Math 210C. A non-closed commutator subgroup

1. Introduction

In Exercise 3(i) of HW7 we saw that every element of SU(2) is a commutator (i.e., has the form $xyx^{-1}y^{-1}$ for $x, y \in \mathrm{SU}(2)$), so the same holds for its quotient SO(3). We have seen in Exercise 2 of HW8 that if G is a connected compact Lie group then its commutator subgroup G' is closed.

This can be pushed a bit further. The arguments in Exercise 2 of HW8 show there is a finite set of subgroups G_i of the form SU(2) or SO(3) such that the multiplication map of manifolds $G_1 \times \cdots \times G_n \to G'$ is surjective on tangent spaces at the identity, and hence (by the submersion theorem) has image containing an open neighborhood U of e in G'. Since U generates G' algebraically (as for an open neighborhood of the identity in any connected Lie group), G' is covered by open sets of the form $Uu_1 \dots u_r$ for finite sequences $\{u_1, \dots, u_r\}$ in U. But any open cover of G' has a finite subcover since G' is compact, so there is a *finite* upper bound on the number of commutators needed to express any element of G'.

It is natural to wonder about the non-compact case. Suppose G is an arbitrary connected Lie group and $H_1, H_2 \subset G$ are normal closed connected Lie subgroups. Is the commutator subgroup (H_1, H_2) closed? It is an important theorem in the theory of linear algebraic groups that if G is a Zariski-closed subgroup of some $\mathrm{GL}_n(\mathbf{C})$ with $H_1, H_2 \subset G$ also Zariski-closed then the answer is always affirmative, with the commutator even Zariski-closed. But counterexamples occur for the non-compact $G = \mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{R})$! Before discussing such counterexamples, we discuss important cases in which the answer is affirmative.

2. The compact case and the simply connected case

For compact G, we know from HW9 that G' is closed in G. Let's push that a bit further before we explore the non-compact case.

Proposition 2.1. *If G is a compact connected Lie group and H_1, H_2 are normal closed connected Lie subgroups then (H_1, H_2) is a semisimple compact subgroup.*

In Corollary 2.6, we will see that if G is semisimple then the closedness hypothesis can be dropped: any normal connected Lie subgroup of G is automatically closed.

Proof. If $Z_i \subset H_i$ is the maximal central torus in H_i then it is normal in G since G-conjugation on H_i is through automorphisms and clearly Z_i is carried onto itself under any automorphism of H_i. But a normal torus T in a connected compact Lie group is always central since its n-torsion $T[n]$ subgroups are collectively dense and each is a *finite* normal subgroup of G (so central in G by discreteness). Thus, each Z_i is central in G.

By Exercise 4(ii) in HW9, $H_i = Z_i \cdot H_i'$ with H_i' a semisimple compact connected Lie group. The central Z_i in G is wiped out in commutators, so $(H_1, H_2) = (H_1', H_2')$. Thus, we can replace H_i with H_i' so that each H_i is semisimple. Now $H_i \subset G'$, so we can replace G with G' to arrange that G is semisimple. If $\{G_1, \dots, G_n\}$ is the set of pairwise commuting simple factors of G in the sense of Theorem 1.1 in the handout "Simple Factors" then each H_i is a generated by some of the G_j's due to normality, so the isogeny $\prod G_j \to G$ implies that (H_1, H_2) is generated by the G_j's contained in H_1 and H_2. ■

Our main aim is to prove the following result, which addresses general connected Lie groups and highlights a special feature of the simply connected case. It rests on an important Theorem of Ado in the theory of Lie algebras that lies beyond the level of this course and will never be used in our study of compact Lie groups.

Theorem 2.2. *Suppose G is a connected Lie group, and H_1, H_2 are normal connected Lie subgroups. The subgroup (H_1, H_2) is a connected Lie subgroup with Lie algebra $[\mathfrak{h}_1, \mathfrak{h}_2]$, and if G is simply connected then the H_i and (H_1, H_2) are closed in G.*

In particular, $G = G'$ if and only if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

To prove this theorem, we first check that each $\mathfrak{h}_i$ is a "Lie ideal" in $\mathfrak{g}$ (i.e., $[\mathfrak{g}, \mathfrak{h}_i] \subset \mathfrak{h}_i$). More generally:

Lemma 2.3. *If $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{g}$ and N is the corresponding connected Lie subgroup of G then N is normal in G if and only if $\mathfrak{n}$ is a Lie ideal in $\mathfrak{g}$.*

Proof. Normality of N is precisely the assertion that for $g \in G$, conjugation $c_g : G \to G$ carries N into itself. This is equivalent to $\mathrm{Lie}(c_g) = \mathrm{Ad}_G(g)$ carries $\mathfrak{n}$ into itself for all g. In other words, N is normal if and only if $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ lands inside the closed subgroup of linear automorphisms preserving the subspace $\mathfrak{n}$. Since G is connected, it is equivalent that the analogue hold on Lie algebras. But $\mathrm{Lie}(\mathrm{Ad}_G) = \mathrm{ad}_{\mathfrak{g}}$, so the Lie ideal hypothesis on $\mathfrak{n}$ is precisely this condition. ■

Returning to our initial setup, since each $\mathfrak{h}_i$ is a Lie ideal in $\mathfrak{g}$, it follows from the Jacobi identity (check!) that the $\mathbf{R}$-span $[\mathfrak{h}_1, \mathfrak{h}_2]$ consisting of commutators $[X_1, X_2]$ for $X_i \in \mathfrak{h}_i$ is a Lie ideal in $\mathfrak{g}$. Thus, by the lemma, it corresponds to a normal connected Lie subgroup N in G. We want to prove that $N = (H_1, H_2)$, with N closed if G is simply connected.

Let $\widetilde{G} \to G$ be the universal cover (as in Exericse 3(iv) in HW9), and let $\widetilde{H}_i$ be the connected Lie subgroup of $\widetilde{G}$ corresponding to the Lie ideal $\mathfrak{h}_i$ inside $\mathfrak{g} = \widetilde{\mathfrak{g}}$. The map $\widetilde{H}_i \to G$ factors through the connected Lie subgroup H_i via a Lie algebra isomorphism since this can be checked on Lie algebras, so $\widetilde{H}_i \to H_i$ is a covering space.

Remark 2.4. Beware that $\widetilde{H}_i$ might *not* be the entire preimage of H_i in $\widetilde{G}$. Thus, although it follows that (H_1, H_2) is the image of $(\widetilde{H}_1, \widetilde{H}_2)$, this latter commutator subgroup of $\widetilde{G}$ might not be the full preimage of (H_1, H_2). In particular, closedness of $(\widetilde{H}_1, \widetilde{H}_2)$ will not imply closedness of (H_1, H_2) (and it really cannot, since we will exhibit counterexamples to such closedness later when G is not simply connected).

Letting $\widetilde{N} \subset \widetilde{G}$ be the normal connected Lie subgroup corresponding to the Lie ideal to $[\mathfrak{h}_1, \mathfrak{h}_2] \subset \widetilde{\mathfrak{g}}$, it likewise follows that $\widetilde{N} \to G$ factors through a covering space map onto N, so if $\widetilde{N} = (\widetilde{H}_1, \widetilde{H}_2)$ then $N = (H_1, H_2)$. Hence, it suffices to show that if G is simply connected then the H_i and N are closed in G and $N = (H_1, H_2)$. For the rest of the argument, we therefore may and do *assume G is simply connected.*

Lemma 2.5. *For simply connected G and any Lie ideal $\mathfrak{n}$ in $\mathfrak{g}$, the corresponding normal connected Lie subgroup N in G is closed.*

This is false without the simply connected hypothesis; consider a line with "irrational angle" in the Lie algebra of $S^1 \times S^1$ (contrasted with the analogue for its universal cover $\mathbf{R}^2$!).

Proof. There is an important Theorem of Ado (proved as the last result in Chapter I of Bourbaki's *Lie Groups and Lie Algebras*), according to which *every* finite-dimensional Lie algebra over a field F is a Lie subalgebra of $\mathfrak{gl}_n(F)$; we only need the case when F has characteristic 0 (more specifically $F = \mathbf{R}$). Thus, every finite-dimensional Lie algebra over $\mathbf{R}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbf{R})$, and so occurs as a (not necessarily closed!) connected Lie subgroup of $\mathrm{GL}_n(\mathbf{R})$. In particular, all finite-dimensional Lie algebras occur as Lie algebras of connected Lie groups! Hence, there is a connected Lie group Q such that $\mathrm{Lie}(Q) \simeq \mathfrak{g}/\mathfrak{n}$.

Since G is *simply connected*, so it has no nontrivial connected covering spaces (by HW9, Exercise 3(ii)!), the discussion near the end of §4 in the Frobenius theorem handout ensures that the Lie algebra map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n} = \mathrm{Lie}(Q)$ arises from a Lie group homomorphism $\varphi : G \to Q$. Consider $\ker \varphi$. By Proposition 4.4 in the Frobenius Theorem handout, this is a closed Lie subgroup of G and $\mathrm{Lie}(\ker \varphi)$ is identified with the kernel of the map $\mathrm{Lie}(\varphi)$ that is (by design) the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$. In other words, $\ker \varphi$ is a closed Lie subgroup of G whose Lie subalgebra of $\mathfrak{g}$ is $\mathfrak{n}$, so the same also holds for its identity component $(\ker \varphi)^0$. Thus, we have produced a *closed* connected subgroup of G whose Lie subalgebra coincides with a given Lie ideal $\mathfrak{n}$. It must therefore coincide with N by uniqueness, so N is closed. ■

Corollary 2.6. *If G is a semisimple compact connected Lie group and N is a normal connected Lie subgroup then N is closed.*

Proof. The case of simply connected G is already settled. In general, compactness ensures that the universal cover $\widetilde{G} \to G$ is a finite-degree covering space. The normal connected Lie subgroup $\widetilde{N} \subset \widetilde{G}$ corresponding to the Lie ideal $\mathfrak{n} \subset \mathfrak{g} = \widetilde{\mathfrak{g}}$ is therefore closed, yet $\widetilde{N} \to G$ must factor through N via a covering space map, so N is the image of the compact $\widetilde{N}$ and hence it is compact. Thus, N is closed. ■

We have constructed a normal closed connected Lie subgroup N in G such that $\mathfrak{n} = [\mathfrak{h}_1, \mathfrak{h}_2]$. It remains to prove that $N = (H_1, H_2)$. First we show that $(H_1, H_2) \subset N$. Since $\mathfrak{n}$ is a Lie ideal, we know by the preceding arguments that N is normal in G. Thus, the quotient map $G \to \overline{G} := G/N$ makes sense as a homomorphism of Lie groups, and $\mathrm{Lie}(\overline{G}) = \mathfrak{g}/\mathfrak{n} =: \overline{\mathfrak{g}}$. Let $\overline{\mathfrak{h}}_i \subset \overline{\mathfrak{g}}$ denote the image of $\mathfrak{h}_i$, so it arises from a unique connected (perhaps not closed) Lie subgroup $\overline{H}_i \hookrightarrow \overline{G}$. The map $H_i \to \overline{G}$ factors through $\overline{H}_i$ since this holds on Lie algebras, so to prove that $(H_1, H_2) \subset N$ it suffices to show that $\overline{H}_1$ and $\overline{H}_2$ commute inside $\overline{G}$.

By design, $[\overline{\mathfrak{h}}_1, \overline{\mathfrak{h}}_2] = 0$ inside $\overline{\mathfrak{g}}$ since $[\mathfrak{h}_1, \mathfrak{h}_2] = \mathfrak{n}$, so we will prove that in general a pair of connected Lie subgroups $\overline{H}_1$ and $\overline{H}_2$ inside a connected Lie group $\overline{G}$ commute with each other if their Lie algebras satisfy $[\overline{\mathfrak{h}}_1, \overline{\mathfrak{h}}_2] = 0$ inside $\overline{\mathfrak{g}}$. For any $\overline{h}_1 \in \overline{H}_1$, we want to prove that $c_{\overline{h}_1}$ on $\overline{G}$ carries $\overline{H}_2$ into itself via the identity map. It suffices to prove that $\mathrm{Ad}_{\overline{G}}(\overline{h}_1)$ on $\overline{\mathfrak{g}}$ is the identity on $\overline{\mathfrak{h}}_2$.

In other words, we claim that $\mathrm{Ad}_{\overline{G}} : \overline{H}_1 \to \mathrm{GL}(\overline{\mathfrak{g}})$ lands inside the subgroup of automorphisms that restrict to the identity on the subspace $\overline{\mathfrak{h}}_2$. By connectedness of $\overline{H}_1$, this holds if it does on Lie algebras. But $\mathrm{Lie}(\mathrm{Ad}_{\overline{G}}) = \mathrm{ad}_{\overline{\mathfrak{g}}}$, so we're reduced to checking that $\mathrm{ad}_{\overline{\mathfrak{g}}}$ on $\overline{\mathfrak{h}}_1$

restricts to 0 on $\overline{\mathfrak{h}}_2$, and that is precisely the vanishing hypothesis on Lie brackets between $\overline{H}_1$ and $\overline{H}_2$. This completes the proof that $(H_1, H_2) \subset N$.

To prove that the subgroup (H_1, H_2) exhausts the connected Lie group N, it suffices to prove that it contains an open neighborhood of the identity in N. By design, $\mathfrak{n} = [\mathfrak{h}_1, \mathfrak{h}_2]$ is spanned by finitely many elements of the form $[X_i, Y_i]$ with $X_i \in \mathfrak{h}_1$ and $Y_i \in \mathfrak{h}_2$ $(1 \leq i \leq n)$. Consider the C^∞ map $f : \mathbf{R}^{2n} \to G$ given by

$$f(s_1, t_1, \dots, s_n, t_n) = \prod_{i=1}^{n} (\exp_G(s_i X_i), \exp_G(t_i Y_i)) = \prod_{i=1}^{n} (\alpha_{X_i}(s_i), \alpha_{Y_i}(t_i)).$$

This map is valued inside $(H_1, H_2) \subset N$ and so by the Frobenius theorem (applied to the integral manifold N in G to a subbundle of the tangent bundle of G) it factors smoothly through N.

Rather generally, for any $X, Y \in \mathfrak{g}$, consider the C^∞ map $F : \mathbf{R}^2 \to G$ defined by

$$F(s,t) = (\exp_G(sX), \exp_G(tY)) = \exp_G(\mathrm{Ad}_G(\alpha_X(s))(tY)) \exp_G(-tY).$$

For fixed s_0, the parametric curve $F(s_0, t)$ passes through e at $t = 0$ with velocity vector $\mathrm{Ad}_G(\alpha_X(s_0))(Y) - Y$. Hence, if $s_0 \neq 0$ then $t \mapsto F(s_0, t/s_0)$ sends 0 to e with velocity vector at $t = 0$ given by

$$(1/s_0)(\mathrm{Ad}_G(\alpha_X(s_0))(Y) - Y).$$

But

$$\mathrm{Ad}_G(\alpha_X(s_0)) = \mathrm{Ad}_G(\exp_G(s_0 X)) = e^{\mathrm{ad}_{\mathfrak{g}}(s_0 X)}$$

as linear automorphisms of $\mathfrak{g}$, and since $\mathrm{ad}_{\mathfrak{g}}(s_0 X) = s_0 \mathrm{ad}_{\mathfrak{g}}(X)$, its exponential is $1 + s_0[X, \cdot] + O(s_0^2)$ (with implicit constant depending only on X), so

$$\mathrm{Ad}_G(\alpha_X(s_0))(Y) - Y = s_0[X,Y] + O(s_0^2).$$

Hence, $F(s_0, t/s_0)$ has velocity $[X,Y] + O(s_0)$ at $t = 0$ (where $O(s_0)$ has implied constant depending just on X and Y).

For fixed $s_i \neq 0$, we conclude that the map $\mathbf{R}^n \to (H_1, H_2) \subset N$ given by

$$(t_1, \dots, t_n) \mapsto f(s_1, t_1/s_1, \dots, s_n, t_n/s_n)$$

carries $(0, \dots, 0)$ to e with derivative there carrying $\partial_{t_i}|_0$ to $[X_i, Y_i] + O(s_i) \in \mathfrak{n}$ with implied constant depending just on X_i and Y_i. The elements $[X_i, Y_i] \in \mathfrak{n}$ form a spanning set, so likewise for elements $[X_i, Y_i] + O(s_i)$ if we take the s_i's sufficiently close to (but distinct from) 0. Hence, we have produced a map $\mathbf{R}^n \to N$ valued in (H_1, H_2) carrying 0 to e with derivative that is *surjective*. Thus, by the submersion theorem this map is open image near e, which gives exactly that (H_1, H_2) contains an open neighborhood of e inside N, as desired. This completes the proof of Theorem 2.2.

3. Non-closed examples

We give two examples of non-closed commutators. One is an example with (G, G) not closed in G, but in this example G is not built as a "matrix group" over $\mathbf{R}$ (i.e., a subgroup of some $\mathrm{GL}_n(\mathbf{R})$ that is a zero locus of polynomials over $\mathbf{R}$ in the matrix entries). The second example will use the matrix group $G = \mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{R})$ and provide closed connected Lie subgroups H_1 and H_2 for which (H_1, H_2) is not closed.

Such examples are "best possible" in the sense that (by basic results in the theory of linear algebraic groups over general fields) if G is a matrix group over $\mathbf{R}$ then relative to any chosen realization of G as a zero locus of polynomials in some $\mathrm{GL}_n(\mathbf{R})$, if H_1, H_2 are any two subgroups *Zariski-closed* in G relative to such a matrix realization of G then (H_1, H_2) is closed in G, in fact closed of finite index in a possibly disconnected Zariski-closed subgroup of G (the Zariski topology may be too coarse to detect distinct connected components of a Lie group, as for $\mathbf{R}^\times_{>0} \subset \mathbf{R}^\times$ via the matrix realization as $\mathrm{GL}_1(\mathbf{R})$). In particular, for any such matrix group G, (G, G) is always closed in G.

Example 3.1. Let $H \to \mathrm{SL}_2(\mathbf{R})$ be the universal cover, so it fits into an exact sequence

$$1 \to \mathbf{Z} \xrightarrow{\iota} H \to \mathrm{SL}_2(\mathbf{R}) \to 1.$$

Since $\mathfrak{h} = \mathfrak{sl}_2(\mathbf{R})$ is its own commutator subalgebra, we have $(H, H) = H$.

Choose an injective homomorphism $j : \mathbf{Z} \to S^1$, so $j(n) = z^n$ for $z \in S^1$ that is not a root of unity. Then $\mathbf{Z}$ is naturally a closed discrete subgroup of $S^1 \times H$ via $n \mapsto (j(-n), \iota(n))$, so we can form the quotient

$$G = (S^1 \times H)/\mathbf{Z}.$$

There is an evident connected Lie subgroup inclusion $H \to G$. The commutator subgroup (G, G) is equal to $(H, H) = H$, and we claim that this is not closed in G.

Suppose to the contrary that H were closed in G. By normality we could then form the Lie group quotient G/H that is a quotient of S^1 in which the image of z is trivial. But the kernel of $f : S^1 \twoheadrightarrow G/H$ would have to be a closed subgroup, so containment of z would imply that it contains the closure of the subgroup generated by z, which is dense in S^1. In other words, $\ker f = S^1$, which forces $G/H = 1$, contradicting that obviously $H \neq G$.

Example 3.2. The following example is an explicit version of a suggestion made by Tom Goodwillie on Math Overflow (as I discovered via Google search; try "Commutator of closed subgroups" to find it). Let $G = \mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{R})$. Let

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

inside the space $\mathfrak{sl}_2(\mathbf{R})$ of traceless 2×2 matrices. Obviously $\alpha_X(\mathbf{R})$ is the closed subgroup of diagonal elements in $\mathrm{SL}_2(\mathbf{R})$ with positive entries (a copy of $\mathbf{R}^\times_{>0}$), and $\alpha_Y(\mathbf{R})$ is a conjugate of this since $Y = gYg^{-1}$ for $g = \left(\begin{smallmatrix} 1 & -1/2 \\ 1 & 1/2 \end{smallmatrix}\right)$. But $[X, Y] = 2\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ has pure imaginary eigenvalues, so its associated 1-parameter subgroup is a circle.

Note that $(X, 0)$ and $(0, X)$ commute inside $\mathfrak{g}$, so they span of subspace of $\mathfrak{g}$ that exponentiates to a closed subgroup isomorphic to $\mathbf{R}^2$. In particular, (X, sX) exponentiates to a closed subgroup of G isomorphic to $\mathbf{R}$ for any $s \in \mathbf{R}$. We choose $s = 1$ to define H_1. Likewise,any (Y, cY) exponentiates to a closed subgroup of G isomorphic to $\mathbf{R}$, and we choose $c \in \mathbf{R}$ to be irrational to define H_2. Letting $Z = [X, Y]$, we likewise have that $(Z, 0)$ and $(0, Z)$ span a subspace of $\mathfrak{g}$ that exponentiates to a closed subgroup isomorphic to $S^1 \times S^1$. The commutator $[\mathfrak{h}_1, \mathfrak{h}_2]$ is (Z, cZ), so by the irrationality of c this exponentiates to a densely wrapped line L in the torus $S^1 \times S^1$. Hence, $(H_1, H_2) = L$ is not closed in G.