Math 210C. Homework 1

1. Read the handout on quaternions.

2. Exercise 2 in I.1 (show that $\mathrm{O}(n)$ is a closed submanifold of $\mathrm{GL}_n(\mathbf{R})$). Hint: to prove $f : \mathrm{Mat}_n(\mathbf{R}) \to S$ given by $A \mapsto A^{\mathrm{T}}A$ has surjective differential at all $M \in \mathrm{O}(n)$, prove $\mathrm{d}f(M) : \mathrm{Mat}_n(\mathbf{R}) \to S$ is $N \mapsto M^T \cdot N + N^T \cdot M$ by expanding out $f(A+\varepsilon)$ for $\varepsilon = (\varepsilon_{ij})$ (and note that $N^T \cdot M$ is the transpose of $M^T \cdot N$).

Apply the same method for $\mathrm{SL}_n(\mathbf{R})$ using $\det : \mathrm{GL}_n(\mathbf{R}) \to \mathbf{R}^\times$ and for $\mathrm{SU}(n)$ using $\det : \mathrm{U}(n) \to S^1$.

3. (i) Show that if a subgroup Γ of a Lie group G has the discrete topology as its subspace topology then it is necessary closed in G. (Hint: consider $\gamma'\gamma^{-1}$ for suitable $\gamma, \gamma' \in \Gamma$.) These are called the *discrete* subgroups of G. (For example, $\mathbf{Z}^n$ is discrete in $\mathbf{R}^n$ and $\mathrm{GL}_n(\mathbf{Z})$ is discrete in $\mathrm{GL}_n(\mathbf{R})$.)

(ii) To appreciate the significance of (i), give a non-closed subset of $\mathbf{R}$ whose induced topology is discrete.

4. Let G be a topological group.

(i) Show that any open subgroup H of G is closed. (Hint: Consider H-cosets in G.)

(ii) Using (i), show that if G is *connected* and U is an open subset of G containing the identity then U generates G as a group (i.e., every element of G is a "word" in elements of U and their inverses). This is Exercise 4 in I.1.

5. Exercise 5 in I.1. (Hint: If G is the given connected Lie group and Γ is the given discrete *normal* subgroup, consider the effect on Γ by g-conjugation for g in a path through the identity.)

6. Exercise 8 in I.1 (where the definition of D has the typo that its definition should use $\mathrm{GL}_n(\mathbf{R})$ rather than $\mathrm{SL}_n(\mathbf{R})$). Hint: After establishing bijectivity, use left and right multiplications to reduced the tangential isomorphism property to a computation at the identity points. For this use that at (e,e), the differential of $m : G \times G \to G$ is addition on $\mathrm{T}_e(G)$, as shown in the handout on smoothness of inversion.

7. (i) Exercise 13 in I.1, and read the Wikipedia page on "Quaternions and spatial rotation".

(ii) Consider the rotations r and r' by 90 degrees counterclockwise around the lines spanned by respective independent unit vectors $u, u' \in \mathbf{R}^3$. As an application of (i), give a nonzero vector on the line about which $r \circ r'$ is a rotation (your answer should involve addition and cross product on $\mathbf{R}^3$ with u and u'). Can you determine the angle of rotation around this line in a clean way too?