Math 210C. Homework 10

1. Let G be a connected compact Lie group, and $T \subset G$ a maximal torus.

(i) If G is semisimple and $\Phi = \Phi(G,T)$ is irreducible, prove $\mathrm{Lie}(G)_{\mathbf{C}}$ is G-irreducible. (Hint: Show a nonzero G-subrepresentation V is spanned by V^T and T-root lines. Using that $N_G(T)/T = W(\Phi)$ and $\mathrm{Lie}(T) = \mathrm{X}_*(T)_{\mathbf{R}}$, via the end of Exercise 1(ii) in HW9 deduce $V \supset \mathfrak{t}_{\mathbf{C}} \oplus (\mathfrak{g}_{\mathbf{C}})_b \oplus (\mathfrak{g}_{\mathbf{C}})_{-b}$ for some $b \in \Phi$, so V contains the coroot line in $\mathfrak{t}_{\mathbf{C}}$ for *every* $a \in \Phi$.)

(ii) Using Theorem 4.4 in Chapter III, prove that every finite-dimensional representation of G over $\mathbf{C}$ trivial on Z_G occurs inside a positive tensor power of the adjoint representation.

2. Let T be a maximal torus in a connected compact Lie group G, $r = \dim T$ (the "rank" of G). Recall that $t \in T$ is *regular* if $t^a \neq 1$ for all $a \in \Phi(G,T)$.

(i) Prove that $t \in T$ is regular if and only if T is the unique maximal torus of G containing t. (Hint: Study the T-action on the Lie algebra of $Z_G(t)^0$.) Describe these elements explicitly inside the "standard" maximal torus of $\mathrm{Sp}(n)$ ($n \geq 1$), $\mathrm{SO}(2m)$ ($m \geq 2$), and $\mathrm{SO}(2m+1)$.

(iii) Define $g \in G$ to be *regular* if g lies in a unique maximal torus. If $g \in T$, prove this is equivalent to $g^a \neq 1$ for all $a \in \Phi(G,T)$. In general, show the characteristic polynomial of $\mathrm{Ad}_G(g)$ on $\mathfrak{g}$ is $(X-1)^r p_G(X)$ where p_G is monic of degree $\dim G - r$ with $p_G(0) = (-1)^r$ and coefficients smooth in g, and $p_G(1) \neq 0$ if and only if g is regular. Deduce that the regular locus is *open and non-empty*, and in $\mathrm{SU}(n)$ consists of g with no repeated eigenvalues.

3. The *Killing form* of a connected Lie group G is the symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ defined by $\kappa(X,Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$. (This is *not* functorial in G; think about it.)

(i) For $G = \mathrm{SL}_n(\mathbf{R})$, prove $\kappa(X,Y) = 2n\mathrm{Tr}(XY)$. (Hint: One can compute with bases. Or instead, show that $\mathrm{Ad}_{\mathrm{SL}_n(\mathbf{R})}$ is absolutely irreducible on $\mathfrak{sl}_n(\mathbf{R})$, and deduce that the space of $\mathbf{R}$-valued $\mathrm{SL}_n(\mathbf{R})$-equivariant symmetric bilinear forms on $\mathfrak{sl}_n(\mathbf{R})$ is at most 1-dimensional.)

(ii) Assume G is semisimple and compact. Identifying S^1 with $\mathbf{R}/\mathbf{Z}$ via $e^{2\pi i\theta}$, prove for a coroot $b^\vee : S^1 \to T$ that the adjoint action of $\mathrm{Lie}(b^\vee)(\partial_\theta|_1) \in \mathfrak{t}$ vanishes on $\mathfrak{t}_{\mathbf{C}}$ and on $(\mathfrak{g}_{\mathbf{C}})_a$ is multiplication by $2\pi i\langle a, b^\vee\rangle$. Deduce that κ is *negative-definite*. (Hint: show every $X \in \mathfrak{g}$ lies in $\mathfrak{t}$ for a suitable choice of T.)

Remark. Conversely, if $\#Z_G < \infty$, $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, and κ is (negative-)definite then G is compact. An idea to prove this is to use that the map $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ with finite kernel Z_G factors through the compact $\mathrm{SO}(q)$ where $q(X) = \pm\kappa(X,X)$, but closedness of its image is not at all clear! One can overcome this topological problem either by using the theory of linear algebraic groups (over $\mathbf{R}$) or by using curvature results in differential geometry.

4. (extra credit exercise, uses Lecture 27 material!)

(i) Via the Weyl dimension formula and inspecting highest weights, prove the fundamental representations of $\mathrm{SU}(n)$ ($n \geq 2$) are $\wedge^j(V)$ for $1 \leq j \leq n-1$ and the standard $V = \mathbf{C}^n$.

(ii) Let $\{a,b\}$ be a basis of the root system for $(\mathrm{Sp}(2),T)$ with a short and b long. Using $\mathrm{Sp}(2) \subset \mathrm{Sp}_4(\mathbf{C})$ (see the end of 1.12 in Chapter I) to define the standard 4-dimensional $\mathbf{C}$-linear representation V_4 of $\mathrm{Sp}(2)$, deduce via the Weyl dimension formula that $V_4 = V_{a+b/2}$. Show V_{a+b} is the 5-dimensional quotient of $\wedge^2(V_4)$ orthogonal to the line in $\wedge^2(V_4^*) = \wedge^2(V_4)^*$ spanned by the standard symplectic form on V_4 from the definition of $\mathrm{Sp}_4(\mathbf{C})$.

(iii) With notation as in (ii), prove that V_{2a+b} is the 10-dimensional $\mathrm{Sym}^2(V_4)$ and $V_{3a+(3/2)b}$ is the 20-dimensional $\mathrm{Sym}^3(V_4)$.