Math 210C. The remarkable SU(2)

Let G be a non-commutative connected compact Lie group, and assume that its rank (i.e., dimension of maximal tori) is 1; equivalently, G is a compact connected Lie group of rank 1 that has dimension > 1. In class we have seen a natural way to make such G, namely $G = Z_H(T_a)/T_a$ for a non-commutative connected compact Lie group H, a maximal torus T in H, a root $a \in \Phi(H,T)$, and the codimension-1 subtorus $T_a := (\ker a)^0 \subset T$; this G has maximal torus T/T_a. (If $\dim T = 1$ then $T_a = 1$.)

There are two examples of such G that we have seen: SO(3) and its connected double cover SU(2). These Lie groups are not homeomorphic, as their fundamental groups are distinct. Also, by inspecting the adjoint action of a maximal torus, SU(2) has center $\{\pm 1\}$ of order 2 whereas SO(3) has trivial center (see HW7, Exercise 1(iii)), so they are not isomorphic as abstract groups.

The main aim of this handout is to prove that there are *no other examples*. Once that is proved, we use it to describe the structure of $Z_G(T_a)$ in the general case (without a rank-1 assumption). This is a crucial building block in the structure theory of general G. In Chapter V, pages 186–188 of the course text you'll find two proofs that any rank-1 non-commutative G is isomorphic to SO(3) or SU(2): a topological proof using higher homotopy groups ($\pi_2(S^m) = 1$ for $m > 2$) and an algebraic proof that looks a bit "unmotivated" (for a beginner). Our approach is also algebraic, using the representation theory of $\mathfrak{sl}_2(\mathbf{C})$ to replace hard group-theoretic problems with easier Lie algebra problems.

1. Rank 1

Fix a maximal torus T in G and an isomorphism $T \simeq S^1$. Consider the representation of T on $\mathfrak{g}$ via Ad_G. By the handout on Frobenius' theorem we know that the subspace $\mathfrak{g}^T$ of T-invariants is $\mathrm{Lie}(Z_G(T))$, and this is $\mathfrak{t}$ since $Z_G(T) = T$ (due to the maximality of T in G). On HW7 Exercise 5, you show that the (continuous) representation theory of compact Lie groups on finite-dimensional $\mathbf{R}$-vector spaces is completely reducible, and in particular that the non-trivial irreducible representations of $T = S^1$ over $\mathbf{R}$ are all 2-dimensional and indexed by integers $n \geq 1$: these are $\rho_n : S^1 = \mathbf{R}/\mathbf{Z} \to \mathrm{GL}_2(\mathbf{R})$ via n-fold counterclockwise rotation: $\rho_n(\theta) = r_{2\pi n\theta}$. (This makes sense for $n < 0$ via clockwise $|n|$-fold rotation, and $\rho_{-n} \simeq \rho_n$ by choosing an orthonormal basis with the opposite orientation.) Note that $(\rho_n)_{\mathbf{C}} = \chi^n \oplus \chi^{-n}$ where $\chi : S^1 \to \mathbf{C}^\times$ is the standard embedding.

As $\mathbf{R}$-linear T-representations,

$$\mathfrak{g} = \mathfrak{t} \oplus (\oplus_{n \geq 1} \mathfrak{g}(n))$$

where $\mathfrak{g}(n)$ denotes the ρ_n-isotypic subspace. In particular, each $\mathfrak{g}(n)$ is even-dimensional and so has dimension at least 2 if it is nonzero. Passing to the complexfication $\mathfrak{g}_{\mathbf{C}}$ and using the decomposition of $(\rho_n)_{\mathbf{C}}$ as a direct sum of *reciprocal* characters with weights n and $-n$, as a $\mathbf{C}$-linear representation of T we have

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus (\oplus_{a \in \Phi} (\mathfrak{g}_{\mathbf{C}})_a)$$

where the set $\Phi \subset \mathrm{X}(T) = \mathrm{X}(S^1) = \mathbf{Z}$ of nontrivial T-weights is *stable under negation* and $\dim(\mathfrak{g}_{\mathbf{C}})_a = \dim(\mathfrak{g}_{\mathbf{C}})_{-a}$ (this common dimension being $(1/2)\dim_{\mathbf{R}} \mathfrak{g}(n)$ if $a : t \mapsto t^n$).

Since the action of T on $(\mathfrak{g}_{\mathbf{C}})_a$ via Ad_G is given by the character $a : S^1 = T \to S^1 \subset \mathbf{C}^\times$ (some power map), the associated action $X \mapsto [X, \cdot]_{\mathfrak{g}_{\mathbf{C}}}$ of $\mathfrak{t} = \mathbf{R} \cdot \partial_\theta$ on $(\mathfrak{g}_{\mathbf{C}})_a$ via $\mathrm{Lie}(\mathrm{Ad}_G) = (\mathrm{ad}_{\mathfrak{g}})_{\mathbf{C}} = \mathrm{ad}_{\mathfrak{g}_{\mathbf{C}}}$ is given by multiplication against $\mathrm{Lie}(a)(\partial_\theta|_{\theta=1}) \in \mathbf{Z} \subset \mathbf{R}$. Thus, the $\mathfrak{t}_{\mathbf{C}}$-action on $(\mathfrak{g}_{\mathbf{C}})_a$ via the Lie bracket on $\mathfrak{g}_{\mathbf{C}}$ is via multiplication by the same integer. This visibly scales by $c \in \mathbf{R}^\times$ if we replace ∂_θ with $c\partial_\theta$. Hence, we obtain:

Lemma 1.1. *Let H be a nonzero element in the line $\mathfrak{t}$. The action of $\mathrm{ad}(H) = [H, \cdot]$ on $\mathfrak{g}_{\mathbf{C}}$ has as its nontrivial weight spaces exactly the subspaces $(\mathfrak{g}_{\mathbf{C}})_a$, with eigenvalue $(\mathrm{Lie}(a))(H)$.*

Since $\Phi(G, T) \subset \mathrm{X}(T) - \{0\} = \mathbf{Z} - \{0\}$ is a non-empty subset stable under negation, it contains a unique highest element, say $a \in \mathbf{Z}_{>0}$. The stability of $\Phi(G, T)$ under negation implies that $-a$ is the unique lowest weight. For any $b, b' \in \Phi(G, T)$ and $v \in (\mathfrak{g}_{\mathbf{C}})_b$, $v' \in (\mathfrak{g}_{\mathbf{C}})_{b'}$ we have

$$[v, v'] \subset (\mathfrak{g}_{\mathbf{C}})_{b+b'}$$

since applying $\mathrm{Ad}_G(t)$ to $[v, v']$ carries it to $[t^b v, t^b v'] = t^{b+b'}[v, v']$. (We allow the case $b + b' = 0$, the 0-weight space being $\mathfrak{t}_{\mathbf{C}}$.) In particular, since $(\mathfrak{g}_{\mathbf{C}})_{\pm 2a} = 0$ (due to the nonzero a and $-a$ being the respective highest and lowest T-weights for the T-action on $\mathfrak{g}_{\mathbf{C}}$), each $(\mathfrak{g}_{\mathbf{C}})_{\pm a}$ is a *commutative Lie subalgebra* of $\mathfrak{g}_{\mathbf{C}}$ and

$$[(\mathfrak{g}_{\mathbf{C}})_a, (\mathfrak{g}_{\mathbf{C}})_{-a}] \subseteq \mathfrak{t}_{\mathbf{C}}.$$

Hence, this latter bracket pairing is either 0 or exhausts the 1-dimensional $\mathfrak{t}_{\mathbf{C}}$.

Lemma 1.2. *For any $a \in \Phi(G, T)$, $[(\mathfrak{g}_{\mathbf{C}})_a, (\mathfrak{g}_{\mathbf{C}})_{-a}] = \mathfrak{t}_{\mathbf{C}}$.*

Proof. Suppose otherwise, so this bracket vanishes. Hence, $V_a := (\mathfrak{g}_{\mathbf{C}})_a \oplus (\mathfrak{g}_{\mathbf{C}})_{-a} = V_{-a}$ is a *commutative* Lie subalgebra of $\mathfrak{g}_{\mathbf{C}}$. By viewing a as an element of $\mathrm{X}(T) = \mathbf{Z}$, we see that $V_a \simeq (\rho_a)_{\mathbf{C}}^{\oplus d}$ as $\mathbf{C}$-linear T-representations, where d is the common $\mathbf{C}$-dimension of $(\mathfrak{g}_{\mathbf{C}})_{\pm a}$. Clearly $V_a = \mathfrak{g}(a)_{\mathbf{C}}$ inside $\mathfrak{g}_{\mathbf{C}}$, where $\mathfrak{g}(a)$ denotes the ρ_a-isotypic part of $\mathfrak{g}$ as an $\mathbf{R}$-linear representation of T, so $\mathfrak{g}(a)$ is a *commutative* Lie subalgebra of $\mathfrak{g}$ with dimension at least 2.

Choose $\mathbf{R}$-linearly independent $X, Y \in \mathfrak{g}(a)$, so $\alpha_X(\mathbf{R})$-conjugation on G leaves the map $\alpha_Y : \mathbf{R} \to G$ *invariant* since by connectedness of $\mathbf{R}$ such invariance may be checked on the map $\mathrm{Lie}(\alpha_Y) = \alpha_Y'(0) : \mathbf{R} \to \mathfrak{g}$ sending $c \in \mathbf{R}$ to $c\alpha_Y'(0) = cY$ by using $\mathrm{Ad}_G(\alpha_X(\mathbf{R}))$ (and noting that $\mathrm{Lie}(\mathrm{Ad}_G \circ \alpha_X) = \mathrm{ad}_{\mathfrak{g}}(\alpha_X'(0)) = [X, \cdot]$ and $[X, Y] = 0$). Hence, the closure

$$\overline{\alpha_X(\mathbf{R}) \cdot \alpha_Y(\mathbf{R})}$$

is a connected commutative closed subgroup of G whose Lie algebra contains $\alpha_X'(0) = X$ and $\alpha_Y'(0) = Y$, so its dimension is at least 2. But connected compact commutative Lie groups are necessarily tori, and by hypothesis G has no tori of dimension larger than 1! ■

Now we may choose $X_\pm \in (\mathfrak{g}_{\mathbf{C}})_{\pm a}$ such that the element $H := [X_+, X_-] \in \mathfrak{t}_{\mathbf{C}}$ is *nonzero*. Clearly $[H, X_\pm] = \mathrm{Lie}(\pm a)(H)X_\pm = \pm\mathrm{Lie}(a)(H)X_\pm$ with $\mathrm{Lie}(a)(H) \in \mathbf{C}^\times$. If we replace X_+ with cX_+ for $c \in \mathbf{C}^\times$ then H is replaced with $H' := cH$, and $[H', cX_+] = \mathrm{Lie}(a)(H')(cX_+)$, $[H', X_-] = -\mathrm{Lie}(a)(H')X_-$ with $\mathrm{Lie}(a)(H') = c\,\mathrm{Lie}(a)(H)$. Hence, using such scaling with a suitable c allows us to arrange that $\mathrm{Lie}(a)(H) = 2$, so $\{X_+, X_-, H\}$ span an $\mathfrak{sl}_2(\mathbf{C})$ as a Lie subalgebra of $\mathfrak{g}_{\mathbf{C}}$. The restriction of $\mathrm{ad}_{\mathfrak{g}_{\mathbf{C}}}$ to this copy of $\mathfrak{sl}_2(\mathbf{C})$ makes $\mathfrak{g}_{\mathbf{C}}$ into a $\mathbf{C}$-linear representation space for $\mathfrak{sl}_2(\mathbf{C})$ such that the H-weight spaces for this $\mathfrak{sl}_2(\mathbf{C})$-representation are $\mathfrak{t}_{\mathbf{C}}$ for the trivial weight and each $(\mathfrak{g}_{\mathbf{C}})_b$ (on which H acts as scaling by $\mathrm{Lie}(b)(H) \in \mathbf{C}$).

The highest weight for $\mathfrak{g}_{\mathbf{C}}$ as an $\mathfrak{sl}_2(\mathbf{C})$-representation is $\mathrm{Lie}(a)(H) = 2$ (due to how a was chosen!), so the only other possible weights are $\pm 1, 0, -2$, and the entire weight-0 space for the action of H is a single $\mathbf{C}$-line $\mathfrak{t}_{\mathbf{C}}$. Our knowledge of the finite-dimensional representation theory of $\mathfrak{sl}_2(\mathbf{C})$ (e.g., its complete reducibility, and the determination of each irreducible representation via its highest weight) shows that the adjoint representation of $\mathfrak{sl}_2(\mathbf{C})$ on itself is the *unique* irreducible representation with highest weight 2. This representation contains a line with H-weight 0, so as an $\mathfrak{sl}_2(\mathbf{C})$-representation we see that the highest weight of 2 cannot occur in $\mathfrak{g}_{\mathbf{C}}$ with multiplicity larger than 1 (as otherwise $\mathfrak{g}_{\mathbf{C}}$ would contain multiple independent copies of the adjoint representation $\mathfrak{sl}_2(\mathbf{C})$, contradicting that the weight-0 space for the H-action on $\mathfrak{g}_{\mathbf{C}}$ is only 1-dimensional).

It follows that $\mathfrak{g}_{\mathbf{C}}$ is $\mathfrak{sl}_2(\mathbf{C})$-equivariantly isomorphic to a direct sum of $\mathfrak{sl}_2(\mathbf{C})$ and copies of the "standard 2-dimensional representation" (the only irreducible option with highest weight < 2 that doesn't introduce an additional weight-0 space for H). We conclude that $\Phi(G,T)$ is either $\{\pm a\}$ or $\{\pm a, \pm a/2\}$ and that $\dim(\mathfrak{g}_{\mathbf{C}})_{\pm a} = 1$.

It remains to rule out that possibility that the weights $\pm a/2$ also occur as T-weights on $\mathfrak{g}_{\mathbf{C}}$. Let's suppose these weights do occur, so $a \in 2\mathrm{X}(T)$, and let $b = a/2$ for typographical simplicity. Choose a nonzero $X \in (\mathfrak{g}_{\mathbf{C}})_b$, and let $\overline{X}$ be the complex conjugate of X inside the complex conjugate $(\mathfrak{g}_{\mathbf{C}})_{-b}$ of $(\mathfrak{g}_{\mathbf{C}})_b$ (this "makes sense" since the T-action on $\mathfrak{g}_{\mathbf{C}}$ respects the $\mathbf{R}$-structure $\mathfrak{g}$, and the complex conjugate of t^b is t^{-b} for $t \in T = S^1$). Although we cannot argue that $(\mathfrak{g}_{\mathbf{C}})_{\pm b}$ is commutative, since b isn't the highest weight, we can nonetheless use:

Lemma 1.3. *The element $[X, \overline{X}]$ is nonzero.*

Proof. The elements $v := X + \overline{X}$ and $v := i(X - \overline{X})$ in $(\mathfrak{g}_{\mathbf{C}})_b \oplus (\mathfrak{g}_{\mathbf{C}})_{-b} \subset \mathfrak{g}_{\mathbf{C}}$ are visibly nonzero (why?) and linearly indepedent over $\mathbf{R}$, and they are invariant under complex conjugation on $\mathfrak{g}_{\mathbf{C}}$, so v and v' lie in $\mathfrak{g}$. If $[X, \overline{X}] = 0$ then clearly $[v, v'] = 0$, so v and v' would span a *two-dimensional* commutative Lie subalgebra of $\mathfrak{g}$. But we saw above via 1-parameter subgroups that such a Lie subalgebra creates a torus inside G with dimension at least 2, contradicting the assumption that G has rank 1. ∎

Since $H' := [X, \overline{X}] \in \mathfrak{t}_{\mathbf{C}}$, we can use X, $\overline{X}$, and H' (after preliminary $\mathbf{C}^{\times}$-scalings) to generate *another* Lie subalgebra inclusion $\mathfrak{sl}_2(\mathbf{C}) \hookrightarrow \mathfrak{g}_{\mathbf{C}}$ such that the diagonal subalgebra of $\mathfrak{sl}_2(\mathbf{C})$ is $\mathfrak{t}_{\mathbf{C}}$. Let H_0 be the "standard" diagonal element of $\mathfrak{sl}_2(\mathbf{C})$, so our new embedding of $\mathfrak{sl}_2(\mathbf{C})$ into $\mathfrak{g}_{\mathbf{C}}$ identifies $\mathrm{Lie}(b)$ with the weight 2 for the action of H_0. Since $a = 2b$ we see that $\mathfrak{g}_{\mathbf{C}}$ as an $\mathfrak{sl}_2(\mathbf{C})$-representation has H_0-weights 0, ± 2, ± 4, with the weight-0 space just the line $\mathfrak{t}_{\mathbf{C}}$ and the highest-weight line (for weight 4) equal to $(\mathfrak{g}_{\mathbf{C}})_a$ that we have already seen is 1-dimensional. Consequently, as an $\mathfrak{sl}_2(\mathbf{C})$-representation, $\mathfrak{g}_{\mathbf{C}}$ must contain a copy of the 5-dimensional irreducible representation V_4 with highest weight 4.

If $W \subset \mathfrak{g}_{\mathbf{C}}$ is any $\mathfrak{sl}_2(\mathbf{C})$-subrepresentation that contains the weight-0 line then $W \cap V_4$ is nonzero and thus coincdes with V_4 (since V_4 is irreducible); i.e., $V_4 \subset W$. This V_4 contains a weight-0 line that must be $\mathfrak{t}_{\mathbf{C}}$, and it also exhausts the lines $(\mathfrak{g}_{\mathbf{C}})_{\pm a}$ with weights ± 4. But the new copy of $\mathfrak{sl}_2(\mathbf{C})$ that we have built as a Lie subalgebra of $\mathfrak{g}_{\mathbf{C}}$ is a V_2, which has a weight-0 line and thus contains the unique copy of V_4, an absurdity.

This contadiction shows that the case $\Phi(G,T) = \{\pm a, \pm a/2\}$ cannot occur, so in the rank-1 non-commutative case we have shown $\dim G = 3$, as desired. (In class we deduced from the condition $\dim G = 3$ that G is isomorphic to either SO(3) or SU(2).)

2. Centralizers in higher rank

Now consider a pair (G,T) with maximal torus $T \neq G$ (equivalently, $\Phi := \Phi(G,T) \neq \emptyset$), and $\dim T > 0$ arbitrary. Choose $a \in \Phi$, so $Z_G(T_a)/T_a$ is either SO(3) or SU(2). Hence, $Z_G(T_a)$ sits in the middle of a short exact sequence

$$1 \to T_a \to Z_G(T_a) \to H \to 1$$

with H non-commutative of rank 1. In this section, we shall explicit describe *all* Lie group extensions of such an H by a torus. This provides information about the structure of the group $Z_G(T_a)$ that we shall later use in our proof that the commutator subgroup $G' = [G,G]$ of G is closed and perfect (i.e., $(G')' = G'$) in general.

Lemma 2.1. *Let $q : \widetilde{H} \to H$ be an isogeny between connected Lie groups with $\pi_1(\widetilde{H}) = 1$. For any connected Lie group G and isogeny $f : G \to H$, there is a unique Lie group homomorphism $F : \widetilde{H} \to G$ over H; i.e., a unique way to fill in a commutative diagram*

$$\begin{array}{ccc} \widetilde{H} & \xrightarrow{F} & G \\ & {\scriptstyle q}\searrow & \downarrow{\scriptstyle f} \\ & & H \end{array}$$

Moreover, F is an isogeny.

For our immediate purposes, the main case of interest is the degree-2 isogeny $q : \mathrm{SU}(2) \to \mathrm{SO}(3)$. Later we will show that if H is a connected compact Lie group with finite center then such a q always exists with $\widetilde{H}$ compact too. The lemma then says that such an $\widetilde{H}$ *uniquely* sits "on top" of all isogenous connected covers of H.

Proof. Consider the pullback $\widetilde{f} : \widetilde{G} := G \times_H \widetilde{H} \to \widetilde{H}$ of the isogeny f along q, as developed in HW7 Exercise 4, so $\widetilde{f}$ is surjective with kernel $\ker f$ that 0-dimensional and central in $\widetilde{G}$. In particular, $\mathrm{Lie}(\widetilde{f})$ is surjective with kernel $\mathrm{Lie}(\ker f) = 0$, so $\mathrm{Lie}(\widetilde{f})$ is an isomorphism. Hence, $\widetilde{f}^0 : \widetilde{G}^0 \to \widetilde{H}$ is a map between *connected* Lie groups that is an isomorphism on Lie algebras, so it is surjective and its kernel Γ is also 0-dimensional and central. Thus, as we saw on HW5 Exercise 4(iii), there is a surjective homomorphism $1 = \pi_1(\widetilde{H}) \twoheadrightarrow \Gamma$, so $\Gamma = 1$ and hence $\widetilde{f}^0$ is an isomorphism. Composing its inverse with $\mathrm{pr}_1 : \widetilde{G} \to G$ defines an F fitting into the desired commutative diagram, and $\mathrm{Lie}(F)$ is an isomorphism since f and q are isogenies. Thus, F is also an isogeny.

It remains to prove the uniqueness of an F fitting into such a commutative diagram. If $F' : \widetilde{H} \to G$ is a map fitting into the commutative diagram then the equality $f \circ F' = f \circ F$ implies that for all $\widetilde{h} \in \widetilde{H}$ we have $F'(\widetilde{h}) = \phi(\widetilde{h})F(\widetilde{h})$ for a unique $\phi(\widetilde{h}) \in \ker f$. Since $\ker f$ is central in G, it follows that $\phi : \widetilde{H} \to \ker f$ is a homomorphism since F and F' are, and ϕ is continuous since F' and F are continuous (and G is a topological group). But $\ker f$ is discrete and $\widetilde{H}$ is connected, so ϕ must be constant and therefore trivial; i.e., $F' = F$. ■

Proposition 2.2. *Consider an exact sequence of compact connected Lie groups* $1 \to S \to G \to H \to 1$ *with* S *a central torus in* G *and* H *non-commutative of rank* 1*.*

(1) *The commutator subgroup* G' *is closed in* G *and* $G' \to H$ *is an isogeny, with the given exact sequence split group-theoretically if and only if* $G' \to H$ *is an isomorphism, in which case there is a spliting via* $S \times H = S \times G' \to G$ *(using multiplication).*

(2) *The isogeny* $G' \to H$ *is an isomorphism if* $H \simeq \mathrm{SU}(2)$ *or* $G' \simeq \mathrm{SO}(3)$*, and otherwise* $S \cap G'$ *coincides with the order-*2 *center* μ *of* $G' \simeq \mathrm{SU}(2)$ *and* $G = (S \times G')/\mu$

Proof. We know that H is isomorphic to either SO(3) or SU(2). First we treat the case $H = \mathrm{SU}(2)$, and then we treat the case $H = \mathrm{SO}(3)$ via a pullback argument to reduce to the case of SU(2). Assuming $H = \mathrm{SU}(2)$, we get an exact sequence of Lie algebras

$$0 \to \mathfrak{s} \to \mathfrak{g} \to \mathfrak{su}(2) \to 0 \tag{2.1}$$

with $\mathfrak{s}$ in the Lie-algebra center $\ker \mathrm{ad}_{\mathfrak{g}}$ of $\mathfrak{g}$ since $\mathrm{ad}_{\mathfrak{g}} = \mathrm{Lie}(\mathrm{Ad}_G)$ and S is central in G.

We shall prove that this is uniquely split as a sequence of Lie algebras This says exactly that there is a unique Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ such that $\mathfrak{g}' \to \mathfrak{h}$ is an isomorphism, as then the vanishing of the bracket between $\mathfrak{s}$ and $\mathfrak{g}'$ implies that $\mathfrak{g} = \mathfrak{s} \times \mathfrak{g}'$ as Lie algebras, providing the unique splitting of the sequence (2.1) using the isomorphism of $\mathfrak{g}'$ onto $\mathfrak{h}$.

Since $H \simeq \mathrm{SU}(2)$, it follows that $\mathfrak{h} \simeq \mathfrak{su}(2)$ is its own commutator subalgebra (see HW4 Exercise 3(ii)). Hence, if there is to be a splitting of (2.1) as a direct product of $\mathfrak{s}$ and a Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ mapping isomorphically onto $\mathfrak{h}$ then *necessarily* $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. Our splitting assertion for Lie algebras therefore amounts to the claim that the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ maps isomorphically onto $\mathfrak{h}$. This isomorphism property is something which is necessary and sufficient to check after extension of scalars to $\mathbf{C}$, so via the isomorphism $\mathfrak{su}(2)_{\mathbf{C}} \simeq \mathfrak{sl}_2(\mathbf{C})$ it is sufficient to show that over a field k of characteristic 0 (such as $\mathbf{C}$) *any* "central extension" of Lie algebras

$$0 \to \mathfrak{s} \to \mathfrak{g} \to \mathfrak{sl}_2 \to 0 \tag{2.2}$$

(i.e., $\mathfrak{s}$ is killed by $\mathrm{ad}_{\mathfrak{g}}$) is split, as once again this is precisely the property that $[\mathfrak{g}, \mathfrak{g}]$ maps isomorphically onto $\mathfrak{sl}_2$ (which is its own commutator subalgebra, by HW4 Exercise 3(iii)).

Now we bring in the representation theory of $\mathfrak{sl}_2$. Consider the Lie algebra representation

$$\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) := \mathrm{End}(\mathfrak{g}).$$

This factors through the quotient $\mathfrak{g}/\mathfrak{s} = \mathfrak{sl}_2$ since $\mathfrak{s}$ is central in $\mathfrak{g}$, and so as such defines a representation of $\mathfrak{sl}_2$ on $\mathfrak{g}$ lifting the adjoint representation of $\mathfrak{sl}_2$ on itself (check!). Note that since $\mathfrak{s}$ is in the Lie-algebra center of $\mathfrak{g}$, it is a direct sum of copies of the trivial representation (the action via "zero") of $\mathfrak{sl}_2$, so (2.2) presents the $\mathfrak{sl}_2$-representation $\mathfrak{g}$ as an extension of the irreducible adjoint representation by a direct sum of copies of the trivial representation.

By the complete reducibility of the finite-dimensional representation theory of $\mathfrak{sl}_2$, we conclude that the given central extension of Lie algebras admits an $\mathfrak{sl}_2$-equivariant splitting as a representation space, which is to say that there is a k-linear subspace $V \subset \mathfrak{g}$ stable under the $\mathfrak{sl}_2$-action that is a linear complement to $\mathfrak{s}$. We need to show that V a Lie subalgebra of $\mathfrak{g}$. By the very definition of the $\mathfrak{sl}_2$-action on $\mathfrak{g}$ via the *central quotient* presentation $\mathfrak{g}/\mathfrak{s} \simeq \mathfrak{sl}_2$, it follows that V is stable under the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$ of $\mathfrak{g}$ on itself, so certainly V is stable under Lie bracket against itself (let alone against the entirety of $\mathfrak{g}$).

Now returning to (2.1) that we have split, we have found a copy of $\mathfrak{h} = \mathfrak{su}(2)$ inside $\mathfrak{g}$ lifting the quotient $\mathfrak{h}$ of $\mathfrak{g}$. But H is *connected* and $\pi_1(H) = 1$, so by the Frobenius handout this inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ "integrates" to a Lie group homomorphism $H \to G$, and its composition $H \to H$ with the quotient map $G \to H$ is the identity map (as that holds on the level of Lie algebras). Thus, we have built a Lie group section $H \to G$ to the quotient map, and since S is central in G it follows that the multiplication map $S \times H \to G$ is a Lie group homomorphism. This latter map is visibly bijective, so it is an isomorphism of Lie groups. This provides a splitting when $H = \mathrm{SU}(2)$, and in such cases the commutator subgroup G' of G is visibly this direct factor H (forcing uniqueness of the splitting) since S is central and $\mathrm{SU}(2)$ *is its own commutator subgroup* (HW7, Exercise 3(i)). In particular, G' is closed in G.

For the remainder of the proof we may assume $H = \mathrm{SO}(3)$. Let $\widetilde{H} = \mathrm{SU}(2)$ equipped with the degree-2 isogeny $q : \widetilde{H} \to H$ in the usual manner. Now form the pullback central extension of Lie groups (see HW7 Exercise 4 for this pullback construction)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S & \longrightarrow & E & \longrightarrow & \widetilde{H} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle q} & & \\ 1 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 1 \end{array}$$

We may apply the preceding considerations to the top exact sequence as long as E is compact and connected. Since E is an S-fiber bundle over $\widetilde{H}$, E inherits connectedness from $\widetilde{H}$ and S, and likewise for compactness. We conclude that the top exact sequence is *uniquely split*. In particular, the commutator subgroup E' is closed in E and maps isomorphically onto $\widetilde{H} = \mathrm{SU}(2)$.

The compact image of E' in G is visibly the commutator subgroup, so G' is closed with the surjective $E' \to G'$ sandwiching G' in the middle of the degree-2 covering $\widetilde{H} \to H$! Thus, the two maps $\widetilde{H} = E' \to G'$ and $G' \to H$ are isogenies whose degrees have product equal to the degree 2 of $\widetilde{H}$ over H. A degree-1 isogeny is an isomorphism, so either $G' \to H$ is an isomorphism or $G' \to H$ is identified with $\widetilde{H} \to H$ (and in this latter case the identification "over H" is unique, by Lemma 2.1).

Since $G' \to H = \mathrm{SO}(3)$ is an isogeny, the maximal tori of G' map isogenously onto those of H and so have dimension 1. Hence, abstractly G' is isomorphic to $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. In the latter case G' has trivial center (as $\mathrm{SO}(3)$ has trivial center; see HW7 Exercise 1(iii)), so the isogeny $G' \to H$ cannot have nontrivial kernel and so must be an isomorphism. In other words, G' is isomorphic to $\mathrm{SO}(3)$ abstractly as Lie groups if and only if the isogeny $G' \to H$ is an isomorphism. In such cases, the multiplication homomorphism of Lie groups $S \times G' \to G$ is visibly bijective, hence a Lie group isomorphism, so the given exact sequence is split as Lie groups (and the splitting is unique since $G = \mathrm{SO}(3)$ is its own commutator subgroup).

Now we may and do suppose G' is abstractly isomorphic to $\mathrm{SU}(2)$ as Lie groups, so the isogeny $G' \to H \simeq \mathrm{SO}(3)$ must have nontrivial central kernel, yet $\mathrm{SU}(2)$ has center $\{\pm 1\}$ of order 2. Hence, the map $G' \to H$ provides a specific identification of H with the quotient $\mathrm{SU}(2)/\{\pm 1\} = \mathrm{SO}(3)$ of $\mathrm{SU}(2)$ modulo its center. In these cases the given exact sequence cannot be split even group-theoretically, since a group-theoretic splitting of G as a direct product of the commutative S and the perfect H would force the commutator subgroup of G to coincide with this copy of the centerless H, contradicting that in the present circumstances $G' = \mathrm{SU}(2)$ has nontrivial center.

Consider the multiplication map $S \times G' \to G$ that is not an isomorphism of groups (since we have seen that there is no group-theoretic splitting of the given exact sequence). This is surjective since $G' \to H = G/S$ is surjective, so its kernel must be nontrivial. But the kernel is $S \cap G'$ anti-diagonally embedded via $s \mapsto (s, 1/s)$, and this has to be a nontrivial *central* subgroup of $G' = \mathrm{SU}(2)$. The only such subgroup is the order-2 center (on which inversion has no effect), so we are done. ■