Math 210C. Representation ring and algebraicity of compact Lie groups

1. Introduction

Let G be a nontrivial connected compact Lie group that is semisimple and simply connected (e.g., $\mathrm{SU}(n)$ for $n \geq 2$, $\mathrm{Sp}(n)$ for $n \geq 1$, or $\mathrm{Spin}(n)$ for $n \geq 3$). Let T be a maximal torus, with $r = \dim T > 0$, and let $\Phi = \Phi(G, T)$ the associated root system.

Let $B = \{a_i\}$ be a basis of Φ, so $B^\vee := \{a_i^\vee\}$ is a basis for $\Phi^\vee$. In particular, $B^\vee$ is a $\mathbf{Z}$-basis for the coroot lattice $\mathbf{Z}\Phi^\vee$ that is dual to the weight lattice P. Since G is simply connected, $\mathbf{Z}\Phi^\vee = \mathrm{X}_*(T)$. Passing to $\mathbf{Z}$-duals, we have $\mathrm{X}(T) = P$. Since $B^\vee$ is a $\mathbf{Z}$-basis for $\mathbf{Z}\Phi^\vee = \mathrm{X}_*(T)$, the dual lattice $\mathrm{X}(T)$ has a corresponding dual basis $\{\varpi_i\}$ characterized by the condition $\langle \varpi_i, a_j^\vee \rangle = \delta_{ij}$. We call these ϖ_i's the *fundamental weights* with respect to (G, T, B).

The closure $\overline{K(B)}$ of the Weyl chamber $K(B)$ in $\mathrm{X}(T)_{\mathbf{R}}$ corresponding to B is characterized by the property of having non-negative pairing against each $a_j^\vee$, so

$$\mathrm{X}(T) \cap \overline{K(B)} = P \cap \overline{K(B)} = \sum_i \mathbf{Z}_{\geq 0} \cdot \varpi_i;$$

these are called the *dominant weights* with respect to B.

By the Theorem of the Highest Weight, any irreducible (continuous finite-dimensional $\mathbf{C}$-linear) representation V of G has a unique "highest weight" χ that is dominant, the weight space for which is 1-dimensional, and every χ arises in this way from a unique such V, denoted as V_χ. The representations $V_{\varpi_1}, \dots, V_{\varpi_r}$ are the *fundamental representations* for G.

Example 1.1. Consider $G = \mathrm{SU}(n)$ with maximal torus T given by the diagonal. Then $\mathrm{X}(T) = \mathbf{Z}^{\oplus n}/\Delta$ (where Δ is the diagonal copy of $\mathbf{Z}$), and its dual $\mathrm{X}_*(T) \subset \mathbf{Z}^{\oplus n}$ is the "hyperplane" defined by $\sum x_j = 0$. Letting $\{e_i\}$ denote the standard basis of $\mathbf{Z}^{\oplus n}$, and $\overline{e}_i$ the image of e_i in the quotient $\mathrm{X}(T)$ of $\mathbf{Z}^{\oplus n}$, we have $\Phi = \{\overline{e}_i - \overline{e}_j \mid i \neq j\}$ with a basis

$$B = \{a_i = \overline{e}_i - \overline{e}_{i+1} \mid 1 \leq i \leq n-1\}.$$

The reader can check that $B^\vee = \{a_i^\vee = e_i^* - e_{i+1}^*\}$, where $\{e_i^*\}$ is the dual basis in $\mathbf{Z}^{\oplus n}$ to the standard basis $\{e_i\}$ of $\mathbf{Z}^{\oplus n}$, and $\overline{K(B)} = \{\sum x_i \overline{e}_i \mid x_i \in \mathbf{R}, x_1 \geq \cdots \geq x_n\}$. From this it follows that

$$\varpi_i = \overline{e}_1 + \cdots + \overline{e}_i$$

for $1 \leq i \leq n-1$. In HW10 Exercise 4 one finds that the corresponding representation V_{ϖ_i} is $\wedge^i(\rho_{\mathrm{std}})$ where ρ_{std} is the standard n-dimensional representation of G.

Example 1.2. For the group $\mathrm{Spin}(2n)$ with $n \geq 4$, the diagram of type D_n has two "short legs" whose extremal vertices (as elements of a basis of the root system) have corresponding fundamental weights ϖ_{n-1} and ϖ_n whose associated fundamental representations of $\mathrm{Spin}(2n)$ called the *half-spin* representations. These do *not* factor through the central quotient $\mathrm{SO}(2n)$, and they are constructed explicitly via Clifford-algebra methods.

For any dominant weight $\lambda = \sum_i n_i \varpi_i$ (with integers $n_1, \dots, n_r \geq 0$), we saw in class that the irreducible representation V_λ of G with highest weight λ occurs exactly once as a subrepresentation of

$$V(n_1, \dots, n_r) := V_{\varpi_1}^{\otimes n_1} \otimes \cdots \otimes V_{\varpi_r}^{\otimes n_r}$$

due to the Theorem of the Highest Weight.

Recall that the commutative *representation ring*

$$R(G) = \bigoplus \mathbf{Z}[\rho]$$

is the free abelian group generated by the irreducible representations ρ of G and it has the ring structure given by $[\rho]\cdot[\rho'] = \sum c_\sigma [\sigma]$ where $\rho \otimes \rho' = \bigoplus \sigma^{\oplus c_\sigma}$. The ring $R(G)$ is a subring of the ring of $\mathbf{C}$-valued class functions on G via the formation of characters (e.g., $[\rho] \mapsto \chi_\rho$).

We saw long ago via Weyl's theorems on maximal tori that for *any* connected compact Lie group G (not necessarily semisimple or simply connected) and maximal torus $T \subset G$, restriction of class functions to T (or of G-representations to T-representations) makes $R(G)$ naturally a subring of $R(T)^W$, where $W := W(G,T) = N_G(T)/T$ acts in the natural way on $R(T)$. The aim of this handout is to use the Theorem of the Highest Weight and fundamental representations to prove that $R(G) = R(T)^W$ in general (using reduction to the simply connected semisimple case, where deeper structural results are available such as the Theorem of the Highest Weight), and to discuss some further results in the representation theory of compact Lie groups.

Remark 1.3. For each finite-dimensional representation V of G, the associated character χ_V is an element of $R(G)$. The elements obtained in this way constitute the subset $R_{\text{eff}}(G) \subset R(G)$ of "effective" elements: the $\mathbf{Z}_{\geq 0}$-linear combinations of $[\rho]$'s for irreducible ρ. (If $V \simeq \bigoplus_\sigma \sigma^{e_\sigma}$ for pairwise distinct irreducible σ then $\chi_V = \sum e_\sigma \chi_\sigma$, with $\chi_\sigma = [\sigma]$ in $R(G)$.)

The subset $R_{\text{eff}}(T) \subset R(T)$ is W-stable and $R_{\text{eff}}(G) \subset R_{\text{eff}}(T)^W$. Our proof of the equality $R(G) = R(T)^W$ will involve much use of subtraction, so the proof certainly does not address how much bigger $R_{\text{eff}}(T)^W$ is than $R_{\text{eff}}(G)$. In general $R_{\text{eff}}(T)^W$ is much larger than $R_{\text{eff}}(G)$, so it is not obvious how to determine in terms of the language of $R(T)^W$ whether a given element of $R(G)$ comes from an actual representation of G or not.

For example, if $G = \mathrm{SU}(2)$ and T is the diagonal torus then $R(T)$ is equal to the Laurent polynomial ring $\mathbf{Z}[t, t^{-1}]$ on which the nontrivial element of $W = \mathfrak{S}_2$ acts through swapping t and t^{-1}, so

$$R(T)^W = \mathbf{Z}[t + t^{-1}] = \mathbf{Z} + \sum_{n>0} \mathbf{Z}(t^n + t^{-n}) \supset \mathbf{Z}_{\geq 0} + \sum_{n>0} \mathbf{Z}_{\geq 0}(t^n + t^{-n}) = R_{\text{eff}}(T)^W.$$

But HW6 Exercise 4(ii) gives $R(G) = \mathbf{Z}[t + t^{-1}]$, and inside here $R_{\text{eff}}(G) = \mathbf{Z}_{\geq 0}[t + t^{-1}]$. The latter does not contain $t^m + t^{-m}$ for any $m > 1$ since such containment would yield an identity of Laurent polynomials

$$t^m + t^{-m} = c_0 + \sum_{1 \leq i \leq m} c_i (t + t^{-1})^i$$

with integers $c_i \geq 0$, and by the positivity of binomial coefficients it is clear that no such identity is possible.

The upshot is that using $R(T)^W$ to describe $R(G)$ does not easily keep track of even very basic questions related to identifying characters of actual (rather than just virtual) representations of G.

2. The equality $R(G) = R(T)^W$

As a warm-up to showing that $R(G)$ exhausts $R(T)^W$, we first show that if G is semisimple and simply connected then $R(G)$ is a polynomial ring on the fundamental representations:

Proposition 2.1. *Assume G is semisimple and simply connected. The map $\mathbf{Z}[Y_1, \dots, Y_r] \to R(G)$ defined by $Y_j \mapsto [V_{\varpi_j}]$ is an isomorphism.*

Proof. First we prove surjectivity, by showing $[\rho]$ is hit for each irreducible ρ. The case of trivial ρ is obvious (as $[1]$ is the identity element of $R(G)$), so suppose ρ is nontrivial. It is given by V_λ for $\lambda = \sum n_i \varpi_i$ with $n_i \geq 0$ not all 0. Note that $V := \bigotimes V_{\varpi_i}^{\otimes n_i}$ is typically not irreducible, but it contains ρ as an irreducible subrepresentation *with multiplicity one* and all other dominant T-weights occurring in V are $< \lambda$ with respect to the lexicographical order on $\mathrm{X}(T) = \bigoplus \mathbf{Z}\varpi_i$ (by the very meaning of λ being the "highest weight" of ρ).

By the Theorem of the Highest Weight, $\prod [V_{\varpi_i}]^{n_i}$ differs from $[\rho]$ by the $\mathbf{Z}$-span of "lower-weight" irreducible representations (i.e., those whose highest weights are $< \lambda$), so induction with respect to the lexicographical order on $\mathrm{X}(T) = \bigoplus \mathbf{Z}\varpi_i$ yields surjectivity. (In effect, we build up an expression for $[\rho]$ as a $\mathbf{Z}$-linear combination of monomials in the Y_j's.) This completes the proof of surjectivity.

To prove injectivity, consider a hypothetical non-trivial $\mathbf{Z}$-linear dependence relation in $R(G)$ on pairwise distinct monomials in the $[V_{\varpi_i}]$'s. In this relation, some unique monomial term is maximal for the lexicographical ordering. Rewrite the dependence relation as an equality with non-negative coefficients on both sides:

$$c \prod [V_{\varpi_i}]^{n_i} + \dots = \dots$$

in $R(G)$, where c is a positive integer and the omitted terms involve monomials in the V_{ϖ_i}'s which are strictly smaller in the lexicographical order. Such an equality of elements of $R(G)$ with non-negative coefficients corresponds to an *isomorphism of representations* since it expresses an equality of the corresponding characters as class functions on G, so we get an isomorphism of representations

$$V(n_1, \dots, n_r)^{\oplus c} \oplus V' \simeq V''$$

where the irreducible representation $\rho := V_{\sum n_i \varpi_i}$ occurs once in $V(n_1, \dots, n_r)$ (by the Theorem of the Highest Weight) and not at all in V' or V'' (since all of the irreducible constituents of V' and V'' have highest weights $< \sum n_i \varpi_i$ by design). Comparing complete reducibility on both sides, ρ occurs with multiplicity c on the left side and not at all on the right side, a contradiction. ■

The preceding result for semisimple G that is simply connected yields a result in general:

Corollary 2.2. *For any connected compact Lie group G, the inclusion $R(G) \subset R(T)^W$ is an equality.*

Proof. First we treat the case when G is semisimple and simply connected. Fix a basis B of $\Phi = \Phi(G,T)$ to define a notion of "dominant weight". The inclusion $\mathrm{X}(T) \subset P$ into the weight lattice P of the root system $\Phi(G,T)$ is an equality since G is simply connected, so the representation ring $R(T) = \mathbf{Z}[\mathrm{X}(T)]$ is equal to the group ring $\mathbf{Z}[P]$ compatibly with the natural action of $W = W(G,T) = W(\Phi)$ throughout. Hence, viewing $R(T)$ as the free $\mathbf{Z}$-module on the set P, $R(T)^W$ is a free $\mathbf{Z}$-module with basis given by the sums (without multiplicity!) along each W-orbit in P (why?).

Since W acts simply transitively on the set of Weyl chambers, and $\mathrm{X}(T)_{\mathbf{R}}$ is covered by the closures of the Weyl chambers, $\mathrm{X}(T)_{\mathbf{R}}$ is covered by the members of the W-orbit of $\overline{K(B)}$, where $K(B)$ is the Weyl chamber corresponding to the basis B of Φ. Thus, each W-orbit of an element of $P = \mathrm{X}(T)$ meets $\overline{K(B)}$. But $\overline{K(B)} \cap P = \bigoplus \mathbf{Z}_{\geq 0} \cdot \varpi_i$, so each W-orbit in P contains a dominant weight.

If V is a (finite-dimensional continuous $\mathbf{C}$-linear) representation of G then a 1-dimensional character of T has multiplicity in the T-restriction $V|_T$ equal to that of any member of its W-orbit in $\mathrm{X}(T)$ since $W = N_G(T)/T$. In particular, $V_{\varpi_i}|_T$ viewed in $R(T)^W$ involves the sum (without multiplicity!) of the W-orbit of ϖ_i *exactly once*, due to the multiplicity aspect of the Theorem of the Highest Weight, and all other W-orbits occurring in $V_{\varpi_i}|_T$ (if any arise) are for dominant weights *strictly smaller* than ϖ_i.

We conclude that under the composite map

$$\mathbf{Z}[Y_1, \dots, Y_r] \simeq R(G) \hookrightarrow R(T)^W,$$

if we arrange W-orbits in P according to the biggest dominant weight that occurs in each (there might be more than dominant weight in an orbit, for weights on the boundary of the Weyl chamber $K(B)$) then for a given element of $R(T)^W$ we can subtract off a unique $\mathbf{Z}$-multiple of a unique monomial in the Y_j's to reduce the unique biggest weight that remains under the W-orbit decomposition (or arrive at 0). In this manner, we see by descending inductive considerations that $\mathbf{Z}[Y_1, \dots, Y_r] \to R(T)^W$ is surjective, forcing $R(G) = R(T)^W$. That settles the case when G is semisimple and simply connected.

The rest of the argument is an exercise in the general structure theory of compact Lie groups via tori and semisimple groups, as follows. The case $G = T$ (so $W = 1$) is trivial. In general, if $H \to G'$ is the finite-degree universal cover of the semisimple derived group of G and Z is the maximal central torus of G then

$$G = (Z \times H)/\mu$$

for a finite central closed subgroup $\mu \subset Z \times H$.

If G_1 and G_2 are (possibly disconnected) compact Lie groups then the natural map

$$R(G_1) \otimes_{\mathbf{Z}} R(G_2) \to R(G_1 \times G_2)$$

is an isomorphism. Indeed, this expresses the fact that irreducible representations of $G_1 \times G_2$ are precisely $(g_1, g_2) \mapsto \rho_1(g_1) \otimes \rho_2(g_2)$ for uniquely determined irreducible representations (V_i, ρ_i) of G_i, a result that is well-known for finite groups and carries over without change to compact Lie groups since the latter have a similarly robust character theory. Since the maximal tori of $G_1 \times G_2$ are exactly $T = T_1 \times T_2$ for maximal tori $T_i \subset G_i$, and the corresponding Weyl group $W = W(G, T)$ is naturally $W_1 \times W_2$ for $W_i = W(G_i, T_i)$, we have

$$R(T_1 \times T_2)^W = (R(T_1) \otimes_{\mathbf{Z}} R(T_2))^{W_1 \times W_2} = R(T_1)^{W_1} \otimes_{\mathbf{Z}} R(T_2)^{W_2}$$

(the final equality using that representation rings are torsion-free and hence **Z**-flat). Hence, if the equality "$R(G) = R(T)^W$" holds for a pair of connected compact Lie groups then it holds for their direct product. In particular, the desired equality holds for $Z \times H$ by the settled cases of tori and simply connected semisimple compact Lie groups.

It remains to show that if $G = \mathscr{G}/\mu$ is a central quotient of a connected compact Lie group $\mathscr{G}$, so $T = \mathscr{T}/\mu$ for a unique maximal torus $\mathscr{T} \subset \mathscr{G}$ and the natural map

$$W(\mathscr{G}, \mathscr{T}) = N_{\mathscr{G}}(\mathscr{T})/\mathscr{T} \to N_G(T)/T = W(G, T)$$

is an equality (easy from the definitions, as we saw in class), then the equality $R(\mathscr{G}) = R(\mathscr{T})^W$ implies that $R(G) = R(T)^W$. In other words, consider the commutative diagram

$$\begin{tikzcd} R(\mathscr{G}) \arrow[r] & R(\mathscr{T})^W \\ R(G) \arrow[u] \arrow[r] & R(T)^W \arrow[u] \end{tikzcd}$$

whose vertical maps are natural inclusions (composing representations of G and T with the quotient maps $\mathscr{G} \twoheadrightarrow G$ and $\mathscr{T} \twoheadrightarrow T$ respectively) and horizontal maps are natural inclusions (restricting $\mathscr{G}$-representations to $\mathscr{T}$-representations and restricting G-representations to T-representations). We claim that if the top is an equality then so is the bottom.

Thinking in terms of class functions, it is clear via respective **Z**-bases of characters of irreducible representations that the subset $R(T) \subset R(\mathscr{T})$ is the **Z**-span of the irreducible characters trivial on Z, and likewise (using Schur's Lemma!) for $R(G) \subset R(\mathscr{G})$. Hence, $R(T)$ consists of the elements of $R(\mathscr{T})$ that as class functions on $\mathscr{T}$ are invariant under Z-translation on $\mathscr{T}$, and likewise for $R(G)$ in relation to $R(\mathscr{G})$. (Keep in mind that Z is central in $\mathscr{G}$, so composing with translation by any $z \in Z$ carries class functions to class functions.) Thus, passing to W-invariants, $R(T)^W$ consists of elements of $R(\mathscr{T})^W = R(\mathscr{G})$ that as class functions on $\mathscr{T}$ are invariant under Z-translation on $\mathscr{T}$. So it remains to check that if a class function f on $\mathscr{G}$ has $\mathscr{T}$-restriction that is invariant under Z-translation on $\mathscr{G}$ then f is also Z-invariant on $\mathscr{G}$. But a class function on $\mathscr{G}$ is *uniquely determined* by its $\mathscr{T}$-restriction (why?), and $Z \subset \mathscr{T}$, so indeed the Z-invariance of a class function on G can be checked on its restriction to $\mathscr{T}$. ■

3. Further refinements

In Chapter III of the course text, functional analysis is used to obtain some important results concerning Hilbert-space representations of compact Lie groups G (i.e., continuous homomorphism $G \to \mathrm{Aut}(H)$ into the topological group of bounded linear automorphisms of a Hilbert space H). This provides the inspiration for many considerations in the more subtle non-compact case. The following two big theorems are proved in Chapter III:

Theorem 3.1. *Every irreducible Hilbert representation H of a compact Lie group G (i.e., no nonzero proper closed G-stable subspaces) is finite-dimensional.*

This result explains the central importance of the finite-dimensional case in the representation theory of compact groups. A natural infinite-dimensional Hilbert representation of G is $L^2(G, \mathbf{C})$ with the right regular action $g.f = f \circ r_g$, where $r_g : x \mapsto xg$ (this is a left action on $L^2(G, \mathbf{C})$!). For finite G this is just the group ring $\mathbf{C}[G]$ on which G acts through right multiplication composed with inversion on G (as the point mass $[g] \in \mathbf{C}[G]$ is carried to the point mass $[gh^{-1}]$ under the right-regular action of $h \in G$). Generalizing the well-known decomposition of $\mathbf{C}[G]$ as a G-representation for finite G, one has the much deeper:

Theorem 3.2 (Peter–Weyl)**.** *Let G be a compact Lie group. Equip the space $C^0(G)$ of continuous $\mathbf{C}$-valued functions on G with a hermitian structure $\langle f_1, f_2 \rangle = \int_G f_1(g)\overline{f}_2(g)\mathrm{d}g$ for the volume-1 Haar measure $\mathrm{d}g$ on G, having completion $L^2(G, \mathbf{C})$.*

(i) *Every irreducible finite-dimensional continuous $\mathbf{C}$-linear representation V of G occurs inside $L^2(G, \mathbf{C})$ with finite multiplicity $\dim(V)$.*

Upon equipping each such V with its G-invariant inner product that is unique up to a scaling factor (by irreducibility), $L^2(G, \mathbf{C})$ is the Hilbert direct sum of the representations $V^{\oplus \dim(V)}$.

(ii) *The group G admits a faithful finite-dimensional representation over $\mathbf{C}$; i.e., there is an injective continuous linear representation $G \to \mathrm{GL}_n(\mathbf{C})$ for some n, or equivalently G arises as a compact subgroup of some $\mathrm{GL}_n(\mathbf{C})$.*

(iii) *The pairwise-orthogonal characters χ_V of the irreducible representations of G span a dense subspace of the Hilbert space of continuous $\mathbf{C}$-valued class functions on G. In particular, a continuous class function $f : G \to \mathbf{C}$ that is nonzero must satisfy $\langle f, \chi_V \rangle \neq 0$ for some irreducible V.*

Moreover, the $\mathbf{C}$-subalgebra of $C^0(G)$ generated by the finitely many "matrix coefficients" $a_{ij} : G \to \mathbf{C}$ for a single faithful continuous representation $\rho : G \to \mathrm{GL}_n(\mathbf{C})$ is dense.

The proof of the Weyl character formula in the course text, based on analytic methods, uses (iii). There is a purely algebraic proof of the Weyl character formula (usually expressed in terms of semisimple Lie algebras over $\mathbf{C}$, and given in books on Lie algebras such as by Humphreys), but that involves infinite-dimensional algebraic tools (such as Verma modules).

We conclude our discussion by highlighting the use of "matrix coefficients" from (iii) to algebraize the theory of compact Lie groups. The starting point is the following interesting property of the collection of matrix coefficients $a_{ij} : G \to \mathbf{C}$ for a single representation $\rho : G \to \mathrm{GL}_n(\mathbf{C})$: the identity $\rho(gh) = \rho(g)\rho(h)$ for $g, h \in G$ says $a_{ij}(gh) = \sum_k a_{ik}(g)a_{kj}(h)$ for all i, j, or equivalently

$$a_{ij} \circ r_h = \sum_k a_{kj}(h) \cdot a_{ik}$$

for all $h \in G$ and i, j. In other words, under the right-regular representation of G on $C^0(G)$ we have

$$g.a_{ij} = \sum_k a_{kj}(g) a_{ik}$$

for all $g \in G$ and all i, j. Hence, inside the representation $C^0(G)$ of G, the vectors $a_{ij} \in C^0(G)$ are "G-finite" in the sense that the G-orbit of a_{ij} is contained in the finite-dimensional $\mathbf{C}$-vector space $\sum_k \mathbf{C} \cdot a_{ik}$.

It is clear that sums and products of G-finite vectors in $C^0(G)$ are G-finite, so by (iii) above we see that the subspace of G-finite vectors in $L^2(G)$ is *dense*. Note that the functions a_{ij} are given by $g \mapsto e_i^*(g.e_j)$ where $\{e_1, \dots, e_n\}$ is the standard basis of the representation space $\mathbf{C}^n$ for ρ and $\{e_1^*, \dots, e_n^*\}$ is the dual basis. In Proposition 1.2 of Chapter III of the course text, one finds a proof of the remarkable converse result that *every* G-finite vector in $C^0(G)$ is a matrix coefficient $g \mapsto \ell(g.v)$ for some finite-dimensional continuous $\mathbf{C}$-linear representation V of G, some vector $v \in V$, and some linear form $\ell \in V^*$.

This brings us to the core of the algebraization of the theory, *Tannaka–Krein Duality* (proved in Chapter III of the course text). Consider the $\mathbf{R}$-subspace $A(G) \subset C^0(G, \mathbf{R})$ consisting of the G-finite vectors. This is an $\mathbf{R}$-subalgebra (why?), and the existence of a faithful finite-dimensional representation of G over $\mathbf{C}$ can be used to prove that $A(G)$ is finitely generated as an $\mathbf{R}$-algebra. Moreover, in Chapter III it is proved that the natural map

$$A(G_1) \otimes_{\mathbf{R}} A(G_2) \to A(G_1 \times G_2)$$

an isomorphism for compact Lie groups G_1 and G_2. Thus, the map of topological spaces $m : G \times G \to G$ give by multiplication induces a map of $\mathbf{R}$-algebras

$$m^* : A(G) \to A(G \times G) = A(G) \otimes_{\mathbf{R}} A(G)$$

such that the resulting composition law on the topological space

$$G^{\mathrm{alg}} = \mathrm{Hom}_{\mathbf{R}\text{-alg}}(A(G), \mathbf{R})$$

via

$$G^{\mathrm{alg}} \times G^{\mathrm{alg}} = \mathrm{Hom}_{\mathbf{R}\text{-alg}}(A(G) \otimes_{\mathbf{R}} A(G), \mathbf{R}) \xrightarrow{m^*} G^{\mathrm{alg}}$$

is a continuous group law. Most remarkably of all, G^{alg} can be identified with a (smooth) Zariski-closed subgroup of some $\mathrm{GL}_n(\mathbf{R})$ (so it is a "matrix Lie group" over $\mathbf{R}$) with the natural map $G \to G^{\mathrm{alg}}$ carrying g to evaluation ev_g at g an isomorphism of Lie groups. In this sense, G is recovered from the $\mathbf{R}$-algebra $A(G)$ consisting of matrix coefficients for representations of G over $\mathbf{R}$.

Example 3.3. The $\mathbf{R}$-algebra $A(\mathrm{SO}(n))$ is $\mathbf{R}[x_{ij}]/(XX^{\mathrm{T}} = 1)$.

The following amazing theorem, beyond the level of the course text, expresses the precise link between compact Lie groups and purely algebro-geometric notions over $\mathbf{R}$.

Theorem 3.4. *The functor $G \rightsquigarrow \mathrm{Spec}(A(G))$ is an equivalence of categories from the category of compact Lie groups to the category of smooth affine group schemes $\mathscr{G}$ over $\mathbf{R}$ such that $\mathscr{G}$ does not contain GL_1 as a Zariski-closed $\mathbf{R}$-subgroup and $\mathscr{G}(\mathbf{R})$ meets every Zariski-connected component of $\mathscr{G}$. The inverse functor is $\mathscr{G} \rightsquigarrow \mathscr{G}(\mathbf{R})$, and G^0 goes over to $\mathscr{G}^0$.*

This explains why all examples of compact Lie groups that we have seen in this course, *and* all Lie group homomorphisms between them, are given by polynomial constructions in matrix entries (possibly decomposed into real and imaginary parts when the compact Lie groups are presented as closed subgroups of $\mathrm{GL}_n(\mathbf{C})$'s).