

1. INTRODUCTION

Let  $G$  be a connected compact Lie group, and  $S$  a torus in  $G$  (not necessarily maximal). In class we saw that the centralizer  $Z_G(S)$  of  $S$  in  $G$  is connected *conditional* on Weyl's general conjugacy theorem for maximal tori in compact connected Lie groups. The normalizer  $N_G(S)$  is a closed subgroup of  $G$  that contains  $S$ . (Read the discussion just below the statement of Corollary 4.6 in the handout on local/global Frobenius theorems for subtleties concerning normalizer subgroups of closed subgroups that have infinite component group.)

For any compact Lie group  $H$ , the component group  $\pi_0(H) = H/H^0$  is finite since it is compact and discrete (as  $H^0$  is open in  $H$ , due to the local connectedness of manifolds). Thus, if  $H$  is a compact Lie group then a connected closed subgroup  $H'$  exhausts the identity component  $H^0$  precisely when  $H/H'$  is finite. (Indeed, in such cases  $H^0/H'$  with its quotient topology is a subspace of the Hausdorff quotient  $H/H'$  that is finite and hence *discrete*. Thus,  $H'$  is open in  $H$ , yet also visibly closed, so its finitely many cosets contained in the compact connected  $H^0$  constitute a pairwise disjoint open cover, contradicting connectedness of  $H^0$  unless the cover is a singleton; i.e.,  $H' = H^0$ .)

Note that since  $Z_G(S)$  is normal in  $N_G(S)$  (why?), its identity component  $Z_G(S)^0$  (which we don't know is equal to  $Z_G(S)$  until the Conjugacy Theorem is proved) is also normal in  $N_G(S)$ . In particular,  $N_G(S)/Z_G(S)^0$  makes sense as a compact Lie group. In this handout, we will show that  $W(G, S) := N_G(S)/Z_G(S)$  is finite, which is equivalent to the finiteness of the compact Lie group  $N_G(S)/Z_G(S)^0$  since  $\pi_0(Z_G(S))$  is finite. This finiteness holds if and only if  $Z_G(S)^0 = N_G(S)^0$  (why?). The group  $W(G, S)$  is the *Weyl group* of  $G$  relative to  $S$ , and its primary interest is when  $S$  is a maximal torus in  $G$  (in which case  $Z_G(S) = S$  *conditional* on the Conjugacy Theorem, as we saw in class).

We will first look at some instructive examples when  $S$  is maximal (by far the most important case for the study of connected compact Lie groups, for which  $G$ -conjugation permutes all choices of such a torus, *conditional* on the Conjugacy Theorem). Then we will prove two results: the finiteness of  $W(G, S)$  and that  $T = Z_G(T)^0$  when  $T$  is maximal (so  $N_G(T)/T$  is also finite for maximal  $T$ , without needing to know that  $Z_G(T) = T$ ). These two proofs will have no logical dependence on the Conjugacy Theorem, an important point because the proof of the Conjugacy Theorem *uses* the finiteness of  $N_G(T)/T$  for maximal  $T$ .

2. EXAMPLES

Let  $G = U(n) \subset GL_n(\mathbf{C})$  with  $n > 1$  and let  $T = (S^1)^n$  be the diagonal maximal torus in  $G$ . There are some evident elements in  $N_G(T)$  outside  $T$ , namely the standard permutation matrices that constitute an  $S_n$  inside  $G$ . We first show:

**Lemma 2.1.** *The centralizer  $Z_G(T)$  is equal to  $T$  and the subgroup  $S_n$  of  $G$  maps isomorphically onto  $W(G, T) = N_G(T)/T$ .*

*Proof.* We shall work “externally” by appealing to the vector space  $\mathbf{C}^n$  on which  $G = U(n)$  naturally acts. The diagonal torus  $T = (S^1)^n$  acts linearly on  $\mathbf{C}^n$  in this way, with the standard basis lines  $\mathbf{C}e_j$  supporting the *pairwise distinct* characters  $\chi_j : t \mapsto t_j \in \mathbf{C}^\times$ .

This gives an intrinsic characterization of these lines via the completely reducible finite-dimensional  $\mathbf{C}$ -linear representation theory of  $T$ . Thus, conjugation on  $T$  by any  $g \in N_G(T)$  must *permute* these lines! Likewise, an element of  $N_G(T)$  that centralizes  $T$  must act on  $\mathbf{C}^n$  in a manner that *preserves* each of these lines, and so is diagonal in  $\mathrm{GL}_n(\mathbf{C})$ . In other words,  $Z_G(T)$  is the diagonal subgroup of the compact subgroup  $U(n) \subset \mathrm{GL}_n(\mathbf{C})$ . But by compactness we therefore have  $Z_G(T) = T$ , since entry-by-entry we can apply the evident fact that inside  $\mathbf{C}^\times = S^1 \times \mathbf{R}_{>0}^\times$  any compact subgroup lies inside  $S^1$  (as  $\mathbf{R}_{>0}^\times$  has no nontrivial compact subgroups).

Returning to an element  $g \in N_G(T)$ , whatever its permutation effect on the standard basis lines may be, there is visibly a permutation matrix  $s \in S_n \subset G$  that achieves the same effect, so  $s^{-1}g$  preserves each of the standard basis lines, which is to say  $s^{-1}g$  is diagonal as an element in the ambient  $\mathrm{GL}_n(\mathbf{C})$  and so it lies in  $Z_G(T) = T$ . Hence,  $g \in sT$ , so every class in  $W(G, T)$  is represented by an element of  $S_n$ . It follows that  $S_n \rightarrow W(G, T)$  is surjective.

For injectivity, we just have to note that a nontrivial permutation matrix must move some of the standard basis lines in  $\mathbf{C}^n$ , so its conjugation effect on  $T = (S^1)^n$  is to permute the  $S^1$ -factors accordingly. That is a nontrivial automorphism of  $T$  since  $n > 1$ , so  $S_n \rightarrow W(G, T)$  is indeed injective.  $\blacksquare$

The preceding lemma has an additional wrinkle in its analogue for  $G' = \mathrm{SU}(n)$  in place of  $G = \mathrm{U}(n)$ , as follows. In  $G'$ , consider the diagonal torus  $T'$  of dimension  $n - 1$ . This is the kernel of the map  $(S^1)^n \rightarrow S^1$  defined by  $(z_1, \dots, z_n) \mapsto \prod z_j$ , and the diagonally embedded  $S^1$  is the maximal central torus  $Z$  in  $G = \mathrm{U}(n)$ . It is clear by inspection that  $Z \cdot T' = T$ ,  $Z \cdot G' = G$ , and  $T' = T \cap G'$ , so  $Z_{G'}(T') = Z_{G'}(T) = G' \cap Z_G(T) = G' \cap T = T'$ . Thus,  $T'$  is a maximal torus in  $G'$ ,  $N_{G'}(T') = Z \cdot N_{G'}(T')$ , and the inclusion  $Z \subset T$  implies  $Z \cap N_{G'}(T') = Z \cap G' = Z[n] \subset T'$ . Hence, the Weyl group  $W(G', T')$  is naturally identified with  $W(G, T) = S_n$ .

Under this identification of Weyl groups, the effect of  $S_n$  on the codimension-1 subtorus  $T' \subset T = (S^1)^n$  via the identification of  $S_n$  with  $N_{G'}(T')/T'$  is the restriction of the usual  $S_n$ -action on  $(S^1)^n$  by permutation of the factors. This is reminiscent of the representation of  $S_n$  on the hyperplane  $\{\sum x_j = 0\}$  in  $\mathbf{R}^n$ , and literally comes from this hyperplane representation if we view  $T$  as  $(\mathbf{R}/\mathbf{Z})^n$  (and thereby write points of  $T$  as  $n$ -tuples  $(e^{2\pi i x_j})$ ).

A key point is that, in contrast with  $\mathrm{U}(n)$ , in general the Weyl group quotient of  $N_{G'}(T')$  does *not* isomorphically lift into  $N_{G'}(T')$  as a subgroup (equivalently,  $N_{G'}(T')$  is not a semi-direct product of  $T'$  against a finite group). In particular, the Weyl group is generally only a quotient of the normalizer of a maximal torus and cannot be found as a subgroup of the normalizer. The failure of such lifting already occurs for  $n = 2$ :

*Example 2.2.* Let's show that for  $G' = \mathrm{SU}(2)$  and  $T'$  the diagonal maximal torus, there is no order-2 element in the nontrivial  $T'$ -coset of  $N_{G'}(T')$ . A representative for this non-identity coset is the “standard Weyl representative”

$$\nu = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

that satisfies  $\nu^2 = -1$ . For any  $t' \in T'$ , the representative  $\nu t'$  for the nontrivial class in the order-2 group  $W(G', T')$  satisfies

$$(\nu t')(\nu t') = (\nu t' \nu^{-1})(\nu^2 t') = t'^{-1}(-1)t' = -1$$

regardless of the choice of  $t'$ . Hence, no order-2 representative exists.

For  $n \geq 3$  the situation is more subtle, since (for  $G' = \text{SU}(n)$  and its diagonal maximal torus  $T'$ ) the order-2 elements of  $W(G', T')$  do always lift (non-uniquely) to order-2 elements of  $N_{G'}(T')$ . We will build such lifts and use them to show that the Weyl group doesn't lift isomorphically to a subgroup of  $N_{G'}(T')$ . Inside  $W(G', T') = S_n$  acting in its usual manner on the codimension-1 subtorus  $T' \subset T = (S^1)^n$ , consider the element  $\sigma = (12)$ . This action is the identity on the codimension-1 subtorus

$$T'' := \{(z_2, z_2, z_3, \dots, z_{n-1}, (z_2^2 z_3 \cdots z_{n-1})^{-1}) \in (S^1)^n\} \subset T'$$

of  $T'$  (understood to consist of points  $(z_2, z_2, z_2^{-2})$  when  $n = 3$ ), so *any*  $g' \in N_{G'}(T')$  that lifts  $\sigma \in W(G', T')$  must conjugate  $T'$  by means of  $\sigma$  and thus must *centralize*  $T''$ . Thus, the only place we need to look for possible order-2 lifts of  $\sigma$  is inside  $Z_{G'}(T'')$ . But thinking in terms of eigenlines once again, we see the action of  $T''$  on  $\mathbf{C}^n$  decomposes as a plane  $\mathbf{C}e_1 \oplus \mathbf{C}e_2$  with action through  $z_2$ -scaling and the additional standard basis lines  $\mathbf{C}e_j$  on which  $T''$  acts with pairwise *distinct* non-trivial characters.

Hence, the centralizer of  $T''$  inside the ambient group  $\text{GL}_n(\mathbf{C})$  consists of elements that (i) preserve the standard basis lines  $\mathbf{C}e_j$  for  $j > 2$  and (ii) preserve the plane  $\mathbf{C}e_1 \oplus \mathbf{C}e_2$ . This is  $\text{GL}_2(\mathbf{C}) \times (S^1)^{n-2}$ . Intersecting with  $\text{SU}(n)$ , we conclude that

$$Z_{G'}(T'') = (\text{SU}(2) \cdot S_0) \times S$$

where this  $\text{SU}(2)$  is built inside  $\text{GL}(\mathbf{C}e_1 \oplus \mathbf{C}e_2)$ ,  $S$  is the codimension-1 subtorus of  $(S^1)^{n-2}$  (in coordinates  $z_3, \dots, z_n$ ) whose coordinates have product equal to 1, and

$$S_0 = \{(z, z, 1, \dots, 1, z^{-2})\}$$

(so  $\text{SU}(2) \cap S_0 = \{\pm 1\}$ ).

Clearly

$$N_{G'}(T') \cap Z_{G'}(T'') = (N_{\text{SU}(2)}(T_0) \cdot S_0) \times S$$

where  $T_0 \subset \text{SU}(2)$  is the diagonal torus. Thus, an order-2 element of  $N_{G'}(T')$  whose effect on  $T'$  is induced by  $\sigma = (12)$  has the form  $(\nu, s)$  for some 2-torsion  $\nu \in N_{\text{SU}(2)}(T_0) \cdot S_0$  acting on  $T_0$  through inversion and some 2-torsion  $s \in S$ . But  $\nu$  cannot be trivial, and  $s$  centralizes  $T'$ , so it is equivalent to search for order-2 lifts of  $\sigma$  subject to the hypothesis  $s = 1$ . This amounts to the existence of an order-2 element  $\nu \in N_{\text{SU}(2)}(T_0) \cdot S_0$  that acts on  $T_0$  through inversion. Writing  $\nu = \nu_0 \cdot s_0$  for  $s_0 \in S_0$  and  $\nu_0 \in N_{\text{SU}(2)}(T_0)$ , necessarily  $\nu_0$  acts on  $T_0$  through inversion (so  $\nu_0^2 \in Z_{\text{SU}(2)}(T_0) = T_0$ ) and  $\nu_0^2 = s_0^{-2} \in S_0 \cap T_0 = \langle \tau \rangle$  for  $\tau = (-1, -1, 1, \dots, 1)$ .

A direct calculation in  $\text{SU}(2)$  shows that every element of  $N_{\text{SU}(2)}(T_0) - T_0$  has square equal to the unique element  $-1 \in T_0 \simeq S^1$  with order 2, so  $s_0^{-2} = \nu_0^2 = \tau$ . This says  $s_0 = (i, i, 1, \dots, 1, -1)$  for some  $i = \pm\sqrt{-1}$ , so if  $n = 3$  (so  $S = 1$ ) then the order-2 lifts of

$\sigma = (12)$  are exactly

$$\begin{pmatrix} 0 & it & 0 \\ -i/t & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

for some  $i = \pm\sqrt{-1}$  and  $t \in S^1$ . In particular, these all have trace equal to  $-1$  when  $n = 3$ .

With a general  $n \geq 3$ , assume we have a subgroup of  $N_{G'}(T')$  isomorphically lifting the Weyl group. That identifies the lift with a subgroup  $S_n \subset N_{G'}(T') \subset G' \subset \mathrm{GL}_n(\mathbf{C})$  acting on the standard basis lines through the standard permutation because this subgroup has conjugation effect on  $X(T = T' \cdot Z_G) = \mathbf{Z}^n$  through the standard permutation. (We do *not* claim the effect of this action permutes the set of standard basis vectors; it merely permutes the set of lines they span.) In particular,  $n$ -cycles in  $S_n$  act by permuting the standard basis lines and not fixing any of them, so such  $n$ -cycles act with trace equal to 0. For  $n = 3$ , inspection of the characters of the 3-dimensional representation of  $S_3$  over  $\mathbf{C}$  shows that the only one with that trace property is the standard representation. But in the standard representation of  $S_3$  the order-2 elements act with trace 1, and we saw above that if  $n = 3$  then an order-2 lift in  $N_{G'}(T')$  of  $\sigma = (12)$  (if one exists) must have trace equal to  $-1$ . This contradiction settles the case  $n = 3$ .

Finally, we can carry out an induction with  $n \geq 4$  (using that the cases of  $n - 1, n - 2 \geq 2$  are settled). Consider the subgroup  $S_{n-1} \subset S_n$  stabilizing  $n$ . This acts on  $\mathbf{C}^n$  preserving the standard basis line  $\mathbf{C}e_n$ , so its action on this line is one of the two 1-dimensional representations of  $S_{n-1}$  (trivial or sign). If trivial then we have

$$S_{n-1} \subset \mathrm{SU}(n-1) \subset \mathrm{GL}(\mathbf{C}e_1 \oplus \cdots \oplus \mathbf{C}e_{n-1})$$

normalizing the diagonal torus, contradicting the inductive hypothesis. Thus, this  $S_{n-1} \subset S_n$  acts through the sign character on  $\mathbf{C}e_n$ .

Now consider the subgroup  $S_{n-2} \subset S_{n-1}$  stabilizing 1, so its action on  $\mathbf{C}^n$  preserves  $\mathbf{C}e_1$ . The action of this subgroup on  $\mathbf{C}e_n$  is through the sign character (as the sign character on  $S_{n-1}$  restricts to that of  $S_{n-2}$ ). If its action on  $\mathbf{C}e_1$  is also through the sign character then by determinant considerations this  $S_{n-2}$  is contained in

$$\mathrm{SU}(n-2) \subset \mathrm{GL}(\mathbf{C}e_2 \oplus \cdots \oplus \mathbf{C}e_{n-1}),$$

contradicting the inductive hypothesis! Hence, its action on  $\mathbf{C}e_1$  is trivial. But  $(1n)$ -conjugation inside  $S_n$  normalizes this subgroup  $S_{n-2}$  and its effect on  $\mathbf{C}^n$  swaps the lines  $\mathbf{C}e_1$  and  $\mathbf{C}e_n$  (perhaps not carrying  $e_1$  to  $e_n$ !). Consequently, this conjugation action on  $S_{n-2}$  must swap the characters by which  $S_{n-2}$  acts on  $\mathbf{C}e_1$  and  $\mathbf{C}e_n$ . But that is impossible since the action on  $\mathbf{C}e_1$  is trivial and the action on  $\mathbf{C}e_n$  is the sign character. Hence, no lift exists.

### 3. CHARACTER AND COCHARACTER LATTICES

We want to show that  $W(G, S) := N_G(S)/Z_G(S)$  is finite for any torus  $S$  in  $G$ , and discuss some applications of this finite group (especially for maximal  $S$ ). The group  $W(G, S)$  with its quotient topology is compact, and we will construct a realization of this topological group inside a discrete group, so we can appeal to the fact that a discrete compact space is finite.

To analyze finiteness in a systematic way, it is useful to introduce a general concept that will pervade the later part of the course.

**Definition 3.1.** The *character lattice*  $X(T)$  of a torus  $T \simeq (S^1)^n$  is the abelian group of (continuous) characters  $\text{Hom}(T, S^1) = \text{Hom}(S^1, S^1)^{\oplus n} = \mathbf{Z}^n$  (where  $(a_1, \dots, a_n) \in \mathbf{Z}^n$  corresponds to the character  $(t_1, \dots, t_n) \mapsto \prod t_i^{a_i}$ ).

Underlying the description of  $X((S^1)^n)$  as  $\mathbf{Z}^n$  is the fact that the power characters  $z \mapsto z^n$  ( $n \in \mathbf{Z}$ ), are the endomorphisms of  $S^1$  as a compact Lie group (as we saw in class earlier, by using  $S^1 = \mathbf{R}/\mathbf{Z}$ ). In an evident manner,  $T \rightsquigarrow X(T)$  is a *contravariant* functor into the category of finite free  $\mathbf{Z}$ -modules, with  $X(T)$  having rank equal to  $\dim T$ .

Often the evaluation of a character  $\chi \in X(T)$  on an element  $t \in T$  is denoted in “exponential” form as  $t^\chi$ , and so correspondingly the group law on the character lattice (i.e., pointwise multiplication of  $S^1$ -valued functions) is denoted *additively*: we write  $\chi + \chi'$  to denote the character  $t \mapsto \chi(t)\chi'(t)$ ,  $0$  denotes the trivial character  $t \mapsto 1$ , and  $-\chi$  denotes the reciprocal character  $t \mapsto 1/\chi(t)$ . This convention allows us to write things like

$$t^{\chi+\chi'} = t^\chi \cdot t^{\chi'}, \quad t^0 = 1, \quad t^{-\chi} = 1/t^\chi.$$

The reason for preferring this exponential notation is due to the following observation. As for any connected commutative Lie group (compact or not), the exponential map of a torus is a canonical surjective homomorphism  $\exp_T : \mathfrak{t} \rightarrow T$  with discrete kernel  $\Lambda \subset \mathfrak{t}$  inducing a canonical isomorphism  $\mathfrak{t}/\Lambda \simeq T$ . The compactness of  $T$  implies that  $\Lambda$  has maximal rank inside the  $\mathbf{R}$ -vector space  $\mathfrak{t}$ , which is to say that that natural map  $\Lambda_{\mathbf{R}} := \mathbf{R} \otimes_{\mathbf{Z}} \Lambda \rightarrow \mathfrak{t}$  is an isomorphism. Thus,

$$X(T) = \text{Hom}(\mathfrak{t}/\Lambda, S^1) = \text{Hom}(\Lambda_{\mathbf{R}}/\Lambda, S^1) \simeq \Lambda^* \otimes_{\mathbf{Z}} \text{Hom}(\mathbf{R}/\mathbf{Z}, S^1)$$

where  $\Lambda^*$  denotes the  $\mathbf{Z}$ -linear dual of  $\Lambda$  and the final isomorphism (right to left) carries  $\ell \otimes f$  to the map  $\mathbf{R} \otimes_{\mathbf{Z}} \Lambda \rightarrow S^1$  given by  $c \otimes \lambda \text{ mod } \Lambda \mapsto f(c) \cdot \ell(\lambda)$ . But  $\mathbf{R}/\mathbf{Z} = S^1$  via  $x \mapsto e^{2\pi i x}$ , so plugging in the identification of the ring  $\text{Hom}(S^1, S^1)$  with  $\mathbf{Z}$  via power maps brings us to the identification

$$\Lambda^* \simeq \text{Hom}(\mathfrak{t}/\Lambda, S^1) = X(T)$$

given by

$$\ell \mapsto (c \otimes \lambda \text{ mod } \Lambda \mapsto e^{2\pi i c \ell(\lambda)} = e^{2\pi i \ell_{\mathbf{R}}(c\lambda)}).$$

Put another way, by identifying the scalar extension  $(\Lambda^*)_{\mathbf{R}}$  with the  $\mathbf{R}$ -linear dual of  $\Lambda_{\mathbf{R}} = \mathfrak{t}$  we can rewrite our identification  $\Lambda^* \simeq X(T)$  as  $\ell \mapsto (v \text{ mod } \Lambda \mapsto e^{2\pi i \ell_{\mathbf{R}}(v)})$  ( $v \in \mathfrak{t}$ ). In this manner, we “see” the character lattice  $X(T)$  occurring inside analytic exponentials, in accordance with which the  $S^1$ -valued pointwise group law on  $X(T)$  literally is addition inside exponents.

*Remark 3.2.* A related functor of  $T$  which comes to mind is the covariant *cocharacter lattice*  $X_*(T) = \text{Hom}(S^1, T)$  (finite free of rank  $\dim T$  also). This is the  $\mathbf{Z}$ -linear dual of the character lattice via the evaluation pairing

$$\langle \cdot, \cdot \rangle : X(T) \times X_*(T) \rightarrow \text{End}(S^1) = \mathbf{Z}$$

defined by  $\chi \circ \lambda : z \mapsto \chi(\lambda(z)) = z^{\langle \chi, \lambda \rangle}$  (check this really is a perfect pairing, and note that the order of appearance of the inputs into the pairing  $\langle \cdot, \cdot \rangle$  – character on the left and cocharacter on the right – matches the order in which we write them in the composition notation  $\chi \circ \lambda$  as functions).

We conclude from this perfect pairing that the covariant functor  $X_*(T)$  in  $T$  is naturally identified with  $X(T)^* = \Lambda^{**} = \Lambda$ , and so its associated  $\mathbf{R}$ -vector space  $X_*(T)_{\mathbf{R}}$  is identified with  $\Lambda_{\mathbf{R}} = \mathfrak{t}$ . Thus, we can expression the exponential parameterization of  $T$  in the canonical form

$$\exp_T : \mathfrak{t}/X_*(T) \simeq T.$$

The preference for  $X(T)$  over  $X_*(T)$  will become apparent later in the course. To emphasize the contravariance of  $X(T)$  in  $T$ , one sometimes denotes the character lattice as  $X^*(T)$ , akin to the convention in topology that the contravariant cohomology functors are denoted with degree superscripts whereas the covariant homology functors are denoted with degree subscripts. We will generally not do this.

Here is an elaborate triviality:

**Lemma 3.3.** *The functor  $T \rightsquigarrow X(T)$  from tori to finite free  $\mathbf{Z}$ -modules is an equivalence of categories. That is, for any tori  $T$  and  $T'$  and any map  $f : X(T') \rightarrow X(T)$  of  $\mathbf{Z}$ -modules there exists exactly one homomorphism  $\phi : T \rightarrow T'$  such that  $X(\phi) = f$ , and every finite free  $\mathbf{Z}$ -module has the form  $X(T)$  for a torus  $T$ .*

*In particular, passing to invertible endomorphisms, for an  $r$ -dimensional torus  $T$  the natural anti-homomorphism*

$$\text{Aut}(T) \rightarrow \text{GL}(X(T)) \simeq \text{GL}_r(\mathbf{Z})$$

*is an anti-isomorphism.*

*Proof.* It is clear that  $\mathbf{Z}^n \simeq X((S^1)^n)$ , so the main issue is bijectivity on Hom-sets. Decomposing  $T$  and  $T'$  into direct products of  $S^1$ 's and using the natural identification  $X(T_1 \times T_2) \simeq X(T_1) \times X(T_2)$ , we may immediately reduce to the case in which  $T, T' = S^1$ . In this case  $X(S^1) = \mathbf{Z}$  via the power characters and the endomorphism ring of  $\mathbf{Z}$  as a  $\mathbf{Z}$ -module is compatibly identified with  $\mathbf{Z}$ . ■

#### 4. FINITENESS

Let's return to tori in a connected compact Lie group  $G$ . For any such  $S$  in  $G$ , there is a natural action of the abstract group  $W(G, S) = N_G(S)/Z_G(S)$  on  $S$  via  $\bar{n}.s = nsn^{-1}$ , and by definition of  $Z_G(S)$  this is a *faithful* action in the sense that the associated action homomorphism  $W(G, S) \rightarrow \text{Aut}(S)$  is injective.

The target group  $\text{Aut}(S)$  looks a bit abstract, but by bringing in the language of character lattices it becomes more concrete as follows. Since the ‘‘character lattice’’ functor on tori is fully faithful and contravariant, it follows that the associated action of  $W(G, S)$  on  $X(S)$  via  $w.\chi : s \mapsto \chi(w^{-1}.s)$  is also faithful: the homomorphism  $W(G, S) \rightarrow \text{GL}(X(S)) \simeq \text{GL}_r(\mathbf{Z})$  is injective (with  $r = \dim S = \text{rank}_{\mathbf{Z}}(X(S))$ ).

Now we are finally in position to prove:

**Proposition 4.1.** *The group  $W(G, S)$  is finite.*

As we noted at the outset, this finiteness is equivalent to the equality  $Z_G(S)^0 = N_G(S)^0$ . (We cannot say  $Z_G(S) = N_G(S)^0$  until we know that  $Z_G(S)$  is connected, which rests on the Conjugacy Theorem.)

*Proof.* Since  $W(G, S)$  with its quotient topology is compact, it suffices to show that the natural injective map  $W(G, S) \rightarrow \mathrm{GL}(X(S)) = \mathrm{GL}_r(\mathbf{Z})$  is continuous when the target is given the discrete topology (as then the image of this *injective* map would be compact and discrete, hence finite). Of course, it is equivalent (and more concrete) to say that  $N_G(S) \rightarrow \mathrm{GL}(X(S))$  is continuous.

By using duality to pass from automorphisms of  $X(S)$  to automorphisms of  $X_*(S)$ , it is equivalent to check that the injective homomorphism  $N_G(S) \rightarrow \mathrm{GL}(X_*(S))$  arising from the action of  $N_G(S)$  on  $X_*(S)$  defined by  $(n.\lambda)(z) = n \cdot \lambda(z) \cdot n^{-1}$  is continuous when the target is viewed discretely. Equivalently, this says that the action map

$$N_G(S) \times X_*(S) \rightarrow X_*(S)$$

is continuous when we give  $X_*(S)$  the discrete topology and the left side the product topology.

But the discrete topology on  $X_*(S)$  is the subspace topology from naturally sitting inside the finite-dimensional  $\mathbf{R}$ -vector space  $X_*(S)_{\mathbf{R}}$ , so it is equivalent to check that the natural action map

$$f_S : N_G(S) \times X_*(S)_{\mathbf{R}} \rightarrow X_*(S)_{\mathbf{R}}$$

is continuous relative to the natural manifold topologies on the source and target.

Recall that for any torus  $T$  whatsoever, the “realification”  $X_*(T)_{\mathbf{R}}$  of the cocharacter lattice is naturally identified by means of  $\exp_T$  with the Lie algebra  $\mathrm{Lie}(T)$ . Identifying  $X_*(S)_{\mathbf{R}}$  with  $\mathrm{Lie}(S)$  in this way, we can express  $f_S$  as a natural map

$$N_G(S) \times \mathrm{Lie}(S) \rightarrow \mathrm{Lie}(S).$$

What could this natural map be? By using the *functoriality* of the exponential map of Lie groups with respect to the inclusion of  $S$  into  $G$ , and the definition of  $\mathrm{Ad}_G$  in terms the effect of conjugation on the tangent space at the identity point of  $G$ , one checks (do it!) that this final map carries  $(n, v)$  to  $\mathrm{Ad}_G(n)(v)$ , where we note that  $\mathrm{Ad}_G(n)$  acting on  $\mathfrak{g}$  preserves the subspace  $\mathrm{Lie}(S)$  since  $n$ -conjugation on  $G$  preserves  $S$ . (The displayed commutative diagram just above (1.2) in Chapter IV gives a nice visualization for the automorphism  $\varphi = \mathrm{Ad}_G(n)$  of the torus  $S$ .)

So to summarize, our map of interest is induced by the restriction to  $N_G(S) \times \mathrm{Lie}(S)$  of the action map

$$G \times \mathfrak{g} \rightarrow \mathfrak{g}$$

given by  $(g, X) \mapsto \mathrm{Ad}_G(g)(X)$ . The evident continuity of this latter map does the job. ■

*Remark 4.2.* What is “really going on” in the above proof is that since the automorphism group of a torus is a “discrete” object, a continuous action on  $S$  by a *connected* Lie group  $H$  would be a “connected family of automorphisms” of  $S$  (parameterized by  $H$ ) containing the identity automorphism and so would have to be the constant family: a trivial action on  $S$  by every  $h \in H$ . That alone forces  $N_G(S)^0$ , whatever it may be, to act trivially on  $S$ . In other words, this forces  $N_G(S)^0 \subset Z_G(S)$ , so  $N_G(S)^0 = Z_G(S)^0$ .

The reason that this short argument is not a rigorous proof as stated is that we have not made a precise logical connection between the idea of “discreteness” for the automorphism group of  $S$  and the connectedness of the topological space  $N_G(S)^0$ ; e.g., we have not discussed the task of putting a natural topology on the set of automorphisms of a Lie group.

(The proof in IV, 1.5 that  $N_G(T)^0 = T$  is incomplete because of exactly this issue, so we prove it below using Proposition 4.1, whose proof made this discrete/connected dichotomy rigorous in some situations.) There is a systematic and useful way to justify this intuition by means of the idea of “Lie group of automorphisms of a connected Lie group”, but it is more instructive for later purposes to carry out the argument as above using the crutch of the isomorphism  $X_*(S)_{\mathbf{R}} \simeq \text{Lie}(S)$  rather than to digress into the topic (actually not very hard) of equipping the automorphism group of a connected Lie group with a structure of (possibly very disconnected) Lie group in a useful way.

**Proposition 4.3.** *Let  $G$  be a compact connected Lie group,  $T$  a maximal torus. The identity component  $N_G(T)^0$  is equal to  $T$  (so  $N_G(T)/T$  is finite).*

*Remark 4.4.* The proof of the Conjugacy Theorem works with the finite group  $N_G(T)/T$ , and it is essential in that proof that we know  $N_G(T)/T$  is *finite* (but only after the proof of the Conjugacy Theorem is over and connectedness of torus centralizers thereby becomes available can we conclude that  $N_G(T)/T = N_G(T)/Z_G(T) =: W(G, T)$ ).

*Proof.* To prove that  $N_G(T)^0 = T$  for any choice of maximal torus  $T$  in a compact connected Lie group  $G$ , we first note that  $N_G(T)^0 = Z_G(T)^0$  by Proposition 4.1. To show that  $Z_G(T)^0 = T$ , if we rename  $Z_G(T)^0$  with  $G$  (as we may certainly do!) then we find ourselves in the case that  $T$  is *central* in  $G$ , and in such cases we wish to prove that  $G = T$ .

It suffices to show that the inclusion of Lie algebras  $\mathfrak{t} \hookrightarrow \mathfrak{g}$  is an equality. Choose  $X \in \mathfrak{g}$  and consider the associated 1-parameter subgroup  $\alpha_X : \mathbf{R} \rightarrow G$ . Clearly  $\alpha_X(\mathbf{R})$  is a connected commutative subgroup of  $G$ , so  $\alpha_X(\mathbf{R}) \cdot T$  is also a connected commutative subgroup of  $G$  (as  $T$  is central in  $G$ ). Its closure  $H$  is therefore a connected commutative compact Lie group, but we know that the only such groups are tori. Thus,  $H$  is a torus in  $G$ , but by design it contains  $T$  which is a maximal torus of  $G$ . Thus,  $H = T$ , so  $\alpha_X(\mathbf{R}) \subseteq T$ . Hence,  $\alpha_X : \mathbf{R} \rightarrow G$  factors through  $T$ , so  $X = \alpha'_X(0) \in \mathfrak{t}$ . This proves that  $\mathfrak{g} = \mathfrak{t}$  (so  $G = T$ ). ■

## 5. SUBTORI VIA CHARACTER LATTICES

We finish this handout with a discussion of how to use the character lattice to keep track of subtori and quotient tori. This is very convenient in practice, and clarifies the sense in which the functor  $T \rightsquigarrow X(T)$  is similar to duality for finite-dimensional vector spaces over a field. (This is all subsumed by the general Pontryagin duality for locally compact Hausdorff topological abelian groups, but our treatment of tori is self-contained without such extra generalities.) As an application, we shall deduce by indirect means that any torus contains a dense cyclic subgroup.

Let  $T$  be a torus. For any closed (not necessarily connected) subgroup  $H \subset T$ , the quotient  $T/H$  is a compact connected commutative Lie group, so it is a torus. The pullback map  $X(T/H) \rightarrow X(T)$  (composing characters with the quotient map  $T \rightarrow T/H$ ) is an injection between finitely generated  $\mathbf{Z}$ -modules. For example, if  $T = S^1$  and  $H = \mu_n$  is the cyclic subgroup of  $n$ th roots of unity ( $n > 0$ ) then the map  $f : T \rightarrow S^1$  defined by  $f(t) = t^n$  identifies  $T/H$  with  $S^1$  and the subgroup  $X(T/H) = X(S^1) = \mathbf{Z}$  of  $X(T) = X(S^1) = \mathbf{Z}$  via composition of characters with  $f$  is the injective map  $\mathbf{Z} \rightarrow \mathbf{Z}$  defined by multiplication by  $n$ . (Check this!)



**Proposition 5.1.** *For any subtorus  $T' \subset T$ , the induced map  $X(T) \rightarrow X(T')$  is surjective with kernel  $X(T/T')$ . In particular,  $X(T/T')$  is a saturated subgroup of  $X(T)$  in the sense that the quotient  $X(T)/X(T/T') = X(T')$  is torsion-free.*

*Proof.* The elements  $\chi \in X(T)$  killed by restriction to  $T'$  are precisely those  $\chi : T \rightarrow S^1$  that factor through  $T/T'$ , which is to say  $\chi \in X(T/T')$  inside  $X(T)$ . This identifies the kernel, so it remains to prove surjectivity. That is, we have to show that every character  $\chi' : T' \rightarrow S^1$  extends to a character  $T \rightarrow S^1$  (with all characters understood to be homomorphisms of Lie groups, or equivalently continuous). We may assume  $T' \neq 1$ .

Choose a product decomposition  $T' = (S^1)^m$ , so  $X(T')$  is generated as an abelian group by the characters  $\chi'_i : (z_1, \dots, z_n) \mapsto z_i$ . (Check this!) It suffices to show that each  $\chi'_i$  extends to a character of  $T$ . Let  $T'' = (S^1)^{m-1}$  be the product of the  $S^1$ -factors of  $T'$  apart from the  $i$ th factor. In an evident manner,  $\chi'_i$  is identified with a character of  $T'/T'' = S^1$  and it suffices to extend this to a character of  $T/T''$  (as such an extension composed with  $T \twoheadrightarrow T/T''$  is an extension of  $\chi'_i$ ). Thus, it suffices to treat the case  $T' = S^1$  and to extend the identity map  $S^1 \rightarrow S^1$  to a character of  $T$ .

Choosing a product decomposition  $T = (S^1)^n$ , the given abstract inclusion  $S^1 = T' \hookrightarrow T = (S^1)^n$  is a map  $\iota : z \mapsto (z^{e_1}, \dots, z^{e_n})$  for some integers  $e_1, \dots, e_n$ , and the triviality of  $\ker \iota$  implies that  $\gcd(e_i) = 1$ . Hence, there exist integers  $r_1, \dots, r_n$  so that  $\sum r_j e_j = 1$ . Hence,  $\chi : T \rightarrow S^1$  defined by

$$\chi(z_1, \dots, z_n) = \prod z_j^{r_j}$$

satisfies  $\chi(\iota(z)) = z^{\sum r_j e_j} = z$ , so  $\chi$  does the job. ■

**Corollary 5.2.** *A map  $f : T' \rightarrow T$  between tori is a subtorus inclusion (i.e., isomorphism of  $f$  onto a closed subgroup of  $T$ ) if and only if  $X(f)$  is surjective.*

*Proof.* We have already seen that the restriction map to a subtorus induces a surjection between character groups. It remains to show that if  $X(f)$  is surjective then  $f$  is a subtorus inclusion. For any homomorphism  $f : G' \rightarrow G$  between compact Lie groups, the image is closed, even compact, and the surjective map  $G' \rightarrow f(G')$  between Lie groups is a Lie group isomorphism of  $G' / (\ker f)$  onto  $f(G')$ . Thus, if  $\ker f = 1$  then  $f$  is an isomorphism onto a closed Lie subgroup of  $G$ . Hence, in our setting with tori, the problem is to show  $\ker f = 1$ .

By the surjectivity hypothesis, every  $\chi' \in X(T')$  has the form  $\chi \circ f$  for some  $\chi \in X(T)$ , so  $\chi'(\ker f) = 1$ . Hence, for any  $t' \in \ker f$  we have  $\chi'(t') = 1$  for all  $\chi' \in X(T')$ . But by choosing a product decomposition  $T' \simeq (S^1)^m$  it is clear that for any nontrivial  $t' \in T'$  there is some  $\chi' \in X(T')$  such that  $\chi'(t') \neq 1$  (e.g., take  $\chi'$  to be the projection from  $T' = (S^1)^m$  onto an  $S^1$ -factor for which  $t'$  has a nontrivial component). Thus, we conclude that any  $t' \in \ker f$  is equal to 1, which is to say that  $\ker f = 1$ . ■

**Corollary 5.3.** *The map  $T' \mapsto X(T/T')$  is a bijection from the set of subtori of  $T$  onto the set of saturated subgroups of  $X(T)$ . Moreover, it is inclusion-reversing in both directions in the sense that  $T' \subseteq T''$  inside  $T$  if and only if  $X(T/T'') \subseteq X(T/T')$  inside  $X(T)$ .*

*Proof.* Let  $j' : T' \hookrightarrow T$  and  $j'' : T'' \hookrightarrow T$  be subtori, so the associated maps  $X(j')$  and  $X(j'')$  on character lattices are surjections from  $X(T)$ . Clearly if  $T' \subseteq T''$  inside  $T$  then we get a

quotient map  $T/T' \twoheadrightarrow T/T''$  as torus quotients of  $T$ , so passing to character lattices gives that  $X(T/T'') \subseteq X(T/T')$  inside  $X(T)$ .

Let's now show the reverse implication: assuming  $X(T/T'') \subseteq X(T/T')$  inside  $X(T)$  we claim that  $T' \subseteq T''$  inside  $T$ . Passing to the associated quotients of  $X(T)$ , we get a map  $f : X(T'') \rightarrow X(T')$  respecting the surjections  $X(j'')$  and  $X(j')$  onto each side from  $X(T)$ . By the categorical equivalence in Lemma 3.3, the map  $f$  has the form  $X(\phi)$  for a map of tori  $\phi : T'' \rightarrow T'$ , and  $j'' \circ \phi = j'$  since we can check the equality at the level of character groups (by how  $\phi$  was constructed). This says exactly that  $T' \subseteq T''$  inside  $T$ , as desired.

As a special case, we conclude that  $X(T/T'') = X(T/T')$  inside  $X(T)$  if and only if  $T' = T''$  inside  $T$ . Thus, the saturated subgroup  $X(T/T')$  inside  $X(T)$  determines the subtorus  $T'$  inside  $T$ .

It remains to show that if  $\Lambda \subseteq X(T)$  is a saturated subgroup then  $\Lambda = X(T/T')$  for a subtorus  $T' \subseteq T$ . By the definition of saturatedness,  $X(T)/\Lambda$  is a finite free  $\mathbf{Z}$ -module, so it has the form  $X(T')$  for an abstract torus  $T'$ . By the categorical equivalence in Lemma 3.3, the quotient map  $q : X(T) \twoheadrightarrow X(T)/\Lambda = X(T')$  has the form  $X(\phi)$  for a map of tori  $\phi : T' \rightarrow T$ . By Corollary 5.2,  $\phi$  is a subtorus inclusion since  $X(\phi) = q$  is surjective, so  $T'$  is identified with a subtorus of  $T$  identifying the  $q$  with the restriction map on characters. Hence, the kernel  $\Lambda$  of  $q$  is identified with  $X(T/T')$  inside  $X(T)$  as desired. ■

**Corollary 5.4.** *For a map  $f : T \rightarrow S$  between tori, the kernel is connected (equivalently, is a torus) if and only if  $X(f)$  has torsion-free cokernel.*

*Proof.* We may replace  $S$  with the torus  $f(T)$ , so  $f$  is surjective and hence  $S = T/(\ker f)$ . Thus,  $X(f)$  is injective. If  $T' := \ker f$  is a torus then

$$\text{coker}(X(f)) = X(T)/X(S) = X(T)/X(T/T') = X(T')$$

is torsion-free. It remains to prove the converse.

Assume  $\text{coker}(X(f))$  is torsion-free, so the image of  $X(S)$  in  $X(T)$  is saturated. Thus, this image is  $X(T/T')$  for a subtorus  $T' \subset T$ . The resulting composite map

$$X(f) : X(S) \twoheadrightarrow X(T/T') \hookrightarrow X(T)$$

arises from a diagram of torus maps factoring  $f$  as a composition

$$T \twoheadrightarrow T/T' \hookrightarrow S$$

of the natural quotient map and a subtorus inclusion. It follows that  $\ker f = T'$  is a torus. ■

Since  $X(T)$  has only *countably many* subgroups (let alone saturated ones), Corollary 5.3 implies that  $T$  contains only *countably many* proper subtori  $\{T_1, T_2, \dots\}$ . As an application, we get the existence of (lots of)  $t \in T$  for which the cyclic subgroup  $\langle t \rangle$  generated by  $t$  is dense, as follows. Pick any  $t \in T$  and let  $Z$  be the closure of  $\langle t \rangle$  (which we want to equal  $T$  for suitable  $t$ ), so  $Z^0$  is a subtorus of  $T$ . This has finite index in  $Z$ , so  $t^n \in Z^0$  for some  $n \geq 1$ . We need to rule out the possibility that  $Z^0 \neq T$ ; i.e., that  $Z^0$  is a proper subtorus. There are countably many proper subtori of  $T$ , say denoted  $T_1, T_2, \dots$ , and we just need to avoid the possibility  $t \in [n]^{-1}(T_m)$  for some  $n \geq 1$  and some  $m$ . Put another way, we just have to show that  $T$  is not the union of the preimages  $[n]^{-1}(T_m)$  for all  $n \geq 1$  and all  $m$ . This is immediate via the Baire Category Theorem, but here is a more elementary approach.

If some  $t^n$  (with  $n > 0$ ) lies in a proper subtorus  $T' \subset T$  then  $t^n$  is killed by the quotient map  $T \rightarrow T/T'$  whose target  $T/T'$  is a *non-trivial* torus (being commutative, connected, and compact). Writing  $T/T'$  as a product of copies of  $S^1$  and projecting onto one of those factors, we get a quotient map  $q : T \twoheadrightarrow S^1 = \mathbf{R}/\mathbf{Z}$  killing  $t^n$ . If we identify  $T$  with  $\mathbf{R}^d/\mathbf{Z}^d$ ,  $t$  is represented by  $(t_1, \dots, t_d) \in \mathbf{R}^d$  and  $q : \mathbf{R}^d/\mathbf{Z}^d \twoheadrightarrow \mathbf{R}/\mathbf{Z}$  is induced by  $(x_1, \dots, x_d) \mapsto \sum a_j x_j$  for some  $a_1, \dots, a_d \in \mathbf{Z}$  not all zero. Thus, we just have to pick  $(t_1, \dots, t_d) \in \mathbf{R}^d$  avoiding the possibility that for some  $n > 0$  and some  $a_1, \dots, a_d \in \mathbf{Z}$  not all 0,  $\sum a_j (nt_j) \in \mathbf{Z}$ . Replacing  $a_j$  with  $na_j$ , it suffices to find  $t \in \mathbf{R}^d$  so that  $\sum b_j t_j \notin \mathbf{Z}$  for all  $b_1, \dots, b_d \in \mathbf{Z}$  not all zero.

Suppose for some  $b_1, \dots, b_d \in \mathbf{Z}$  not all zero that  $\sum b_j t_j = c \in \mathbf{Z}$ . Then  $\sum b_j t_j + (-c) \cdot 1 = 0$ , so  $\{t_1, \dots, t_d, 1\}$  is  $\mathbf{Q}$ -linearly dependent. Hence, picking a  $\mathbf{Q}$ -linearly independent subset of  $\mathbf{R}$  of size  $\dim(T) + 1$  (and scaling throughout so 1 occurs in the set) provides such a  $t$ . In this way we see that there are very many such  $t$ .