Math 210C. Calculation of some root systems

1. Introduction

The *classical groups* are certain non-trivial compact connected Lie groups with finite center: $\mathrm{SU}(n)$ for $n \geq 2$, $\mathrm{SO}(2m+1)$ for $m \geq 1$, $\mathrm{Sp}(n)$ for $n \geq 1$, and $\mathrm{SO}(2m)$ for $m \geq 2$. For the even special orthogonal groups one sometimes requires $m \geq 3$ (because in HW6 we showed $\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2$, so it is isogenous to a direct product of smaller such groups, whereas in all other cases the group is (almost) simple in the sense that its only proper normal closed subgroups are the subgroups of the finite center).

For each of these groups G, in the earlier handout on Weyl group computations we have specified a "standard" maximal torus T (for the purpose of doing computations) and we explicitly computed $W(G,T) = N_G(T)/T$ as a finite group equipped with its action on $\mathrm{X}(T)$ (or on $\mathrm{X}(T)_{\mathbf{Q}}$ if one wishes to not worry about isogeny issues). For example, in $G = \mathrm{SU}(n)$ we have $T = \{(z_1, \dots, z_n) \in (S^1)^n \mid \prod z_j = 1\}$ and $\mathrm{X}(T) = \mathbf{Z}^n/\Delta(\mathbf{Z})$ (quotient by the diagonally embedded $\mathbf{Z}$) with $W(G,T) = S_n$ acting through coordinate permutation in the usual way.

Remark 1.1. Working rationally, we can identify $\mathrm{X}(T)_{\mathbf{Q}}$ canonically as a direct summand of the rationalized character lattice $\mathbf{Q}^n$ of the diagonal maximal torus $(S^1)^n$ of $\mathrm{U}(n)$, namely $\mathrm{X}(T)_{\mathbf{Q}} = \{\vec{x} \in \mathbf{Q}^n \mid \sum x_j = 0\}$. This hyperplane maps isomorphically onto the quotient of $\mathbf{Q}^n$ modulo its diagonal copy of $\mathbf{Q}$, but the same is not true for the $\mathbf{Z}$-analogue: the map

$$\{\vec{x} \in \mathbf{Z}^n \mid \sum x_j = 0\} \to \mathbf{Z}^n/\Delta(\mathbf{Z}) = \mathrm{X}(T)$$

is injective with cokernel $\mathbf{Z}/n\mathbf{Z}$ of size n. (This corresponds to the quotient torus $T \twoheadrightarrow T/\mu_n$ modulo the center μ_n of $\mathrm{SU}(n)$; this quotient is a maximal torus of the centerless quotient $\mathrm{SU}(n)/Z_{\mathrm{SU}(n)}$.) But upon dualizing, the $\mathbf{Z}$-hyperplane $\mathrm{X}_*(T) \subset \mathbf{Z}^n$ is the locus defined by $\sum x_j = 0$ (check!).

We have also computed $\Phi(G,T) \subset \mathrm{X}(T)$ and the coroot $a^\vee \in \mathrm{X}_*(T) = \mathrm{Hom}(\mathrm{X}(T), \mathbf{Z})$ associated to each root $a \in \Phi(G,T)$ for $G = \mathrm{SU}(n)$. Explicitly, the roots are the classes modulo $\Delta(\mathbf{Z})$ of the differences $a_i - a_j$ among the standard basis vectors $\{a_1, \dots, a_n\}$ of $\mathbf{Z}^n$ (which conveniently do lie in the $\mathbf{Z}$-hyperplane $\sum x_j = 0$ in $\mathbf{Z}^n$). For $a = a_i - a_j \bmod \Delta(\mathbf{Z})$ the coroot $a^\vee \in \mathrm{X}_*(T) \subset \mathbf{Z}^n$ is $a_i^* - a_j^*$. The standard quadratic form $q = \sum x_i^2$ on $\mathbf{Q}^n$ is invariant under the standard action of S_n, so restricting this to the hyperplane $\sum x_i = 0$ gives an explicit positive-definite quadratic form that is Weyl-invariant. Under this quadratic form the roots have squared-length equal to 2, so the resulting identification $a^\vee = 2a/q(a) = a$ works out very cleanly. In particular, the root system is self-dual. For $n \geq 2$, the root system $\Phi(\mathrm{SU}(n), T)$ of rank $n-1$ is called A_{n-1} (or "type A" if the rank is not essential to the discussion).

In this handout, we work out analogous computations for the other classical groups, beginning with the case of $\mathrm{SO}(2m)$ ($m \geq 2$). Then we move on to $\mathrm{SO}(2m+1)$ ($m \geq 2$), which is only a tiny bit more work beyond the case of $\mathrm{SO}(2m)$. (We may ignore $\mathrm{SO}(3)$ since it is the central quotient $\mathrm{SU}(2)/\{\pm 1\}$ and we have already handled $\mathrm{SU}(2)$; recall that central isogeny has no effect on the root system $\Phi = \Phi(G,T)$ by Exercise 1 in HW7 – and so it also does not affect $W(\Phi)$, which we shall soon prove is equal to $W(G,T)$ – though it affects the character lattice of the maximal torus as a lattice inside the $\mathbf{Q}$-vector space of the root system.) We finish by treating $\mathrm{Sp}(n)$ for $n \geq 2$; we can ignore $\mathrm{Sp}(1)$ since it is $\mathrm{SU}(2)$ by another name.

Remark 1.2. The viewpoint of Dynkin diagrams will eventually explain why treating $\mathrm{SO}(3)$ separately from all other odd special orthogonal groups and treating $\mathrm{Sp}(1)$ separately from all other compact symplectic groups is a very natural thing to do. It is not just a matter of bookkeeping convenience.

2. Even special orthogonal groups

Let $G = \mathrm{SO}(2m)$ with $m \geq 2$. This has Lie algebra $\mathfrak{so}(2m)$ consisting of the skew-symmetric matrices in $\mathfrak{gl}_{2m}(\mathbf{R})$, of dimension $(2m)(2m-1)/2 = m(2m-1)$. We write the quadratic form underlying the definition of G as

$$\sum_{j=1}^{m}(x_j^2 + x_{-j}^2)$$

and view $\mathbf{R}^{2m}$ as having ordered basis denoted $\{e_1, e_{-1}, e_2, e_{-2}, \ldots, e_m, e_{-m}\}$. We take the standard maximal torus T to consist of the block matrix consisting of 2×2 rotation matrices r_{θ_j} on the plane $\mathbf{R}e_j \oplus \mathbf{R}e_{-j}$ for $1 \leq j \leq m$. This is a direct product of m circles in an evident manner, so $\mathrm{X}(T) = \mathbf{Z}^m$ on which the index-2 subgroup $W(G,T) \subset (\mathbf{Z}/2\mathbf{Z})^m \rtimes S_m$ acts via S_m-action in the standard way and $(\mathbf{Z}/2\mathbf{Z})^m$ acting through negation on the m coordinate lines.

To compute $\Phi(G,T) \subset \mathbf{Z}^m$ and $\Phi(G,T)^\vee \subset \mathrm{X}_*(T) = \mathrm{X}(T)^* = \mathbf{Z}^m$ we seek to find the weight spaces for the T-action on $\mathfrak{so}(2m)_{\mathbf{C}} \subset \mathfrak{gl}_{2m}(\mathbf{C})$ with nontrivial weights. The total number of roots is $m(2m-1) - m = 2m^2 - 2m$, so once we find that many distinct nontrivial weights we will have found all of the roots.

Let's regard a $2m \times 2m$ matrix as an $m \times m$ matrix in 2×2 blocks, so in this way T naturally sits as a subgroup of the m such blocks along the "diagonal" of $\mathrm{SO}(2m)$, and so likewise for $\mathfrak{t}$ inside the space $\mathfrak{so}(2m)$ of skew-symmetric matrices. We focus our attention on the 2×2 blocks "above" that diagonal region, since the corresponding transposed block below the diagonal is entirely determined via negation-transpose.

For $1 \leq j < j' \leq m$ consider the 2×2 block $\mathfrak{gl}_2(\mathbf{R})$ in the jj'-position within the $2m \times 2m$ matrix. (Such pairs (j, j') exist precisely because $m \geq 2$.) Working inside $\mathfrak{so}(2m) \subset \mathfrak{gl}_{2m}(\mathbf{R})$, the effect of conjugation by

$$(r_{\theta_1}, \ldots, r_{\theta_m}) \in (S^1)^m = T$$

on the 2×2 block $\mathfrak{gl}_2(\mathbf{R})$ is readily computed to be

$$M \mapsto [r_{\theta_j}]M[r_{\theta_{j'}}]^{-1}$$

where $[r_\theta] \in \mathrm{GL}_2(\mathbf{R})$ is the standard matrix for counterclockwise rotation by θ on $\mathbf{R}^2$.

Working over $\mathbf{C}$, to find the weight spaces we recall the standard diagonalization of rotation matrices over $\mathbf{C}$:

$$[r_\theta] = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}$$

and note that the elementary calculation

$$\mathrm{diag}(\lambda, 1/\lambda)\begin{pmatrix} x & y \\ z & w \end{pmatrix}\mathrm{diag}(1/\mu, \mu) = \begin{pmatrix} (\lambda/\mu)x & (\lambda\mu)y \\ (\lambda\mu)^{-1}z & (\mu/\lambda)w \end{pmatrix}$$

identifies the weight spaces as the images under $\left(\begin{smallmatrix} 1 & 1 \\ -i & i \end{smallmatrix}\right)$-conjugation of the "matrix entry" lines. In this way, we see that the four nontrivial characters $T \to S^1$ given by $t \mapsto t_j/t_{j'}, t_{j'}/t_j, t_j t_{j'}, (t_j t_{j'})^{-1}$ are roots with respective weight spaces in $\mathfrak{so}(2m)_{\mathbf{C}} \subset \mathfrak{gl}_{2m}(\mathbf{C})$ given inside the 2×2 block in the jj'-position by the respective 1-dimensional $\mathbf{C}$-spans

$$\mathbf{C}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \ \mathbf{C}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \ \mathbf{C}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \ \mathbf{C}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

Varying across all such pairs $j < j'$, we arrive at $4m(m-1)/2 = 2m^2 - 2m$ roots. But this is the number that we were looking for, so

$$\Phi(G, T) = \{\pm a_j \pm a_{j'} \mid 1 \le j < j' \le m\} \subset \mathbf{Z}^m$$

(with all 4 sign options allowed for each (j, j')). Since the standard quadratic form $q = \sum z_j^2$ on $\mathbf{Z}^m$ is invariant under the action of the explicitly determined $W(G,T)$, we can use the formula $a^\vee = 2a/q(a)$ to compute the coroots in the dual lattice $\mathrm{X}_*(T) = \mathbf{Z}^m$: each root $\pm a_j \pm a_{j'}$ has squared-length 2, so $a^\vee = a$ under the self-duality, whence $(\pm a_j \pm a_{j'})^\vee = \pm a_j^* \pm a_{j'}^*$. This determines $\Phi(G,T)^\vee$, and makes evident that the root system is self-dual. This root system of rank m is called D_m (or "type D" if the rank is not essential to the discussion).

Since SO(4) is an isogenous quotient of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, we have $\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$ (for the evident notion of direct product of root systems: direct sum of vector spaces and disjoint union of roots embedded in their own factor spaces, consistent with direct product of pairs (G,T)). Likewise, $\mathrm{D}_3 = \mathrm{A}_3$ by inspection, encoding an exceptional isomorphism $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ that we might discuss later if time permits. Thus, the most interesting cases for even special orthogonal groups $\mathrm{SO}(2m)$ are $m \ge 4$; the case $m = 4$, which is to say SO(8), is especially remarkable (because the Dynkin diagram for type D_4 possesses more symmetry than any other Dynkin diagram; see Remark 2.1).

Inside $\mathrm{X}(T) = \mathbf{Z}^m$ it is easy to check that the $\mathbf{Z}$-span $\mathbf{Z}\Phi$ is the index-2 subgroup $\{\vec{x} \mid \sum x_j \equiv 0 \bmod 2\}$; this index of 2 corresponding to the size of the center μ_2 of $\mathrm{SO}(2m)$. The $\mathbf{Z}$-dual $(\mathbf{Z}\Phi^\vee)' \subset \mathbf{Q}^m$ is $\mathrm{X}(T) + (1/2)(1,1,\dots,1)$, which contains $\mathrm{X}(T)$ with index 2; this index of 2 corresponds to the fact that the simply connected cover $\mathrm{Spin}(2m)$ of $\mathrm{SO}(2m)$ is a degree-2 covering. A very interesting question now arises: the quotient $\Pi := (\mathbf{Z}\Phi^\vee)'/(\mathbf{Z}\Phi)$ has order 4, but is it cyclic of order 4 or not? This amounts to asking if it is 2-torsion or not, which amounts to asking if twice $(1/2)(1,1,\dots,1)$, which is to say $(1,1,\dots,1)$, lies in $\mathbf{Z}\Phi$. This latter vector has coordinate sum m that vanishes mod 2 precisely for m even.

We conclude that Π is cyclic of order 4 when m is odd and is $(\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$ when m is even. By Exercise 1(iii) in HW7, this says that $\mathrm{Spin}(2m)$ has center that is μ_4 (cyclic) when m is odd and is $\mu_2 \times \mu_2$ when m is even. (For example, when $m = 2$ we recall from HW6 that SO(4) is the quotient of the simply connected $\mathrm{SU}(2) \times \mathrm{SU}(2)$ modulo the diagonal μ_2, so $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ with center visibly $\mu_2 \times \mu_2$. The exceptional isomorphism

$\mathrm{SU}(4) \simeq \mathrm{Spin}(6)$ – which we might discuss later in the course if time permits – gives a concrete way to see that for $m = 3$ the center of $\mathrm{Spin}(2m)$ is μ_4.)

It turns that among *all* "simple" compact connected Lie groups with finite center (not just the classical ones), the groups $\mathrm{Spin}(2m)$ for even $m \geq 4$ are the *only* ones whose center is non-cyclic. For any $m \geq 2$, the proper isogenous quotients of $\mathrm{Spin}(2m)$ larger than the centerless quotient $\mathrm{Spin}(2m)/Z_{\mathrm{Spin}(2m)} = \mathrm{SO}(2m)/\langle -1 \rangle$ are the quotients of $\mathrm{Spin}(2m)$ by the proper nontrivial subgroups of $Z_{\mathrm{Spin}(2m)}$, so $\mathrm{SO}(2m)$ is the *unique* such intermediate quotient when m is odd but there are *two additional* intermediate quotients when m is even. This dichotomy in the structure of the center sometimes causes properties of $\mathrm{SO}(2m)$ or $\mathfrak{so}(2m)$ to be sensitive to the parity of m (i.e., $\mathrm{SO}(n)$ can exhibit varying behavior depending on $n \bmod 4$, not just $n \bmod 2$).

Remark 2.1. For any connected Lie group G, its universal cover (equipped with a base point $\widetilde{e}$ over the identity e of G) admits a unique compatible structure of Lie group, denoted $\widetilde{G}$; this is called the *simply connected cover* of G and will be discussed in Exercise 3 on HW9 (e.g., if $G = \mathrm{SO}(n)$ then $\widetilde{G} = \mathrm{Spin}(n)$ for $n \geq 3$). It will be shown in that exercise that every homomorphism $G \to H$ between connected Lie groups uniquely lifts to a homomorphism $\widetilde{G} \to \widetilde{H}$ between their universal covers. In particular, $\mathrm{Aut}(G)$ is a naturally a subgroup of $\mathrm{Aut}(\widetilde{G})$ containing the normal subgroup $\widetilde{G}/Z_{\widetilde{G}} = G/Z_G$ of inner automorphisms. The image of $\mathrm{Aut}(G)$ in $\mathrm{Aut}(\widetilde{G})$ consists of the automorphisms of $\widetilde{G}$ that preserve the central discrete subgroup $\ker(\widetilde{G} \twoheadrightarrow G)$ (and so descend to automorphisms of the quotient G).

Understanding the gap between $\mathrm{Aut}(\widetilde{G})$ and $\mathrm{Aut}(G)$ amounts to understanding the effect of the outer automorphism group $\mathrm{Out}(\widetilde{G}) := \mathrm{Aut}(\widetilde{G})/(\widetilde{G}/Z_{\widetilde{G}}) = \mathrm{Aut}(\widetilde{G})/(G/Z_G)$ on the set of subgroups of $Z_{\widetilde{G}}$, especially on the central subgroup $\ker(\widetilde{G} \twoheadrightarrow G)$. For example, when $Z_{\widetilde{G}}$ is finite cyclic then the outer automorphisms must preserve each such subgroup (since a finite cyclic group has each subgroup uniquely determined by its size) and hence $\mathrm{Aut}(G) = \mathrm{Aut}(\widetilde{G})$ for all quotients G of $\widetilde{G}$ by central subgroups in such cases. For example, $\mathrm{Aut}(\mathrm{SO}(2m)) = \mathrm{Aut}(\mathrm{Spin}(2m))$ for odd $m \geq 2$.

A closer analysis is required for even $m \geq 2$ since (as we saw above) $Z_{\mathrm{Spin}(2m)}$ is non-cyclic for even m, but the equality of automorphism groups remains true as for odd m *except* when $m = 4$. The key point (which lies beyond the level of this course) is that the outer automorphism group of any simply connected compact connected Lie group with finite center is naturally isomorphic to the automorphism group of its Dynkin diagram, and in this way one can show that $\mathrm{Out}(\mathrm{Spin}(2m))$ has order 2 for all $m \geq 2$ *except* for $m = 4$ (when it is S_3 of order 6). Using this determination of the outer automorphism group of $\mathrm{Spin}(2m)$, it follows that if $m \geq 2$ with $m \neq 4$ then inside $\mathrm{Aut}(\mathrm{Spin}(2m))$ the subgroup $\mathrm{Spin}(2m)/Z_{\mathrm{Spin}(2m)} = \mathrm{SO}(2m)/Z_{\mathrm{SO}(2m)}$ of inner automorphisms has index 2, whence $\mathrm{Aut}(\mathrm{SO}(2m)) = \mathrm{Aut}(\mathrm{Spin}(2m))$ provided that $\mathrm{SO}(2m)$ has *some* non-inner automorphism.

But there is an easy source of a non-inner automorphism of $\mathrm{SO}(2m)$ for *any* $m \geq 1$: conjugation by the non-identity component of $\mathrm{O}(2m)$! Indeed, if $g \in \mathrm{O}(2m) - \mathrm{SO}(2m)$ and g-conjugation on $\mathrm{SO}(2m)$ is inner then replacing g with an $\mathrm{SO}(2m)$-translate would provide such g that centralizes $\mathrm{SO}(2m)$. A direct inspection shows that for the standard maximal torus T, its centralizer in $\mathrm{O}(2m)$ is still equal to T (the analogue for $\mathrm{O}(2m+1)$

is false since $\mathrm{O}(2m+1) = \mathrm{SO}(2m+1) \times \langle -1 \rangle)$, so $\mathrm{Out}(\mathrm{SO}(2m))$ is always nontrivial and hence $\mathrm{Aut}(\mathrm{SO}(2m)) = \mathrm{Aut}(\mathrm{Spin}(2m))$ for all $m \geq 2$ with $m \neq 4$. In contrast, additional work shows that $\mathrm{Aut}(\mathrm{Spin}(8))$ contains $\mathrm{Aut}(\mathrm{SO}(8))$ as a non-normal subgroup of index 3 (underlying the phenomenon called *triality*).

3. Odd special orthogonal groups

Consider $G = \mathrm{SO}(2m+1)$ with $m \geq 2$, in which a maximal torus T is given by the "standard" maximal torus of $\mathrm{SO}(2m)$ used above. We regard G as associated to the quadratic form $x_0^2 + \sum_{j=1}^m (x_j^2 + x_{-j}^2)$ in which $\mathbf{R}^{2m+1}$ has ordered basis

$$\{e_1, e_{-1}, e_2, e_{-2}, \dots, e_m, e_{-m}, e_0\}.$$

The dimension of G is $(2m+1)(2m)/2 = m(2m+1)$, so we are searching for $2m^2 + m - m = 2m^2$ root spaces inside $\mathfrak{so}(2m+1)_{\mathbf{C}} \subset \mathfrak{gl}_{2m+1}(\mathbf{C})$.

Inside the upper left $2m \times 2m$ submatrix we get $\mathfrak{so}(2m)_{\mathbf{C}}$ in which we have already found $2m^2 - 2m$ roots. We seek $2m$ additional roots, and we will find them inside $\mathfrak{gl}_{2m+1}(\mathbf{C}) = \mathrm{Mat}_{2m+1}(\mathbf{C})$ by looking at the right column and bottom row viewed as vectors in $\mathbf{C}^{2m+1}$. Taking into account the skew-symmetry of $\mathfrak{so}(2m+1) \subset \mathfrak{gl}_{2m+1}(\mathbf{R})$, we may focus on the right column. Since T viewed inside $\mathrm{SO}(2m+1) \subset \mathrm{GL}_{2m+1}(\mathbf{R})$ has lower-right entry equal to 1, the conjugation action of $T \subset \mathfrak{gl}_{2m}(\mathbf{R}) \subset \mathfrak{gl}_{2m+1}(\mathbf{R})$ preserves the right column with trivial action along the bottom entry and action on the remaining part of the right column in accordance accordance with how $\mathrm{GL}_{2m}(\mathbf{R})$ acts on $\mathbf{R}^{2m}$. This provides a direct sum of planes $\mathbf{R}e_j \oplus \mathbf{R}e_{-j}$ on which $t \in (S^1)^m = T$ acts through the standard rotation representation of S^1 on $\mathbf{R}^2$ via the jth component projection $t \mapsto t_j \in S^1$.

To summarize, $\Phi(\mathrm{SO}(2m+1), T)$ is the union of $\Phi(\mathrm{SO}(2m), T)$ along with the additional roots $t \mapsto t^{\pm a_j} := t_j^{\pm 1}$ (the standard basis vectors of $\mathrm{X}(T) = \mathbf{Z}^m$ and their negatives), and the corresponding root spaces are given by the eigenlines $\mathbf{C}(e_j \pm i e_{-j})$ for the standard rotation action of S^1 on $\mathbf{R}e_j \oplus \mathbf{R}e_{-j}$. Explicitly, $e_j \mp i e_{-j}$ spans the root space for the root $\pm a_j : t \mapsto t_j^{\pm 1}$, the dichotomy between i and $-i$ arising from how we view S^1 inside $\mathbf{C}^\times$ for the purpose of regarding a character $T \to S^1$ as giving a homomorphism of T into $\mathbf{C}^\times$.

To compute the additional coroots in $\mathrm{X}_*(T)$, we observe that the standard positive-definite quadratic form $q = \sum z_j^2$ is invariant under $W(\mathrm{SO}(2m+1), T)$ since this Weyl group is the full $(\mathbf{Z}/2\mathbf{Z})^m \rtimes S_m$ that we have already noted (in our treatment of even special orthogonal groups) preserves this quadratic form. Hence, we can use the formula $a^\vee = 2a/q(a)$ to compute the coroots associated to this new roots.

Now we meet a new phenomenon: the additional roots (beyond $\Phi(\mathrm{SO}(2m), T)$) have q-length 1 rather than q-length $\sqrt{2}$ as for all roots in $\Phi(\mathrm{SO}(2m), T)$, so we have two distinct root lengths. The coroot $a^\vee$ is equal to $2a$ in this way for each additional root, or in other words $(\pm a_j)^\vee = \pm 2a_j^*$ in the dual $\mathbf{Z}^m$. Thus, $\Phi(\mathrm{SO}(2m+1), T)$ is *not* a self-dual root system, due to the distinct root lengths: there are $2m$ short roots and $2m^2 - 2m$ long roots. (Strictly speaking, for this to be a valid proof of non-self-duality we have to show that in this case the $\mathbf{Q}$-vector space of Weyl-invariant quadratic forms on $\mathrm{X}(T)_{\mathbf{Q}}$ is 1-dimensional, so the notion of "ratio of root lengths" is intrinsic. This will be addressed in Exercise 1(ii) on HW9.)

The rank-m root system $\Phi(\mathrm{SO}(2m+1), T)$ is called B_m (or "type B" if the rank is not essential to the discussion). For $m = 2$, it recovers the rank-2 example called by the name B_2 in class.

Remark 3.1. The additional roots $\pm a_j$ ensure that the inclusion $\mathbf{Z}\Phi \subset \mathrm{X}(T)$ is an *equality*, in contrast with $\mathrm{SO}(2m)$. By Exercise 1(iii) in HW7, this implies that $\mathrm{SO}(2m+1)$ has trivial center. (The element $-1 \in \mathrm{O}(2m+1)$ is not in $\mathrm{SO}(2m+1)$ since it has determinant $(-1)^{2m+1} = -1$, so there is no "obvious guess" for nontrivial elements in the center, and the equality of root lattice and character lattice ensures that the center really is trivial.)

The $\mathbf{Z}$-dual $(\mathbf{Z}\Phi^\vee)' \subset \mathrm{X}(T)_{\mathbf{Q}} = \mathbf{Q}^m$ to the coroot lattice coincides with the lattice

$$\mathbf{Z}^m + (1/2)(1, 1, \ldots, 1)$$

computed for $\mathrm{SO}(2m)$ since the additional coroots $\pm 2a_j^*$ lie in the $\mathbf{Z}$-span of the coroots for $(\mathrm{SO}(2m), T)$. Hence, $(\mathbf{Z}\Phi^\vee)'$ contains $\mathrm{X}(T)$ with index 2, corresponding to the fact that the simply connected $\mathrm{Spin}(2m+1)$ is a degree-2 cover of $\mathrm{SO}(2m+1)$.

4. Symplectic groups

Finally, we treat $\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{GL}_n(\mathbf{H})$ for $n \geq 2$. Viewed inside $\mathrm{U}(2n) \subset \mathrm{GL}_{2n}(\mathbf{C})$, this is the group of matrices

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \in \mathrm{U}(2n),$$

and the standard maximal torus $T = (S^1)^n$ of G consists of diagonal matrices

$$\mathrm{diag}(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n)$$

for $z_j \in S^1$. The Lie algebra $\mathfrak{sp}(n)$ consists of the matrices

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

with $A, B \in \mathfrak{gl}_n(\mathbf{C})$ satisfying ${}^\top(\overline{A}) = -A$ and ${}^\top B = B$. In particular, it has $\mathbf{R}$-dimension $2n^2 + n$. Since $\dim T = n$, we are looking for $2n^2$ roots.

Working with blocks of $n \times n$ matrices over $\mathbf{C}$, the adjoint action of $t = (z, \overline{z}) = (z, z^{-1}) \in T \subset \mathrm{GL}_{2n}(\mathbf{C})$ (for $z \in (S^1)^n$) on $\mathfrak{sp}(n) \subset \mathfrak{gl}_{2n}(\mathbf{C})$ is given by

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \mapsto \begin{pmatrix} zAz^{-1} & zBz \\ -z^{-1}\overline{B}z^{-1} & z^{-1}\overline{A}z \end{pmatrix}.$$

Beware that in this formula we use the Lie algebra $\mathfrak{sp}(n)$ over $\mathbf{R}$ and not its complexification. This is relevant to the appearance of complex conjugation in the matrix expression, since the $\mathbf{R}$-linear inclusions $\mathbf{C} \to \mathbf{C} \times \mathbf{C}$ via $u \mapsto (u, \pm\overline{u})$ define 2-dimensional $\mathbf{R}$-subspaces and hence after complexification provide 2-dimensional $\mathbf{C}$-subspaces of $\mathbf{C} \otimes_{\mathbf{R}} (\mathbf{C} \times \mathbf{C})$. Thus, the roles of B and $-\overline{B}$, as well as of A and $\overline{A}$, provide twice as much $\mathbf{C}$-dimension in the complexification as one might have naively expected based on reasoning solely inside $\mathfrak{gl}_{2n}(\mathbf{C})$ (where one has to recognize that the maps $u \mapsto (u, \pm\overline{u})$ at the level of certain matrix entries are *not* $\mathbf{C}$-linear).

Consider the B-part. The weight space decomposition there is for the action $(z, B) \mapsto z.B := zBz$ of $(S^1)^n$ on the space of symmetric $n \times n$ matrices over $\mathbf{C}$ *viewed as an* $\mathbf{R}$*-vector* *space*. This has each "off-diagonal matrix entry" paired with its transposed identical matrix entry as T-stable $\mathbf{R}$-subspace that is a copy of $\mathbf{C}$; the jj'-entry scaled by the character $z \mapsto z_j z_{j'}$ for $1 \leq j \neq j' \leq n$ (regardless of which of j or j' is larger). This is at the $\mathbf{R}$-linear level, and passing to the complexification entails using the $\mathbf{C}$-linear isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathbf{C} \times \mathbf{C}$ defined by $u \otimes u' \mapsto (uu', u\overline{u}')$ (where $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ is a $\mathbf{C}$-vector space via the left tensor factor and $\mathbf{C} \times \mathbf{C}$ is a $\mathbf{C}$-vector space via the diagonal scaling action) which turns the $\mathbf{R}$-linear scaling $u' \mapsto zu'$ on the right tensor factor for $z \in S^1 \subset \mathbf{C}^\times$ into $(y_1, y_2) \mapsto (zy_1, \overline{z}y_2) = (zy_1, (1/z)y_2)$. Hence, each off-diagonal entry of the symmetric B (as a 2-dimensional $\mathbf{R}$-vector space) contributes two root spaces in the complexification $\mathfrak{sp}(n)_{\mathbf{C}}$, with roots $z \mapsto z_j z_{j'}$ and $z \mapsto 1/(z_j z_{j'})$ for $1 \leq j < j' \leq n$. That is $2(n(n-1)/2) = n^2 - n$ roots. The jj-entries of B provide two additional roots in $\mathfrak{sp}(n)_{\mathbf{C}}$, namely $z \mapsto z_j^{\pm 2}$. That provides $2n$ additional roots, for a total of $n^2 + n$ roots.

For the A-part, we are considering the standard conjugation action of $(S^1)^n$ on $\mathfrak{gl}_n(\mathbf{C})$ restricted to the $\mathbf{R}$-subspace of skew-hermitian matrices. We can focus our attention on the strictly upper triangular part, since the diagonal vanishes and the strictly lower triangular part is determined by the strictly upper triangular part via the $\mathbf{R}$-linear negated conjugation operation. This is exactly the content of the computation of the root system for $\mathrm{SU}(n)$ via the identification $\mathfrak{su}(n)_{\mathbf{C}} = \mathfrak{sl}_n(\mathbf{C})$ except that once again we are viewing this as an $\mathbf{R}$-vector space representation of T, so complexifying turns each $\mathbf{C}$-linear eigenline (an $\mathbf{R}$-plane!) for the $\mathbf{C}$-linear action of T on $\mathfrak{sl}_n(\mathbf{C})$ into a pair of root lines in $\mathfrak{sp}(n)_{\mathbf{C}}$ having opposite roots. That is, for $1 \leq j < j' \leq n$ the jj'-entry of A contributes root lines for the weights $z \mapsto z_j/z_{j'}, z_{j'}/z_j$ in the complexfication of $\mathfrak{sp}(n)$. This provides $2((n^2 - n)/2) = n^2 - n$ additional roots. Together with the ones we already found, we have found all $2n^2$ roots.

To summarize, inside $\mathrm{X}(T) = \mathbf{Z}^n$ the set of roots is

$$\Phi(G, T) = \{\pm(a_j + a_{j'}) \,|\, 1 \leq j \leq j' \leq n\} \cup \{\pm(a_j - a_{j'}) \,|\, 1 \leq j < j' \leq n\} \subset \mathbf{Z}^n.$$

The Weyl group is $(\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n$ acting on $\mathrm{X}(T) = \mathbf{Z}^n$ exactly as for $\mathrm{SO}(2n+1)$, so once again the standard quadratic form $q = \sum x_j^2$ on $\mathbf{Z}^m$ is Weyl-invariant and thereby enables us to compute the coroots $a^\vee = 2a/q(a)$ via the induced standard self-duality of $\mathrm{X}(T)_{\mathbf{Q}} = \mathbf{Q}^m$. We conclude that for $1 \leq j \leq n$ and $j < j' \leq n$,

$$(\pm(a_j + a_{j'}))^\vee = \pm(a_j^* + a_{j'}^*), \;\; (\pm(a_j - a_{j'}))^\vee = \pm(a_j^* - a_{j'}^*), \;\; (\pm 2a_j)^\vee = \pm a_j^*$$

inside the dual lattice $\mathrm{X}_*(T) = \mathrm{X}(T)^* = \mathbf{Z}^n$. There are once again two root lengths, as for $\Phi(\mathrm{SO}(2n+1), T)$, but now there are $2n$ long roots and $2n^2 - 2n$ short roots. A bit of inspection reveals that $\Phi(\mathrm{Sp}(n), T)$ is the dual root system to $\Phi(\mathrm{SO}(2n+1), T)$! We call this rank-$n$ root system C_n (or "type C" if the rank is not essential to the discussion).

Remark 4.1. There is a remarkably special feature of the C_n root systems: there are roots (the long ones) that are nontrivially divisible (in fact, by 2) in $\mathrm{X}(T)$. Among *all* irreducible (and reduced) root systems (V, Φ) over $\mathbf{Q}$, not just the classical types, the only cases in which there is a $\mathbf{Z}$-submodule X intermediate between the root lattice $Q := \mathbf{Z}\Phi$ and the so-called weight lattice $P := (\mathbf{Z}\Phi^\vee)'$ such that some elements of Φ are nontrivially divisible in X is the case of type C_n (allowing $n = 1$, via the convention that $\mathrm{C}_1 = \mathrm{A}_1$ that is reasonable since $\mathrm{Sp}(1) = \mathrm{SU}(2)$) with $\mathrm{X} = P$. This corresponds precisely to the compact symplectic groups $\mathrm{Sp}(n)$ ($n \geq 1$) among *all* "simple" compact connected Lie groups with finite center.

5. Computation of a weight lattice

Later we will see the topological significance of the *weight lattice* $P = (\mathbf{Z}\Phi^\vee)' \subset \mathrm{X}(T)_{\mathbf{Q}}$ for the root system $(\mathrm{X}(T)_{\mathbf{Q}}, \Phi)$ attached to a connected compact Lie group G with finite center. (Recall that if V is a finite-dimensional $\mathbf{Q}$-vector space and $L \subset V^*$ is a lattice in the dual space then the *dual lattice* $L' \subset V$ consists of those $v \in V$ such that $\ell(v) \in \mathbf{Z}$ for all $\ell \in L$.) We shall see later that P is the character lattice for the simply connected universal cover $\widetilde{G}$ of G (that will be proved to be a compact finite-degree cover of G).

In this final section, for the root system Φ of type $\mathrm{B}_2 = \mathrm{C}_2$ (corresponding to the double cover $\mathrm{Spin}(5)$ of $\mathrm{SO}(5)$) we compute P. Explicitly, Φ viewed inside $\mathbf{R}^2$ equipped with the standard inner product (identifying $\mathbf{R}^2$ with its own dual) consists of the vertices and edge midpoints for a square centered at the origin with sides parallel to the coordinate axes; a is an edge midpoint and b is the vertex counterclockwise around by the angle $3\pi/4$. We have $\|b\|/\|a\| = \sqrt{2}$, $\langle b, a^\vee\rangle = -2$, and $\langle a, b^\vee\rangle = -1$.

Let's scale so that $\|a\| = \sqrt{2}$, as then $a^\vee = 2a/\|a\|^2 = a$ (and likewise for the coroots associated to all short roots). Now $\|b\| = 2$, so $b^\vee = 2b/\|b\|^2 = b/2$ (and likewise for the coroots associated to all long roots). Using the standard dot product, we have $a \cdot b = -2$ (reality check: $-2 = \langle b, a^\vee\rangle = 2(b\cdot a)/\|a\|^2 = b \cdot a$).

The description of the coroots associated to short and long roots (all viewed inside $\mathbf{R}^2$ via the self-duality arising from the standard dot product) gives by inspection that $\mathbf{Z}\Phi^\vee = \mathbf{Z}a + \mathbf{Z}(b/2)$. Hence, the dual lattice $P = (\mathbf{Z}\Phi^\vee)' \subset \mathbf{R}^2$ is the set of vectors $xa + yb$ ($x, y \in \mathbf{R}$) such that $(xa + yb)\cdot a \in \mathbf{Z}$ and $(xa+yb)\cdot(b/2) \in \mathbf{Z}$. This says $2x - 2y =: n \in \mathbf{Z}$ and $-x + 2y := m \in \mathbf{Z}$, or in other words $x = n + m, y = m + n/2$ for $m, n \in \mathbf{Z}$. Equivalently,

$$P = \{(n+m)a + (m + n/2)b \,|\, m, n \in \mathbf{Z}\} = \mathbf{Z}(a + b/2) + \mathbf{Z}(a+b) = \mathbf{Z}a + \mathbf{Z}(b/2).$$

This lattice contains the root lattice $\mathbf{Z}\Phi$ with index 2, by inspection.

Remark 5.1. The determination of P from Φ can be done by more "combinatorial" means without reference to Euclidean structures or $\mathbf{R}$, using only the relations $\langle b, a^\vee\rangle = -2$ and $\langle a, b^\vee\rangle = -1$. The key point is that since $\{a, b\}$ rationally span the $\mathbf{Q}$-structure for the root system, elements of the dual lattice $P = (\mathbf{Z}\Phi^\vee)'$ are precisely $xa + yb$ for $x, y \in \mathbf{Q}$ such that $n := \langle xa + yb, a^\vee\rangle \in \mathbf{Z}$ and $m := \langle xa + yb, b^\vee\rangle \in \mathbf{Z}$. Since the definitions give $n = 2x - 2y$ and $m = -x + 2y$ respectively, we have arrived at exactly the same equations obtained in the Euclidean considerations above (and solved via elementary algebraic manipulations).

In terms of the visualization with a square, if Φ "corresponds" to a given square S with an edge midpoint a and a vertex b as above (where a is short of length $\sqrt{2}$ and b is long of length 2) then via the self-duality of the ambient plane as used above we have that $\Phi^\vee$ "corresponds" to a square S' centered inside S that is tilted by $\pi/4$ with size $1/\sqrt{2}$ that of S (so half the area of S), where the edge midpoint $a = a^\vee$ is a vertex b' of S' and an edge midpoint a' of S' is given by the vector $b/2$ halfway along b. If we try to iterate this again then we don't get a yet smaller square S'' half the size of S since for the preceding geometric considerations with self-duality of $\mathbf{R}^2$ it was crucial that S has side-length $2\sqrt{2}$ (so $a^\vee = a$, etc.); this is consistent with that $(\Phi^\vee)^\vee = \Phi$. More specifically,

$$a'^\vee = 2a'/\|a'\|^2 = 2(b/2)/1^2 = b, \;\; b'^\vee = 2b'/\|b'\|^2 = 2a/(\sqrt{2})^2 = a.$$