Homework 4

1. Read §4 in the handout on local and global Frobenius theorems (skip the proof of Theorem 4.3 if you haven't read the earlier sections of the handout). Note the discussion at the end concerning bijectivity of $\mathrm{Hom}(G, G') \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}')$ for connected G with $\pi_1(G) = 1$.

2. (i) For a Lie group G and left-invariant top-degree ω, prove $(r_{g^{-1}})^*(\omega) = (\det \mathrm{Ad}_G(g))\omega$ (so $|\det \mathrm{Ad}_G|$ is the modulus character Δ_G). Deduce ω is right-invariant for $\mathrm{GL}_n(\mathbf{R}), \mathrm{SL}_n(\mathbf{R})$.

(iii) Let $G \subset \mathrm{GL}_2(\mathbf{R})$ be the "$ax + b$ group" consisting of invertible matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$. Show that G is the semidirect product $\mathbf{R} \rtimes \mathbf{R}^\times$ defined via the $\mathbf{R}^\times$-action on $\mathbf{R}$ given by $a.b = ab$ ($a \in \mathbf{R}^\times$, $b \in \mathbf{R}$). Check that the 2-form $(\mathrm{d}a \wedge \mathrm{d}b)/a^2$ on G is left-invariant whereas the 2-form $(\mathrm{d}a \wedge \mathrm{d}b)/a$ on G is right-invariant. What is $\Delta_G(a, b)$?

3. (i) Let V be a finite-dimensional $\mathbf{R}$-vector space and L a discrete subgroup. For $G = V/L$, identify $\mathfrak{g}$ with $\mathrm{Lie}(V) = V$ in the natural manner. Show that $\exp_G : V = \mathfrak{g} \to G = V/L$ is thereby identified with the canonical projection $v \mapsto v \bmod L$. Compute $\alpha_v : \mathbf{R} \to G$.

(ii) Show that $H = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $X = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and $Y = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$ are an $\mathbf{R}$-basis of $\mathfrak{su}(2) \subset \mathfrak{sl}_2(\mathbf{C})$ and that $\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{su}(2) \to \mathfrak{sl}_2(\mathbf{C})$ is an isomorphism. Check that $[H, X] = 2Y$, $[H, Y] = -2X$, and $[X, Y] = 2H$, and compute the 1-parameter subgroups $\alpha_H, \alpha_X, \alpha_Y$ in $\mathrm{SU}(2)$.

(iii) Show that the elements $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ are an $\mathbf{R}$-basis of $\mathfrak{sl}_2(\mathbf{R})$. Check $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$, and compute the 1-parameter subgroups $\alpha_H, \alpha_E, \alpha_F$ in $\mathrm{SL}_2(\mathbf{R})$.

4. (i) For $n \geq 1$, show any $g \in \mathrm{U}(n)$ has all eigenvalues in S^1, and can be $\mathrm{U}(n)$-conjugated into the "diagonal torus". (Hint: hermitian Gramm-Schmidt.) Do the same for $\mathrm{SU}(n)$.

(ii) Prove that any representation of $\mathrm{SU}(n)$ is determined up to isomorphism by its restriction to a representation of the diagonal torus T. For $n = 2$, show that each $\mathrm{diag}(t, 1/t) \in T$ ($t \in S^1$) is $\mathrm{SU}(2)$-conjugate to its inverse, and deduce that the T-restriction of any $\mathrm{SU}(2)$-representation has character of the form $c_0 + \sum_{m>0} c_m(t^m + t^{-m})$ for integers $c_j \geq 0$.

(iii) For $m \geq 0$, let V_m be the irreducible representation of $\mathrm{SU}(2)$ on the $(m+1)$-dimensional space of degree-m homogenous polynomials via $(g.P)(z_1, z_2) = P((z_1, z_2)g)$ (viewing (z_1, z_2) as a 1×2 matrix). Show that the character of V_m has T-restriction $\sum_{-m \leq j \leq m,\, j \equiv m\,(2)} t^j$.

(iv) For any representation W of $\mathrm{SU}(2)$, show there exist direct sums V and V' among the V_m's so that $V \oplus W \simeq V'$. Deduce that every irreducible representation of $\mathrm{SU}(2)$ is a V_m!

5. For $\ell \geq 0$, let H_ℓ be the space of degree-ℓ homogenous $\mathbf{C}$-valued harmonic polynomials in x_1, x_2, x_3. This is naturally a representation of $\mathrm{SO}(3) = \mathrm{SU}(2)/\{1, -1\}$ and hence of $\mathrm{SU}(2)$.

(i) Show that every element in $\mathrm{SO}(3)$ can be conjugated into the "standard" 1-dimensional torus T consisting of rotations around the line spanned by the first standard basis vector.

(ii) Read Lemma 5.6 in Chapter II and find an eigenbasis of H_2 for the action of the torus in (i), computing the eigencharacters of T that arise in this way.

(iii) Show $(x_2 + ix_3)^\ell \in H_\ell$ and that it's a T-eigenvector with eigencharacter $t \mapsto t^{-\ell}$ for $t \in T = S^1$. Deduce (via 4(iv)) that the H_ℓ's are the irreducible representations of $\mathrm{SO}(3)$.

(iv) Establish the *Clebsch-Gordan formula* for $\mathrm{SO}(3)$-representations: for $1 \leq k \leq \ell$,

$$H_k \otimes H_\ell \simeq H_{\ell-k} \oplus H_{\ell-k+1} \oplus \cdots \oplus H_{\ell+k}.$$

Make this explicit for $k = \ell = 1$.