Math 210C. Quaternions

1. Introduction

Inside the **C**-algebra $\mathrm{Mat}_n(\mathbf{C})$ there is the **R**-subalgebra $\mathrm{Mat}_n(\mathbf{R})$ with the property that the natural map of **C**-algebras

$$\mathbf{C} \otimes_{\mathbf{R}} \mathrm{Mat}_n(\mathbf{R}) \to \mathrm{Mat}_n(\mathbf{C})$$

(satisfying $c \otimes M \mapsto cM$) is an isomorphism. (Proof: compare **C**-bases on both sides, using the standard **R**-basis of $\mathrm{Mat}_n(\mathbf{R})$ and the analogous **C**-basis for $\mathrm{Mat}_n(\mathbf{C})$.) There are many other **R**-subalgebras $A \subset \mathrm{Mat}_n(\mathbf{C})$ with the property that the natural map of **C**-algebras $\mathbf{C} \otimes_{\mathbf{R}} A \to \mathrm{Mat}_n(\mathbf{C})$ is an isomorphism: $A = g\mathrm{Mat}_n(\mathbf{R})g^{-1}$ for any $g \in \mathrm{GL}_n(\mathbf{C})$. This is a bit "fake" since such an **R**-subalgebra A is just $\mathrm{Mat}_n(\mathbf{R})$ embedded into $\mathrm{Mat}_n(\mathbf{C})$ via applying an automorphism of $\mathrm{Mat}_n(\mathbf{C})$ (namely, g-conjugation) to the usual copy of $\mathrm{Mat}_n(\mathbf{R})$ inside $\mathrm{Mat}_n(\mathbf{C})$.

But are there any fundamentally different A, such as one that is *not isomorphic* to $\mathrm{Mat}_n(\mathbf{R})$ as an **R**-algebra? Any such A would have to have **R**-dimension equal to n^2. In the mid-19th century, Hamilton made the important discovery that for $n = 2$ there *is* a very different choice for A. This exotic 4-dimensional **R**-algebra is denoted **H** in his honor, called the *quaternions*.

2. Basic construction

Define $\mathbf{H} \subset \mathrm{Mat}_2(\mathbf{C})$ to be the **R**-span of the elements

$$\mathbf{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{i} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{j} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{k} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Explicitly, for $a, b, c, d \in \mathbf{R}$ we have

$$a \cdot \mathbf{1} + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

for $\alpha = a + bi, \beta = c + di \in \mathbf{C}$. It follows that such an **R**-linear combination vanishes if and only if $\alpha, \beta = 0$, which is to say $a, b, c, d = 0$, so $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is **R**-linearly independent; we call it the *standard basis* for **H**. These calculations also show that **H** can be alternatively described as the set of elements of $\mathrm{Mat}_2(\mathbf{C})$ admitting the form

$$M = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

for $\alpha, \beta \in \mathbf{C}$.

It is easy to verify by direct calculation (do it!) that the following relations are satisfied:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \;\; \mathbf{ij} = \mathbf{k} = -\mathbf{ji}$$

and likewise

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \;\; \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

For any two quaternions

$$h = a \cdot \mathbf{1} + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}, \;\; h' = a' \cdot \mathbf{1} + b' \cdot \mathbf{i} + c' \cdot \mathbf{j} + d' \cdot \mathbf{k}$$

with $a, b, c, d, a', b', c', d' \in \mathbf{R}$, the product $hh' \in \mathrm{Mat}_2(\mathbf{C})$ expands out as an $\mathbf{R}$-linear combination in the products $\mathbf{e}\mathbf{e}'$ for $\mathbf{e}, \mathbf{e}'$ in the standard basis. But we just saw that all products among pairs from the standard basis are in $\mathbf{H}$, establishing the first assertion in:

Proposition 2.1. *The quaternions are an* $\mathbf{R}$*-subalgebra of* $\mathrm{Mat}_2(\mathbf{C})$*, and the natural map of* $\mathbf{C}$*-algebras* $\mu : \mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \to \mathrm{Mat}_2(\mathbf{C})$ *is an isomorphism.*

The stability of $\mathbf{H}$ under multiplication could also be checked using the description as matrices of the form $\left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$ with $\alpha, \beta \in \mathbf{C}$.

Proof. The source and target of μ are 4-dimensional over $\mathbf{C}$, so for the isomorphism assertion it suffices to check injectivity. More specifically, this is the assertion that the standard basis of $\mathbf{H}$ (viewed inside $\mathrm{Mat}_2(\mathbf{C})$) is even linearly independent over $\mathbf{C}$ (not just over $\mathbf{R}$).

Taking a, b, c, d from $\mathbf{C}$, we have

$$a \cdot \mathbf{1} + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k} = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix},$$

so if this vanishes then we have $a \pm bi = 0 = \pm c + di$ with $a, b, c, d \in \mathbf{C}$ (not necessarily in $\mathbf{R}$!). It is then clear that $a = 0$, so $b = 0$, and likewise that $di = 0$, so d and c vanish too. ■

The *center* of an associative ring with identity is the subset of elements commuting with everything under multiplication. This is a commutative subring (with the same identity).

Corollary 2.2. *The center of* $\mathbf{H}$ *coincides with* $\mathbf{R} = \mathbf{R} \cdot \mathbf{1}$.

Proof. Let $Z \subset \mathbf{H}$ be the center, so $\mathbf{R} \subset Z$. To prove equality it suffices to show $\dim_{\mathbf{R}} Z \leq 1$. But $\mathbf{C} \otimes_{\mathbf{R}} Z$ is certainly contained in the center of $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \simeq \mathrm{Mat}_2(\mathbf{C})$, and the center of the latter is just the evident copy of $\mathbf{C}$. This shows that $\dim_{\mathbf{R}} Z = \dim_{\mathbf{C}}(\mathbf{C} \otimes_{\mathbf{R}} Z) \leq 1$. ■

3. Conjugation and norm

For $h = a \cdot \mathbf{1} + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}$, define its *conjugate* to be

$$\overline{h} = a \cdot \mathbf{1} - b \cdot \mathbf{i} - c \cdot \mathbf{j} - d \cdot \mathbf{k},$$

so clearly $\overline{\overline{h}} = h$. We call h a *pure quaternion* if $a = 0$, or equivalently $\overline{h} = -h$. Although multiplication in $\mathbf{H}$ is not commutative, in a special case commutativity holds:

Proposition 3.1. *The products* $h\overline{h}$ *and* $\overline{h}h$ *coincide and are equal to* $a^2 + b^2 + c^2 + d^2$*. This is also equal to* $\det(h)$ *viewing* h *inside* $\mathrm{Mat}_2(\mathbf{C})$.

There is also a much easier identity: $h + \overline{h} = 2a = \mathrm{Tr}(h)$.

Proof. The expression $a^2 + b^2 + c^2 + d^2$ is unaffected by replacing h with $\overline{h}$, so if we can prove $h\overline{h}$ is equal to this expression in general then applying that to $\overline{h}$ gives the same for $\overline{h} \cdot \overline{\overline{h}} = \overline{h}h$. Hence, we focus on $h\overline{h}$. Writing h as a 2×2 matrix, we have

$$h = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

for $\alpha = a + bi$ and $\beta = c + di$. Since $a - bi = \overline{\alpha}$ and $-c - di = -\beta$, $\overline{h}$ corresponds to the analogous matrix using $\overline{\alpha}$ in place of α and $-\beta$ in place of β. Hence,

$$h\overline{h} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix} = \begin{pmatrix} \alpha\overline{\alpha} + \beta\overline{\beta} & 0 \\ 0 & \overline{\beta}\beta + \alpha\overline{\alpha} \end{pmatrix}.$$

This is $\mathbf{1}$ multiplied against the real scalar $|\alpha|^2 + |\beta|^2 = a^2 + b^2 + c^2 + d^2$. ■

We call $a^2 + b^2 + c^2 + d^2$ the *norm* of h, and denote it as $\mathrm{N}(h)$; in other words,

$$\mathrm{N}(h) = h\overline{h} = \overline{h}h$$

by viewing $\mathbf{R}$ as a subring of $\mathbf{H}$ via $c \mapsto c\mathbf{1}$.

It is clear by inspection of the formula that if $h \neq 0$ then $\mathrm{N}(h) \in \mathbf{R}^\times$, so in such cases $\overline{h}/\mathrm{N}(h)$ is a multiplicative inverse to h! Hence,

$$\mathbf{H}^\times = \mathbf{H} - \{0\};$$

we say $\mathbf{H}$ is a *division algebra* (akin to a field, but without assuming multiplication is commutative; multiplicative inverses do work the same on both sides). The $\mathbf{R}$-algebra $\mathbf{H}$ is very different from $\mathrm{Mat}_2(\mathbf{R})$ since the former is a division algebra whereas the latter has lots of zero-divisors!

Remark 3.2. The $\mathbf{R}$-linear operation $h \mapsto \overline{h}$ is an "anti-automorphism" of $\mathbf{H}$: it satisfies $\overline{hh'} = \overline{h'} \cdot \overline{h}$ for any $h, h' \in \mathbf{H}$. One way to see this quickly is to note that the cases $h = 0$ or $h' = 0$ are easy, and otherwise it suffices to check equality after multiplying on the left against $hh' \in \mathbf{H}^\times$. But

$$(hh')\overline{hh'} = \mathrm{N}(hh') = \det(hh') = \det(h)\det(h') = \mathrm{N}(h)\mathrm{N}(h') = h\overline{h}\mathrm{N}(h') = h\mathrm{N}(h')\overline{h} = hh'\overline{h'} \cdot \overline{h}$$

(the second to last equality using that $\mathbf{R}$ is central in $\mathbf{H}$, and the final equality using associativity).