Math 210C. Simple factors

1. Introduction

Let G be a nontrivial connected compact Lie group, and let $T \subset G$ be a maximal torus. Assume Z_G is finite (so $G = G'$, and the converse holds by Exercise 4(ii) in HW9). Define $\Phi = \Phi(G,T)$ and $V = \mathrm{X}(T)_{\mathbf{Q}}$, so (V, Φ) is a nonzero root system.

By the handout "Irreducible components of root systems", (V, Φ) is *uniquely* a direct sum of irreducible components $\{(V_i, \Phi_i)\}$ in the following sense. There exists a unique collection of nonzero $\mathbf{Q}$-subspaces $V_i \subset V$ such that for $\Phi_i := \Phi \cap V_i$ the following hold: each pair (V_i, Φ_i) is an irreducible root system, $\oplus V_i = V$, and $\coprod \Phi_i = \Phi$.

Our aim is to use the unique irreducible decomposition of the root system (which involves the choice of T, unique up to G-conjugation) and some Exercises in HW9 to prove:

Theorem 1.1. *Let $\{G_j\}_{j \in J}$ be the set of minimal non-trivial connected closed normal subgroups of G.*

(i) *The set J is finite, the G_j's pairwise commute, and the multiplication homomorphism*
$$\prod G_j \to G$$
is an isogeny.

(ii) *If $Z_G = 1$ or $\pi_1(G) = 1$ then $\prod G_i \to G$ is an isomorphism (so $Z_{G_j} = 1$ for all j or $\pi_1(G_j) = 1$ for all j respectively).*

(iii) *Each connected closed normal subgroup $N \subset G$ has finite center, and $J' \mapsto G_{J'} = \langle G_j \rangle_{j \in J'}$ is a bijection between the set of subsets of J and the set of connected closed normal subgroups of G.*

(iv) *The set $\{G_j\}$ is in natural bijection with the set $\{\Phi_i\}$ via $j \mapsto i(j)$ defined by the condition $T \cap G_j = T_{i(j)}$. Moreover, $T_{i(j)}$ is a maximal torus in G_j and $\Phi_{i(j)} = \Phi(G_j, T_{i(j)})$.*

We emphasize that the collection of G_j's does *not* involve a choice of T (and though T is only unique up to conjugacy, by normality each G_j is preserved by G-conjugation). Parts (i) and (iv) (along with Exercise 4(ii) in HW9) are the reason that in the study of general connected compact Lie groups H, by far the most important case is that of semisimple H with an irreducible root system.

Note that by (i) and (iii), if N is a connected closed normal subgroup of G then every connected closed normal subgroup of N is normalized by *every* G_j and hence is normal in G! Thus, normality is *transitive* for connected closed subgroups of a semisimple connected compact Lie group (with all such subgroups also semisimple, by (iii)). In particular, if $J' \subset J$ corresponds to N as in (iii) then $\{G_j\}_{j \in J'}$ is also the set of minimal nontrivial connected closed normal subgroups of N. As a special case, each G_j has no nontrivial connected closed normal subgroups; for this reason, one often says that each G_j is "almost simple" and the G_j are called the "simple factors" of G (even though they are generally not direct factors of G except when $Z_G = 1$ or $\pi_1(G) = 1$, by (ii)).

Remark 1.2. By Corollary 2.6 in the handout "A non-closed commutator subgroup", every normal connected Lie subgroup of a semisimple connected compact Lie group is *automatically* closed (so closedness can be dropped as a hypothesis on N above). The proof of that rests on Ado's theorem in the theory of Lie algebras, which in turn lies rather beyond the level of this course. Thus, in this handout we never make use of that closedness result.

It is nonetheless of interest to record one consequence: G is almost simple if and only if the non-commutative Lie algebra $\mathfrak{g}$ is simple over $\mathbf{R}$ (i.e., is non-commutative and has no nonzero proper Lie ideal). Indeed, by Corollary 4.6 in the "Frobenius Theorem" handout we know that Lie ideals in $\mathfrak{g}$ are in bijective correspondence with connected Lie subgroups $H \subset G$ such that $\mathrm{Lie}(N_G(H)) = \mathfrak{g}$, or equivalently (by dimension reasons and the connectedness of G) that $N_G(H) = G$, and this equality is precisely the normality of H in G. Thus, $\mathfrak{g}$ is simple if and only if G has no nontrivial connected normal proper Lie subgroups, and that is the "almost simple" property as defined above since every connected normal Lie subgroup of the semisimple compact connected G is *closed*.

By using the theory of root systems for semisimple Lie algebras over $\mathbf{C}$, one can go further and deduce that when $\mathfrak{g} = \mathrm{Lie}(G)$ is simple over $\mathbf{R}$ then $\mathfrak{g}_{\mathbf{C}}$ is simple over $\mathbf{C}$. Such "absolute simplicity" for $\mathfrak{g}$ is a special feature of the simple Lie algebras arising from *compact* (connected semisimple) Lie groups. That is, if H is a general connected Lie group for which $\mathfrak{h}$ is simple over $\mathbf{R}$ then $\mathfrak{h}_{\mathbf{C}}$ is generally not simple over $\mathbf{C}$; examples of such (non-compact) H are provided by the underlying real Lie group of many important connected matrix groups over $\mathbf{C}$ such as $\mathrm{SL}_n(\mathbf{C})$ ($n \geq 2$) and $\mathrm{Sp}_{2n}(\mathbf{C})$ ($n \geq 1$).

Part (iv) in Theorem 1.1 makes precise the connection between almost-simple factors of G and the irreducible components of Φ, and yields that G is almost simple if and only if Φ is irreducible.

The proof of Theorem 1.1 will involve initially constructing the G_j's in a manner very different from their initial definition, using the Φ_i's. It will not be apparent at the outset that the G_j's built in this alternative way are normal (or even independent of T!), but the construction in terms of the Φ_i's will be essential for showing that these subgroups pairwise commute and generate G (from which we will eventually deduce that they do recover the G_j's as defined at the start of Theorem 1.1, so are normal and independent of T).

2. Constructions via roots

We begin by using the $\mathbf{Q}$-subspaces $V_i \subset V = \mathrm{X}(T)_{\mathbf{Q}}$ to build subtori $T_i \subset T$. This might initially seem surprising since the character-lattice functor is contravariant, but keep in mind that we have not just a single V_i but an entire collection of V_i's giving a direct-sum decomposition of V, so we also have projections $V \twoheadrightarrow V_i$:

Lemma 2.1. *There are unique subtori $T_i \subset T$ such that $\prod T_i \to T$ is an isogeny and the resulting isomorphism $\prod \mathrm{X}(T_i)_{\mathbf{Q}} \simeq \mathrm{X}(T)_{\mathbf{Q}} = V$ identifies $\mathrm{X}(T_i)_{\mathbf{Q}}$ with V_i.*

Proof. A subtorus $S \subset T$ is "the same" as a short exact sequence of tori

$$1 \to S \to T \to \overline{T} \to 1$$

and so is "the same" as a quotient map $\mathrm{X}(T) \twoheadrightarrow L = \mathrm{X}(S)$ onto a finite free $\mathbf{Z}$-module. But such quotient maps correspond to saturated subgroups $\Lambda \subset \mathrm{X}(T)$ (i.e., subgroups with torsion-free cokernel), and as such are in bijective correspondence with subspaces $W \subset V$ (via the inverse operations $\Lambda \mapsto \Lambda_{\mathbf{Q}}$ and $W \mapsto W \cap \mathrm{X}(T)$). This defines a bijective correspondence between the sets of subtori $S \subset T$ and quotient maps $\pi : V \twoheadrightarrow \overline{V}$. (Explicitly, from π we form the saturated subgroup $\Lambda = \mathrm{X}(T) \cap \ker(\pi)$ of $\mathrm{X}(T)$, and by saturatedness the common kernel S of all characters in Λ is connected; i.e., a subtorus of T. Conversely, to a subtorus $S \subset T$ we associate the quotient $\mathrm{X}(S)_{\mathbf{Q}}$ of $V = \mathrm{X}(T)_{\mathbf{Q}}$.)

Thus, to each projection $V = \bigoplus_i V_i \twoheadrightarrow V_{i_0}$ we get an associated subtorus $T_{i_0} \subset T$ for which $\mathrm{X}(T_{i_0})_{\mathbf{Q}}$ is identified with V_{i_0} as a quotient of $\mathrm{X}(T)_{\mathbf{Q}} = V$. The resulting multiplication map $f : \prod T_i \to T$ induces a map of character lattices $\mathrm{X}(T) \to \prod \mathrm{X}(T_i)$ whose rationalization is precisely the sum $V \to \oplus V_i$ of the projection maps (why?), and this sum is an isomorphism by design (why?). Hence, $\mathrm{X}(f)_{\mathbf{Q}}$ is an isomorphism, so f is an isogeny. This shows that the T_i's as just built do the job. The method of construction also yields the desired uniqueness of the collection of T_i's (check!). ■

For each $i \in I$, let $S_i \subset T$ be the subtorus generated by the T_k's for $k \neq i$. Keeping in mind the desired goal in Theorem 1.1(iv), for each $i \in I$ we are motivated to define

$$G_i = Z_G(S_i)'.$$

By Exercise 4(ii) in HW9 applied to the connected compact Lie group $H = Z_G(S_i)$, we see that each G_i is its own derived group and more specifically has finite center.

Beware that at this stage the subgroups $G_i \subset G$ might depend on T, and it is not at all apparent if they are normal in G. In particular, we do not yet know if these groups coincide with the ones as defined at the start of Theorem 1.1! We will prove that the G_i's as just defined satisfy all of the desired properties in the first 3 parts of Theorem 1.1, from which we will deduce that they do recover the G_j's as at the start of Theorem 1.1 (in accordance with the recipe in part (iv)).

The first step in the analysis of the G_i's is to link them to the Φ_i's in a direct manner:

Lemma 2.2. *For each i, $T_i \subset G_i$ and T_i is maximal as a torus in G_i (so $T \cap G_i = T_i$). Moreover, $\Phi(G_i, T_i) = \Phi_i$ inside $\mathrm{X}(T_i)_{\mathbf{Q}} = V_i$ and G_i is generated by the 3-dimensional subgroups G_a associated to pairs of opposite roots in $\Phi_i \subset \Phi$.*

Proof. The Lie algebra $\mathfrak{h}_i$ of $H_i := Z_G(S_i)$ inside $\mathfrak{g}$ has complexification $\mathfrak{g}_{\mathbf{C}}^{S_i}$. This is the span of $\mathfrak{t}_{\mathbf{C}}$ and the root lines for roots trivial on the subtorus $S_i \subset T$ generated by the T_k's for $k \neq i$.

The design of the T_k's implies that a root $a \in \Phi$ is trivial on T_k if and only if $a \notin \Phi_k$ (check!). Thus, $a|_{S_i} = 1$ if and only if $a \in \Phi_i$. Thus, $(\mathfrak{h}_i)_{\mathbf{C}}$ is spanned by $\mathfrak{t}_{\mathbf{C}}$ and the root lines for roots in Φ_i. In particular, for each $a \in \Phi_i$ the 3-dimensional connected Lie subgroup $G_a \subset G$ associated to $\pm a$ is contained in H_i because $\mathrm{Lie}(G_a) \subset \mathfrak{h}_i$ (as we may check after complexification, with $\mathrm{Lie}(G_a)_{\mathbf{C}}$ generated as a Lie algebra by the $\pm a$-root lines). Each G_a is perfect (being $\mathrm{SU}(2)$ or $\mathrm{SO}(3) = \mathrm{SU}(2)/\langle -1 \rangle$) and has maximal torus $a^\vee(S^1)$.

We conclude that the derived group G_i of H_i contains every such G_a, so the coroot $a^\vee : S^1 \to T$ for each $a \in \Phi_i$ factors through H_i. Such coroots generate T_i inside T: it is equivalent (why?) to show that they rationally span $\mathrm{X}_*(T_i)_{\mathbf{Q}}$ inside $\mathrm{X}_*(T)_{\mathbf{Q}} = V^*$, and the dual space $V_i^* = \mathrm{X}_*(T_i)_{\mathbf{Q}}$ for the root system dual to Φ_i is spanned by the associated coroots (as for dual root systems in general). This proves that $T_i \subset G_i$ for every i. But G_i has finite center, so its root system has rank equal to the common dimension of its maximal tori. The roots for $G_i = (H_i)'$ relative to the maximal torus $T \cap G_i$ coincides with the T-roots of H_i, and we have seen above that $\Phi(H_i, T) = \Phi_i$ inside $X(T)_{\mathbf{Q}}$. This has $\mathbf{Q}$-span $V_i = X(T_i)_{\mathbf{Q}}$ of dimension $\dim T_i$, so the maximal tori of G_i have dimension $\dim T_i$. Hence, the torus $T_i \subset G_i$ is maximal! This argument also identifies Φ_i with $\Phi(G_i, T_i)$ in the desired manner.

Every semisimple connected compact Lie group is generated by the 3-dimensional subgroups associated to pairs of opposite roots. Such subgroups for G_i are precisely the G_a's from G for $a \in \Phi_i$ since the T_i-root lines in $(\mathfrak{g}_i)_{\mathbf{C}}$ are precisely the T-root lines in $\mathfrak{g}_{\mathbf{C}}$ for the T-roots in Φ_i. ■

We now use the preceding lemma to deduce that the G_i's pairwise commute, so there is then a multiplication homomorphism $\prod G_i \to G$. A criterion for connected Lie subgroups to commute is provided by:

Lemma 2.3. *Connected Lie subgroups H_1 and H_2 of a Lie group G commute if and only if their Lie algebras $\mathfrak{h}_i$ commute inside $\mathfrak{g}$.*

Note that we do not assume closedness of the H_j's in G.

Proof. First assume H_1 and H_2 commute. For any $h_1 \in H_1$, the conjugation c_{h_1} of h_1 on G is trivial on H_2, so $\mathrm{Ad}_G(h_1)$ is trivial on $\mathfrak{h}_2$. That is, the representation $\mathrm{Ad}_G|_{H_1}$ of H_1 on $\mathfrak{g}$ is trivial on the subspace $\mathfrak{h}_2$. By visualizing this in terms of matrices via a basis of $\mathfrak{g}$ extending a basis of $\mathfrak{h}_2$, differentiating gives that $\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{h}_1}$ vanishes on $\mathfrak{h}_2$. This says that $[X_1, \cdot]$ kills $\mathfrak{h}_2$ for all $X_1 \in \mathfrak{h}_1$, which is the desired conclusion that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ commute.

Now assume conversely that $[X_1, X_2] = 0$ for all $X_i \in \mathfrak{h}_i$. We want to deduce that H_1 and H_2 commute. Since H_i is generated by any open neighborhood of the identity due to connectedness, it suffices to check neighborhoods of the identity in each H_i commute with each other. Thus, letting exp denote $\exp_G$, it suffices to show that for any $X_i \in \mathfrak{h}_i$ we have

$$\exp(X_1)\exp(tX_2)\exp(-X_1) = \exp(tX_2)$$

for all $t \in \mathbf{R}$ (as then by setting $t = 1$ we conclude by choosing X_i in an open neighborhood of 0 in $\mathfrak{h}_i$ on which $\exp_{H_i} = \exp_G|_{H_i}$ is an isomorphism onto an open image in H_i).

Both sides of this desired identity are 1-parameter subgroups $\mathbf{R} \to G$, so they agree if and only if their velocities at $t = 0$ are the same. The right side has velocity X_2 at $t = 0$, and by the Chain Rule (check!) the left side has velocity

$$(\mathrm{Ad}_G(\exp(X_1)))(X_2).$$

But $\mathrm{Ad}_G(\exp(X)) \in \mathrm{GL}(\mathfrak{g})$ coincides with $e^{[X,\cdot]}$ for any $X \in \mathfrak{g}$ (proof: replace X with tX for $t \in \mathbf{R}$ to make this a comparison of 1-parameter subgroups $\mathbf{R} \to \mathrm{GL}(\mathfrak{g})$, and compare velocities at $t = 0$ using that $\mathrm{Lie}(\mathrm{Ad}_G) = \mathrm{ad}_{\mathfrak{g}}$), so

$$(\mathrm{Ad}_G(\exp(X_1)))(X_2) = (e^{[X_1,\cdot]})(X_2).$$

Since $[X_1, \cdot]$ kills X_2 by hypothesis, the final expression on the right side collapses to X_2 (due to the expansion $e^T = \mathrm{id} + T + T^2/2! + T^3/3! + \ldots$ for any linear endomorphism T of a finite-dimensional $\mathbf{R}$-vector space). ■

Let's now check that the $\mathfrak{g}_i$'s pairwise commute inside $\mathfrak{g}$. It suffces to check this after extending scalars to $\mathbf{C}$. The advantage of such scalar extension is that $(\mathfrak{g}_i)_{\mathbf{C}}$ is generated as a Lie algebra by the T_i-root lines (recall that by considerations with SU(2) and SO(3), any pair of opposite root lines have Lie bracket that is a nonzero vector in the corresponding coroot line over $\mathbf{C}$), so to check the pairwise-commuting property it suffices to check that the root lines for roots $a \in \Phi_i$ and $b \in \Phi_{i'}$ commute with each other for $i \neq i'$. By T-weight considerations, the bracket $[(\mathfrak{g}_{\mathbf{C}})_a, (\mathfrak{g}_{\mathbf{C}})_b]$ is contained in the T-weight space for the character $a+b$. But $a+b \neq 0$ and $a+b$ is not a root since a, b lie in *distinct* irreducible components of Φ. This completes the proof that the G_i's pairwise commute.

Now consider the multiplication homomorphism

$$m : \prod G_i \to G.$$

This restriction to the multiplication map $\prod T_i \to T$ between maximal tori that is an isogeny by design of the T_i's, so by dimension considerations with maximal tori we see that the connected normal subgroup $(\ker m)^0$ contains no nontrivial torus (why not?). Hence, the connected compact Lie group $(\ker m)^0$ is trivial, which is to say $\ker m$ is finite. But $\mathrm{Lie}(m)$ is also surjective since its complexification hits every root line in $\mathfrak{g}_{\mathbf{C}}$ (as well as hits the entirety of $\mathfrak{t}_{\mathbf{C}}$), so $\mathrm{Lie}(m)$ is an isomorphism. Thus, m is an isogeny.

Since m is surjective, it follows that every $g \in G$ is a commuting product of some elements $g_i \in G_i$ (one per i). But the effect of conjugation on G by $\prod g_i$ clearly preserves each G_{i_0} (as G_{i_0} commutes with G_i for all $i \neq i_0$), so each G_i is *normal* in G. Applying g-conjugation carries the entire construction resting on T to the one resting on gTg^{-1}, so normality of the G_i's and the conjugacy of maximal tori in G thereby implies that the collection of subgroups $\{G_i\}$ is *independent* of T.

We have not completed the proof of Theorem 1.1(i) because we have not yet proved that $\{G_i\}$ coincides with the set of minimal nontrivial connected closed normal subgroups of G. But conditional on that, (i) is proved and (ii) is then immediate: if $\pi_1(G) = 1$ then the connected covering space $\prod G_i \to G$ is an isomorphism (see the end of Exercise 3(ii) in HW9), and if $Z_G = 1$ then each $Z_{G_i} = 1$ (as the surjectivity of $\prod G_i \to G$ forces $Z_{G_i} \subset Z_G$) and so the central kernel of the isogeny $\prod G_i \to G$ is trivial.

3. Normal subgroups

We next prove that the G_i's are built above fulfill the requirements in (iii), from which we will deduce that they are precisely the minimal nontrivial connected closed normal subgroups of G. This will finally bring us in contact with the G_j's as defined at the start of Theorem 1.1, and complete the proofs of all 4 parts of the theorem.

Let N be a connected closed normal subgroup of G. The key task is to show that if $N \neq 1$ then N contains *some* G_i. Once this is proved, we can pass to $N/G_i \subset G/G_i$ (with root system $\oplus_{i' \neq i} \Phi_{i'}$ relative to the maximal torus T/T_i) to then conclude by dimension induction that in general N is generated by some of the G_i's (so in particular Z_N is finite, so $N = N'$!).

Let $\widetilde{N} = m^{-1}(N)^0$. Since N is connected and m is surjective, the natural map $\widetilde{N} \to N$ is surjective (so $\widetilde{N} \neq 1$) and $\widetilde{N}$ is a connected closed *normal* subgroup of $\prod G_i$. Thus, to prove that the subgroup $N \subset G$ contains some G_i it suffices to prove the same for the subgroup $\widetilde{N} \subset \prod G_i$. In other words, for this step we may rename $\prod G_i$ as G (as $\prod T_i$ as T) so that $G = \prod G_i$ and $T = \prod T_i$.

Let $p_i : G \to G_i$ be the projection. By nontriviality of N, some $p_i(N)$ is nontrivial. But $p_i(N)$ is certainly a connected closed normal subgroup of G_i. Thus, *provided that* G_i has no nontrivial proper connected closed normal subgroups, it would follows that $p_i(N) = G_i$, so since $G_i = G_i'$ and $p_i|_{G_i}$ is the identity map on G_i we see that p_i carries (N, G_i) *onto* G_i. But this commutator subgroup is contained in each of N and G_i since both N and G_i are normal in G, so the identity map $p_i : G_i \to G_i$ identifies (N, G_i) with G_i. We conclude that $G_i = (N, G_i) \subset N$, as desired, once we prove:

Lemma 3.1. *Each G_i contains no nontrivial proper connected closed normal subgroup.*

Proof. Since G_i has finite center and an irreducible root system, we may rename G_i as G to reduce to the case when Φ is irreducible. In this case we want to show that a nontrivial connected closed normal subgroup $N \subset G$ must coincide with G.

Let $S \subset N$ be a maximal torus, so $S \neq 1$ since $N \neq 1$, and let $T \subset G$ be a maximal torus containing S. Define $\Phi = \Phi(G, T)$; by hypothesis, Φ is irreducible. Since $T \cap N = S$ (since $Z_N(S) = S$), the action of the normalizer $N_G(T)$ on N preserves S. Hence, $\mathrm{X}(S)_{\mathbf{Q}}$ is a nonzero quotient representation of $\mathrm{X}(T)_{\mathbf{Q}}$ relative to the action of the Weyl group $W(G, T) = W(\Phi)$.

Finiteness of Z_G ensures that $\mathrm{X}(T)_{\mathbf{Q}}$ is the $\mathbf{Q}$-span of Φ, so by Exercise 1(ii) in HW9 and the irreducibility of Φ the action of $W(\Phi)$ on $\mathrm{X}(T)_{\mathbf{Q}}$ is (absolutely) irreducible! Thus, the quotient representation $\mathrm{X}(S)_{\mathbf{Q}}$ of $\mathrm{X}(T)_{\mathbf{Q}}$ must be full, which is to say $S = T$ (for dimension reasons). Hence, N contains some maximal torus T of G. But every element of G lies in a conjugate of T, and so by normality of N in G we conclude that $N = G$. ■

We have proved that every connected closed subgroup N of G has the form

$$N = G_{I'} := \langle G_{i'} \rangle_{i' \in I'}$$

for some subset $I' \subset I$. Since the multiplication map $m : \prod G_i \to G$ is an isogeny, it is clear that $N \times G_{I-I'} \to G$ is an isogeny for Lie algebra (or other) reasons. Hence, $N \cap G_{I-I'}$ is finite, so I' is uniquely characterized in terms of N as the set of $i \in I$ such that $G_i \subset N$. In particular, the G_i's are precisely the minimal nontrivial connected closed normal subgroups of G! Thus, the G_i's are exactly the G_j's as defined at the start of Theorem 1.1, and we have proved all of the desired properties for this collection of subgroups.