Math 210C. Homework 3

1. Let G be a Lie group and H a closed Lie subgroup. This exercise puts the computations in (2.15)–(2.19) in Chapter I into a wider setting.

(i) For $v \in \mathfrak{g}$, show that $v \in \mathfrak{h}$ if and only if $\exp_G(tv) \in H$ for $t \in \mathbf{R}$ with $|t|$ small.

(ii) Using (i) with $G = \mathrm{GL}_n(\mathbf{R})$, show that inside $\mathfrak{gl}_n(\mathbf{R}) = \mathrm{Mat}_n(\mathbf{R})$, $\mathfrak{sl}_n(\mathbf{R})$ is the traceless matrices, $\mathfrak{o}(n) = \mathfrak{so}(n)$ is the skew-symmetric matrices, and $\mathfrak{sp}_{2n}(\mathbf{R})$ is the space of $n \times n$ matrices A satisfying $J \cdot {}^{\top}A \cdot J = A$ where $J = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$.

(iii) Using $\mathrm{GL}_n(\mathbf{C})$ or $\mathrm{GL}_n(\mathbf{H})$, show that $\mathfrak{u}(n) \subseteq \mathrm{Mat}_n(\mathbf{C})$ is the space of skew-Hermitian A (i.e., ${}^{\top}\overline{A} = -A$) and $\mathfrak{sp}(n) \subset \mathrm{Mat}_n(\mathbf{H})$ is the space of skew-Hermitian quaternionic A.

2. Consider the group $\mathrm{SU}(2)$ of norm-1 quaternions (i.e., $h\overline{h} = 1$).

(i) Show $\mathfrak{su}(2) \hookrightarrow \mathrm{Lie}(\mathbf{H}^{\times}) = \mathbf{H}$ has image $\mathbf{H}_0 := \mathbf{R}i + \mathbf{R}j + \mathbf{R}k$. Deduce that $\mathrm{Ad}_{\mathrm{SU}(2)}$ defines $f : \mathrm{SU}(2) \to \mathrm{SO}(3)$ identifying $\mathrm{SU}(2)/\{1, -1\}$ with $\mathrm{SO}(3)$ (hence $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$) and carrying $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$ to counterclockwise rotation by 2θ around the i-axis in $\mathbf{H}_0$.

(ii) Show that multiplying $\mathfrak{su}(2) \simeq \mathbf{H}_0$ in (i) against $1/2$ identifies $\mathfrak{su}(2)$ with $\mathbf{R}^3$ carrying the Lie bracket to the cross product. Relate $\mathrm{Lie}(f)$ to the explicit φ in Exercise 8 of I.2.

3. This exercise addresses a step in the proof of the Local Formula for $[X, Y]_G$ (Thm. 2.1 in the adjoint representation handout), expanding on Exer. 9 in I.2. Let $U \subset \mathbf{R}^2$ be open containing $(-c, c) \times (-c, c)$ with $c > 0$, and $a : U \to M$ a C^{∞} map with $a(s, 0) = m_0 \in M$ when $|s| < c$. For such s, $a_s : t \mapsto a(s, t)$ has velocity at $t = 0$ denoted $(\partial_t a)(s, 0) \in \mathrm{T}_{m_0}(M)$.

(i) Prove that the map $(-c, c) \to \mathrm{T}_{m_0}(M)$ defined by $s \mapsto (\partial_t a)(s, 0)$ is C^{∞}.

(ii) For the parametric curve in $\mathrm{T}_{m_0}(M)$ given by the map in (i), show that its velocity vector in $\mathrm{T}_{m_0}(M)$ at $s = 0$ has associated point derivation on C^{∞} functions near m_0 given by $\varphi \mapsto (\partial_s \partial_t(\varphi \circ a))(0, 0)$. (Hint: using coordinates on M, reduce to the special case $M = \mathbf{R}^n$ with $m_0 = 0$ and φ either constant or linear on M.)

4. Inside each of the connected compact Lie groups $\mathrm{SO}(n)$ ($n \geq 2$), $\mathrm{U}(n)$ ($n \geq 1$), and $\mathrm{Sp}(n)$ ($n \geq 1$), find a torus $(S^1)^r$ (with $r = \lfloor n/2 \rfloor, n, n$ respectively) equal to its own centralizer and hence not inside any strictly larger torus. Hint: look near the diagonal.

5. (i) Show that if $f : M' \to M$ is a surjective submersion of C^{∞} manifolds then a C^{∞} map $g : M' \to N$ that factors through f set-theoretically does so smoothly (i.e., the associated $M \to N$ is C^{∞}). Deduce that if H is a closed subgroup of a Lie group G then $G \to G/H$ is universal for C^{∞} maps $G \to M$ that are invariant under the right H-action on G.

(ii) For closed subgroups H and H' in respective Lie groups G and G', prove that the natural map $(G \times G')/(H \times H') \to (G/H) \times (G'/H')$ is a diffeomorphism. Deduce that the action map $G' \times (G'/H') \to G'/H'$ is C^{∞}, and that if H is normal in G then the group law on G/H is C^{∞} (so G/H is then a Lie group).

(iii) Using Sard's theorem (see Guillemin & Pollack or Wikipedia), prove that if $f : M' \to M$ is a surjective map of C^{∞} manifolds (always understood to be second countable!!) then it is a submersion at *some* (in fact "most") points of M'. Deduce that if $G \times X \to X$ is a C^{∞} transitive action on a C^{∞} manifold X then for any $x \in X$ and its closed stabilizer subgroup $G_x \subseteq G$, the orbit map $G \to X$ through x induces a *diffeomorphism* $G/G_x \to X$. In particular, for a Lie group homomorphism $f : G \to G'$, show that if f is surjective then it is a submersion and that if f is bijective then it is a diffeomorphism.