Math 210C. More features and applications of the exponential map

This handout addresses smoothness of the exponential map and two applications: continuous homomorphisms between Lie groups are C^∞ and a neat proof of the Fundamental Theorem of Algebra. We also discuss another context in which an exponential-like expression naturally arises: Lie's local formula for vector flow in the real-analytic case.

1. Smoothness

Let G be a Lie group, and $\exp_G : \mathfrak{g} \to G$ the exponential map. By definition, $\exp_G(v) = \alpha_v(1)$ is the flow 1 unit of time along the integral curve through e for the global left-invariant vector field $\widetilde{v}$ extending v at e. As we noted in class, it isn't immediately evident from general smoothness properties of solutions to ODE's (in terms of dependence on initial conditions and/or auxiliary parameters) that $\exp_G$ is C^∞. Of course, it is entirely unsurprising that such smoothness should hold. We now justify this smoothness, elaborating a bit on the proof given as Proposition 3.1 in Chapter I of the course text.

Consider the manifold $M = G \times \mathfrak{g}$, and the set-theoretic global vector field X on M given by

$$X(g, v) = (\widetilde{v}(g), 0) = (d\ell_g(e)(v), 0) \in T_{(g,v)}(M).$$

Note that X is C^∞ because $g \mapsto d\ell_g(e)$ is the adjoint representation $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ that we know is C^∞. We are going to directly construct integral curves to X and see by inspection that these are defined for all time. Hence, the associated flow is defined for all time. The general smoothness property for such flow on its domain of definition (see Theorem 5.12 in the handout on integral curves, whose proof involved some real work) will then yield what we want.

For each $m = (g, v) \in M$, consider the mapping $c_m : \mathbf{R} \to M$ defined by $c_m(t) = (g \cdot \alpha_v(t), v)$. This is a C^∞ mapping since the 1-parameter subgroup $\alpha_v : \mathbf{R} \to G$ is C^∞. Let's calculate the velocity vector at each time for this parametric curve in M: by the Chain Rule we have

$$c_m'(t) = ((d\ell_g(\alpha_v(t)))(\alpha_v'(t)), 0) = ((d\ell_g(\alpha_v(t)))(\widetilde{v}(\alpha_v(t))), 0)$$

where the final equality uses the identity $\alpha_v'(t) = \widetilde{v}(\alpha_v(t))$ which expresses that α_v is an integral curve to $\widetilde{v}$ (as holds by design of α_v). But $\widetilde{v}$ is a left-invariant vector field on G by design, so

$$(d\ell_g(h))(\widetilde{v}(h)) = \widetilde{v}(gh)$$

for all $h \in G$. Setting h to be $\alpha_v(t)$, we obtain

$$(d\ell_g(\alpha_v(t)))(\widetilde{v}(\alpha_v(t))) = \widetilde{v}(g \cdot \alpha_v(t)),$$

so

$$c_m'(t) = (\widetilde{v}(g \cdot \alpha_v(t)), 0) = X(g \cdot \alpha_v(t), v) = X(c_m(t)).$$

We have shown that c_m is an integral curve to the global smooth vector field X on M with value $c_m(0) = (g, v) = m$ at $t = 0$. In other words, c_m is an integral curve to X through m at time 0. But by design the smooth parametric curve c_m in M is defined on the entire real line, so we conclude that the open domain in $\mathbf{R} \times M$ for the flow associated to X is the entirety of $\mathbf{R} \times M$. In other words, the global flow is a mapping

$$\Phi : \mathbf{R} \times M \to M$$

and (as we noted already) this flow is known to always be C^∞ on its domain. That is, Φ is a C^∞ map. By design, $\Phi(t, m)$ is the flow to time t of the integral curve to X that is at m at time 0; i.e., $\Phi(t, m) = c_m(t)$.

We conclude that the restriction of Φ to the closed C^∞-submanifold $\mathbf{R} \times \{1\} \times \mathfrak{g}$ is also C^∞. This map

$$\mathbf{R} \times \mathfrak{g} \to M = G \times \mathfrak{g}$$

is $(t, v) \mapsto c_{(1,v)}(t) = (\alpha_v(t), v)$. Composing with projection to G, we conclude that the map $\mathbf{R} \times \mathfrak{g} \to G$ defined by $(t, v) \mapsto \alpha_v(t)$ is C^∞. Now restricting this to the slice $t = 1$ gives a further C^∞ map $\mathfrak{g} \to G$, and this is exactly the map $v \mapsto \alpha_v(1) = \exp_G(v)$ that we wanted to show is C^∞.

2. Continuous homomorphisms

Let $f : G' \to G$ be a continuous homomorphism between Lie groups. We aim to prove that f is necessarily C^∞. Consider the graph map

$$\Gamma_f : G' \to G' \times G$$

defined by $g' \mapsto (g', f(g'))$. This is a continuous injection with inverse given by the continuous restriction of pr_1, so it is a homeomorphism onto its image. Moreover, this image is closed, since it is the preimage of the diagonal in $G \times G$ under the continuous map $G' \times G \to G \times G$ defined by $(g', g) \mapsto (f(g'), g)$. The homomorphism property of f ensures that Γ_f is a subgroup inclusion, so Γ_f identifies G' topologically with a closed subgroup of $G' \times G$.

As we mentioned in class (and is proved in Theorem 3.11 of Chapter I of the course text, using the exponential map in an essential manner), a closed subgroup of a Lie group is necessarily a closed smooth submanifold. Thus, the graph has a natural smooth manifold structure, and we shall write $H \subseteq G' \times G$ to denote this manifold structure on $\Gamma_f(G')$.

Observe that topologically, f is the composition of the *inverse* of the homeomorphism $\mathrm{pr}_1 : H \to G$ with the other projection $\mathrm{pr}_2 : H \to G$. Since the inclusion of H into $G' \times G$ is C^∞ and $\mathrm{pr}_2 : G' \times G \to G$ is C^∞, it follows that $\mathrm{pr}_2 : H \to G$ is C^∞. Hence, to deduce that f is C^∞ it suffices to show that the C^∞ homeomorphism $\mathrm{pr}_1 : H \to G$ is a diffeomorphism (so its inverse is C^∞). But this latter map is a bijective Lie group homomorphism, so by an application of Sard's theorem and *homogeneity* developed in Exercise 5(iii) of HW3 it is necessarily a diffeomorphism! (Beware that in general a C^∞ homeomorphism between smooth manifolds need not be a diffeomorphism, as illustrated by $x \mapsto x^3$ on $\mathbf{R}$.)

3. Fundamental Theorem of Algebra

Now we present a beautiful proof of the Fundamental Theorem of Algebra that was discovered by Witt (and rediscovered by Benedict Gross when he was a graduate student). Let F be a finite extension of $\mathbf{C}$ with degree $d \geq 1$. Viewing F as an $\mathbf{R}$-vector space, $F \simeq \mathbf{R}^{2d}$ with $2d \geq 2$. Our aim is to prove $d = 1$.

Topologically, $F - \{0\} = \mathbf{R}^{2d} - \{0\}$ is connected. For any finite-dimensional associative $\mathbf{R}$-algebra (equipped with its natural manifold structure as an $\mathbf{R}$-vector space), the open submanifold of units is a Lie group. So $F^\times$ is a *connected* commutative Lie group.

In class we have seen via the exponential map that every connected commutative Lie group is a product of copies of S^1 and $\mathbf{R}$. Thus, $F^\times \simeq (S^1)^r \times \mathbf{R}^{2d-r}$ as Lie groups, for some $0 \leq r \leq 2d$. Necessarily $r > 0$, as the vector space factor $\mathbf{R}^{2d-r}$ is torsion-free whereas $F^\times$ has nontrivial torsion (any of the nontrivial roots of unity in $\mathbf{C}^\times$). By contracting the factor $\mathbf{R}^{2d-r}$ to a point, we see that $F^\times$ retracts onto $(S^1)^r$, and so by the homotopy-invariance and direct product functoriality of the fundamental group it follows that $\pi_1(F^\times) = \pi_1((S^1)^r)) = \pi_1(S^1)^r = \mathbf{Z}^r \neq 0$. But we noted that $F^\times = \mathbf{R}^{2d} - \{0\}$, and for $n > 1$ the group $\pi_1(\mathbf{R}^n - \{0\})$ is nontrivial only for $n = 2$, so $2d = 2$ and hence $d = 1$.

4. Lie's exponential flow formula

We know that if X is a C^∞ vector field on a C^∞ manifold M then the associated domain of flow $\Omega \subset \mathbf{R} \times M$ is an open subset containing $\{0\} \times M$ such that the flow $\Phi : \Omega \to M$ (also written as $\Phi_t(m) := \Phi(t, m))$ is C^∞ and satisfies $\Phi_0 = \mathrm{id}_M$ and

$$\Phi_{t'} \circ \Phi_t = \Phi_{t'+t}$$

locally on M for t, t' near 0 (depending on a small region in M on which we work). By considerations with several complex variables (to handle convergence issues for power series via differentiability conditions), one can refine this to see that if M and X are real-analytic then so is Φ.

The procedure can be run essentially in reverse if we focus on $t \approx 0$ (and hence don't fret about the maximality property of Ω): if M is a C^∞-manifold and $\Omega \subset \mathbf{R} \times M$ is an open subset containing $\{0\} \times M$ on which a C^∞-map $\Phi : \Omega \to M$ is given that satisfies $\Phi_0 = \mathrm{id}_M$ and $\Phi_{t'} \circ \Phi_t = \Phi_{t'+t}$ locally on M for t, t' near 0 (nearness to 0 depending on a small region in M on which we work) then near $\{0\} \times M$ we claim that Φ *is* the flow for an associated C^∞ vector field! Indeed, by smoothness of Φ the set-theoretic vector field

$$m \mapsto X(m) = \Phi_0'(m) \in \mathrm{T}_m(M)$$

is easily seen to be C^∞, and we claim that for each $m \in M$ the parametric curve $t \mapsto \Phi_t(m)$ (defined for t near 0 since Ω is open containing $\{0\} \times M$, and passing through $\Phi_0(m) = m$ at $t = 0$) is an integral curve to X on the open subset $\Omega \cap (\mathbf{R} \times \{m\}) \subset \mathbf{R}$ that contains 0 (but might be disconnected). To see this, we consider the identity

$$\Phi(t', \Phi(t, m)) = \Phi(t + t', m)$$

for t, t' near 0. Following Lie, we fix t to make each side a C^∞ parametric curve in t' and compute the velocity vector at $t' = 0$. Using the Chain Rule and the hypothesis $\Phi(0, \cdot) = \mathrm{id}_M$, this becomes $\Phi_0'(\Phi_t(m)) = \Phi_t'(m)$ for t near 0. By definition of X this says

$$X(\Phi_t(m)) = \Phi_t'(m),$$

or in other words $t \mapsto \Phi_t(m)$ is an integral curve to X for t near 0, as desired.

Thus, as long as we're willing to work in a small region $U \subset M$ and stick to time near 0, two viewpoints are interchangeable: the vector field X on U and the "action" Φ on U by a small interval in $\mathbf{R}$ around 0 (the perspective of "local Lie group" that was all Lie ever considered). In the real-analytic case (which is all Lie ever considered) the flow is also real-analytic as we noted above, and we claim that the flow Φ_t can then be described in terms of X in rather concrete terms. This rests on Lie's precursor to the Baker-Campbell-Hausdorff formula, a kind of "exponential map" at the level of operators on functions to reconstruct the flow for small time from the associated vector field:

Proposition 4.1 (Lie)**.** *For any point $m \in M$ and real-analytic f defined near m, we have*

$$f \circ \Phi_t = (e^{tX})(f) := \sum_{j=0}^{\infty} \frac{t^j}{j!} X^j(f)$$

on a small open neighborhood of m for all t near 0*.*

This "exponential operator" notation e^{tX} is just suggestive. The proof below ensures the convergence since we work real-analytically throughout.

Proof. The left side is real-analytic on $I \times U$ for some open interval $I \subset \mathbf{R}$ around 0 and some open subset $U \subset M$ around m. Thus, it admits a convergent power series expansion $\sum_{j \geq 0} c_j(t^j/j!)$ for t near 0 and *real-analytic functions* c_j defined on a common small open neighborhood of m in M. By analytic continuation, it suffices to show that for each j we have $c_j = X^j(f)$ near m.

Considering $f \circ \Phi_t$ as an $\mathbf{R}$-valued real-analytic function on some domain $I \times U$, if we differentiate it with respect to t then by the Chain Rule we get the real-analytic function on U given by

$$((df)(\Phi_t(u)))(\Phi_t'(u)) = ((df)(\Phi_t(u)))(X(\Phi_t(u))),$$

where the equality uses that the parametric curve $t \mapsto \Phi_t(u)$ is an integral curve to X (by design of X). The right side is exactly $(Xf)(\Phi_t(u))$, so we conclude that

$$\partial_t(f \circ \Phi_t) = (Xf) \circ \Phi_t$$

on $I \times U$. Now iterating this inductively,

$$\partial_t^j(f \circ \Phi_t) = (X^j f) \circ \Phi_t.$$

Setting t to be 0 on the right collapses this to be $X^j f$ since $\Phi_0 = \mathrm{id}_M$, so the coefficient functions in $C^\omega(U)$ for the series expansion of $f \circ \Phi_t$ in t are exactly the $X^j f$'s as desired. ■

If $\{x_1, \ldots, x_n\}$ are local real-analytic coordinates on M on a small open neighborhood U of a point $m \in M$, we conclude that for t near 0 the flow Φ_t near m is given in these coordinates by

$$\Phi_t(u) = (y_1(t, u), \ldots, y_n(t, u))$$

where

$$y_i(t, u) = e^{tX}(x_i) := \sum_{j=0}^{\infty} (X^j(x_i))(u) \frac{t^j}{j!}.$$

This is Lie's explicit description of the flow along X for small time in a small region of M when M and X are real-analytic.