Math 210C. Fundamental groups of Lie groups

Let G be a connected Lie group, and Γ a discrete normal (hence closed and central) subgroup. Let $G' = G/\Gamma$. Equip each of these connected manifolds with its identity point as the base point for computing its fundamental group. In HW5 you constructed a natural surjection $\pi_1(G') \twoheadrightarrow \Gamma$ and showed it is an isomorphism when $\pi_1(G) = 1$. The aim of this handout is to prove in general that the natural map $\pi_1(G) \to \pi_1(G')$ is injective and the resulting diagram of groups

$$1 \to \pi_1(G) \to \pi_1(G') \to \Gamma \to 1$$

is short exact; i.e., $\pi_1(G)$ maps isomorphically onto $\ker(\pi_1(G') \twoheadrightarrow \Gamma)$. Along the way, we'll construct the surjective homomorphism $\pi_1(G') \twoheadrightarrow \Gamma$ whose kernel is identified with $\pi_1(G)$. These matters can be nicely handled by using the formalism of universal covering spaces (especially that the universal cover $\widetilde{H}$ of a connected Lie group H is uniquely equipped with a Lie group structure compatible with one on H upon choosing a point $\widetilde{e} \in \widetilde{H}$ over the identity $e \in H$ to serve as the identity of the group law on $\widetilde{H}$). In this handout we give a more hands-on approach that avoids invoking universal covering spaces.

Proposition 0.1. *The natural map $\pi_1(G) \to \pi_1(G')$ is injective.*

Proof. Suppose $\sigma : (S^1, 1) \to (G, e)$ is a loop whose composition with the quotient map $q : G \to G'$ is homotopic to the constant loop based at e'. View σ as a continuous map $[0,1] \to G$ carrying 0 and 1 to e, and let

$$F : [0,1] \times [0,1] \to G'$$

be such a homotopy, so $F(\cdot, 0) = \sigma$, $F(x,1) = e'$, and $F(0,t) = F(1,t) = e'$ for all $t \in [0,1]$. Letting S be the 3/4-square $\partial_{\mathbf{R}^2}([0,1]^2) - (0,1) \times \{1\}$ obtained by removing the right edge, $F|_S : S \to G'$ lifts to the continuous map $\widetilde{F}_S : S \to G$ defined by $\widetilde{F}_S(x,0) = \sigma(x)$, $\widetilde{F}_S(0,t) = \widetilde{F}_S(1,t) = e$.

By the homotopy lifting lemma in HW5 Exercise 3(iii), $\widetilde{F}_S$ extends to a lift $\widetilde{F} : [0,1]^2 \to G$ of F; i.e., $q \circ \widetilde{F} = F$. In particular, $\widetilde{F}$ gives a homotopy between $\widetilde{F}(\cdot, 0) = \sigma$ and the path $\tau := \widetilde{F}(\cdot, 1)$ in G which lifts the constant path $F(\cdot, 1) = \{e'\}$. Hence, $\tau : [0,1] \to G$ is a path joining $\tau(0) = \widetilde{F}(0,1) = e$ to $\tau(1) = \widetilde{F}(1,1) = e$ in G and supported entirely inside the fiber $q^{-1}(e') = \Gamma$. But Γ is *discrete*, so the path τ valued in Γ must be constant. Since $\tau(0) = e$, it follows that τ is the constant path $\tau(x) = e$, so F is a homotopy between σ and the constant path in G based at e. In other words, the homotopy class of the initial σ is trivial, and this is precisely the desired injectivity. ■

Next, we construct a surjective homomorphism $\pi_1(G') \to \Gamma$. For any continuous map $\sigma : (S^1, 1) \to (G', e')$, by using compactness and connectedness of $[0,1]$ the method of proof of the homotopy lifting lemma gives that σ admits a lifting $\widetilde{\sigma} : [0,1] \to G$ with $\widetilde{\sigma}(0) = e$. In fact, since $G \to G'$ is a covering space (as Γ is discrete), the connectedness of $[0,1]$ implies that such a lift $\widetilde{\sigma}$ is *unique*. The terminal point $\widetilde{\sigma}(1)$ is an element of $q^{-1}(e') = \Gamma$. If $\sigma' \sim \sigma$ is a homotopic loop then the homotopy lifting argument in the previous paragraph adapts to show that a homotopy $F : [0,1]^2 \to G$ from σ to σ' lifts to a continuous map $\widetilde{F} : [0,1]^2 \to G'$ satisfying $\widetilde{F}(\cdot, 0) = \widetilde{\sigma}$, $\widetilde{F}(0,t) = \widetilde{\sigma}(0) = e$, and $\widetilde{F}(1,t) = \widetilde{\sigma}(1)$. Consequently,

$\widetilde{F}(\cdot,1) : [0,1] \to G$ is a continuous lift of $F(\cdot,1) = \sigma'$ that begins at $\widetilde{F}(0,1) = e$. By the *uniqueness* of the lift $\widetilde{\sigma}'$ of σ' beginning at e, it follows that $\widetilde{F}(\cdot,1) = \widetilde{\sigma}'$. In particular, $\widetilde{\sigma}'(1) = \widetilde{F}(1,1) = \widetilde{\sigma}(1)$ and $\widetilde{\sigma}'$ is homotopic to $\widetilde{\sigma}$ (as paths in G with initial point e and the same terminal point). Hence, $\widetilde{\sigma}(1)$ only depends on the homotopy class $[\sigma]$ of σ, so we get a well-defined map of sets

$$f : \pi_1(G') \to \Gamma$$

via $[\sigma] \mapsto \widetilde{\sigma}(1)$.

The map f is surjective. Indeed, choose $g_0 \in \Gamma = q^{-1}(e')$ and a path $\tau : [0,1] \to G$ linking e to g_0 (as we may do since G is path-connected). Define $\sigma := q \circ \tau : [0,1] \to G'$, visibly a *loop* based at e'. We have $\widetilde{\sigma} = \tau$ due to the uniqueness of the lift $\widetilde{\sigma}$ of σ beginning at $\tau(0) = e$. Consequently, $f([\sigma]) = \widetilde{\sigma}(1) = \tau(1) = g_0$, so f is surjective. From the definition of f it is clear that $f(\pi_1(G)) = \{e\}$ (since if σ is the image of a loop in G based on e then this latter *loop* must be $\widetilde{\sigma}$ and hence its terminal point $\widetilde{\sigma}(1)$ is equal to e). Conversely, if $\widetilde{\sigma}(1) = e$ then $\widetilde{\sigma}$ is a loop $(S^1, 1) \to (G, e)$ whose projection into $G' = G/\Gamma$ is σ, so $f^{-1}(e) = \pi_1(G)$. Thus, if f is a group homomorphism then it is surjective with kernel $\pi_1(G)$, so we would be done.

It remains to show that f is a homomorphism. For loops $\sigma_1, \sigma_2 : (S^1, 1) \rightrightarrows (G', e)$, we want to show that

$$\widetilde{\sigma}_1(1)\widetilde{\sigma}_2(1) = \widetilde{\sigma_1 * \sigma_2}(1)$$

in Γ, where the left side uses the group law in G and $\sigma_1 * \sigma_2 : S^1 \to G'$ is the loop made via concatenation (and time reparameterization). In other words, we wish to show that the unique lift of $\sigma_1 * \sigma_2$ to a path $[0,1] \to G$ beginning at e has as its terminal point exactly the product $\widetilde{\sigma}_1(1)\widetilde{\sigma}_2(1)$ computed in the group law of G.

Consider the two paths $\widetilde{\sigma}_2 : [0,1] \to G$ and $\widetilde{\sigma}_1(\cdot)\widetilde{\sigma}_2(1) : [0,1] \to G$. The first of these lifts σ_2 with initial point e and terminal point $\widetilde{\sigma}_2(1)$, and the second is the right-translation by $\widetilde{\sigma}_2(1) \in q^{-1}(e')$ of the unique path lifting σ_1 with initial point e and terminal point $\widetilde{\sigma}_1(1)$. Hence, $\widetilde{\sigma}_1(\cdot)\widetilde{\sigma}_2(1)$ is the unique lift of σ_1 with initial point $\widetilde{\sigma}_2(1)$ that is the terminal point of $\widetilde{\sigma}_2$, and its terminal point is $\widetilde{\sigma}_1(1)\widetilde{\sigma}_2(1)$. Thus, the concatenation path

$$(\widetilde{\sigma}_1(\cdot)\widetilde{\sigma}_2(1)) * \widetilde{\sigma}_2$$

is the unique lift of $\sigma_1 * \sigma_2$ with initial point e, so it is $\widetilde{\sigma_1 * \sigma_2}$ (!), and its terminal point is $\widetilde{\sigma}_1(1)\widetilde{\sigma}_2(1)$ as desired.