Math 210C. Size of fundamental group

1. Introduction

Let (V, Φ) be a nonzero root system. Let G be a connected compact Lie group that is semisimple (equivalently, Z_G is finite, or $G = G'$; see Exercise 4(ii) in HW9) and for which the associated root system is isomorphic to (V, Φ) (so $G \neq 1$). More specifically, we pick a maximal torus T and suppose an isomorphism $(V, \Phi) \simeq (\mathrm{X}(T)_{\mathbf{Q}}, \Phi(G, T))$ is chosen. Where can $\mathrm{X}(T)$ be located inside V? It certainly has to contain the *root lattice* $Q := \mathbf{Z}\Phi$ determined by the root system (V, Φ). Exercise 1(iii) in HW7 computes the index of this inclusion of lattices: $\mathrm{X}(T)/Q$ is dual to the finite abelian group Z_G. The root lattice is a "lower bound" on where the lattice $\mathrm{X}(T)$ can sit inside V.

This handout concerns the deeper "upper bound" on $\mathrm{X}(T)$ provided by the $\mathbf{Z}$-dual

$$P := (\mathbf{Z}\Phi^\vee)' \subset V^{**} = V$$

of the coroot lattice $\mathbf{Z}\Phi^\vee$ (inside the dual root system $(V^*, \Phi^\vee)$). This lattice P is called the *weight lattice* (the French word for "weight" is "poids") because of its relation with the representation theory of any possibility for G (as will be seen in our discussion of the Weyl character formula).

To explain the significance of P, recall from HW9 that semisimplicity of G implies $\pi_1(G)$ is a finite abelian group, and more specifically if $f : \widetilde{G} \to G$ is the finite-degree universal cover and $\widetilde{T} := f^{-1}(T) \subset \widetilde{G}$ (a maximal torus of $\widetilde{G}$) then

$$\pi_1(G) = \ker f = \ker(f : \widetilde{T} \to T) = \ker(\mathrm{X}_*(\widetilde{T}) \to \mathrm{X}_*(T))$$

is dual to the inclusion $\mathrm{X}(T) \hookrightarrow \mathrm{X}(\widetilde{T})$ inside V. Moreover, by HW8, the equality $V = \mathrm{X}(T)_{\mathbf{Q}} = \mathrm{X}(\widetilde{T})_{\mathbf{Q}}$ identifies $\mathrm{X}(\widetilde{T})$ with an intermediate lattice between $\mathrm{X}(T)$ and the weight lattice P. Thus, $\#\pi_1(G) | [P : \mathrm{X}(T)]$, with equality if and only if the inclusion $\mathrm{X}(\widetilde{T}) \subseteq P$ is an equality.

Our aim in this handout is to prove that $\mathrm{X}(\widetilde{T})$ is "as large as possible":

Theorem 1.1. *For any (G, T) as above, $\mathrm{X}(\widetilde{T}) = P$.*

This amounts to saying that the $\mathbf{Z}$-dual $\mathbf{Z}\Phi^\vee$ of P is equal to the $\mathbf{Z}$-dual $\mathrm{X}_*(\widetilde{T})$ of the character lattice of $\widetilde{T}$. In §7 of Chapter V of the course text you will find a proof of this result which has the same mathematical content as what follows, but is presented in a different style; see Proposition 7.16(ii) in Chapter V.

Since $\mathbf{Z}\Phi^\vee$ has as a $\mathbf{Z}$-basis the set $B^\vee = \{a^\vee \,|\, a \in B\}$ of "simple positive coroots" associated to a choice of basis B of (V, Φ), it would follow that the map $\prod_{a \in B} S^1 \to \widetilde{T}$ defined by $(z_a) \mapsto \prod_{a \in B} a^\vee(z_a)$ is an isomorphism. This is "dual" to the fact that $\mathrm{X}(T/Z_G) = Q = \mathbf{Z}\Phi$, which says the map $T/Z_G \to \prod_{a \in B} S^1$ defined by $t \mapsto (t^a)_{a \in B}$ is an isomorphism.

To summarize, upon choosing a basis B for the root system, the set B of simple positve roots is a basis for the character lattice of the "adjoint torus" T/Z_G whereas the set $B^\vee$ of simple positive coroots is a basis for the cocharacter lattice of the torus $\widetilde{T}$. This gives a very hands-on description for the maximal torus at the two extremes (simply connected or centerless).

In view of the containments $Q \subset \mathrm{X}(T) \subset \mathrm{X}(\widetilde{T}) \subset P$ where the first two inclusions have respective indices $\# Z_G$ and $\#\pi_1(G)$, yet another way to express that $\mathrm{X}(\widetilde{T}) \stackrel{?}{=} P$ is

$$\# Z_G \cdot \#\pi_1(G) \stackrel{?}{=} [P : Q]$$

for all semisimple G (or even just one G) with root system (V, Φ). Informally, this says that there is always a balancing act between the size of the center and the size of the fundamental group for semisimple connected compact Lie groups with a specified root system. (Note that $[P : Q]$ is instrisic to the root system (V, Φ), without reference to the lattice $\mathrm{X}(T)$!)

2. An application to isomorphism classes

This section will not be used later in this handout or anywhere in the course, but it is sufficiently important in practice and interesting for its own sake that we now digress on it before taking up the proof of Theorem 1.1.

Note that every sublattice of $\mathrm{X}(\widetilde{T})$ containing $\mathrm{X}(\widetilde{T}/Z_{\widetilde{G}}) = \mathrm{X}(T/Z_G) = Q$ has the form $\mathrm{X}(\widetilde{T}/Z)$ for a unique subgroup $Z \subset Z_{\widetilde{G}}$, and all subgroups Z of the finite group $Z_{\widetilde{G}}$ obviously arise in this way. Thus, once we show that $\mathrm{X}(\widetilde{T}) = P$, so the set of intermediate lattices between $\mathrm{X}(\widetilde{T}) = P$ and Q corresponds to the set of subgroups of P/Q (a set that is intrinsic to the root system), it follows that the set of such subgroups corresponds bijectively to the set of isogenous quotients of $\widetilde{G}$, or equivalently to the set of connected compact Lie groups with a specified universal cover $\widetilde{G}$.

The structure theory of semisimple Lie algebras over $\mathbf{R}$ and $\mathbf{C}$ (beyond the level of this course) implies that the Lie algebra of a semisimple connected compact Lie group is determined up to isomorphism by the root system, so since a connected Lie group that is *simply connected* is determined up to isomorphism by its Lie algebra (due to the Frobenius theorem and the link between π_1 and covering spaces in Exercise 3 of HW9), it follows that $\widetilde{G}$ is uniquely determined up to isomorphism by (V, Φ). Thus, the set of subgroups of the finite group P/Q associated to the root system (V, Φ) parameterizes the set of *all* possibilities for G with root system (V, Φ) without reference to a specific $\widetilde{G}$. (Explicitly, to any G we associate the subgroup $\mathrm{X}(T)/Q \subset P/Q$.)

There is a subtlety lurking in this final "parameterization". The argument shows that the possibilities for (G, T) with a given root system (V, Φ) are labeled by the finite subgroups of $P/Q = \mathrm{Hom}(Z_{\widetilde{G}}, S^1)$, which is to say that as we vary through the subgroups Z of $Z_{\widetilde{G}} \simeq \mathrm{Hom}(P/Q, S^1)$ the quotients $\widetilde{G}/Z$ exhaust all possibilities for G. However, this description rests on a *choice* of how to identify the universal cover of G with a fixed simply connected semisimple compact Lie group $\mathbf{G}$ having a given (V, Φ) as its root system; if $\mathbf{G}$ admits nontrivial automorphisms them in principle there might be several ways to associate a subgroup of P/Q to a possibility for G (via several isogenies $\mathbf{G} \to G$ not related through automorphisms of G)

More specifically, we have not accounted for the possibility that distinct $Z_1 \neq Z_2$ inside $Z_{\widetilde{G}}$ might yield quotients $G_i = \widetilde{G}/Z_i$ that are abstractly isomorphic. Since an isomorphism between connected Lie groups uniquely lifts to an isomorphism between their universal covers (combine parts (iii) and (iv) in Exercise 3 of HW9), the situation we need to look for is an *automorphism* f of $\widetilde{G}$ that descends to an isomorphism $G_1 \simeq G_2$, which is to say $f(Z_1) = Z_2$.

To summarize, the subtlety we have passed over in silence is the possibility that $\mathrm{Aut}(\widetilde{G})$ might act nontrivially on the set of subgroups of the finite center $Z_{\widetilde{G}}$. Obviously the normal subgroup $\mathrm{Inn}(\widetilde{G}) := \widetilde{G}/Z_{\widetilde{G}}$ of inner automorphisms acts trivially on the center, so it is really the effect on $Z_{\widetilde{G}}$ by the outer automorphism group

$$\mathrm{Out}(\widetilde{G}) := \mathrm{Aut}(\widetilde{G})/\mathrm{Inn}(\widetilde{G})$$

that we need to understand. This outer automorphism group for simply connected groups turns out to coincide with the automorphism group of the Dynkin diagram Δ (a fact whose proof requires relating Lie algebra automorphisms to diagram automorphisms, and so lies beyond the level of this course). Since P has a $\mathbf{Z}$-basis that is $\mathbf{Z}$-dual to $B^\vee$, and Q has B as a $\mathbf{Z}$-basis with B the set of vertices of Δ, there is an evident action of $\mathrm{Aut}(\Delta)$ on P/Q which computes exactly the dual of the action of $\mathrm{Out}(G)$ on $Z_{\widetilde{G}}$.

Thus, the uniform answer to the problem of parameterizing the set of isomorphism classes of semisimple compact connected G with a given root system (V, Φ) is the set of $\mathrm{Aut}(\Delta)$-orbits in the set of subgroups of P/Q. This is messy to make explicit when (V, Φ) is reducible: by Exercise 5 on HW9, it corresponds to the case $\widetilde{G} = \prod \widetilde{G}_i$ for several "simple factors" $\widetilde{G}_i$ (all of which are simply connected), and thereby gets mired in unraveling the effect on the set of subgroups of $\prod Z_{\widetilde{G}_i}$ by factor-permutation according to isomorphic $\widetilde{G}_i$'s.

The case of *irreducible* (V, Φ) has a very elegant description, as follows. Note that whenever $Z_{\widetilde{G}}$ is cyclic, each of its subgroups is determined by the size and hence is preserved under any automorphism of $Z_{\widetilde{G}}$, so in such cases there is no overlooked subtlety. It is a convenient miracle of the classification of irreducible root systems that the only cases for which P/Q is not cyclic are type D_{2m} for $m \geq 2$, which is to say $\widetilde{G} = \mathrm{Spin}(4m)$ with $m \geq 2$. In this case there are three proper nontrivial subgroups of $Z_{\widetilde{G}} = \mu_2 \times \mu_2$, corresponding to $\mathrm{SO}(4m)$ and two other degree-2 quotients of $\mathrm{Spin}(4m)$. For $m \neq 2$ the diagram has only a single order-2 symmetry, and as an automorphism of $Z_{\mathrm{Spin}(4m)}$ this turns out to preserve $\ker(\mathrm{Spin}(4m) \to \mathrm{SO}(4m))$ and swap the other two order-2 subgroups of $Z_{\mathrm{Spin}(4m)}$. In contrast, the D_4 diagram has automorphism group S_3 that transitively permutes all order-2 subgroups of $Z_{\mathrm{Spin}(8)}$, so in this case all three degree-2 quotients of $\mathrm{Spin}(8)$ are abstractly isomorphic!

3. The regular locus

As we have already noted above, Theorem 1.1 is equivalent to the assertion that the inequality $\#\pi_1(G) \leq [P : \mathrm{X}(T)]$ is an equality. By passing to $\mathbf{Z}$-dual lattices, this index is equal to that of the coroot lattice $\mathbf{Z}\Phi^\vee$ in the cocharacter lattice $\mathrm{X}_*(T)$. Thus, it suffices to show $\pi_1(G) \geq [\mathrm{X}_*(T) : \mathbf{Z}\Phi^\vee]$. We will achieve this by constructing a connected covering space of degree $[\mathrm{X}_*(T) : \mathbf{Z}\Phi^\vee]$ over an open subset of G whose complement is sufficiently thin. In this section, we introduce and study this open subset.

In Exercise 2 of HW10, we define $g \in G$ to be *regular* if g lies in a unique maximal torus of G, and it is shown there that the locus G^{reg} of regular elements of G is always open and non-empty. Although G^{reg} does not rest on a choice of maximal torus of G, it has a nice description in terms of such a choice:

Proposition 3.1. *The map $q : (G/T) \times T \to G$ defined by $(g \bmod T, t) \mapsto gtg^{-1}$ has all finite fibers of size $\#W$, and the fiber-size $\#W$ occurs precisely over G^{reg}. The restriction $q^{-1}(G^{\mathrm{reg}}) \to G^{\mathrm{reg}}$ is a (possibly disconnected) covering space with all fibers of size $\#W$.*

Here and below, $W = W(G,T)$.

Proof. For any $g_0 \in G$, if we let $f_{g_0} : (G/T)\times T \simeq (G/T)\times T$ be defined by left g_0-translation on G/T and the identity map on T then $q \circ f_{g_0} = c_{g_0} \circ q$ where $c_{g_0} : G \simeq G$ is g_0-conjugation. Thus, the q-fiber over $g \in G$ is identified via f_{g_0} with the q-fiber of the g_0-conjugate of g, so for the purpose of studying fiber-size and relating it to G^{reg} we may restrict our attention to $g \in T$ (as every element of G has a conjugate in T, by the Conjugacy Theorem).

The q-fiber over $t_0 \in T$ consists of points $(g \bmod T, t)$ such that $gtg^{-1} = t_0$. We saw earlier in the course that $T/W \to \mathrm{Conj}(G)$ is injective (even bijective), or more specifically that elements of T are conjugate in G if and only if they are in the same orbit under the natural action of $W = N_G(T)/T$ on T (via $N_G(T)$-conjugation). Thus, for any such (g,t), necessarily $t = w.t_0$ for some $w \in W$. Hence, $q^{-1}(t_0)$ consists of points $(g \bmod T, w.t_0)$ for $w \in W$ and $g \in G/T$ such that $gn_w \in Z_G(t_0)$ (where $n_w \in N_G(T)$ is a fixed representative of w. In other words, $g \bmod T \in (Z_G(t_0)/T).w$ via the natural right W-action on G/T using right-translation by representatives in $N_G(T)$.

We have shown that $\#q^{-1}(t_0) \geq \#W$ with equality if and only if $(Z_G(t_0)/T).W = W$ inside G/T, which is to say $Z_G(t_0) \subset N_G(T)$. Hence, we need to show that T is the unique maximal torus containing t_0 if and only if $Z_G(t_0) \subset N_G(T)$ (and then in all other cases $\dim Z_G(t_0) > \dim T$, so $q^{-1}(t_0)$ is infinite). If T is the only maximal torus containing t_0 then for any $g \in Z_G(t_0)$ the maximal torus gTg^{-1} containing $gt_0g^{-1} = t_0$ must equal T, which is to say $g \in N_G(T)$. Conversely, if $Z_G(t_0) \subset N_G(T)$ and T' is a maximal torus of G containing t_0 then T and T' are both maximal tori of the connected compact Lie group $Z_G(t_0)^0$, so they are conjugate under $Z_G(t_0)^0$. But any such conjugation preserves T since $Z_G(t_0) \subset N_G(T)$ by hypothesis, so $T' = T$.

It remains to prove that the map $q^{-1}(G^{\mathrm{reg}}) \to G^{\mathrm{reg}}$ whose fibers all have size $\#W$ is a covering map. In our proof of the Conjugacy Theorem, we showed rather generally that for any $(g \bmod T, t) \in (G/T) \times T$ and suitable bases of tangent spaces, the map $\mathrm{d}q(g \bmod T, t)$ has determinant $\det(\mathrm{Ad}_{G/T}(t^{-1}) - 1)$, where $\mathrm{Ad}_{G/T}$ is the T-action on $\mathrm{Tan}_e(G/T)$ induced by T-conjugation on G. Explicitly, we saw by considering the complexified tangent space that $\mathrm{Ad}_{G/T}(t^{-1}) - 1$ diagonalizes over $\mathbf{C}$ with eigenlines given by the root spaces in $\mathfrak{g}_{\mathbf{C}}$, on which the eigenvalues are the numbers $t^a - 1$. Hence,

$$\det(\mathrm{Ad}_{G/T}(t^{-1}) - 1) = \prod_{a \in \Phi(G,T)} (t^a - 1),$$

and this is nonzero precisely when $t^a \neq 1$ for all roots $a \in \Phi(G,T)$. By Exercise 2 of HW10, this is precisely the condition that $t \in T \cap G^{\mathrm{reg}}$, and that in turn is exactly the condition that the point $gtg^{-1} = q(g \bmod T, t)$ lies in G^{reg}. Hence, on $q^{-1}(G^{\mathrm{reg}})$ the map q is a local diffeomorphism. The following lemma concludes the proof. ∎

Lemma 3.2. *If $q : X \to Y$ is a continuous map between Hausdorff topological spaces and it is a local homeomorphism with finite constant fiber size $d > 0$ then it is a covering map.*

Proof. Choose $y \in Y$ and let $\{x_1, \ldots, x_d\} = q^{-1}(y)$. By the local homeomorphism property, each x_i admits an open set U_i mapping homeomorphically onto an open set $V_i \subset Y$ around y. By the Hausdorff property, if we shrink the U_i's enough (and then the V_i's correspondingly) we can arrange that they are pairwise disjoint. Let $V = \cap V_i$, and let $U'_i = U_i \cap q^{-1}(V)$. Since V is an open subset of V_i containing y, U'_i is an open subset of U_i containing x_i such that q carries U'_i homeomorphically onto V. But the U'_i's are pairwise disjoint (since the U_i's are), so $\coprod U'_i$ is an open subset of $q^{-1}(V)$ with each U'_i carried homeomorphically onto V.

All q-fibers have the same size d by hypothesis, and each of the d pairwise disjoint U'_i's contributes a point to each q-fiber over V, so this must exhaust all such fibers. That is, necessarily $\coprod U'_i = q^{-1}(V)$. In other words, we have identified the entire preimage $q^{-1}(V)$ with a disjoint union of d topological spaces U'_i that are each carried homeomorphically onto V by q, so q is a covering map. ■

We will show G^{reg} is connected and $\pi_1(G^{\mathrm{reg}}) \to \pi_1(G)$ is an isomorphism, and build a *connected* covering space of G^{reg} with degree $[\mathrm{X}_*(T) : \mathbf{Z}\Phi^\vee]$. The link between covering spaces and π_1 that was developed in Exercise 3 of HW9 then implies $\#\pi_1(G^{\mathrm{reg}}) \geq [\mathrm{X}_*(T) : \mathbf{Z}\Phi^\vee]$, so we would be done.

To show G^{reg} is connected and $\pi_1(G^{\mathrm{reg}}) \to \pi_1(G)$ is an isomorphism, we have to understand the effect on connectedness and on π_1 upon removing a closed subset from a connected manifold (as G^{reg} is obtained from G by removing the closed subset $G - G^{\mathrm{reg}}$ whose elements are called *singular*, as "singular" is an archaic word for "special"). If we remove a point from $\mathbf{R}^2$ then we retain connectedness (in contrast with removing a point from $\mathbf{R}$) but we increase the fundamental group. If $n \geq 3$ and we remove a point from $\mathbf{R}^n$ then the fundamental group does not change. More generally, if V is a subspace of $\mathbf{R}^n$ with codimension c then $\mathbf{R}^n - V$ is connected if $c \geq 2$ and $\pi_1(\mathbf{R}^n - V)$ is trivial if $c \geq 3$: we can change linear coordinates to arrange that $V = \{0\} \times \mathbf{R}^{n-c}$, so $\mathbf{R}^n - V = (\mathbf{R}^c - \{0\}) \times \mathbf{R}^{n-c}$, which retracts onto $\mathbf{R}^c - \{0\}$, whic is connected if $c \geq 2$ and has trivial π_1 if $c \geq 3$. Roughly speaking, connectedness is insensitive to closed subset of "codimension ≥ 2" and π_1 is insensitive to closed subsets of "codimension ≥ 3". To exploit this, we need to understand how thin $G - G^{\mathrm{reg}}$ is inside G.

Lemma 3.3. *The set $G - G^{\mathrm{reg}}$ is the image under a C^∞ map $M \to G$ for a non-empty compact C^∞ manifold M of dimension $\dim G - 3$.*

We know $\dim G \geq 3$ since $Z_G(T_a)/T_a$ is $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$ for any $a \in \Phi(G,T)$ (and G has root system (V, Φ) that was assumed to be non-zero; i.e., $\Phi \neq \emptyset$).

Proof. Fix a maximal torus T in G, so by the Conjugacy Theorem every $g \in G$ is conjugate to an element of T. For each $a \in \Phi(G,T)$ the kernel $K_a = \ker(a : T \to S^1)$ is a subgroup whose identity component is the codimension-1 torus T_a killed by a, and the singular set meets T in exactly the union of the K_a's (by Exercise 2 in HW10). Hence, $G - G^{\mathrm{reg}}$ is the union of the G-conjugates of the K_a's for varying $a \in \Phi(G,T)$.

In other words, $G - G^{\mathrm{reg}}$ is the union of the images of the the C^∞ conjugation maps $c_a : (G/Z_G(T_a)) \times K_a \to G$ defined by $(g \bmod Z_G(T_a), k) \mapsto gkg^{-1}$. For $r := \dim T$, we have $\dim K_a = \dim T_a = r - 1$ and (by complexified Lie algebra considerations) $\dim Z_G(T_a) = 3 + (r-1) = r+2$, so $\dim G/Z_G(T_a) = \dim G - r - 2$. Thus, $(G/Z_G(T_a)) \times K_a$ has dimension

$\dim G - 3$, so we can take M to be the union of the finitely many compact manifolds $(G/Z_G(T_a)) \times K_a$. ■

Corollary 3.4. *The regular locus G^{reg} inside G is connected and $\pi_1(G^{\mathrm{reg}}) \to \pi_1(G)$ is an isomorphism.*

Proof. This is part of Lemma 7.3 in Chapter V, and it amounts to some general facts concerning deformations of C^∞ maps to C^∞ manifolds, for which references are provided in the course text. Here we sketch the idea.

Consider a C^∞ map $f : Y \to X$ between non-empty C^∞ compact manifolds with X *connected*, Y possibly disconnected but with all connected components of the dimension at most some $d \le \dim X - 2$. (We have in mind the example $M \to G$ from Lemma 3.3.) Although $f(Y)$ may be rather nasty inside X, informally we visualize it as having "codimension" at least $\dim X - d \ge 2$, even though we have not made a rigorous definition of codimension for general closed subsets of X. (It is crucial that f is C^∞ – or at least differentiable – and not merely continuous, in view of space-filling curves. We also need the compactness of Y to avoid situations like a densely-wrapped line in $S^1 \times S^1$.)

Consider points $x, x' \in X - f(Y)$. They can be joined by a path in X, since X is path-connected. The first key thing to show, using the "codimension-2" property for $f(Y)$, is that any path $\sigma : [0,1] \to X$ with endpoints away from $f(Y)$ can be continuously deformed without moving the endpoints so that it entirely avoids $f(Y)$. The method goes as follows. By compactness of $[0,1]$, we can deform σ to be C^∞ without moving the endpoints, so suppose σ is C^∞. Intuitively, $\sigma([0,1])$ and $f(Y)$ are closed subsets of X whose "dimensions" add up to $< \dim X$, and σ carries the boundary $\{0,1\}$ of $[0,1]$ away from $f(Y)$, so a small perturbation of σ *leaving it fixed at the boundary* should provide a smooth path $[0,1] \to X - f(Y)$ entirely avoiding $f(Y)$ and having the same endpoints as σ. (As motivation, consider a pair of *affine linear* subspaces V, V' in $\mathbf{R}^n$ with $\dim V + \dim V' < n$. It is clear that by applying a tiny translation to V, we make these affine spaces disjoint, such as for a pair of lines in $\mathbf{R}^3$.) It follows that $X - f(Y)$ is (path-)connected, and likewise that $\pi_1(X - f(Y)) \to \pi_1(X)$ is surjective (using a base point in $X - f(Y)$).

Now consider the injectivity question for π_1's, assuming $d \le \dim X - 3$. For a path $\gamma : [0,1] \to X - f(Y)$ carrying 0 and 1 to points $x_0, x_1 \notin f(Y)$, suppose γ is homotopic to another path γ' linking x_0 to x_1, with the homotopy leaving the endpoints fixed. (The relevant case for π_1's is $x_0 = x_1$.) This homotopy is a continuous map $F : [0,1] \times [0,1] \to X$ carrying the boundary square S into $X - f(Y)$. By compactness of $[0,1] \times [0,1]$, we can deform F to be C^∞ without moving $F(0,t) = x_0$ or $F(1,t) = x_1$ and keeping $F(S)$ away from the closed $f(Y)$. Informally, $F([0,1] \times [0,1])$ and $f(Y)$ have "dimensions" adding up to $< \dim X$, and $F(S)$ avoids $f(Y)$, so a small perturbation of F *leaving it fixed on the boundary* should be a smooth map $H : [0,1] \times [0,1] \to X - f(Y)$ entirely avoiding $f(Y)$ and agreeing with F on S. Then $H(0,t) = F(0,t) = x_0$, $H(1,t) = F(1,t) = x_1$, $H(\cdot, 0) = F(\cdot, 0) = \gamma$, and $H(\cdot, 1) = F(\cdot, 1) = \gamma'$, so H is a homotopy from γ to γ' entirely within $X - f(Y)$. ■

Now our problem is reduced to constructing a connected covering space of G^{reg} with degree $[\mathrm{X}_*(T) : \mathbf{Z}\Phi^\vee]$.

4. Construction of covering spaces

Let's revisit the covering space $q^{-1}(G^{\text{reg}}) \to G^{\text{reg}}$ built above. We saw that this has fibers of size $\#W$. But this fiber-count can be refined in a useful way by bringing in group actions. Note that $W = N_G(T)/T$ acts on $(G/T) \times T$ on the right via the formula

$$(g \bmod T, t).w = (gn_w \bmod T, w^{-1}.t)$$

using the natural left W-action on T (pre-composed with inversion on W) and the right action of W on G/T via right $N_G(T)$-translation on G. By design, this action leaves q invariant: $q(x.w) = q(x)$ for all $x \in (G/T) \times T$ and $w \in W$. Hence, the W-action on $(G/T) \times T$ preserves each q-fiber, so in particular W acts on $q^{-1}(G^{\text{reg}})$ permuting all q-fibers in here.

For $t \in T \cap G^{\text{reg}}$, the q-fiber $q^{-1}(t)$ consists of the points $(w^{-1}, w.t)$ for $w \in W$, and these are the points in the W-orbit of $(1, t)$. Thus, by counting we see that W acts simply transitivity on the q-fiber over each point of $T \cap G^{\text{reg}}$, and so likewise on the entirety of G^{reg} since G-conjugation on G^{reg} brings any point into $T \cap G^{\text{reg}}$ (as we recall that $c_{g_0} \circ q = q \circ f_{g_0}$ by a suitable automorphism f_{g_0} of $(G/T) \times T$).

To summarize, not only is $q^{-1}(G^{\text{reg}}) \to G^{\text{reg}}$ a covering space with fibers of size $\#W$, but there is a W-action on $q^{-1}(G^{\text{reg}})$ leaving q invariant and acting simply transitively on all fibers. This leads us to a useful notion for *locally connected* topological spaces (i.e., topological spaces for which every point admits a base of open neighborhoods that are connected, such as any topological manifold):

Definition 4.1. Let Γ be a group, and X a locally connected non-empty topological space. A Γ-*space* over X is a covering space $f : E \to X$ equipped with a right Γ-action on E leaving f invariant (i.e., $f(\gamma.e) = f(e)$ for all $e \in E$ and $\gamma \in \Gamma$) such that Γ acts simply transitively on all fibers.

In practice, Γ-spaces tend to be disconnected.

Proposition 4.2. *Let X be non-empty and locally connected, $E \to X$ a covering space equipped with a right action over X by a group Γ, and Γ' a normal subgroup of Γ.*

1. *If E is a Γ-space then E/Γ' with the quotient topology is a Γ/Γ'-space over X.*
2. *If the Γ'-action makes E a Γ'-space over some Y then $E/\Gamma' \simeq Y$, making $Y \to X$ a covering space. If moreover the resulting Γ/Γ'-action on Y makes it a Γ/Γ'-space over X then E is a Γ-space over X.*
3. *If X is a connected and Γ' acts transitively on the set of connected components of X then E/Γ' is connected.*

Proof. The first assertion is of local nature over X, which is to say that it suffices to check it over the constituents of an open covering of X. Thus, we choose open sets over which the covering space splits, which is to say that we may assume $E = \coprod_{i \in I} U_i$ for pairwise disjoint open subsets U_i carried homeomorphically onto X. By shrinking some more we can arrange that X is (non-empty and) connected, so every U_i is connected. In other words, the U_i's have an intrinsic meaning relative to E having nothing to do with the covering space map down to X: they are the connected components of E.

Consequently, the action on E over X by each $\gamma \in \Gamma$ must permute the U_i's. Moreover, since Γ acts simply transitively on each fiber, and each fiber meets a given U_i in exactly one point (!), it follows that Γ acts simply transitively on the set of U_i's. That is, if we choose some U_{i_0} then for each $i \in I$ there is a unique $\gamma_i \in \Gamma$ such that γ_i carries U_i onto U_{i_0}, and moreover $i \mapsto \gamma_i$ is a bijection from I to Γ. In other words, the natural map $U_{i_0} \times \Gamma \to E$ over X defined by $(u, \gamma) \mapsto u.\gamma$ is a homeomorphism (giving Γ the discrete topology and $U_{i_0} \times \Gamma$ the product topology).

Identifying U_{i_0} with X via the covering map $E \to X$, we can summarize our conclusion as saying that when X is connected and the covering space is split, a Γ-space over X is simply $X \times \Gamma$ (mapping to X via pr_1) equipped with its evident right Γ-action; this is the *trivial* Γ-space. Now it is clear that passing to the quotient by Γ' yields a Γ/Γ'-equivariant homeomorphism $E/\Gamma' \simeq X \times (\Gamma/\Gamma')$, so by inspecting the right side we see that we have a Γ/Γ'-space. This completes the proof of (1).

For the proof of (2), we note that in (1) the case $\Gamma' = \Gamma$ recovers the base space as the quotient. Hence, in the setting of (2), we immediately get that $Y = E/\Gamma'$. Thus, there is a natural map $Y = E/\Gamma' \to X$, and by hypothesis this is a Γ/Γ'-space. We want to show that $E \to X$ is a Γ-space. This is of local nature on X, so we can shrink X to be connected with $E/\Gamma' = X \times (\Gamma/\Gamma')$. Since $E \to E/\Gamma'$ is a Γ'-space, any point of E/Γ' has a base of connected open neighborhoods over which the preimage in E is a trivial Γ'-space. The topological description of E/Γ' as $X \times (\Gamma/\Gamma')$ provides a copy of $X \times \{1\}$ inside E/Γ' and by working on sufficiently small connected open sets in there we can arrange by shrinking X that the preimage of $X \times \{1\}$ under $E \to E/\Gamma'$ is a disjoint union E' of copies of X permutted simply transitively by Γ'. But then the Γ-action on E permutes a single connected component E'_0 of E' simply transitively to all connected components of E (the ones over $X \times \{1\} \subset E/\Gamma'$ constitute a Γ'-orbit, and the ones over the other Γ/Γ'-translates of $X \times \{1\}$ inside E/Γ' exhaust the set of components without repetition). Hence, $E \to X$ is identified with $E'_0 \times \Gamma$ where E'_0 maps homeomorphically onto X, so $E = X \times \Gamma$. This establishes the Γ-space property, and so proves (2).

Now we prove (3). Since X is locally connected by hypothesis, so is its covering space E, so the connected components of E (which are always closed, as for any topological space) are also *open*. In other words, $E = \coprod E_i$ for the set $\{E_i\}$ of connected components of E. Fix some i_0, so if Γ' acts transitively on the set of connected components of E then each E_i is a Γ'-translate of E_{i_0}. Hence, the natural continuous map $E_{i_0} \to E/\Gamma'$ is surjective, so the target inherits connectedness from E_{i_0}. ∎

We will construct a (highly disconnected) $W \ltimes \mathrm{X}_*(T)$-space $E \to G^{\mathrm{reg}}$ (with W acting on $\mathrm{X}_*(T)$ on the right by composing its usual left action with inversion on W) for which the normal subgroup $W \ltimes \mathbf{Z}\Phi^\vee$ acts transitively on the set of connected components of E. By the preceding proposition, this then yields a *connected* covering space over G^{reg} that is a $\mathrm{X}_*(T)/\mathbf{Z}\Phi^\vee$-space, so it has degree equal to the size of this finite quotient group. That would establish the desired lower bound on the size of $\pi_1(G^{\mathrm{reg}}) = \pi_1(G)$.

The construction of E uses the exponential map for T as follows. By construction, $q^{-1}(G^{\mathrm{reg}}) = (G/T) \times T^{\mathrm{reg}}$ inside $(G/T) \times T$, where $T^{\mathrm{reg}} = T \times G^{\mathrm{reg}}$ is the set of $t \in T$ such that $t^a \neq 1$ for all $a \in \Phi(G, T)$. The exponential map $\exp_T : \mathfrak{t} \to T$ is an $\mathrm{X}_*(T)$-space (check!), so

$$\mathrm{id} \times \exp_T : (G/T) \times \mathfrak{t} \to (G/T) \times T$$

is also an $\mathrm{X}_*(T)$-space. Its restriction over the connected open $q^{-1}(G^{\mathrm{reg}}) = (G/T) \times T^{\mathrm{reg}}$ is then an $\mathrm{X}_*(T)$-space over $q^{-1}(G^{\mathrm{reg}})$. Explicitly, this $\mathrm{X}_*(T)$-space is $E := (G/T) \times \mathfrak{t}^{\mathrm{reg}}$ where $\mathfrak{t}^{\mathrm{reg}}$ is the set of points $v \in \mathfrak{t} = \mathrm{X}_*(T)_{\mathbf{R}}$ such that $(\exp_T(v))^a \neq 1$ for all $a \in \Phi(G,T)$. Each root $a : T \to S^1 = \mathbf{R}/\mathbf{Z}$ is obtained by exponentiating its Lie algebra map $\mathrm{Lie}(a) : \mathfrak{t} \to \mathbf{R}$, which is to say $\exp_T(v)^a = e^{2\pi i \mathrm{Lie}(a)(v)}$. Hence, $\exp_T(v)^a \neq 1$ if and only if $\mathrm{Lie}(a)(v) \notin \mathbf{Z}$. Thus,

$$\mathfrak{t}^{\mathrm{reg}} = \mathfrak{t} - \bigcup_{a \in \Phi, n \in \mathbf{Z}} H_{a,n} \tag{4.1}$$

where $H_{a,n}$ is the affine hyperplane in $\mathfrak{t}$ defined by the equation $\mathrm{Lie}(a)(v) = n$. The affine hyperplane $H_{a,n}$ is a translate of $H_{a,0} = \mathrm{Lie}(T_a) \subset \mathfrak{t}$. Since Φ is finite, it is easy to see that the collection $\{H_{a,n}\}$ of affine hyperplanes inside $\mathfrak{t}$ is locally finite.

Composing with the W-space map $q^{-1}(G^{\mathrm{reg}}) \to G^{\mathrm{reg}}$ yields a covering space map

$$E = (G/T) \times \mathfrak{t}^{\mathrm{reg}} \to G^{\mathrm{reg}}.$$

The source has a natural action by $\mathrm{X}_*(T)$ via translation on $\mathfrak{t} = \mathrm{X}_*(T)_{\mathbf{R}}$, and it also has a natural right action by $W = N_G(T)/T$ via right translation on G/T and via pre-composition of inversion on W with the natural left action on $\mathfrak{t}$ (akin to the W-action used to make $q^{-1}(G^{\mathrm{reg}}) \to G^{\mathrm{reg}}$ into a W-space). The W-action and $\mathrm{X}_*(T)$-action are compatible via the fact that the W-action on $\mathfrak{t}$ preserves the lattice $\mathrm{X}_*(T)$.

In this way we get an action by $W \ltimes \mathrm{X}_*(T)$ on $(G/T) \times \mathfrak{t}^{\mathrm{reg}}$ extending the $\mathrm{X}_*(T)$-space structure over $q^{-1}(G^{\mathrm{reg}})$ and inducing upon

$$q^{-1}(G^{\mathrm{reg}}) = E/\mathrm{X}_*(T)$$

its W-space structure as built above. Consequently, by Proposition 4.2(2), E has been equipped with a structure of $W \ltimes \mathrm{X}_*(T)$-space over G^{reg}. Since G^{reg} is a connected manifold, by Proposition 4.2(3) we just need to show that $W \ltimes (\mathbf{Z}\Phi^\vee)$ acts transitively on the set of connected components of $E = (G/T) \times \mathfrak{t}^{\mathrm{reg}}$.

There is an evident continuous left G-action on E through the first factor G/T, and this *commutes* with the right action of $W \ltimes (\mathbf{Z}\Phi^\vee)$ since this right action affects G/T through right $N_G(T)$-translation by G (and right translations commute with left translations on any group). The connectedness of G forces its action on E to preserve each connected component of E, so we may focus our attention on the action of $W \ltimes (\mathbf{Z}\Phi^\vee)$ on the quotient space $G\backslash E$. But the natural map $\mathfrak{t}^{\mathrm{reg}} \to G\backslash E$ is a homeomorphism under which the right $W \ltimes (\mathbf{Z}\Phi^\vee)$-action on E induces upon $\mathfrak{t}^{\mathrm{reg}}$ the right action given by $\mathbf{Z}\Phi^\vee$-translation and pre-composition of inversion on W with the usual left action by W (check this W-aspect!).

To summarize, upon looking back at (4.1), our task has been reduced to showing:

Proposition 4.3. *In the space $\mathfrak{t}^{\mathrm{reg}}$ obtained by removing from $\mathfrak{t}$ the locally finite collection of affine hyperplanes $H_{a,n}$, the action by $W \ltimes \mathbf{Z}\Phi^\vee$ is transitive on the set of connected components.*

These connected components are called *alcoves*; they are intrinsic to the root system (over $\mathbf{R}$). Further study of root systems shows that each Weyl chamber K contains a single alcove with 0 in its closure, and that alcove has an additional wall beyond the hyperplanes defined by $B(K)$, using the unique "highest root" relative to the basis $B(K)$. For example, in type A_3 these are open triangles. (See Figure 24 in §7 of Chapter V of the course text for pictures of the alcoves for several semisimple groups of low rank.) In general $W \ltimes \mathbf{Z}\Phi^\vee$ acts simply transitively on the set of alcoves, and it is called the *affine Weyl group* of the root system.

Proof. Recall that the proof of transitivity of the action of $W(\Phi)$ on the set of Weyl chambers proved more generally (with help from Exercise 1 in HW8) that for *any* locally finite set $\{H_i\}$ of affine hyperplanes in a finite-dimensional inner product space V over $\mathbf{R}$, the subgroup of affine transformations in $\mathrm{Aff}(V)$ generated by the orthogonal reflections in the H_i's acts transitively on the set of connected components of $V - (\cup H_i)$. Thus, equipping $\mathfrak{t}$ with a W-invariant inner product, it suffices check that group of affine-linear transformations of $\mathfrak{t}$ provided by $W \ltimes \mathbf{Z}\Phi^\vee$ (which permutes the set of affine hyperplanes $H_{a,n}$, as does even the bigger group $W \ltimes \mathrm{X}_*(T)$) contains the subgroup of $\mathrm{Aff}(\mathfrak{t})$ generated by the orthogonal reflections in the $H_{a,n}$'s.

Since $W = W(\Phi)$ (!) and we use a W-invariant inner product, the W-action on $\mathfrak{t}$ provides exactly the subgroup of $\mathrm{Aff}(\mathfrak{t})$ generated by orthogonal reflections in the hyperplanes $H_{a,0} = \mathrm{Lie}(T_a)$ through 0. We have to show that by bringing in translations by elements of the coroot lattice, we obtain the reflections in the $H_{a,n}$'s for any $n \in \mathbf{Z}$.

For ease of notation, let $a' = \mathrm{Lie}(a) \in \mathfrak{t}^*$. The orthogonal reflection $r_{a,n}$ in $H_{a,n}$ is given as follows. We know $r_{a,0}(\lambda) = \lambda - \langle \lambda, a\rangle a^\vee$, and the inclusion $\mathrm{X}_*(T) \hookrightarrow \mathrm{X}_*(T)_{\mathbf{R}} \simeq \mathfrak{t}$ makes $a'(b^\vee) = \langle a, b^\vee\rangle$ (apply the Chain Rule to a composition $S^1 \to T \to S^1$), so the vector $(n/2)a^\vee$ lies in $H_{a,n}$ and lies in the line through 0 orthogonal to this affine hyperplane since $H_{a,n}$ is a translate of $H_{a,0}$ and the W-invariant inner product identifies a' with a scalar multiple of dot product against $a^\vee$. Thus, to compute orthogonal reflection about $H_{a,n}$ we can translate by $-(n/2)a^\vee$ to move to $H_{a,0}$, then apply $r_{a,0}$, and finally translate by $(n/2)a^\vee$. This gives the formula

$$r_{a,n}(\lambda) = r_{a,0}(\lambda - (n/2)a^\vee) + (n/2)a^\vee = r_{a,0}(\lambda) + na^\vee,$$

so $r_{a,n} = r_{a,0} + na^\vee$. ■