Math 210C. Irreducible decomposition of root systems

Let (V, Φ) be a root system over a field k of characteristic 0, with $V \neq 0$ (so $\Phi \neq \emptyset$). If it is reducible, then by definition there is a decomposition $\Phi = \Phi_1 \coprod \Phi_2$ so that $V = V_1 \oplus V_2$ with V_i the k-span of Φ_i and each (V_i, Φ_i) a root system. Continuing in this way, eventually we arrive at an expression for (V, Φ) as a direct sum of finitely many *irreducible* root systems. Also, in view of how the canonical $\mathbf{Q}$-structure of a root system is made, it is clear that irreducibility or not of a root system is insensitive to ground field extension and can be checked on the $\mathbf{Q}$-structure.

A (nonzero) root system with connected Dynkin diagram is certainly irreducible (as the diagram of a direct sum of root systems is easily seen to be the disjoint union of the diagrams of the factors). The converse is also true:

Lemma 0.1. *An irreducible root system has connected Dynkin diagram.*

This connectedness is an ingredient in the classification of possible Dynkin diagrams for irreducible root systems.

Proof. We consider (V, Φ) a root system with disconnected Dynkin diagram. We seek to show that (V, Φ) is reducible. Let B be a basis (which yields the diagram, as does any basis), let B_1 be the subset of B corresponding to the vertices in one connected component of the diagram, and let $B_2 = B - B_1 \neq \emptyset$. Upon choosing a $W(\Phi)$-invariant positive-definite quadratic form $q : V \to \mathbf{Q}$, the elements of B_1 are q-orthogonal to the elements of B_2, so their respective spans V_i are q-orthogonal. Thus, V_i is stable under the reflections on B_i and for $a_i \in B_i$ the reflections in a_1 and a_2 commute. But $W(\Phi)$ is generated by the reflections in B and $\Phi = W(\Phi).B$. It follows that $\Phi = \Phi_1 \coprod \Phi_2$ with $\Phi_i = \Phi \cap V_i$, so (V_i, Φ_i) is a root system (Exercise 3(iii), HW8) and $(V_1, \Phi_1) \times (V_2, \Phi_2) \simeq (V, \Phi)$. ■

Our main aim is to prove a strong uniqueness result concerning the "irreducible decomposition" of a nonzero root system. Unlike representations of finite groups (in characteristic zero), for which the irreducible subrepresentations are not uniquely determined as subspaces in case of multiplicities, the subspaces will be unique in the case of irreducible decomposition of a root system. More precisely:

Theorem 0.2. *Let $\{(V_i, \Phi_i)\}_{i \in I}$ be a finite collection of irreducible root systems, and let (V, Φ) be their direct sum (i.e., $V = \oplus V_i$ with $\Phi = \coprod \Phi_i$). If $\{(W_j, \Psi_j)\}_{j \in J}$ is a finite collection of irreducible root sytems and*

$$f : \prod (W_j, \Psi_j) \simeq (V, \Phi)$$

is an isomorphism then there must be a bijection $\sigma : J \simeq I$ and isomorphisms $f_j : (W_j, \Psi_j) \simeq (V_{\sigma(j)}, \Phi_{\sigma(j)})$ whose direct sum coincides with f.

Proof. Let B_j be a basis of (W_j, Ψ_j), so $\coprod B_j$ is carried by f onto a basis B for (V, Φ). Since f is an isomorphism of root systems, so it induces an isomorphism between the associated *finite* Weyl groups, the uniqueness of reflections inside a finite group in terms of the line that is negated implies that f is compatible with the reflections associated to the elements in the bases on both sides. That is, if $a \in B_j$ then f intertwines r_a and $r_{f(a)}$. These reflections determined the Cartan integers among pairs in the bases on both sides, so the connectedness of the Dynkin diagrams for (W_j, Ψ_j) implies that the elements in $f(B_j)$ are "orthogonal" (in the sense of Cartan integers) to those in $f(B_{j'})$ for any $j' \neq j$ whereas any two elements $a, a' \in f(B_j)$ can be linked to each other in finitely many steps $a = a_0, a_1, \dots, a_m = a'$ using elements of $f(B_j)$ in such a way that the Cartan integers n_{a_r, a_s} are nonzero.

As we vary j, it follows that the sets $f(B_j)$ constitute connected components of the diagram for (V, Φ, B). Since the decomposition $\prod(V_i, \Phi_i)$ of (V, Φ) provides a basis for (V, Φ) as a disjoint union of bases for the (V_i, Φ_i)'s and identifies $W(\Phi)$ with $\prod W(\Phi_i)$, we see from the Weyl-transitivity on the set of bases that *every* basis of (V, Φ) is a disjoint union of bases of the (V_i, Φ_i)'s. Thus, B has such a disjoint union decomposition: $B = \coprod B_i$ for a basis B_i of (V_i, Φ_i). The constituents B_i of this decomposition of B underlie the connected components of the diagram for (V, Φ, B) (in view of how the diagram is defined using Cartan integers), yet $\{f(B_j)\}$ is another such decomposition. Hence, the uniqueness of connected component decomposition for a Dynkin diagram implies that $f(B_j) = B_{\sigma(j)}$ for a bijection $\sigma : J \to I$, so f carries the span W_j of B_j onto the span $V_{\sigma(j)}$ of $B_{\sigma(j)}$.

For $a_j \in B_j$ and the associated root $f(a_j) \in B_{\sigma(j)}$, the reflections r_{a_j} and $r_{f(a_j)}$ must be intertwined by f. As we vary j, these reflections respectively generate $W(\Psi_j)$ and $W(\Phi_{\sigma(j)})$. But $W(\Psi_j).B_j = \Psi_j$ and $W(\Phi_{\sigma(j)}).B_{\sigma(j)} = \Phi_{\sigma(j)}$, so $f(\Psi_j) = \Phi_{\sigma(j)}$. In other words, f carries (W_j, Ψ_j) isomorphically onto $(V_{\sigma(j)}, \Phi_{\sigma(j)})$ for all j. ■