Math 210C. Homework 2

1. Consider $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_{2n}(k)$ for a field k (with $n \geq 1$). Let ψ_n be the standard symplectic form on $V = k^{2n}$. Show that if g preserves ψ_n on V (i.e., $\psi_n(gv, gw) = \psi_n(v, w)$ for all $v, w \in V$) then g is invertible, and that g preserves ψ_n if and only if

$$^\top a \cdot c = {}^\top c \cdot a, \ \ ^\top b \cdot d = {}^\top d \cdot b, \ \ ^\top a \cdot d - {}^\top c \cdot b = 1_n.$$

2 (i) Let $V = \mathbf{R}^{2n}$ and let $\mathrm{Alt}(V)$ be the vector space of alternating bilinear forms $V \times V \to \mathbf{R}$. Define the map $f : \mathrm{End}(V) \to \mathrm{Alt}(V)$ via $A \mapsto \psi_n \circ (A \times A)$. Using the automatic invertibility in Exercise 1, apply the submersion theorem technique to the fiber $\mathrm{Sp}_{2n}(\mathbf{R})$ over $\psi_n \in \mathrm{Alt}(V)$ to show that the closed subset $\mathrm{Sp}_{2n}(\mathbf{R}) \subset \mathrm{Mat}_{2n}(\mathbf{R})$ is a C^∞-submanifold, so $\mathrm{Sp}_{2n}(\mathbf{R})$ is a Lie group. (The same method works for $\mathrm{Sp}_{2n}(\mathbf{C}) \subset \mathrm{Mat}_{2n}(\mathbf{C})$.)

(ii) Let QH_n denote the $\mathbf{R}$-vector space of (not necessarily positive-definite) quaternionic hermitian forms on $\mathbf{H}^n$. Let I_n denote the form $(v, w) \mapsto \sum v_r \overline{w}_r$ in QH_n. Use the map $f : \mathrm{End}_{\mathbf{H}}(\mathbf{H}^n) \to \mathrm{QH}_n$ defined by $f(A) = I_n \circ (A \times A)$ to show that $\mathrm{Sp}(n)$ is a closed C^∞ submanifold of $\mathrm{End}_{\mathbf{H}}(\mathbf{H}^n) \subset \mathrm{Mat}_{2n}(\mathbf{C})$; deduce it is also a closed C^∞-submanifold of $\mathrm{U}(2n)$.

3. Recall from class that $\mathrm{SO}(n)$ is connected for all $n > 0$, so $\mathrm{SL}_n(\mathbf{R})$ is connected due to the diffeomorphism $\mathrm{SL}_n(\mathbf{R}) \simeq \mathrm{SO}(n) \times V$ for a vector space V (proved in Exercise 6 of HW1). This exercise extends the technique to some other Lie groups.

(i) If G is a Lie group and H is a connected closed Lie subgroup such that G/H (with its quotient topology) is connected, show that G is connected.

(ii) For $n > 0$, show that $\mathrm{SU}(n)$ acts transitively on the unit sphere S^{2n-1} in $\mathbf{C}^{2n}$ with stabilizer $\mathrm{SU}(n-1)$ (where $\mathrm{SU}(0) := 1$), and deduce that $\mathrm{SU}(n)$ is connected for all $n > 0$. Why does it follow that $\mathrm{GL}_n(\mathbf{C})$ is connected?

(iii) Adapt the method in (ii) to prove the connectedness of $\mathrm{Sp}(n)$ for all $n > 0$ by considering the unit sphere S^{4n-1} in $\mathbf{H}^n$. (The group $\mathrm{Sp}_{2n}(\mathbf{R})$ is connected, but a proof in the above spirit requires going beyond spheres.)

4. (i) Let G be a Lie group, and choose $v \in \mathrm{T}_e(G)$. Prove rigorously that the unique left-invariant set-theoretic vector field $g \mapsto \mathrm{d}\ell_g(v) \in \mathrm{T}_g(G)$ extending v at e is C^∞.

(ii) Let $G = \mathrm{GL}_n(\mathbf{R})$ viewed as an open submanifold of the vector space $\mathrm{Mat}_n(\mathbf{R})$ (so canonically $\mathrm{Lie}(G) = \mathrm{T}_{1_n}(\mathrm{Mat}_n(\mathbf{R})) = \mathrm{Mat}_n(\mathbf{R})$). Show that the unique left-invariant vector field on G extending $\partial_{x_{ij}}|_e$ at the identity point $e := 1_n \in G$ is $\sum_k x_{ki} \partial_{x_{kj}}$ on G. Using this, show that the "Lie bracket" on $\mathrm{Lie}(G) = \mathrm{Mat}_n(\mathbf{R})$ is the commutator $[A, B] = AB - BA$.

(iii) For $0 \leq i \leq \dim(G)$, a C^∞ differential i-form ω on G is *left-invariant* if $\ell_g^*(\omega) = \omega$ for all $g \in G$. Show it is equivalent that the element $\omega(g) \in \wedge^i(\mathrm{T}_g^*(G)) = (\wedge^i \mathrm{T}_g(G))^*$ satisfies $\omega(g)(v_1, \dots, v_i) = \omega(e)(\mathrm{d}\ell_{g^{-1}}(g)(v_1), \dots, \mathrm{d}\ell_{g^{-1}}(g)(v_i))$ for all $v_1, \dots, v_i \in \mathrm{T}_g(G)$, $g \in G$.

(iv) Show that every $\omega_0 \in \wedge^i(\mathrm{T}_e^*(G))$ uniquely extends to a left-invariant C^∞ differential i-form ω on G (hint: use the formula in (iii)), and for $G = \mathrm{GL}_n(\mathbf{R})$ show the left-invariance of the n^2-form

$$\frac{\mathrm{d}x_{11} \wedge \cdots \wedge \mathrm{d}x_{nn}}{\det(x_{ij})^n}.$$