MATH 210C. COMPACT LIE GROUPS

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This document consists of lectures notes from a course at Stanford University in Spring quarter 2018, along with appendices written by Conrad as supplements to the lectures.

The goal is to cover the structure theory of semisimple *compact* connected Lie groups, an excellent "test case" for the general theory of semisimple connected Lie groups (avoiding many analytic difficulties, thanks to compactness). This includes a tour through the "classical (compact) groups", the remarkable properties of maximal tori (especially conjugacy thereof), some basic ideas related to Lie algebras, and the combinatorial concepts of root system and Weyl group which lead to both a general structure theory in the compact connected case as well as a foothold into the finite-dimensional representation theory of such groups.

For a few topics (such as the manifold structure on G/H and basic properties of root systems) we give a survey in the lectures and refer to the course text [BtD] for omitted proofs. The homework assignments work out some important results (as well as instructive examples), and also develop some topological background (such as covering space theory and its relation to π_1), so one should regard the weekly homework as an essential part of the course. We largely avoid getting into the precise structure theory of semisimple Lie algebras, and develop what we need with $\mathfrak{sl}_2$ at one crucial step.

The main novelty of our approach compared to most other treatments of compact groups is using techniques that adapt to linear algebraic groups over general fields: (i) much more emphasis on the character and cocharacter lattices attached to a maximal torus T than on their scalar extensions to $\mathbf{R}$ (identified with $\mathrm{Lie}(T)^*$ and $\mathrm{Lie}(T)$ respectively) (ii) extensive use of torus centralizers and commutator subgroups, (iii) developing root systems over a general field of characteristic 0 (especially allowing $\mathbf{Q}$, not just $\mathbf{R}$), as in [Bou2].

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Contents

Thanks to Dan Dore and Vivian Kuperberg for note-taking on days when Aaron Landesman was away.

1. Basics of topological groups

1.1. First definitions and examples.

Definition 1.1. A *topological group* is a topological space G with a group structure such that the multiplication map

$$m : G \times G \to G$$

and inversion map

$$i : G \to G$$

are continuous.

Example 1.2. The open subset

$$\mathrm{GL}_n(\mathbf{R}) \subset \mathrm{Mat}_n(\mathbf{R})$$

can be given the structure of a topological group via the usual multiplication and inversion of matrices. We topologize it with the subspace topology (with $\mathrm{Mat}_n(\mathbf{R})$ given the Euclidean topology as a finite-dimensional $\mathbf{R}$-vector space). Matrix multiplication is given in terms of polynomials, and similarly for inversion with rational functions whose denominator is a power of the determinant (Cramer's Formula).

Likewise, $\mathrm{GL}_n(\mathbf{C}) \subset \mathrm{Mat}_n(\mathbf{C})$ is a topological group.

In general, to give a coordinate-free description, for V a finite-dimensional vector space over $\mathbf{R}$ or $\mathbf{C}$, the open subset $\mathrm{GL}(V) \subset \mathrm{End}(V)$ of invertible linear endomorphisms is a topological group with the subspace topology.

The following lemma is elementary to check:

Lemma 1.3. *For topological groups G and G', $G \times G'$ with the product topology is a topological group.*

Example 1.4. We have an isomorphism of topological groups

$$\begin{aligned} \mathbf{R}_{>0} \times S^1 &\to \mathbf{C}^\times = \mathrm{GL}_1(\mathbf{C}) \\ (r, w) &\mapsto rw. \end{aligned}$$

Example 1.5. For G a topological group, any subgroup $H \subset G$ is a topological group with the subspace topology. For us, this will be most useful for closed $H \subset G$.

Example 1.6. For $G = \mathrm{GL}_n(\mathbf{R})$, the *special linear group*

$$\mathrm{SL}_n(\mathbf{R}) := \{g \in G : \det g = 1\}$$

is a closed subset of $\mathrm{Mat}_n(\mathbf{R})$.

Example 1.7. The *orthogonal group*

$$\mathrm{O}(n) := \left\{ g \in G : \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in \mathbf{R}^n, \text{ or equivalently } g^\top g = 1_n \right\}.$$

Note that if $g^\top g = 1_n$ then $\det(g) = \pm 1$. Thus, the subgroup

$$\mathrm{SO}(n) := \mathrm{O}(n)^{\det = 1} \subset \mathrm{O}(n)$$

has finite index, and we even have

$$\mathrm{O}(n) = \mathrm{SO}(n) \rtimes \mathbf{Z}/2\mathbf{Z}$$

where the nontrivial element of $\mathbf{Z}/2\mathbf{Z}$ corresponds to the matrix

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Remark 1.8. In general, for $g \in \mathrm{O}(n)$ each $g(e_i)$ is a unit vector, so g has bounded entries as a matrix. Hence, $\mathrm{O}(n)$ is bounded in $\mathrm{Mat}_n(\mathbf{R})$, and it is also closed in $\mathrm{Mat}_n(\mathbf{R})$ (not only in $\mathrm{GL}_n(\mathbf{R})$!) since the description $g^\top g = 1_n$ implies invertibilty of g, so $\mathrm{O}(n)$ is compact.

1.2. **Non-compact analogues of** $\mathrm{O}(n)$**.** Recall that a *quadratic form* $q : V \to \mathbf{R}$ on a finite-dimensional $\mathbf{R}$-vector space V is a function given in linear coordinates by a degree-2 homogeneous polynomial $\sum_{i \le j} a_{ij} x_i x_j$ (a condition that is clearly insensitive to linear change of coordinates).

Lemma 1.9. *For V a finite-dimensional vector space over $\mathbf{R}$, we have a bijection*

$$\begin{array}{c} \{\text{symmetric bilinear } B : V \times V \to \mathbf{R}\} \\ \big\downarrow \\ \{\text{quadratic forms } q : V \to \mathbf{R}\} \end{array} \tag{1.1}$$

by the assignments $B \mapsto (q_B : v \mapsto B(v,v))$ and

$$q \mapsto (B_q : (v,w) \mapsto \frac{1}{2}\left(q(v+w) - q(v) - q(w)\right)). \tag{1.2}$$

The main point of the proof is to check that each proposed construction lands in the asserted space of objects and that they're inverse to each other. This is left as an exercise. The astute reader will check that this works well over any field not of characteristic 2.

Applying the preceding lemma with B taken to be the standard dot product on $\mathbf{R}^n$, for which q_B is the usual norm-square, we have:

Corollary 1.10.

$$\mathrm{O}(n) = \left\{ g \in \mathrm{GL}(V) : \|g(v)\|^2 = \|v\|^2 \textit{ for all } v \in V = \mathbf{R}^n \right\}.$$

Definition 1.11. A quadratic form q is *non-degenerate* if the associated bilinear form B_q is a perfect pairing, or equivalently its associated symmetric matrix (for a choice of basis of V) is invertible.

Example 1.12. Let $V = \mathbf{R}^n$. We can consider the form

$$q_{r,n-r}(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2.$$

The associated matrix B is a diagonal matrix whose first r entries are 1 and last $n-r$ entries are -1.

The Gram-Schmidt process (adapted to non-degenerate quadratic forms that might have some values ≤ 0) plus a bit of care implies that every (V,q) over $\mathbf{R}$ that is non-degenerate is isomorphic to $(\mathbf{R}^n, q_{r,n-r})$ for a unique $0 \le r \le n$. The idea is to build an orthonormal-like basis using that all positive real numbers have square roots, and the uniqueness needs some care since there is no *intrinsic* "maximal positive-definite subspace" (there are many such). We call the resulting pair $(r, n-r)$ the *signature* of the associated quadratic form q.

Definition 1.13. For q a non-degenerate quadratic form on a finite-dimensional $\mathbf{R}$-vector space V, we define

$$\mathrm{O}(q) := \{g \in \mathrm{GL}(V) : q(gv) = q(v)\} \subset \mathrm{GL}(V),$$

which is a closed subset of $\mathrm{GL}(V)$.

Remark 1.14. When q has signature $(r, n-r)$, we denote $\mathrm{O}(q)$ by $\mathrm{O}(r, n-r)$. For $0 < r < n$, $\mathrm{O}(r, n-r)$ is non-compact. One way to see this is to note that it contains $\mathrm{O}(1,1)$ as a closed subgroup (namely the subgroup that is the identity on all but one of the "positive" e_i's and on all but one of the "negative" e_j's), yet $\mathrm{O}(1,1)$ is non-compact because explicit computation yields

$$\mathrm{O}(1,1) = \left\{ \begin{pmatrix} a & b \\ \pm b & \pm a \end{pmatrix} : a^2 - b^2 = \pm 1 \right\}$$

(same sign choice throughout: either $+$ everywhere, or $-$ everywhere).

Remark 1.15. The determinant-1 subgroup $\mathrm{SO}(1,1)$ of $\mathrm{O}(1,1)$ is given by

$$\mathrm{SO}(1,1) := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 - b^2 = 1 \right\},$$

and this is disconnected since it is just the usual hyperbola in $\mathbf{R}^2$ that is disconnected.

Disconnectedness is somewhat annoying in the setting of Lie groups because it obstructs the use of Lie algebra techniques later on, as Lie algebras will only tell us about the connected component of the identity.

1.3. **Connectedness.** We'll now discuss questions of connectedness, beginning with some examples (to be revisited more fully later).

Question 1.16. Is $\mathrm{GL}_n(\mathbf{R})$ connected?

The answer is negative since we have a surjective continuous determinant map

$$\det : \mathrm{GL}_n(\mathbf{R}) \to \mathbf{R}^\times,$$

and $\mathbf{R}^\times$ is not connected. So, we might ask whether

$$\mathrm{GL}_n^+(\mathbf{R}) = \{g \in \mathrm{GL}_n(\mathbf{R}) : \det g > 0\}$$

is connected. Since $\mathrm{GL}_n^+(\mathbf{R})$ maps under the deteterminant map onto $\mathbf{R}_{>0}$, we actually have an isomorphism of topological groups (check homeomorphsm!)

$$\mathrm{GL}_n(\mathbf{R}) \simeq \mathbf{R}_{>0} \ltimes \mathrm{SL}_n(\mathbf{R}).$$

Hence, $\mathrm{GL}_n^+(\mathbf{R})$ is connected if and only if $\mathrm{SL}_n(\mathbf{R})$ is connected. It turns out $\mathrm{SL}_n(\mathbf{R})$ is connected, though this requires some work that will be taken up later.

Remark 1.17. We saw $\mathrm{O}(n) = \mathrm{SO}(n) \rtimes (\mathbf{Z}/2\mathbf{Z})$ is disconnected, but later we will show that $\mathrm{SO}(n)$ is connected. To appreciate this, we note that for $0 < r < n$ the non-compact $\mathrm{SO}(r, n-r)$ is always disconnected (for reasons related to orientation issues that are addressed in Appendix D, though the case $n = 2$ and $r = 1$ was seen explicitly above).

Example 1.18. The group $\mathrm{GL}_n(\mathbf{C})$ is connected, and we'll see why in HW2.

Definition 1.19. For G a topological group, let G^0 denote the connected component of $1 \in G$.

The subset $G^0 \subset G$ is closed, as for the connected component of any point in a topological space.

Lemma 1.20. *The closed subset $G^0 \subset G$ is a subgroup.*

Proof. The point is that in the commutative diagram

$$\begin{tikzcd} G^0 \times G^0 \arrow[r] \arrow[d] & G \times G \arrow[d, "m"] \\ G^0 \arrow[r] & G \end{tikzcd} \tag{1.3}$$

the point $(1,1) \in G^0 \times G^0$ maps to $1 \in G^0$ and m is continuous (so it carries the connected $G^0 \times G^0$ onto a connected subset of G). $\square$

Construction 1.21. Consider the open subset $\mathrm{GL}(V) \subset \mathrm{End}(V)$ inside an associative finite-dimensional $\mathbf{R}$-algebra. This is the collection of 2-sided units, and we know from linear algebra that a left inverse is the same thing as a right inverse. We now generalize this considerably. Let A be a finite-dimensional associative algebra over $\mathbf{R}$ or $\mathbf{C}$ (with $1 \in A$); a special case is $\mathrm{End}(V)$. Suppose $a \in A$ admits a left inverse: an element $a' \in A$ with $a'a = 1_A$. Then, we claim:

Lemma 1.22. *The element a' is also a right inverse; i.e., $aa' = 1_A$.*

Proof. Consider the natural map

$$\begin{aligned} \varphi : A &\to \mathrm{End}_{\text{v.sp.}}(A) \\ x &\mapsto \ell_x, \end{aligned}$$

where

$$\begin{aligned} \ell_x \colon A &\to A \\ \alpha &\mapsto x\alpha. \end{aligned}$$

The map φ is a map of associative algebras (check!), and we have (by associativity)

$$\ell_{a'} \circ \ell_a = \ell_{a'a} = \ell_{1_A} = \mathrm{id}_A\,.$$

This implies $\ell_a \circ \ell_{a'} = \mathrm{id}_A$ by linear algebra. Applying this to $1_A \in A$, we have $aa' = 1_A$. $\square$

Now it is unambiguous to define

$$A^\times = \{a \in A : \ a \text{ is invertible } \} \subset \mathrm{End}(A),$$

and this is a group by associativity. Moreover, it is an open subset of $\mathrm{End}(A)$ because we claim that the inclusion

$$A^\times \subset A \cap \mathrm{End}(A)^\times$$

is an equality. That is, if $\ell_a : A \to A$ is a linear isomorphism then we claim that a is invertible. Indeed, by surjectivity we have $\ell_a(a') = 1_A$ for some $a' \in A$, which is to say $aa' = 1_A$, so we also know $a'a = 1_A$ and hence $a \in A^\times$.

Example 1.23. The *quaternions* are a specific 4-dimensional associative $\mathbf{R}$-subalgebra

$$\mathbf{H} \subset \mathrm{Mat}_2(\mathbf{C}).$$

This is discussed in detail in Appendix A It is very different from the 4-dimensional associative $\mathbf{R}$-subalgebra $\mathrm{Mat}_2(\mathbf{R})$ because (as is shown in Appendix A) we have $\mathbf{H}^\times = \mathbf{H} - \{0\}$! That is, $\mathbf{H}$ is a *division algebra*. That appendix discusses $\mathbf{H}$ in concrete terms, including how to describe inversion. This example will be rather important for other key examples in the course.

1.4. **Quotients.** Let G be a topological group and $H \subset G$ a subgroup (not necessarily normal). Consider the map

$$\pi : G \to G/H,$$

where the latter has the *quotient topology* (so a subset $U \subset G/H$ is open by definition when $\pi^{-1}(U)$ is open).

Here is an important property that is not generally true of surjective quotient maps of topological spaces.

Lemma 1.24. *The continuous map π is open. That is, if $\Omega \subset G$ is open then $\pi(\Omega)$ is also open.*

Proof. We have

$$\pi^{-1}(\pi(\Omega)) = \bigcup_{h \in H} (\Omega \cdot h),$$

which is a union of open sets, hence open. $\square$

It is an important fact, left as an exercise to check, that the collection of open sets U in G/H is in bijection with the collection of open sets Ω in G stable under right-multiplication by H via the operations

$$U \mapsto \pi^{-1}(U), \quad \Omega \mapsto \pi(\Omega).$$

Corollary 1.25. *For any second such pair (G', H'), the natural continuous bijection*

$$\left(G \times G'\right) / \left(H \times H'\right) \to (G/H) \times \left(G'/H'\right).$$

is a homeomorphism.

Proof. The content is to show that the inverse map is continuous, and this is done by applying the preceding general characterization of open sets in such quotient spaces via open sets in the source that are invariant under suitable right-multiplications. $\square$

Remark 1.26. To appreciate how fortunate we are that the preceding corollary holds, we warn that it is *not* generally true that the formation of a topological quotient space commutes with the formation of direct products (beyond the quotient group setting as above). To make an example, consider the quotient map

$$\pi : X = \mathbf{Q} \twoheadrightarrow Y = X/(n \sim 0)_{n \in \mathbf{Z}}$$

that crushes $\mathbf{Z}$ to a point, where $\mathbf{Q}$ has its usual totally disconnected "archimedean" topology as a subset of $\mathbf{R}$. It turns out that if we form a direct product against $\mathbf{Q}$, the natural continuous bijection

$$(X \times \mathbf{Q})/((n, q) \sim (0, q))_{n \in \mathbf{Z}} \to Y \times \mathbf{Q}$$

is *not* a homeomorphism. A justification is given in [Brown, Example, p. 111] (where Y above is denoted as $\mathbf{Q}/\mathbf{Z}$, not to be confused with the group-theoretic quotient!), whereas in [Brown, 4.3.2] it is shown that such a phenomenon does not arise if one works with locally compact Hausdorff spaces, such as replacing $\mathbf{Q}$ with $\mathbf{R}$ in the preceding construction.

Corollary 1.27. *For $H \subset G$ normal, we have that G/H is a topological group.*

Proof. This is immediate from Corollary 1.25 with $G' = G$ and $H' = H$ (using continuity of the inverse of the bijection there in order to verify continuity of multiplication on G/H). ☐

The first interesting example of G/H as a topological group is:

Example 1.28. We have an isomorphism

$$\begin{aligned} \mathbf{R}/\mathbf{Z} &\to S^1 \\ t &\mapsto e^{2\pi i t} \end{aligned}$$

To illustrate the importance of considering G/H topologically with H not necessarily normal, consider a continuous action

$$G \times X \to X$$

of a topological group G on a topological space X. For a point $x_0 \in X$, consider its stabilizer

$$G_{x_0} := \mathrm{Stab}_G(x_0) \subset G.$$

The G-action on x_0 gives a continuous bijection

$$G/G_{x_0} \to G \cdot x_0 \subset X$$

onto the G-orbit of x_0 equipped with the subspace topology from X.

Is this a homeomorphism? The answer is negative in general, but we will see that it is affirmative in nice situations (and this will be upgraded to respect C^∞-structures later on). Here is one case where the homeomorphism property can be verified directly, and is of immediate utility:

Example 1.29. Fix $n \geq 2$, and let $G = \mathrm{SO}(n)$. For $X = \mathbf{R}^n$ with its usual G-action and $x_0 = e_1$ the first basis vector, we have the orbit $G.x_0 = S^{n-1}$ (since $n \geq 2$; check!). The G-stabilizer of e_1 is $\mathrm{SO}(n-1)$ (on the space spanned by $e_2, \ldots, e_n$); make sure to check this is really correct (i.e., the determinant-1 group $\mathrm{SO}(n-1)$, not $\mathrm{O}(n-1)$ or worse).

Thus, we have a continuous bijection

$$\mathrm{SO}(n)/\mathrm{SO}(n-1) \to S^{n-1}.$$

But the source is compact and the target is Hausdorff, so by everyone's favorite point-set topology exercise this is a homeomorphism! Since S^{n-1} is connected (as $n \geq 2$), this will underlie a later inductive proof of connectedness of $\mathrm{SO}(n)$ beginning with the connectedness of $\mathrm{SO}(1) = 1$.

2. Further classes of compact groups

Definition 2.1. A *Lie group* is a topological group with a C^∞ manifold structure so that $m : G \times G \to G$ and $i : G \to G$ are C^∞.

Remark 2.2. See Appendix B for why the smoothness of inversion is automatic.

Definition 2.3. Let V be a complex vector space. A *hermitian form* on V is a map

$$h : V \times V \to \mathbf{C}$$

that is sesquilinear (i.e., bi-additive and $\mathbf{C}$-linear in the first slot but conjugate-linear in the second slot) and satisfies $h(w, v) = \overline{h(v, w)}$.

Remark 2.4. On $\mathbf{C}^n$, we define the "standard" hermitian form to be

$$\langle v, w \rangle_{\mathrm{std}} = \sum_j v_j \overline{w}_j$$

(denoted $\langle v, w \rangle_{\mathrm{std},n}$ if we need to specify n, and sometimes just as $\langle \cdot, \cdot \rangle$ if the context makes it clear). This particular hermitian form is also non-degenerate, meaning that the conjugate-linear map

$$\begin{aligned} V &\to V^* \\ v &\mapsto h(\cdot, v) \end{aligned}$$

is an isomorphism (equivalently injective, by $\mathbf{C}$-dimension reasons).

Remark 2.5. For h a hermitian form, as a consequence of the conjugate-symmetry we have $h(v, v) \in \mathbf{R}$.

Remark 2.6. Next time, we'll show that for any non-degenerate hermitian h on V there is some (ordered) basis in which it looks like

$$\sum_j \varepsilon_j v_j \overline{w}_j$$

with $\varepsilon_j = \pm 1$.

Definition 2.7. If (V, h) is non-degenerate, define

$$\mathrm{U}(h) := \{ g \in \mathrm{GL}(V) : h(gv, gw) = h(v, w) \text{ for all } v, w \in V \}.$$

Because h is *non-degenerate*, it is easy to check that any $g \in \mathrm{End}(V)$ preserving h in this way has trivial kernel and so is automatically in $\mathrm{GL}(V)$. Thus, the definition of $\mathrm{U}(h)$ is unaffected by replacing $\mathrm{GL}(V)$ with $\mathrm{End}(V)$ (similarly to what we have seen for the definition of $\mathrm{O}(q)$ for non-degenerate quadratic forms q on finite-dimensional $\mathbf{R}$-vector spaces), ensuring that $\mathrm{U}(h)$ is closed in $\mathrm{End}(V)$, not just closed in $\mathrm{GL}(V)$; this will be very important.

Using $V = \mathbf{C}^n$ and $h = \langle \cdot, \cdot \rangle_{\mathrm{std},n}$ yields the group

$$\mathrm{U}(n) := \mathrm{U}(\langle \cdot, \cdot \rangle_{\mathrm{std},n}) = \left\{ g \in \mathrm{GL}_n(\mathbf{C}) : g^\top \overline{g} = 1_n \right\}.$$

Note that as a condition on matrix entries of g these are some complicated quadratic relations in the real and imaginary parts; it is not "complex-algebraic", due to the intervention of complex conjugation.

Remark 2.8. HW1, Exercise 2 shows that the subset $\mathrm{O}(n) \subset \mathrm{GL}_n(\mathbf{R})$ is a closed C^∞ submanifold. One can check that the same method applies to show that the subset $\mathrm{U}(n) \subset \mathrm{GL}_n(\mathbf{C}) \subset \mathrm{GL}_{2n}(\mathbf{R})$ is a closed C^∞ submanifold (*not* a complex submanifold of $\mathrm{GL}_n(\mathbf{C})$ since the defining equations involve complex conjugation).

Lemma 2.9. *For hermitian h on a (finite-dimensional, as always) $\mathbf{C}$-vector space V and $g \in \mathrm{GL}(V)$ we have $h(gv, gw) = h(v, w)$ for all $v, w \in V$ if and only if $h(gv, gv) = h(v, v)$ for all $v \in V$.*

Proof. We essentially want to reconstruct the hermitian form from the values on the diagonal. Observe

$$\begin{aligned} h\left(v+w, v+w\right) &= h(v,v) + h(w,w) + h(v,w) + h(w,v) \\ &= h(v,v) + h(w,w) + 2\mathrm{Re}\left(h(v,w)\right). \end{aligned}$$

Similarly, using iv in place of v and using $\mathbf{C}$-linearity in the first slot, we recover $\mathrm{Im}(h(v,w))$. This shows we can recover the hermitian form from its values on the diagonal and concludes the proof. $\square$

Since the restriction of $\langle \cdot, \cdot \rangle_{\mathrm{std},n}$ to pairs (v, v) for $v \in \mathbf{C}^n$ coincides with the usual length-squared on $\mathbf{C}^n = \mathbf{R}^{2n}$, we obtain:

Corollary 2.10. *We have*

$$\mathrm{U}(n) = \left\{ g \in \mathrm{GL}_n(\mathbf{C}) : \|gv\|^2 = \|v\|^2 \text{ for all } v \in \mathbf{C}^n = \mathbf{R}^{2n} \right\}.$$

Proof. This follows from Lemma 2.9. $\square$

This corollary shows that the $g(e_j)$ are unit vectors, so the closed subset $\mathrm{U}(n) \subset \mathrm{Mat}_n(\mathbf{C})$ is also bounded and hence compact.

Definition 2.11. Let

$$\mathrm{SU}(n) := \ker\left(\mathrm{U}(n) \hookrightarrow \mathrm{GL}_n(\mathbf{C}) \overset{\det}{\to} \mathbf{C}^\times\right) \subset \mathrm{SL}_n(\mathbf{C}) \supset \mathrm{SL}_n(\mathbf{R}).$$

Remark 2.12. We will later see that $\mathrm{SU}(n)$ is a "compact cousin" of $\mathrm{SL}_n(\mathbf{R})$.

2.1. **Compact forms of symplectic groups.** Next, we'll define a compact analogue of symplectic groups by using "quaternionic hermitian" forms. First, let's recall what a symplectic group is. Recall that a bilinear form ψ on a finite-dimensional vector space V over a field k is called *alternating* if $\psi(v, v) = 0$ for all $v \in V$ (which implies the skew-symmetry condition $\psi(w, v) = -\psi(v, w)$ by working with $(v + w, v + w)$, and is equivalent to skew-symmetry when $\mathrm{char}(k) \neq 2$).

Definition 2.13. A *symplectic form* ψ on a finite-dimensional vector space V over a field k is an alternating bilinear form $\psi : V \times V \to k$ that is non-degenerate, meaning that the linear map

$$\begin{aligned} V &\to V^* \\ v &\mapsto \psi(v, \cdot) \end{aligned}$$

is an isomorphism (equivalently injective, by dimension reasons).

Example 2.14. Let $V = k^{2n}$ and consider the bilinear form $\Psi : k^{2n} \times k^{2n} \to k$ with associated matrix

$$[\Psi] = \begin{pmatrix} 0 & 1_n \\ -1_n & 0. \end{pmatrix}$$

This defines a symplectic form, called the *standard symplectic form*.

Fact 2.15. Given a symplectic form ψ, there always exists a "symplectic basis":

$$\{e_1, \ldots, e_{2n}\}$$

(in particular, $\dim V = 2n$ is necessarily even), which is to say a basis relative to which ψ has the matrix as in Example 2.14. That is, for $1 \leq j \leq n$,

$$\psi(e_j, e_k) = \begin{cases} 1 & \text{if } k = j + n \\ -1 & \text{if } k = j - n \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.16. The *symplectic group* of a symplectic space (V, ψ) is

$$\mathrm{Sp}(\psi) = \{g \in \mathrm{GL}(V) : \psi(gv, gw) = \psi(v, w) \text{ for all } v, w \in V\}.$$

In the case of $V = k^{2n}$ and the standard ψ, we denote it as $\mathrm{Sp}_{2n}(k)$.

Once again, as for $\mathrm{O}(n)$ and $\mathrm{U}(n)$, preservation of the non-degenerate ψ implies that the definition of $\mathrm{Sp}(\psi)$ is unaffected if we replace $\mathrm{GL}(V)$ with $\mathrm{End}(V)$. Thus, in terms of $n \times n$ block matrices we have:

$$\mathrm{Sp}_{2n}(k) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{2n}(k) : g^\top \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}. \right\}$$

These yield explicit conditions on the $n \times n$ matrices A, B, C, D given in HW2. These conditions collapse somewhat in the case $n = 1$, recovering $\mathrm{Sp}_2 = \mathrm{SL}_2$ with the usual defining condition on 2×2 matrices.

Remark 2.17. In HW2 it is shown that for $k = \mathbf{R}, \mathbf{C}$, the closed subset $\mathrm{Sp}_{2n}(k) \subset \mathrm{Mat}_{2n}(k)$ is a smooth submanifold (even a complex submanifold when $k = \mathbf{C}$ if you know about such things, but we won't need complex manifolds in this course).

We next seek compact cousins of symplectic groups using a hermitian concept over the quaternions

$$\begin{aligned} \mathbf{H} &:= \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathrm{Mat}_2(\mathbf{C}) \right\} \\ &= \mathbf{R}1 \oplus \mathbf{R}\underline{i} \oplus \mathbf{R}\underline{j} \oplus \mathbf{R}\underline{k} \end{aligned}$$

where

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \underline{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ \underline{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \underline{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Appendix A shows the following facts:

1. $\mathbf{H}$ is an $\mathbf{R}$-subalgebra of $\mathrm{Mat}_2(\mathbf{C})$ with center $\mathbf{R}$,
2. We have the basis multiplication relations

$$\underline{i}\underline{j} = \underline{k},\ \underline{j}\underline{i} = -\underline{k},\ \underline{j}\underline{k} = \underline{i},\ \underline{i}^2 = \underline{j}^2 = \underline{k}^2 = -\mathbf{R}1.$$

(3) For $h = a + b\underline{i} + c\underline{j} + d\underline{k}$ with $a, b, c, d \in \mathbf{R}$, define the *conjugate*
$$\overline{h} := a - b\underline{i} - c\underline{j} - d\underline{k}.$$
The following hold:
$$\overline{h}h = h\overline{h} = a^2 + b^2 + c^2 + d^2 =: N(h)$$
(visibly in $\mathbf{R}^\times$ when $h \neq 0$), so $\mathbf{H}^\times = \mathbf{H} - \{0\}$ with
$$h^{-1} = \frac{h}{N(h)}$$
for $h \neq 0$. Moreover,
$$\overline{h'h} = \overline{h}\overline{h'},$$
from which one obtains
$$N(h')N(h) = N(h'h).$$
(4) We have an isomorphism of associative $\mathbf{C}$-algebras
$$\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \simeq \mathrm{Mat}_2(\mathbf{C}) \simeq \mathbf{C} \otimes_{\mathbf{R}} \mathrm{Mat}_2(\mathbf{R})$$
via the natural maps from the outer terms to the middle term. The content here is that $1, \underline{i}, \underline{j}, \underline{k}$ inside $\mathrm{Mat}_2(\mathbf{C})$ are not just linearly indepent over $\mathbf{R}$, but even linearly independent over $\mathbf{C}$.

Remark 2.18. Using
$$\begin{aligned} \mathbf{C} &\to \mathbf{H} \\ i &\mapsto \underline{i}, \end{aligned}$$
we get $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}\underline{j}$ by writing $\underline{k} = \underline{i}\underline{j}$ to obtain
$$a + b\underline{i} + c\underline{j} + d\underline{k} = (a + b\underline{i}) + (c + d\underline{i})\underline{j}.$$

Consider $\mathbf{H}^n$ as a left $\mathbf{H}$-module. It then makes sense to contemplate $\mathbf{H}$-linear endomorphisms. For
$$T \in \mathrm{End}_{\mathbf{H}}(\mathbf{H}^n),$$
we have
$$\begin{aligned} T\left(\sum_j c_j e_j\right) &= \sum_j c_j T(e_j) \\ &= \sum_j c_j \sum_i h_{ij}(e_i) \\ &= \sum_i \left(\sum_j c_j h_{ij}\right) e_i. \end{aligned}$$
with $c_j \in \mathbf{H}$.

So, for the corresponding "matrix"
$$[T] = \left(h_{ij}\right),$$

that encodes T, it is a straightforward exercise to verify

$$\left[T' \circ T\right] = [T]\left[T'\right];$$

that is,

$$\left[T' \circ T\right]_{ij} = \sum_k h_{kj} h'_{ik}.$$

Warning 2.19. The quaternions are not commutative, so one must be careful about the order of multiplication.

Remark 2.20. The lesson from the above is that, when working over the quaterions, it is often cleaner to avoid matrices if possible and instead use the language of linear maps.

Definition 2.21. Define

$$\mathrm{GL}_n(\mathbf{H}) := \mathrm{End}_{\mathbf{H}}(\mathbf{H}^n)^{\times}$$

(we know this is an open subset of the $\mathbf{R}$-vector space $\mathrm{End}_{\mathbf{H}}(\mathbf{H}^n) = \mathrm{Mat}_n(\mathbf{H})$ by our general discussion about units in finite-dimensional associative $\mathbf{R}$-algebras last time).

Using

$$\mathbf{H} = \mathbf{C} \oplus \mathbf{C}\underline{j},$$

from Remark 2.18, we get

$$\mathbf{H}^n = \mathbf{C}^n \oplus \mathbf{C}^n\underline{j} \simeq \mathbf{C}^{2n}$$

as $\mathbf{C}$-vector spaces (using left multiplication on $\mathbf{H}^n$). Hence, an element of $\mathrm{Mat}_n(\mathbf{H})$ is exactly an element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{2n}(\mathbf{C})$$

(using blocks in $\mathrm{Mat}_n(\mathbf{C})$) that commutes with left multiplication by $\underline{j}$. This leads to the natural question:

Question 2.22. What concrete condition does commuting with left multiplication by $\underline{j}$ impose on the $n \times n$ matrix blocks A, B, C, D over $\mathbf{C}$ above?

Since

$$\underline{j}\left(u + v\underline{j}\right) = -\overline{v} + \overline{u}\underline{j}$$

for any $u, v \in \mathbf{C}$ (an easy consequence of the fact that $\underline{j}$-conjugation on $\mathbf{H}$ restricts to complex conjugation on $\mathbf{C} \subset \mathbf{H}$), one gets

$$\mathrm{Mat}_n(\mathbf{H}) = \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \in \mathrm{Mat}_{2n}(\mathbf{C}). \right\}$$

For $n = 1$ this recovers the initial definition of $\mathbf{H}$ inside $\mathrm{Mat}_2(\mathbf{C})$.

Definition 2.23. Define the "standard quaternionic-hermitian form"

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon \mathbf{H}^n \times \mathbf{H}^n &\to \mathbf{H} \\ (v, w) &\mapsto \sum_r v_r \overline{w}_r. \end{aligned}$$

(Note the order of multiplication in the sum matters since the vector entries are in $\mathbf{H}$.) We have no need for a general theory of quaternionic-hermitian spaces, so rather than make a more abstract definition we will just focus on this particular construction.

Lemma 2.24. *The construction $\langle \cdot, \cdot \rangle$ in Definition 2.23 satisfies the following properties*

1. *It is sesquilinear, meaning $\langle v, hw \rangle = \langle v, w \rangle \overline{h}$ for $h \in \mathbf{H}$, and moreover $\langle w, v \rangle = \overline{\langle v, w \rangle}$.*
2. *It is non-degenerate, meaning $\langle \cdot, v \rangle = 0$ if and only if $v = 0$.*
3. *It is positive-definite, meaning $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.*

The proof is left as an exercise. It is easy to check that $\langle v, v \rangle$ is the usual squared-length on $\mathbf{H}^n = \mathbf{R}^{4n}$, so

$$\|v\| := \sqrt{\langle v, v \rangle}$$

coincides with the usual notion of length on $\mathbf{R}^{4n}$.

We can now define the compact analogue of $\mathrm{Sp}_{2n}(\mathbf{R})$:

Definition 2.25. Define

$$\mathrm{Sp}(n) := \{g \in \mathrm{GL}_n(\mathbf{H}) : \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in \mathbf{H}^n\}$$

Since any $g \in \mathrm{Sp}(n)$ takes unit vectors to unit vectors, the matrix entries of such g are bounded (for the notion of length on $\mathbf{H}^n$). Moreover, the non-degeneracy of $\langle \cdot, \cdot \rangle$ implies similarly to what we have seen for $\mathrm{O}(q)$ with non-degenerate q and $\mathrm{U}(h)$ for non-degenerate h that in the definition of $\mathrm{Sp}(n)$ it is harmless to replace $\mathrm{GL}_n(\mathbf{H})$ with $\mathrm{Mat}_n(\mathbf{H})$. i.e., invertibility is automatic. Thus, $\mathrm{Sp}(n)$ is *closed* in the finite-dimensional $\mathbf{R}$-vector space $\mathrm{Mat}_n(\mathbf{H})$, so by boundedness it is *compact*.

Remark 2.26. A quaternion has "three imaginary parts", and we can adap the trick used in the case of hermitian forms over $\mathbf{C}$ with sesquilinearity to obtain

$$\mathrm{Sp}(n) = \left\{ g \in \mathrm{GL}_n(\mathbf{H}) : \|gv\|^2 = \|v\|^2 \text{ for all } v \right\}.$$

HW2 shows that $\mathrm{Sp}(n)$ is a C^∞ submanifold of $\mathrm{GL}_{2n}(\mathbf{C})$, and [BtD, (1.11)] shows that $\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{GL}_n(\mathbf{H})$ inside $\mathrm{GL}_{2n}(\mathbf{C})$ (exhibiting compactness in another way).

3. Connectedness and vector fields

3.1. **Indefinite hermitian forms.** Let's first address a loose end from last time. Let $h : V \times V \to \mathbf{C}$ be a hermitian form on a finite-dimensional $\mathbf{C}$-vector space V. This yields two maps, $v \mapsto h(\cdot, v)$ and $v \mapsto h(v, \cdot)$. The former map sends v to a linear functional on V but the actual map $V \to V^*$ is conjugate-linear. The latter map sends v to a conjugate-linear functional, so it takes values in the space $\overline{V}^*$ of conjugate-linear forms on V (this is also the linear dual of the space $\overline{V}$ obtained by extending scalars from $\mathbf{C}$ to $\mathbf{C}$ via the complex conjugation map $\mathbf{C} \to \mathbf{C}$), but the actual map $V \to \overline{V}^*$ is $\mathbf{C}$-linear.

These two maps

$$V \to V^*, \;\; V \to \overline{V}^*$$

are "conjugate" to each other in a way that we leave to your imagination, so in particular one is bijective if and only if the other is. The case when bijectivity holds is precisely our definition from last time of h being *non-degenerate*.

Example 3.1. Let $V = \mathbf{C}^n$, $n = r + s$ with $r, s \geq 0$. Then we define a hermitian form $h_{r,s}$ by:

$$h_{r,s}(v, w) = \sum_{j=1}^{r} v_j \overline{w}_j - \sum_{j=1}^{s} v_{r+j} \overline{w}_{r+j}$$

This is easily checked to be non-degenerate.

Given any hermitian form h on V, we may restrict h to the diagonal

$$\{(v, v) \mid v \in V\} \subseteq V \times V$$

to get an $\mathbf{R}$-valued quadratic form $q_h : v \mapsto h(v, v) = \overline{h(v, v)}$ on the underlying real vector space of V. Recall from the previous lecture that we can recover h from q_h, so if $q_h = 0$ then $h = 0$. It is easy to check that h is non-degenerate as a hermitian form if and only if q_h is non-degenerate as an $\mathbf{R}$-valued quadratic form. The examples above give quadratic forms on the underlying real vector space of V with signature $(2r, 2s)$. Hence, by well-definedness of signature, $(\mathbf{C}^n, h_{r,s}) \simeq (\mathbf{C}^{n'}, h_{r',s'})$ if and only if $r = r'$ and $s = s'$.

Now, we have:

Proposition 3.2. *Every non-degenerate hermitian space (V, h) is isomorphic to some $(\mathbf{C}^n, h_{r,s})$ with $r + s = n$ for a* uniquely *determined r, s.*

Proof. Since h is non-zero, the quadratic form q_h is non-zero. Thus, there exists some $v \in V$ with $h(v, v) \in \mathbf{R}^\times$. Scale v by $\mathbf{R}_{>0}$ so that $h(v, v) = \pm 1$. Non-degeneracy of h implies that $(\mathbf{C}v)^\perp := \{w \in V \mid h(w, v) = 0\}$ has a non-degenerate restriction for h and that $\mathbf{C}v \oplus (\mathbf{C}v)^\perp = V$. Then, we may replace (V, h) by $((\mathbf{C}v)^\perp, h|_{(\mathbf{C}v)^\perp})$ and induct on dimension. $\square$

Note that this proof is essentially the Gram-Schmidt process as in the positive-definite case, but some additional care is necessary: in the Gram-Schmidt process, we could choose the nonzero vector v arbitrarily, but here we must ensure that $q_h(v) \neq 0$ so that the non-degeneracy is preserved upon passage to the appropriate orthogonal hyperplane (an issue that is automatic in the positive-definite case).

3.2. **Some more topology.** We will use the following convention for manifolds in this course: all manifolds (whether topological or C^∞ or analytic) are assumed to be Hausdorff, paracompact, and second-countable. (In the presence of the Hausdorff and paracompactness hypotheses, the condition of M being second-countable is equivalent to having a countable set of connected components.)

These mild conditions ensure the existence of C^∞ partitions of unity, so we can define things like $\int_M \omega$ for a differential form ω (when M is oriented) and other such constructions.

Here is a useful point-set topological fact that will often be invoked without comment in later work with coset spaces:

Proposition 3.3. *Let G be a topological group, with $H \subseteq G$ a subgroup. If H is closed then G/H is Hausdorff.*

Proof. Recall that a topological space X is Hausdorff if and only if the image of the diagonal map $\Delta : X \to X \times X$ is closed in $X \times X$.

Since $(G \times G)/(H \times H) \to (G/H) \times (G/H)$ is a homeomorphism (as we discussed last time), so $G \times G \to (G/H) \times (G/H)$ is a topological quotient map, it suffices to check that the preimage of $\Delta(G/H)$ in $G \times G$ is closed. This is the set $\{(g, g') \in G \times G \mid g^{-1}g' \in H\}$, which is the preimage of the *closed* subset $H \subseteq G$ under the continuous map $G \times G \to G$ given by $(g, g') \mapsto g^{-1}g'$, so it is closed. □

Remark 3.4. The converse (that if G/H is Hausdorff then H is closed) is also true (though useless for us) and is left as an easy exercise.

We will use coset spaces to prove that various compact Lie groups seen earlier are connected. Connectedness results are very important in Lie theory: since the Lie algebra captures infinitesimal information at the identity, we can only hope to extract from it information about the connected component of the identity. The group of connected components of a Lie group can be any discrete group (also countable under our countabillity convention on manifolds), so without connectedness assumptions we are confronted with all of the difficulties of general (or "just" countable) abstract groups.

Here is our first connectedness argument, which we sketched last time. It is based on the method of transitive orbits on a connected space.

Example 3.5. We previously constructed a continuous bijection $\varphi : \mathrm{SO}(n)/\mathrm{SO}(n-1) \to S^{n-1}$ for $n \geq 2$ by studying the orbit of $e_1 \in \mathbf{R}^n$ under the standard action of $\mathrm{SO}(n)$. Note that $\mathrm{SO}(n-1)$ is the stabilizer of e_1 in $\mathrm{SO}(n)$. (Some Euclidean geometry is involved in proving that the $\mathrm{SO}(n)$-orbit of e_1 is the entire unit sphere S^{n-1}, assuming $n \geq 2$.) Since $\mathrm{SO}(n)$ and therefore $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ is compact and S^{n-1} is Hausdorff, the continuous bijection φ is actually a homeomorphism. We will see later on that it is actually a C^∞ isomorphism, once we discuss how to endow coset spaces like $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ with C^∞ structures.

In HW2, you will show that if G/H is connected and H is connected, then G is connected. Therefore, we may use the homeomorphism φ to show by induction that $\mathrm{SO}(n)$ is connected for all n because of connectedness of $\mathrm{SO}(1)$ which is the trivial group $\{+1\} \subseteq \mathrm{O}(1) = \{\pm 1\}$. This method for $n = 2$ recovers the familiar isomorphism $\mathrm{SO}(2) \simeq S^1$.

Note that this argument would not work (and in fact, the conclusion is not always true) for $\mathrm{SO}(q)$ for indefinite non-degenerate quadratic forms q over $\mathbf{R}$. This is because such groups do not preserve the unit ball, but rather preserve the set of length-1 vectors with respect to some pseudo-metric. Such a "unit pseudo-ball" is typically disconnected (and non-compact), such as a hyperbola in the plane or a hyperboloid with 2 sheets in $\mathbf{R}^3$.

The technique of argument in the preceding example is robust:

Example 3.6. HW2 uses a similar argument with $S^{2n-1} \subseteq \mathbf{C}^n$ to show $\mathrm{SU}(n)$ is connected for all $n \geq 1$ by using an induction whose base case is the case $n = 1$ with $\mathrm{SU}(1) = \{1\} \subseteq$

$\mathrm{U}(1) = S^1$. In the case $n = 2$ this yields a homeomorphism

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathrm{GL}_2(\mathbf{C}) \mid |\alpha|^2 + |\beta|^2 = 1 \right\} \simeq S^3 \subset \mathbf{C}^2$$

carrying $\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$ to $(\alpha, -\overline{\beta})$.

Example 3.7. In HW2, we similarly use the unit ball $S^{4n-1} \subseteq \mathbf{H}^n$ to show that $\mathrm{Sp}(n)$ is connected for all $n \geq 1$, using the base case $\mathrm{Sp}(1) = \{h \in \mathbf{H}^\times \mid N(h) = 1\} = \mathrm{SU}(2)$.

How might we show (if true?) a similar connectedness result for the non-compact groups $\mathrm{SL}_n(\mathbf{R})$ and $\mathrm{Sp}_{2n}(\mathbf{R})$? We have the lucky fact, shown in HW1, that there is a diffeomorphism $\mathrm{SL}_n(\mathbf{R}) \simeq \mathrm{SO}(n) \times \mathbf{R}^N$ for a suitable N (warning: this is *not* a group isomorphism!). This shows that $\mathrm{SL}_n(\mathbf{R})$ is connected.

The method of studying an appropriate group action and inducting using stabilizers still works in these non-compact cases, but we do not have a natural action of these groups on spheres (the geometry of their natural action on a Euclidean space does not preserve the Euclidean structure), so we must use a more sophisticated space (e.g. Grassmannians or flag varieties). Also, in the non-compact setting we cannot use the trick that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, so to show that the relevant continuous bijection $G/\mathrm{Stab}_G(x) \to X$ is a homeomorphism onto Gx under some mild topological hypotheses on G and X requires some more work.

Since matrix groups are such a ubiquitous source of examples of Lie groups in practice, on the theme of connectedness we can't resist mentioning the following interesting fact (that we will never use in this course):

Theorem 3.8. *For $G \subset \mathrm{GL}_n(\mathbf{R})$ a "Zariski-closed" subgroup (i.e., a group given by the zero-set of some polynomials in $\mathbf{R}[x_{ij}, 1/\det]$), $\pi_0(G)$ is always finite.*

Remark 3.9. In fact, the group setting is irrelevant: it is a general theorem of Whitney that any zero locus of polynomials in any $\mathbf{R}^N$ has only finitely many connected components: for a proof see [Mil, App. A] which rests on [AF, §1, Lemma].

Note that it is possible that G in Theorem 3.8 is Zariski-connected (in the sense that it does not admit a separation whose two members are defined by polynomial equations) but the topological space $G(\mathbf{R})$ is disconnected for its usual topology. This is the case with $\mathrm{GL}_n(\mathbf{R})$ (essentially because using the sign of the determinant to explain the disconnectedness is not a "polynomial condition"). In addition, the groups $\mathrm{SO}(r, n-r) \subset \mathrm{GL}_n(\mathbf{R})$ are always Zariski-connected even though they are disconnected for the usual topology whenever $0 < r < n$. On the other hand, $\mathrm{O}(n) \subset \mathrm{GL}_n(\mathbf{R})$ is not Zariski-connected since the determinant being equal to 1 or being equal to -1 defines disjoint complementary non-empty (closed and open) subsets.

A striking application of the fact that groups such as $\mathrm{GL}_n(\mathbf{R})$ are Zariski-connected is that if we are trying to prove a "purely algebraic" fact for them then we can still profitably use Lie-algebra methods (because there is a purely *algebro-geometric* formulation of the Lie algebra, which however lies beyond the level of this course). In other words, certain disconnected Lie groups can still be studied via Lie-algebra methods for the purpose of sufficiently algebraic questions (provided that a weaker kind of connectedness, called

Zariski-connectedness, holds). For someone who knows only analytic geometry and not algebraic geometry, this must sound very bewildering!

3.3. **Left-invariant vector fields.** As a first step towards developing Lie algebras, we now introduce the notion of *left-invariant vector field*.

Let G be a Lie group. We define $\mathrm{Lie}(G) = \mathfrak{g} := T_1(G)$, i.e. the tangent space of the point $1 \in G$. Let X be a "set-theoretic" vector field on G, i.e. a collection of elements $X(g) \in T_g(G)$ for all $g \in G$. (This is the same thing as a set-theoretic section to the structure map $TG \to G$ for the tangent bundle, for those who prefer such language.)

Definition 3.10. A set-theoretic vector field X on a Lie group G is *left-invariant* if for all $g, g' \in G$, the natural linear isomorphism $d\ell_g(g') : T_{g'}(G) \simeq T_{gg'}(G)$ carries $X(g')$ to $X(gg')$, where $\ell_g : G \to G$ is the diffeomorphism $x \mapsto gx$ given by left translation by g.

By the Chain Rule and the fact that $\ell_{gg'} = \ell_g \circ \ell_{g'}$, it is equivalent to say that there is a vector $v \in \mathfrak{g}$ with $X(g) = d\ell_g(v)$ for all $g \in G$. Thus, we obtain a linear isomorphism between $\mathfrak{g}$ and the $\mathbf{R}$-vector space of left-invariant vector fields on G by sending $v \in \mathfrak{g}$ to $\widetilde{v} : g \mapsto d\ell_g(v)$. On HW2 we will show that such $\widetilde{v}$ are actually C^∞ vector fields, so we will never care about "set-theoretic" vector fields ever again.

Example 3.11. For the open submanifold $G = \mathrm{GL}_n(\mathbf{R}) \subseteq \mathrm{Mat}_n(\mathbf{R})$, we have

$$\mathrm{Lie}(G) = T_1(\mathrm{Mat}_n(\mathbf{R}))$$

yet canonically $\mathrm{Mat}_n(\mathbf{R}) \simeq T_1(\mathrm{Mat}_n(\mathbf{R}))$ as a special case of the general canonical isomorphism $V \simeq T_v(V)$ for the tangent space at any point v in a finite-dimensional $\mathbf{R}$-vector space V viewed as a smooth manifold (this is just the "directional derivative at a point" construction). Thus, we have a canonical isomorphism

$$\mathrm{Mat}_n(\mathbf{R}) \simeq \mathfrak{gl}_n(\mathbf{R}) := \mathrm{Lie}(\mathrm{GL}_n(\mathbf{R})),$$

given by $(a_{ij}) \mapsto \sum_{i,j} a_{ij} \partial_{x_{i,j}}|_1$. (In more coordinate-free terms, this says that *canonically* $\mathrm{End}(V) \simeq \mathrm{Lie}(\mathrm{GL}(V))$ as $\mathbf{R}$-vector spaces.)

This is made more explicit in HW2, where the next proposition is addressed:

Proposition 3.12. *The following two statements hold:*

(i) $\partial_{x_{ij}}|_{1_n}$ *extends to the left-invariant vector field* $\sum_k x_{ki} \partial_{x_{kj}}$.

(ii) For A, B *in* $\mathrm{Mat}_n(\mathbf{R})$, $[\widetilde{A}, \widetilde{B}](1) = AB - BA$ *where* $[\cdot, \cdot]$ *denotes the commutator operation on smooth global vector fields on a manifold.*

4. Lie algebras

We start with an extended parenthetical remark.

Remark 4.1 (Finite groups of Lie type and the relation to algebraic geometry)**.** There is a link between connected (complex) Lie groups and certain finite groups (called "finite groups of Lie type") such as $\mathrm{SL}_n(\mathbf{F}_q), \mathrm{Sp}_{2n}(\mathbf{F}_q)$, etc. The idea is to use algebraic geometry over $\mathbf{F}_q$, applied to $G(\mathbf{F}_q)$, where G is the corresponding affine group variety (which is a "matrix group" such as SL_n or Sp_{2n} that we view as defined by "equations over $\mathbf{Z}$ which have nothing to do with $\mathbf{F}_q$).

We view $G(\mathbf{F}_q)$ as a finite subgroup of something with much richer geometric structure:

$$G(\mathbf{F}_q) \subset \cup_{n\geq 1} G(\mathbf{F}_{q^n}) = G(\overline{\mathbf{F}}_q),$$

where $G(\overline{\mathbf{F}}_q)$ is a smooth affine variety over $\mathbf{F}_q$ (whatever that means) and as such thereby has a useful notion of "tangent space" at each point, such as $T_e(G(\overline{\mathbf{F}}_q))$ on which one can (with enough algebro-geometric technique) make a version of the "left-invariant vector field" construction to define a version of $[\cdot,\cdot]$ on the tangent space at the identity. Even though $G(\mathbf{F}_q)$ has no geometric structure, in many cases $G(\overline{\mathbf{F}}_q)$ has a reasonable notion of connectedness, and one can really harness that together with the Lie algebra on its tangent space at the identity to deduce structural information that in turn tells us non-obvious things about $G(\mathbf{F}_q)$!

The link to Lie groups is that we have maps of commutstive rings

$$\begin{tikzcd} & \mathbf{Z} \arrow[dl] \arrow[dr] & \\ \mathbf{C} & & \mathbf{F}_q \end{tikzcd} \tag{4.1}$$

and one can use these to relate Lie groups over $\mathbf{C}$ to corresponding algebraic groups "over $\mathbf{Z}$" (in the sense of schemes), and then over $\mathbf{F}_q$ via "scalar extension".

One nice example application of this link is the following.

Example 4.2. There are formulas for the size of $G(\mathbf{F}_q)$ as a universal polynomial in q for various G (e.g., the case of SL_n is rather classical), but what *conceptual* explanation may be found for the existence of such universal formulas (i.e., something better than: compute the size and hope it is a universal formula in q)? A very long story going through a lot of algebraic geometry provides such a conceptual explanation for the universal formula by relating it to the purely topological invariants

$$H^*(G(\mathbf{C}), \mathbf{C})$$

that have nothing to do with any specific q!

Coming back to Lie groups G, we know inversion is automatically C^∞ when m is, as shown in Appendix B. More precisely, that appendix shows that for $\mathfrak{g} = T_e(G)$,

$$dm(e,e) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is addition and

$$di(e) : \mathfrak{g} \to \mathfrak{g}$$

is negation.

Last time, we made an $\mathbf{R}$-linear isomorphism

$$\begin{aligned} &\mathfrak{g} \to \{\text{left-invariant } C^\infty \text{ vector fields on } G\} \\ &v \mapsto \widetilde{v} \end{aligned}$$

via

$$\widetilde{v}(g) := d\ell_g(e)(v)$$

with ℓ_g denoting left multiplication by g.

Using this, we defined

$$\begin{aligned}[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (v, w) &\mapsto [\widetilde{v}, \widetilde{w}]\,(e)\end{aligned}$$

(also denoted as $[\cdot,\cdot]_G$ when we need to keep track of G, such as if several Lie groups are being considered at the same time).

Lemma 4.3. *For left-invariant vector fields X, Y on G, the commutator $[X, Y]$ is also left-invariant.*

Proof. To start, we reformulate the condition of being left-invariant. From the definition, X is left-invariant if and only if

$$d\ell_g\left(X(g')\right) = X\left(gg'\right) \text{ for all } g, g'.$$

Viewing $T_g(G)$ as "point-derivations at g" on $C^\infty(G)$, which is to say $\mathbf{R}$-linear maps $v : C^\infty(G) \to \mathbf{R}$ satisfying

$$v(f_1 f_2) = f_1(g)v(f_2) + f_2(g)v(f_1)$$

for all $f_1, f_2 \in C^\infty(G)$, it is a pleasant exercise to check that left-invariance of X is precisely the condition that

$$X \circ \ell_g^* = X \text{ for all } g \in G$$

as maps $C^\infty(G) \to C^\infty(G)$.

But

$$\begin{aligned}[X, Y] \circ \ell_g^* &= X \circ Y\ell_g^* - Y \circ X \circ \ell_g^* \\ &= XY - YX \\ &= [X, Y],\end{aligned}$$

using left-invariance of X and Y, so we're done. □

Remark 4.4. We recorded last time that for the open submanifold $G = \mathrm{GL}_n(\mathbf{R}) \subset \mathrm{Mat}_n(\mathbf{R})$ canonically identifying $\mathfrak{g}$ with $\mathrm{Mat}_n(\mathbf{R})$, the operation $[\cdot,\cdot]$ is the usual commutator. This is proved by a velocity vector calculation in §C.1, where it is noted that the exact same argument works for two other such Lie groups, namely the open submanifolds $\mathrm{GL}_n(\mathbf{C}) \subset \mathrm{Mat}_n(\mathbf{C})$ and $\mathrm{GL}_n(\mathbf{H}) \subset \mathrm{Mat}_n(\mathbf{H})$ (with the corresponding commutator on those matrix algebras).

This is very useful once we show that $[\cdot,\cdot]_G$ is functorial in the group G, since the groups G we care about will typically be closed subgroups of $\mathrm{GL}_n(\mathbf{R})$, $\mathrm{GL}_n(\mathbf{C})$, or $\mathrm{GL}_n(\mathbf{H})$ (so we can then compute the bracket on such $\mathrm{Lie}(G)$ in terms of matrix commutators).

4.1. **Definition of Lie algebras.**

Definition 4.5. A *Lie algebra* over a field k is a k-vector space $\mathfrak{g}$ equipped with a k-bilinear map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

so that

(1) $[\cdot,\cdot]$ is alternating, meaning $[X, X] = 0$ for all X (which implies $[X, Y] = -[Y, X]$, and is equivalent to this in characteristic not 2).

(2) $[\cdot,\cdot]$ satisfies the *Jacobi identity*:
$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

Warning 4.6. The operation $[\cdot,\cdot]$ is usually not associative, as can be seen by the Jacobi identity. Further, there is typically no unit for the bracket operation (aside from silly cases such as $\mathfrak{g}=0$).

4.2. Examples of Lie Algebras.

Example 4.7. The cross product on $\mathbf{R}^3$ is isomorphic to $\mathrm{Lie}(\mathrm{SO}(3))$: see Exercise 2 on HW3.

Example 4.8. Suppose A is an associative k-algebra with identity. Then
$$\left[a',a\right]:=a'a-aa'$$
is easily checked to be a Lie algebra structure. As a special case, one can take $A=\mathrm{End}_k(V)$ for V any k-vector space (even infinite-dimensional).

Example 4.9. By design of $[\cdot,\cdot]_G$, the injective $\mathbf{R}$-linear map
$$\begin{aligned}\mathfrak{g}&\hookrightarrow \mathrm{End}_{\mathbf{R}}\left(C^\infty(G)\right)\\ v&\mapsto \widetilde{v}\end{aligned}$$
is compatible with $[\cdot,\cdot]$ when using Example 4.8 on the target.

4.3. Functoriality of brackets.

Definition 4.10. For any given Lie group, we let the lower-case fraktur font denote the corresponding Lie algebra. So for example,
$$\mathrm{GL}_n(\mathbf{R})\rightsquigarrow\mathfrak{gl}_n(\mathbf{R}),\ \ \mathrm{U}(n)\rightsquigarrow\mathfrak{u}(n),\ \ \mathrm{SL}_n(\mathbf{R})\rightsquigarrow\mathfrak{sl}_n(\mathbf{R}),\ \ \mathrm{Sp}_{2n}(\mathbf{R})\rightsquigarrow\mathfrak{sp}_{2n}(\mathbf{R}).$$
The notion of homomorphism of Lie algebras is the natural one (a map of vector spaces compatible with the Lie bracket). We say $\mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{h}$ is there is an injective homomorphism of Lie algebras $\mathfrak{g}\to\mathfrak{h}$.

Question 4.11. Is $[\cdot,\cdot]_G$ on $\mathfrak{g}$ functorial in G? That is, for $f:G\to H$ a map of Lie groups, is $df(e):\mathfrak{g}\to\mathfrak{h}$ a homomorphism of Lie algebras?

Of course the answer is yes, but it is not obvious because we can't "push forward" vector fields. To prove this, we'll develop a more local description of $[\cdot,\cdot]$.

Here are some side remarks, for general cultural awareness.

Remark 4.12. Every Lie algebra $\mathfrak{g}$ over a field k is naturally a Lie subalgebra of an associative k-algebra $U(\mathfrak{g})$ with identity, known as the *universal enveloping algebra*. For more on this, see [Bou1, Ch. I, §2.7, Cor. 2].

Remark 4.13. Every finite-dimensional Lie algebra $\mathfrak{g}$ over a field k is a Lie subalgebra of some $\mathfrak{gl}_n(k):=\mathrm{Mat}_n(k)$ (equipped with the commutator Lie algebra structure as for any associative k-algebra). This result, called *Ado's Theorem*, lies quite deep (so it would be incorrect to think it is much Ado about nothing). It is proved at the very end of [Bou1, Ch. I].

We now return to showing the bracket is functorial. First, we have an important reformulation of the Jacobi identity, resting on:

Definition 4.14. Let $\mathfrak{g}$ be a Lie algebra over a field k. For any $X \in \mathfrak{g}$ we get a k-linear map

$$[X, \cdot] : \mathfrak{g} \to \mathfrak{g}$$

and so a k-linear map

$$\begin{aligned} \mathrm{ad}_{\mathfrak{g}} \colon \mathfrak{g} &\to \mathrm{End}_k(\mathfrak{g}) \\ X &\mapsto [X, \cdot] \,. \end{aligned}$$

It is a pleasant exercise to check that an alternating bilinear $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfies the Jacobi identity if and only if the associated map $\mathrm{ad}_{\mathfrak{g}}$ built from $[\cdot, \cdot]$ as above intertwines the bracket on the source with the commutator on the target (in which case it is therefore a Lie algebra homomorphism).

For G a Lie group and $\mathfrak{g} := \mathrm{Lie}(G)$, we define the *adjoint representation*

$$\begin{aligned} \mathrm{Ad}_G \colon G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto dc_g(e) \end{aligned}$$

for

$$\begin{aligned} c_g \colon G &\to G \\ x &\mapsto gxg^{-1}. \end{aligned}$$

To justify the "representation" terminology, let's check that Ad_G is a homomorphism:

Lemma 4.15. *Ad_G is a C^∞ homomorphism.*

Proof. Since $c_{g'}c_g = c_{g'g}$, applying the Chain Rule to this at e yields the homomorphism property. The C^∞ property comes down to a calculation in local coordinates near the identity, and is discussed in Appendix C $\square$

In Appendix C it is shown via elementary calculations with velocity vectors that for any $g \in \mathrm{GL}_n(\mathbf{R})$ we have

$$\mathrm{Ad}_{\mathrm{GL}_n(\mathbf{R})}(g) : M \mapsto gMg^{-1}$$

for $M \in \mathrm{Mat}_n(\mathbf{R})$, and similarly for $\mathrm{GL}_n(\mathbf{C})$ and $\mathrm{GL}_n(\mathbf{H})$. We are going to see later that for *connected* Lie groups, understanding the effect of a Lie group homomorphism f between tangent spaces at the identity will exert tremendous control on f. But even without connectedness hypotheses, it is natural to try to understand the induced map

$$d(\mathrm{Ad}_G)(e) : \mathfrak{g} \to T_1\left(\mathrm{GL}(\mathfrak{g})\right) = \mathrm{End}_{\mathbf{R}}(\mathfrak{g}).$$

We have already seen another such map, namely $\mathrm{ad}_{\mathfrak{g}}$, and we next want to discuss some ideas underlying the proof of the following very important result:

Theorem 4.16. *For any Lie group G, $d(\mathrm{Ad}_G)(e) = \mathrm{ad}_{\mathfrak{g}}$ as maps $\mathfrak{g} \to \mathrm{End}_{\mathbf{R}}(\mathfrak{g})$.*

This result is proved Appendix C, and a key ingredient in its proof as well as in the proof of the functoriality of $[\cdot, \cdot]_G$ in G is an entirely different description of $[\cdot, \cdot]_G$ using the notion of "1-parameter subgroup" of G: a homomorphism of Lie groups

$$\alpha : \mathbf{R} \to G$$

which may not be injective (but still called "1-parameter subgroup"!).

Using ODE's and some differential geometry (as we'll see next time), one shows:

Theorem 4.17. *For all $v \in \mathfrak{g}$, there exists a unique 1-parameter subgroup $\alpha_v : \mathbf{R} \to G$ so that $\alpha_v'(0) = v$.*

Example 4.18. If $G = \mathrm{GL}_n(\mathbf{R})$, then (using uniqueness) $\alpha_v(t) = e^{tv}$. The same holds for $\mathrm{GL}_n(\mathbf{C})$ and $\mathrm{GL}_n(\mathbf{H})$.

Example 4.19. If $G = S^1 \times S^1 = (\mathbf{R} \times \mathbf{R})/(\mathbf{Z} \times \mathbf{Z})$ then α_v is a "wrapped line" on the torus.

In Appendix C, 1-parameter subgroups α_v are used to formulate the following totally different description of the bracket that is proved there:

$$[v, w]_G = \frac{d}{dt}|_{t=0} \mathrm{Ad}_G\left(\alpha_v(t)\right)(w).$$

In concrete terms, one can think of $t \mapsto \mathrm{Ad}_G(\alpha_v(t))(w)$ as a parametric curve in the finite-dimensional $\mathbf{R}$-vector space $\mathfrak{g}$, and then we form its velocity at $t = 0$. This description of the bracket is very "local near $e \in G$", in contrast with the initial definition using rather global notions such as $\widetilde{v}$ and $\widetilde{w}$.

Next time we will explain how to use this "local" formula for the bracket to deduce the functoriality of $[\cdot, \cdot]_G$ in G and more importantly we will discuss some of the background in ODE's and differential geometry which underlies the proof of Theorem 4.17.

5. Maximal compact and 1-parameter subgroups

Today, we'll discuss 1-parameter subgroups, which were breifly introduced at the end of last class. But first, to have some sense of how all of our effort on compact Lie groups in this course will not be time wasted relative to what may be learned later in the non-compact case, we shall briefly discuss the remarkable sense in which the compact case plays an essential role in the structure of general Lie groups.

Weyl was the first to carry out a deep general study of the representation theory of Lie groups, focusing on the compact case (where one has techniques such as integration without convergence issues). It is truly amazing that the results in the compact case are important in the non-compact case. To understand this, we mention (without proof) the following general theorem that is good to be aware of before one puts in a lot of effort to learn the structure theory in the compact case.

Theorem 5.1. *Let G be a Lie group with finitely many connected components.*

(i) Every compact subgroup of G is contained in a maximal compact subgroup; i.e., one not contained in a strictly bigger one.

(ii) All maximal compact subgroups $K \subset G$ are conjugate, they meet every connected component, and $K \cap G^0$ is connected and is a maximal compact subgroup of G^0.

(iii) The inclusion of K into G extends to a diffeomorphism of C^∞ manifolds $G \simeq K \times V$ for V a finite dimenaional $\mathbf{R}$-vector space (so $K \hookrightarrow G$ is a deformation retract, and hence G has the same "homotopy type" as K).

Note that in (i) the first assertion is not a consequence of dimension reasons since it isn't evident at the outset why there couldn't be a strictly rising chain of compact subgroups that merely become increasingly disconnected without any increase in dimension. Also, the compact subgroups are necessarily smooth submanifolds (hence "Lie subgroups", a notion we'll discuss in general a bit later) due to an important theorem of Lie to be discussed in a few lectures from now: any closed subgroup of a Lie group is a smooth submanifold.

Theorem 5.1 is proved in [Hoch, Ch. XV, Thm. 3.1]. It is the *only* reference I know that allows the possibility of (finitely) disconnected G, which is important for applications to the group of $\mathbf{R}$-points of an affine group variety over $\mathbf{R}$. I am not aware of any way to deduce the full result from the connected case treated as a black box (e.g., why should there exist a maximal K meeting every connected component?); if you can come up with such a proof, let me know immediately!

Example 5.2. For $G = \mathrm{GL}_n(\mathbf{R})$, we can take K to be $\mathrm{O}(n)$ and

$$\begin{aligned} K \times \mathrm{Sym}_n(\mathbf{R}) &\simeq G \\ (k, T) &\mapsto ke^T \end{aligned}$$

(where $\mathrm{Sym}_n(\mathbf{R})$ is the space of symmetric $n \times n$ matrices over $\mathbf{R}$). This is called the polar decomposition.

Example 5.3. For $G = \mathrm{O}(r,s)$ we can take K to be $\mathrm{O}(r) \times \mathrm{O}(s)$ via an orthogonal decomposition of the quadratic space into an r-dimensional positive-definite space and an s-dimensional negative-definite space. (Note that for any non-degenerate finite-dimensional quadratic space (V, q) over $\mathbf{R}$ we have $\mathrm{O}(q) = \mathrm{O}(-q)$, so the negative-definite case has its orthogonal group identified with $\mathrm{O}(s)$.)

Example 5.4. If $G = \mathrm{U}(r,s)$ then we can take K to be $\mathrm{U}(r) \times \mathrm{U}(s)$.

The three preceding examples are proved in Appendix D: the first is proved by an elementary averaging argument, and the other two are deduced from that via the spectral theorem.

Our main focus for today is to discuss the following fundamental result:

Theorem 5.5. *Let G be a Lie group and choose $v \in \mathfrak{g}$. There exists a unique 1-parameter subgroup*

$$\alpha_v : G \to \mathbf{R}$$

such that $\alpha_v'(0) = v$.

Example 5.6. If $G = \mathrm{GL}_n(\mathbf{F})$ for $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ we have $\alpha_M(t) = e^{tM}$ for $M \in \mathrm{Mat}_n(\mathbf{F})$.

Many G of interest to us will be a closed subgroup of one of the groups as in the preceding example, so this computes α_v in such cases due to the functoriality of 1-parameter subgroups:

Lemma 5.7. *For $f : G \to G'$ a map of Lie groups, suppose the induced map $Lf : \mathfrak{g} \to \mathfrak{g}'$ satisfies $v \mapsto v'$. Then*

$$f \circ \alpha_v = \alpha_{v'}.$$

Proof. By uniqueness, we just have to compute the velocity at $t = 0$, which we do via the Chain Rule:

$$(f \circ \alpha_v)'(0) = (Lf)(\alpha_v'(0)) = (Lf)(v) =: v'.$$

$\square$

Remark 5.8. Before we dive into the proof of Theorem 5.5, as motivation for the effort required we note that Appendix C uses 1-parameter subroups α_v to prove two important results:

- $d(\mathrm{Ad}_G)(e) = \mathrm{ad}_{\mathfrak{g}}$,
- the "Local Formula" for $[\cdot,\cdot]_G$:

$$[v,w]_G = \frac{d}{dt}|_{t=0}\left(\mathrm{Ad}_G\left(\alpha_v(t)\right)(w)\right).$$

This can be proved by hand for $\mathrm{GL}_n(\mathbf{F})$ (see §C.1).

As an application of the above Local Formula, we can answer a question raised earlier:

Proposition 5.9. *For $f : G \to G'$, the map $Lf : \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of Lie algebras.*

Proof. For $v, w \in \mathfrak{g}$ that are carried to $v', w' \in \mathfrak{g}'$ respectively by Lf, we want to show

$$(Lf)\left([v,w]_G\right) \stackrel{?}{=} \left[v',w'\right]_{G'}.$$

We'll basically prove this by plugging it into the Local Formula, using the simple behavior under linear maps (which are their own derivative!) for velocity vectors to a parametric curve in a vector space.

The Local Formula says that $[v,w]_G$ is the velocity at $t=0$ for the parameteric curve

$$t \mapsto \mathrm{Ad}_G(\alpha_v(t))(w) \in \mathfrak{g}.$$

Then, by the Chain Rule $(Lf)([v,w]_G)$ is the velocity at $t=0$ for the parametric curve

$$t \mapsto (Lf)(\mathrm{Ad}_G(\alpha_v(t))(w)) \in \mathfrak{g}'$$

But by definition

$$\mathrm{Ad}_G(\alpha_v(t)) = dc_g(e)$$

for $g = \alpha_v(t)$, and we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ {\scriptstyle c_g}\downarrow & & \downarrow{\scriptstyle c_{f(g)}} \\ G & \xrightarrow{f} & G' \end{array} \tag{5.1}$$

By functoriality of 1-parameter subgroups we have $f(g) = f(\alpha_v(t)) = \alpha_{v'}(t)$, so

$$(Lf)(\mathrm{Ad}_G(\alpha_v(t))(w)) = dc_{\alpha_{v'}(t)}(e')((Lf)(w))$$

and by definition $(Lf)(w) = w'$. Thus, $(Lf)([v,w]_G)$ is the velocity at $t=0$ for

$$t \mapsto dc_{\alpha_{v'}(t)}(e')(w') = \mathrm{Ad}_{G'}(\alpha_{v'}(t))(w'),$$

which is $[v',w']_{G'}$ by the Local Formula for G'. □

5.1. **Proof of existence in Theorem 5.5.** We'll now discuss the proof of existence of α_v; the proof of uniqueness will be much easier, and is deferred to next time.

The key input is the theory of integral curves to C^∞ vector fields on manifolds. The local content amounts to suitable existence and uniqueness results for first order (possibly non-linear) ODE's, as we shall recall below.

We begin with a review of integral curves. Let M be a C^∞ manifold. Let X be a C^∞ vector field on M, and $p \in M$ a point.

Definition 5.10. An *integral curve* to X on M at $p \in M$ is a C^∞ map

$$c : I \to M$$

where I is an open interval in $\mathbf{R}$ containing 0 such that

- $c(0) = p$
- $c'(t) = X(c(t))$ for all $t \in I$ (so $c'(0) = v_0 := X(p)$).

We emphasize that c is a parametric curve, *not* a bare 1-dimensional submanifold of M (as c may well not be injective either, and even if injective it may not be a topological embedding: think of a densely wrapped line in a 2-dimensional torus). Note that we are specifying an actual velocity vector in $T_{c(t)}(M)$ for each t and not merely the "tangent line" inside $T_{c(t)}(M)$ for each t.

Example 5.11. If $M = \mathbf{R}^2 - \{0\}$ and $X = r\partial_\theta$ then the integral curves are constant-speed parametric circles around the origin. The constant speed of each parametric circle decreases as one approaches the origin due to the factor of r in the vector field. Using instead $X = \partial_\theta$, all of these circles would be parametrized at the same speed.

Using local coordinates $\{y_1, \ldots, y_n\}$ near p, the existence of such c on some short interval I becomes a first order ODE result, as follows. Consider $U \subset \mathbf{R}^n$ an open subset containing 0 (corresponding to p in the coordinate domain) and a C^∞ map $f : U \to \mathbf{R}^n$ (in the case of interest this encodes $X|_U = \sum_j f_j \frac{\partial}{\partial y_j}$). Then we seek a smooth map $\phi : I \to U \subset \mathbf{R}^n$ so that

- $\phi(0) = 0$,
- $\phi'(t) = f(\phi(t))$ for all $t \in I$ (so $\phi'(0) = f(0) \in \mathbf{R}^n$).

The existence of such a ϕ is a special case of the usual existence and uniqueness theorem for first order ODE's (where the ODE may be have the rather more general form $\phi'(t) = f(t, \phi(t))$ for some $f : I \times U \to \mathbf{R}^n$ rather than the more special "autonomous" form that arises above).

So the vector-valued 1st-order ODE theory on open subsets of $\mathbf{R}^n$ gives existence on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, and gives uniqueness whenever the integral curve valued in U exists (i.e., for any I on which such an integral curve exists, subject to the integral curve remaining inside the coordinate domain). For a complete treatement of this local ODE existence and uniqueness result on open subsets of $\mathbf{R}^n$, see [Te, §2.4] or Appendices E and F.

We need to push this result inside $\mathbf{R}^n$ to the manifold setting, where the integral curve may not lie entirely inside a coordinate chart around p. Using compactness and connectedness inside the real line, it is explained in Appendix F how to build on the serious work in $\mathbf{R}^n$ to get the following more global version:

(1) There exists a unique maximal integral curve $c_p^{\max} : I(p) \to M$ to X at p (i.e., one defined on an interval $I(p)$ containing all others).
(2) The set

$$\Omega := \{(t, q) \in \mathbf{R} \times M : t \in I(p)\}$$

is *open* in $\mathbf{R} \times M$ and the map

$$\begin{aligned} \Phi \colon \Omega &\to M \\ (t, p) &\mapsto c_p^{\max}(t) =: \Phi_t(p) \end{aligned}$$

is smooth. (This smoothness is very important in what follows, and it requires much more input from the local ODE theory, also proved in the same references given above: smooth dependence on initial conditions, which shouldn't be taken for granted at all.)

Further, the uniqueness of maximal integral curves yields the following bonus fact (as discussed in Appendix F):

Fact 5.12. Suppose $t \in I(p)$ and $t' \in I\left(\Phi_t(p)\right)$. Then, $t + t' \in I(p)$ and

$$\Phi_{t'+t}(p) = \Phi_{t'}\left(\Phi_t(p)\right).$$

Beware that for general (M, X, p), the flow can end in finite time; i.e., we may have $I(p) \neq \mathbf{R}$. For example, this happens in $\mathbf{R}^2 - \{0\}$ with $X = -\frac{\partial}{\partial r}$ (for which the integral curves are constant-speed rays going towards the origin, which all cease to exist in finite time). But in the Lie group setting with $M = G$ and $X = \widetilde{v}$ a left-invariant vector field, the homogeneity buys us much better behavior. For example, since "$d\ell_g(X) = X$", so the entire situation is equivariant for left multiplication by any $g \in G$, we have $I(gp) = I(p)$ for all $g \in G, p \in G$. Hence, by transitivity of the left translation action of G on itself we get $I(p) = I(e) =: I$ for all $p \in G$. Ah, but then Fact 5.12 tells us that the open interval $I \subset \mathbf{R}$ containing 0 is stable under addition, so necessarily $I = \mathbf{R}$! Thus, we have a "global flow"

$$\Phi : \mathbf{R} \times G \to G$$

which satisfies $\Phi_{t'+t}(e) = \Phi_{t'}(\Phi_t(e))$ for all $t, t' \in \mathbf{R}$.

Homogeneity under left multiplication also gives $\Phi_t(gp) = g\Phi_t(p)$ for all $g, p \in G$, so $\Phi_t(g) = g\Phi_t(e)$. Thus,

$$\Phi_{t'+t}(e) = \Phi_{t'}(\Phi_t(e)) = \Phi_t(e)\Phi_{t'}(e).$$

The upshot of this discussion is that $t \mapsto \Phi_t(e)$ is a C^∞ homomorphism $\mathbf{R} \to G$, and being an integral curve to $X = \widetilde{v}$ implies that its velocity at $t = 0$ is $X(e) = v$. We have thereby proved the existence of α_v (namely, define $\alpha_v(t) = \Phi_t(e)$: this is flow to time t along the integral curve through e at time 0 for the vector field $X = \widetilde{v}$).

6. The exponential map

Remark 6.1. For us, all manifolds are second countable so we will be able to apply Sard's theorem.

Last time, we saw existence of 1-parameter subgroups. Today, we'll begin by showing uniqueness of 1-parameter subgroups:

Proposition 6.2. *The 1-parameter subgroup $\alpha : \mathbf{R} \to G$ with $\alpha'(0) = v \in \mathfrak{g}$ is unique.*

Proof. The idea is to use the uniqueness assertion for integral curves (which we saw essentially boils down to the uniqueness theorem for 1st-order ODE's).

For fixed t, define the parametric curve in G:

$$\alpha(t + \tau) := \alpha(t)\alpha(\tau) = \ell_{\alpha(t)}\alpha(\tau).$$

Since $\ell_{\alpha(t)}$ is a C^∞ map having nothing to do with τ, by viewing both sides as parametric curves with parameter τ and applying $\frac{d}{d\tau}$ (i.e., computing velocity vectors on both sides) we get

$$\alpha'(t+\tau) = d\ell_{\alpha(t)}(\alpha(\tau))(\alpha'(\tau))$$

Setting $\tau = 0$, we find

$$\begin{aligned} \alpha'(t) &= d\ell_{\alpha(t)}(e)\left(\alpha'(0)\right) \\ &= d\ell_{\alpha(t)}(e)\,(v) \\ &= \widetilde{v}\left(\alpha(t)\right). \end{aligned}$$

This implies that α is an integral curve to $\widetilde{v}$ with $\alpha(0) = e \in G$, and it is defined on all of $\mathbf{R}$. Now use the uniqueness result for integral curves on the maximal interval of definition to conclude that α is unique. □

Definition 6.3. The *exponential map* $\exp_G \colon \mathfrak{g} \to G$ is defined to be $v \mapsto \alpha_v(1)$, where α_v is the 1-parameter subgroup associated to $v \in \mathfrak{g}$.

Example 6.4. Take $G = \mathrm{GL}_n(\mathbf{F})$ with $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$, so $\mathfrak{g} = \mathrm{Mat}_n(\mathbf{F})$. We then obtain $\alpha_M(t) = e^{tM}$, so $\alpha_M(1) = e^M$ (explaining the "exponential map" name). In particular, we see $\exp_G$ need not be a homomorphism, and typically won't be.

Lemma 6.5. *The exponential map satisfies the following properties:*

(1) $\exp_G(0) = e$;
(2) For all $t \in \mathbf{R}$ *and* $v \in \mathfrak{g}$, *we have* $\exp_G(tv) := \alpha_{tv}(1) = \alpha_v(t)$.
(3) The map $\exp_G$ *is functorial in* G*; i.e., for any map of Lie groups* $f : G \to H$ *the diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{Lf} & \mathfrak{h} \\ {\scriptstyle \exp_G}\downarrow & & \downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow{f} & H \end{array} \tag{6.1}$$

commutes.

Combining the functoriality with Example 6.4 allows us to calculate $\exp_G$ for many G of interest (which arise as a closed subgroup of some $\mathrm{GL}_n(\mathbf{F})$ with $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$).

Proof. We prove the statements in order.

(1) This follows from the definition, since the unique 1-parameter subgroup associated to $v = 0$ is the trivial homomorphism.
(2) For fixed t, $\alpha_{tv}(s)$ and $\alpha_v(ts)$ are 1-parameter subgroups in s with the same velocity at $s = 0$: they both have velocity tv. Hence, by uniqueness of 1-parameter subgroups, they must be equal. Then set $s = 1$ to conclude.
(3) This holds because α_v is functorial in (G, v), as we noted last time.

□

Remark 6.6. It is a natural to expect, especially from the explicit examples with $\mathrm{GL}_n(\mathbf{F})$ as above (and so for closed subgroups thereof), that the map of manifolds $\exp_G : \mathfrak{g} \to G$ is C^∞. This is not an obvious consequence of "smooth dependence on initial conditions" since the definition of α_v is in terms of the global vector field $\widetilde{v}$, making the smooth behavior under variation of v not immediately apparent.

The smoothness of $\exp_G$ for general G is a very important technical point which we will use all over the place in what follows. For a proof, we refer to Appendix G.

Lemma 6.7. *The map*

$$d(\exp_G)(0) : \mathfrak{g} \simeq T_0(\mathfrak{g}) \to T_e(G) =: \mathfrak{g}$$

is the identity map, where the first step is an instance of the canonical isomorphism of a finite-dimensional $\mathbf{R}$*-vector space with its tangent space at any point (via directional derivatives).*

In particular (by the inverse function theorem), $\exp_G$ *is a diffeomorphism between some open neighborhoods of* $0 \in \mathfrak{g}$ *and* $e \in G$.

Proof. We saw above that $\exp_G(tv) = \alpha_v(t)$ as parameteric curves. Computing the velocity at any t, we find

$$((d\exp_G)(tv))(v) = \alpha_v'(t).$$

Setting $t = 0$, we get $((d\exp_G)(0))(v) = \alpha_v'(0) = v$. To conclude, the careful reader should check that the identifications we made in the statement of the lemma agree with the identifications made above in this proof. $\square$

Corollary 6.8. *If* $f : G \to G'$ *is a map of Lie groups and* G *is connected then* f *is uniquely determined by* $Lf : \mathfrak{g} \to \mathfrak{g}'$.

Proof. By functoriality of exp, we have a commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "Lf"] \arrow[d, "\exp_G"'] & \mathfrak{g}' \arrow[d, "\exp_{G'}"] \\ G \arrow[r, "f"] & G'. \end{tikzcd} \tag{6.2}$$

Since $\exp_G$ is a diffeomorphism between open neighborhoods of $0 \in \mathfrak{g}$ and $e \in G$, given Lf we know f on some open $U \ni e$ in G. (In fact, this open U is independent of f, depending only on G, but we don't need that.) Thus, the *homomorphism* f is determined on the open subgroup $H \subset G$ generated by U (here, H consists of all "words" in products of finitely many elements of U and their inverses, the openness of this set of words being an easy exercise). But open subgroups are always closed in any topological group because their nontrivial cosets are each open, so H is also closed in G. Since G is connected, we conclude that $H = G$, so f is uniquely determined. $\square$

Note that connectedness of G cannot be dropped in Corollary 6.8, since Lf clearly exerts no control on what is happening in non-identity components of G (e.g., consider the case when G is a non-trivial discrete group, such as a non-trivial finite group, so $\mathfrak{g} = 0$).

Lemma 6.9. *If* G *is commutative then* $[\cdot,\cdot]_G = 0$.

Proof. This follows from the Local Formula for $[\cdot,\cdot]$ since Ad_G is identically equal to e by commutativity. $\square$

We have a converse in the connected case, via two applications of Corollary 6.8:

Corollary 6.10. *If* G *is connected and* $[\cdot,\cdot]_G = 0$ *then* G *is commutative.*

Clearly the connectedness cannot be dropped here (much as in Corollary 6.8).

Proof. We have $\mathrm{Lie}(\mathrm{Ad}_G) = \mathrm{ad}_{\mathfrak{g}} = 0$, so by Corollary 6.8 applied to $f = \mathrm{Ad}_G$ necessarily

$$\begin{aligned} \mathrm{Ad}_G \colon G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto \mathrm{id}_{\mathfrak{g}} \end{aligned}$$

This says $d(c_g)(e) = \mathrm{id}_{\mathfrak{g}}$ for all $g \in G$, so $c_g = \mathrm{id}_G$ for all $g \in G$ (by Corollary 6.8 applied to $f = c_g$). But that triviality for all g says exactly that G is commutative. $\square$

6.1. **Classification of connected commutative Lie groups.** Let's now describe all connected commutative Lie groups.

Lemma 6.11. *For a connected commutative G,*

$$\exp_G \colon \mathfrak{g} \to G$$

is a surjective homomorphism with discrete kernel.

Proof. To start, let's see why $\exp_G$ is a homomorphism. Commutative G have the following disorienting feature:

$$m : G \times G \to G$$

is a homomorphism (where the source is viewed as a direct product of Lie groups).

Using functoriality of $\exp_G$, and that $\exp_{G\times G} = \exp_G \times \exp_G$ (i.e., exp behaves well for direct product of Lie groups, which is obvious due to the same for integral curves or for a zillion other reasons), we get a commutative diagram

$$\begin{tikzcd} \mathfrak{g} \times \mathfrak{g} \arrow[r, "L(m)"] \arrow[d, "\exp_G \times \exp_G"'] & \mathfrak{g} \arrow[d, "\exp_G"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd} \tag{6.3}$$

Since $L(m)$ is vector addition (as for any Lie group whatsoever), this commutative diagram says exactly that $\exp_G$ is a homomorphism. Furthermore, the subset $\exp_G(\mathfrak{g}) \subset G$ contains an open around e since $\exp_G$ is a diffeomorphism near 0.

But $\exp_G(\mathfrak{g})$ is also a subgroup of G since $\exp_G$ is a homomorphism, so $\exp_G(\mathfrak{g})$ is an open subgroup of G. We have already noted that any open subgroup is always closed, so since G is connected we conclude that $\exp_G(\mathfrak{g}) = G$, which says that $\exp_G$ is surjective.

Finally, to show the kernel is discrete, simply use that $\exp_G$ is a local diffeomorphism near 0. This provides some open neighborhood U of 0 in $\mathfrak{g}$ so that $\ker(\exp_G) \cap U = \{0\}$, which gives discreteness of the kernel. $\square$

To further develop our understanding of connected commutative Lie groups, we need to know the discrete subgroups of $\mathfrak{g} \simeq \mathbf{R}^n$. The following basic result is [BtD, Lemma 3.8, Ch. I] (and we urge a reader who hasn't encountered it before to try to prove the cases $n \leq 2$ on their own, to appreciate the content):

Fact 6.12. Any discrete subgroup $\Gamma \subset \mathbf{R}^n$ is the $\mathbf{Z}$-span of part of a $\mathbf{R}$-basis.

For G any connected commutative Lie group, we conclude via translation considerations that the surjective homomorphism $\exp_G : \mathfrak{g} \to G$ is a local diffeomorphism everywhere due to its behavior as such near $0 \in \mathfrak{g}$. Hence, even without needing a fancy general theory of C^∞-structures on coset spaces for Lie groups modulo closed Lie subgroups (which we'll come to shortly), we obtain a canonical isomorphism of Lie groups

$$G \simeq \mathfrak{g}/\Lambda$$

where $\Lambda \subset \mathfrak{g} \simeq \mathbf{R}^n$ is the $\mathbf{Z}$-span of part of an $\mathbf{R}$-basis. Since $\mathbf{R}/\mathbf{Z} = S^1$, we obtain:

Theorem 6.13. *The connected commutative Lie groups are precisely the groups $(S^1)^r \times \mathbf{R}^m$. In particular, the compact connected commutative Lie groups are precisely the groups $(S^1)^r$.*

Any group of the form $(S^1)^r$ is called a *torus*, due to the picture when $r = 2$.

Remark 6.14. A nice application given in Appendix G is to use the preceding theorem to give a remarkable proof (due to Witt, and rediscovered by B. Gross when he was a graduate student) of the Fundamental Theorem of Algebra (using that if $m > 2$ then $\mathbf{R}^m - \{0\}$ is simply connected). This is not circular because nowhere in our development of Lie theory up to here have we used the Fundamental Theorem of Algebra.

6.2. **Three applications of the exponential map.**

Theorem 6.15 (Closed Subgroup Theorem (E. Cartan))**.** *If $H \subset G$ is a subgroup of a Lie group, then H is closed in G if and only if H is a locally closed smooth submanifold.*

Proof. See [BtD, Theorem 3.11, Ch. I] for a full proof. We now explain the basic idea for the harder implication: if H is closed then it is a C^∞ submanifold. The idea is to cook up a subspace $\mathfrak{h} \subset \mathfrak{g}$ that is a candidate for $\mathrm{Lie}(H)$. One can then use closedness to show that for a suitably small open $U \subset \mathfrak{h}$ around 0, the subset $\exp_G(U) \subset G$ is open in H. Then, $\exp_G(U)$ gives a smooth chart on H near e which is translated around to define a C^∞-structure on H that is seen to do the job. □

As an application of the Closed Subgroup Theorem, we obtain:

Theorem 6.16. *Any continuous homomorphism $f : G \to G'$ between Lie groups is C^∞.*

Proof. For a proof, see Appendix G. The key idea is to consider the graph $\Gamma_f \subset G \times G'$. This is a closed subgroup of the Lie group $G \times G'$ because f is a homomorphism, so by Theorem 6.15 it is even a C^∞-submanifold.

Thus, the two maps

$$\begin{tikzcd} & \Gamma_f \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & G' \end{tikzcd} \tag{6.4}$$

are C^∞, and the first one is a bijection. Although a C^∞ bijection between smooth manifolds is generally not a diffeomorphism (think of x^3 on $\mathbf{R}$), Sard's theorem ensures that there are lots of points where such a map is a submersion. The homogeneity then ensures that for a homomorphism of Lie groups the submersion property holds *everywhere*, and from this one can conclude that π_1 is a diffeomorphism. Hence, $\pi_2 \circ \pi_1^{-1} : G \to G'$ is a C^∞ map, and this is f by another name! □

Theorem 6.17. *For $H \subset G$ a closed subgroup, there exists a unique C^∞-structure on G/H so that*

$$\pi : G \to G/H$$

is a submersion making the natural map $\mathfrak{g} \to T_{\overline{e}}(G/H)$ induce an isomorphism

$$\mathfrak{g}/\mathfrak{h} \simeq T_{\overline{e}}(G/H).$$

Moreover, this is initial among all C^∞ maps of manifolds $G \to M$ that are invariant under right multiplication by H on G.

Proof. A reference for all but the final assertion is [BtD, pp. 34-35] (and the final assertion is Exercise 5(i) in HW3). The idea for the existence of the asserted C^∞-structure on G/H is to choose a linear complement V to $\mathfrak{h}$ in $\mathfrak{g}$ and show that $\exp_G$ applied to a small open around 0 in V provides the required C^∞-structure on G/H near the identity coset $\overline{e}$. This C^∞-structure near $\overline{e}$ is translated around using left multiplication on G/H by elements of G (and is shown to work and be unique making π a submersion with the specified tangent space at $\overline{e}$). □

Remark 6.18. It is a remarkable fact that every C^r Lie group with $1 \leq r \leq \infty$ admits a real-analytic structure making its group law and inversion both real-analytic, and this is unique. More specifically, for $1 \leq r \leq \infty$ the forgetful functor from real-analytic Lie groups to C^r Lie groups is an equivalence of categories.

The essential issue is to reconstruct the group law from the Lie bracket via a universal recipe called the *Baker-Campbell-Hausdorff formula*: one shows for $X, Y \in \mathfrak{g}$ near 0 that $\exp_G(X)\exp_G(Y) = \exp_G(Z(X,Y))$ for a "universal formal series $Z(X,Y)$ in iterated applications of $[\cdot,\cdot]$" that has nothing to do with G (but of course for specific G some of the brackets in this universal formula might happen to vanish). For the definition of the BCH formula as a "formal Lie series" see [Se, Part I, §4.8] or [Bou1, Ch. II, §6.4], and for its convergence near $(0,0)$ see [Se, Part II, §5.4] or [Bou1, Ch. II, §7.2, Prop. 1].

We will never need the real-analytic aspects of Lie groups, but they are very useful in the general representation theory of non-compact Lie groups (whereas in the compact case there is a much stronger feature than unique real-analyticity: a "unique algebraicity" which we will formulate later in the course). If you're interested to learn about the real-analytic enhancement as discussed above then read [Se, Part II] and [Bou1, Ch. III, §1.9, Prop. 18].

7. Subgroup-subalgebra correspondence

As an application of $\exp_G$ and differential geometry, for a Lie group G we want to relate Lie subalgebras of $\mathfrak{g}$ and connected "Lie subgroups" of G.

Definition 7.1. Let G be a Lie group. A *Lie subgroup* of G is a pair (H, j) for a Lie group H and an injective immersion $j : H \to G$.

Example 7.2. The 1-parameter subgroup $\mathbf{R} \to S^1 \times S^1 = (\mathbf{R} \times \mathbf{R})/(\mathbf{Z} \times \mathbf{Z})$ corresponding to a tangent vector in $\mathbf{R} \times \mathbf{R}$ with slope not in $\mathbf{Q}$ is a Lie subgroup that is not a topological embedding.

Remark 7.3. In Appendix H (whose core geometric input is done in [Wa]), we develop a higher-dimensional analogue of integral curves to vector fields: given a rank-d subbundle $E \subset TM$ (this means a specification of d-dimensional subspaces $E(m) \subset T_m(M)$ that "vary smoothly" in $m \in M$), an *integral manifold* in M to E is an injective immersion $N \xrightarrow{j} M$ inducing $T_n(N) \simeq E(j(n)) \subset T_{j(n)}(M)$ for all $n \in N$.

Note that for $d = 1$, such an E is less information than a non-vanishing smooth vector field because we are only giving a (smoothly varying) line in a tangent space at each point of M rather than a (smoothly varying) nonzero vector in the tangent space at each point of M. Also for $d = 1$ an integral manifold is like an integral curve to a smooth vector field except that we drop the parameterization and only keep track of the image. For example, the integral curve to a non-vanishing smooth vector field could be a map from all of $\mathbf{R}$ onto a circle whereas an integral submanifold to the associated line subbundle of TM would such a circle *without* a specified parameterization.

The existence of an integral submanifold N to E through each $m_0 \in M$ requires the vanishing of certain PDE obstructions (which always vanish when $d = 1$ but are typically non-trivial when $d > 1$). The global Frobenius theorem says that this necessary vanishing condition is also sufficient. When M is a Lie group G and E arises from a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (in a manner explained in Appendix H), the stability of $\mathfrak{h}$ under the Lie bracket on $\mathfrak{g}$ implies that the PDE obstructions vanish. That enables one to use the global Frobenius theorem to prove the surjectivity in the following important result which revolutionized the theory of general Lie groups (by connecting it tightly to global differential geometry in a way that had not previously been done):

Theorem 7.4 (Chevalley)**.** *The natural map*

$$\{\text{connected Lie subgroups } H \subset G\} \to \{\text{Lie subalgebras } \mathfrak{h} \subset \mathfrak{g}\} \tag{7.1}$$

is a bijection. Also, for a connected Lie group G', a homomorphism $f : G' \to G$ factors in the C^∞ sense through a connected Lie subgroup $H \hookrightarrow G$ if and only if $Lf : \mathfrak{g}' \to \mathfrak{g}$ lands in $\mathfrak{h} \subset \mathfrak{g}$.

Warning 7.5. Note that $H \to G$ in Theorem 7.4 is usually not an embedding (equivalently, H may not be closed in G)! If one only wanted to get a result for closed H for general G then the proof would be hopeless because it is generally impossible to detect in terms of $\mathfrak{h}$ whether or not H is closed in G (though there are very useful sufficient conditions for this in terms of the structure theory of Lie algebras, with a proof requiring input from the algebro-geometric theory of linear algebraic groups, so lying rather beyond the level of this course).

The proof of Theorem 7.4 is given as Theorem H.4.3 and Proposition H.4.4. Injectivity of the map (7.1) when limited to closed H's is an easy application of exponential maps.

Here is an important consequence that is good to be aware of but which we will never use in this course (and which was known to Lie only in a very localized form in the language of convergent power series and PDE's):

Corollary 7.6 (Lie's 3rd Fundamental Theorem)**.** *Every finite dimensional Lie algebra $\mathfrak{h}$ over $\mathbf{R}$ is isomorphic to* $\mathrm{Lie}(H)$ *for some connected Lie group H.*

Proof. Apply Theorem 7.4 to $G = \mathrm{GL}_n(\mathbf{R})$ for any $\mathfrak{gl}_n(\mathbf{R})$ containing $\mathfrak{h}$ as a Lie subalgebra (provided by Ado's Theorem applied to $\mathfrak{h}$). $\square$

See §H.4 for many more applications of Theorem 7.4. For example, there are applications to computing the Lie subalgebras in $\mathfrak{g}$ corresponding to normalizers, centralizers, intersections, and preimages (under homomorphisms) of closed (sometimes connected) subgroups of connected Lie groups.

7.1. **Applications in representation theory.** Let's consider representations of Lie groups G on finite-dimensional $\mathbf{C}$-vector spaces V. The main case for this course will be compact G, but for now we allow general G. Suppose we have a C^∞ (or equivalently, continuous) map

$$\rho : G \to \mathrm{GL}(V) = \mathrm{GL}_n(\mathbf{C}).$$

In the general compact case, we can use "volume-1 Haar measure" to get results akin to the case G is finite, where one uses the averaging operator

$$\frac{1}{\#G} \sum_{g \in G}$$

to prove results such as complete reducibility of representations. Next time this Haar-measure method will be discussed for compact G. But what about the non-compact case, for which such an averaging operator doesn't exist?

Is there a way to prove complete reducibility results when G is non-compact? We now explain a technique for achieving this in certain situations when G is *connected* by using Theorem 7.4. (Note that for finite G, connectedness forces $G = 1$. Hence, work with connected G tells us nothing interesting about the case of finite groups.)

For connected G, note that $\rho : G \to \mathrm{GL}(V)$ is determined by the map

$$L\rho : \mathfrak{g} \to \mathfrak{gl}(V),$$

which is in turn equivalent to the map

$$\mathfrak{g}_{\mathbf{C}} := \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} \to \mathfrak{gl}(V)$$

of Lie algebras over $\mathbf{C}$. In turn, this map of complex Lie algebras is equivalent to a map

$$\mathfrak{g}_{\mathbf{C}} \times V \to V \tag{7.2}$$

(denoted $(X, v) \mapsto X.v$) which is $\mathbf{C}$-bilinear and satisfies

$$[X, Y].v = X.(Y.v) - Y.(X.v)$$

(such a pairing is called a *representation* of the Lie algebra $\mathfrak{g}_{\mathbf{C}}$ on V).

Lemma 7.7 (Weyl). *Assume G is connected. A $\mathbf{C}$-subspace $V' \subset V$ is G-stable if and only if it is $\mathfrak{g}_{\mathbf{C}}$-stable under the corresponding $\mathfrak{g}_{\mathbf{C}}$-representation on V in the sense of* (7.2)*. In particular, V is completely reducible for G (i.e., a direct sum of irreducible representations) if and only if it is completely reducible for $\mathfrak{g}_{\mathbf{C}}$.*

Proof. Choose a $\mathbf{C}$-basis for V extending one for V', with $d = \dim V'$ and $n = \dim V$. Then V' is G-stable if and only if $\rho : G \to \mathrm{GL}(V) = \mathrm{GL}_n(\mathbf{C})$ lands in the closed Lie subgroup H of invertible $n \times n$ matrices with vanishing lower-right $(n-d) \times d$ block (as that expresses the preservation of V' for a linear automorphism of V). Since G and H are *connected*, this is equivalent to the map $L\rho : \mathfrak{g} \to \mathfrak{gl}(V) = \mathfrak{gl}_n(\mathbf{C})$ landing in $\mathrm{Lie}(H)$ by Theorem 7.4! But $\mathrm{Lie}(H)$ consists of those elements of $\mathfrak{gl}(V) = \mathrm{Mat}_n(\mathbf{C})$ with vanishing lower-right $(n-d) \times d$ block, which in turn corresponds to the elements of $\mathfrak{gl}(V) = \mathrm{End}(V)$ which carry V' into itself, so this is equivalent to the condition $\mathfrak{g}_{\mathbf{C}} \cdot V' \subset V'$. ☐

Example 7.8 (Weyl's unitary trick)**.** Suppose $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}'_{\mathbf{C}}$ for $\mathfrak{g}' = \mathrm{Lie}(G')$ with a connected compact Lie group G'. (We'll see later that this is a very mild hypothesis on $\mathfrak{g}$, satisfied in many interesting cases with non-compact G.) Furthermore *assume* that the resulting representation

$$\mathfrak{g}' \to \mathfrak{g}'_{\mathbf{C}} = \mathfrak{g}_{\mathbf{C}} \xrightarrow{(L\rho)_{\mathbf{C}}} \mathfrak{gl}(V)$$

arises from a Lie group representation

$$\rho' : G' \to \mathrm{GL}(V).$$

(We say the Lie algebra map "integrates" to ρ', since passage from Lie groups to Lie algebras is a kind of "derivative" process.) The existence of ρ' has some topological obstructions in general, but later we will see that such obstructions can often be bypassed by adjusting G' slightly, and in particular are never an issue when G' is simply connected.

By Lemma 7.7, the complete reducibility for (G, ρ) is equivalent to that for (G', ρ'), which holds in turn (by averaging arguments we'll discuss next time) because G' is compact! This is like an out-of-body experience, since:

(i) neither G nor G' live inside the other, so they don't "directly communicate" as Lie groups,
(ii) the contact between G and G' is entirely through their complexified Lie algebras, which aren't visibly the Lie algebras of any complex Lie groups easily attached to G or G' (e.g., G may not have anything to do with algebraic geometry)!

As an example, for $G = \mathrm{SL}_n(\mathbf{R})$ we have $\mathfrak{sl}_n(\mathbf{R})_{\mathbf{C}} = \mathfrak{sl}_n(\mathbf{C}) = \mathfrak{su}(n)_{\mathbf{C}}$ (we'll see the second equality later) and $G' := \mathrm{SU}(n)$ is simply connected (to be proved later). More broadly, it is a general fact that *every* semisimple Lie algebra over $\mathbf{R}$ (of which the $\mathfrak{sl}_n(\mathbf{R})$'s are a special case) has complexification that coincides with the complexification of a (unique) connected compact Lie group that is simply connected. Hence, complete reducibility for representations of connected Lie groups with a semisimple Lie algebra (which constitute a large and important class of Lie groups) can be reduced to the known case of (connected) compact Lie groups! This is called the "unitary trick" since any continuous representation ρ of a compact Lie group K into $\mathrm{GL}_n(\mathbf{C})$ leaves invariant some positive-definite hermitian form on $\mathbf{C}^n$ (by averaging on the compact group $\rho(K)$) and hence lands in $\mathrm{U}(n)$ after a change of basis.

Warning 7.9. It may be that ρ' as in Example 7.8 does not exist if G' is chosen poorly. For example, consider the canonical inclusion

$$\alpha : \mathrm{SU}(2) = \left(\mathbf{H}^\times\right)^1 \hookrightarrow \mathrm{GL}_n(\mathbf{C})$$

and the 2:1 covering map

$$\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$$

(quotient by the central $\{\pm 1\}$) introduced in HW3. The map $\mathrm{Lie}(\pi)$ is an isomorphism and so yields a representation $\rho : \mathfrak{so}(3) \simeq \mathfrak{su}(2) \to \mathfrak{gl}_2(\mathbf{C})$, but there does not exist a map $f : \mathrm{SO}(3) \to \mathrm{GL}_2(\mathbf{C})$ satisfying $Lf = \rho$ since if such an f exists then $f \circ \pi = \alpha$ (as their effects on Lie algebras coincide, and $\mathrm{SU}(2)$ is connected) yet $\ker \alpha = 1$ and $\ker \pi \neq 1$.

Example 7.10 (Spherical harmonics)**.** Consider $\mathrm{SO}(3)$ acting on $C^\infty\left(\mathbf{R}^3;\mathbf{C}\right) \supset \mathbf{C}\left[x_1,x_2,x_3\right]$ through $(g.f)(v) = f(g^{-1}v)$ for $v \in \mathbf{R}^3$. This action commutes with the Laplacian

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$$

(as Δ is the Laplacian attached to the standard Riemannian metric tensor on $\mathbf{R}^3$ which is tautologically invariant under $\mathrm{SO}(3)$) and also clearly preserves homogeneous polynomials of degree n (since it acts through linear substitutions). Hence, $\mathrm{SO}(3)$ acts on

$$W_n := \left\{f \in \mathbf{C}\left[x_1,x_2,x_3\right] : f \text{ is homogeneous of degree } n, \Delta(f) = 0\right\}.$$

(The condition $\Delta(f) = 0$ is called *harmonicity*.)

We'll soon see that the composite representation

$$\mathrm{SU}(2) \to \mathrm{SO}(3) \to \mathrm{GL}(W_n)$$

is

$$\mathrm{Sym}^{2n}(\rho)$$

for the standard 2-dimensional representaiton ρ of $\mathrm{SU}(2)$ (so $\dim W_n = 2n+1$, and hence the W_n's are pairwise non-isomorphic as $\mathrm{SO}(3)$-representations since even their dimensions are pairwise distinct).

In two lectures from now we shall prove:

Theorem 7.11. *These W_n's are precisely the irreducible representations of* $\mathrm{SO}(3)$.

A key ingredient in the proof will be to study the action of a (1-dimensional) maximal torus in $\mathrm{SO}(3)$. As such, it will be a warm-up to the important role of maximal tori in the general structure theory of connected compact Lie groups and their finite-dimensional (continuous) representations over $\mathbf{C}$.

8. Complete reducibility

Today and next time, we'll discuss character theory for compact Lie groups G.

Convention 8.1. When we say "representation" we always mean a continuous finite dimensional linear representation over $\mathbf{C}$ (equivalently, a continuous homomorphism $G \to \mathrm{GL}(V)$ for a finite-dimensional $\mathbf{C}$-vector space V).

Choose a nonzero top-degree left-invariant differential form ω on G. By left-invariance, $\omega(g) \in \wedge^{\dim G}\left(T^*(G)\right)$ is equal to $\ell_{g^{-1}}^*(\omega_0)$ for $\omega_0 \in \wedge^{\dim G}(\mathfrak{g}^*)$. For a Borel set $A \subset G$, we can define $\int_A |\omega|$ without orientation by using $\int |f|$ on coordinate charts. (In global terms, $\int_A |\omega|$ means $\int_M \chi_A|\omega|$ for the characteristic function χ_A of A, where such integration of a "top-degree differential form with locally-L^1 measureable coefficients" is defined using a partition of unity in a familiar way that we leave to your imagination.) This defines a regular Borel measure on G (and works on any smooth manifold with any top-degree differential form).

Remark 8.2. Integration of differential forms needs an oriention because we want $\int_M \omega$ to be linear in ω. But if we only care about defining a measure then we can work with $\int_A |\omega|$, sacrificing linearity in ω with the gain of no longer needing an orientation.

Lemma 8.3. *The regular nonzero Borel measure μ_ω defined by $\mu_\omega(A) = \int_A |\omega|$ is left-invariant (so it is a left Haar measure). Further, this is independent of ω up to scaling by an element of $\mathbf{R}_{>0}$.*

Proof. For $g \in G$ we have $\ell_g : A \simeq gA$ and so

$$\begin{aligned}\mu_\omega(gA) &= \int_{gA} |\omega| \\ &= \int_A \left|\ell_g^* \omega\right| \\ &= \int_A |\omega| \\ &= \mu_\omega(A)\end{aligned}$$

using left-invariance of ω for the second-to-last equality; of course, these integrals might be infinite. That the measure is regular is left to the reader (step 1: remind yourself what regularity means, and then refresh your memory about the Riesz representation theorem).

For another ω' we have $\omega_0' = c\omega_0$ for some $c \in \mathbf{R}^\times$since $\wedge^{\dim G}(\mathfrak{g}^*)$ is a line, and so $\omega' = c\omega$ by left-invariance. Thus, $\mu_{\omega'} = |c|\mu_\omega$. □

Convention 8.4. We write dg to denote such a measure μ_ω (independent of ω up to $\mathbf{R}_{>0}$-scaling); it is *not* a differential form (despite the suggestive notation)!

For $f : G \to \mathbf{C}$ measurable and any $g' \in G$ we have

$$\int_G f(g)dg = \int_G f(g'g)dg$$

in the sense that f is L^1 if and only if $f \circ \ell_{g'}$ is L^1, in which case the two integrals are equal.

Example 8.5. If $G = S^1 \simeq \mathbf{R}/(\mathbf{Z} \cdot \ell)$ (for some $\ell \in \mathbf{R}^\times$, such as $\ell = 1$ or $\ell = 2\pi$ or ...) then one choice for dg is $|d\theta|$. Likewise, for $G = \mathbf{R}^n/\mathbf{Z}^n$ one choice for dg is $|dx_1 \wedge \cdots \wedge dx_n|$. (In practice nobody writes those absolute values, for much the same reason of suggestive but abusive notation by which we write dg for a Haar measure.)

For $G = \mathrm{GL}_n(\mathbf{R})$, by HW2 one choice for dg is

$$\left|\frac{dx_{11} \wedge \cdots \wedge dx_{ij} \wedge \cdots \wedge dx_{nn}}{\det\left(x_{ij}\right)^n}\right|.$$

(Note that this measure is unaffected by the rearrangement of terms in the wedge product, whereas the underlying differential form is sensitive to such a change. There is no "canonical" way to arrange the terms in the wedge product, of course. The theory of smooth group schemes over $\mathbf{Z}$ gives a significance to this particular invariant top-degree form up to $\pm 1 = \mathbf{Z}^\times$.)

For a choice of ω and $g_0 \in G$, $r_{g_0}^*(\omega)$ is also left-invariant and nonzero because r_{g_0} commutes with ℓ_g: this expresses the identity

$$g\left(xg_0\right) = \left(gx\right)g_0$$

in G that is just the associative law. Therefore,

$$r_{g_0}^*(\omega) = c_{g_0} \cdot \omega$$

for some $c_{g_0} \in \mathbf{R}^\times$ which is easily seen to be independent of ω (recall all choices of ω are $\mathbf{R}^\times$-multiples of each other). Clearly the map $G \to \mathbf{R}^\times$ defined by $g_0 \mapsto c_{g_0}$ is a homomorphism.

Lemma 8.6. *For $g_0 \in G$ and any Borel set $A \subset G$ we have*

$$\mu_\omega\left(Ag_0\right) = \mu_{r_{g_0}^*\omega}(A) = |c_{g_0}|\mu_\omega(A).$$

Proof. Snce $r_{g_0} : A \simeq Ag_0$, by the definitions we have

$$\mu_\omega\left(Ag_0\right) = \mu_{r_{g_0}^*\omega}(A) = \mu_{c_{g_0}\omega}(A) = |c_{g_0}|\mu_\omega(A).$$

□

Definition 8.7. We call the map

$$G \to \mathbf{R}_{>0}$$
$$g_0 \mapsto \left|c_{g_0}\right|$$

the *modulus character*, and denote it as Δ_G.

Warning 8.8. If we had worked throughout with right-invariant differential forms and right-invariant measures then the resulting character would be $1/\Delta_G$ (check!). There is much confusion between Δ_G and Δ_G^{-1} due to confusion about whether we are left-scaling on right-invariant measures or right-scaling on left-invariant measures.

On HW4 it is shown that Δ_G is continuous. In fact, something stronger is true upon dropping the absolute value: even the homomorphism

$$G \to \mathbf{R}^\times$$
$$g_0 \mapsto c_{g_0}$$

is continuous. This is easy to see using the definition in terms of a choice of ω. An analogous continuity holds for any locally compact Hausdorff topological group using the modulus character defined in terms of the effect of right translation on left Haar measures, but the proof is more difficult than in our Lie group setting because one doesn't have the concrete description in terms of differential forms.

Since 1 is the only compact subgroup of $\mathbf{R}_{>0}$, we obtain the important:

Theorem 8.9. *For G a compact Lie group we have $\Delta_G = 1$, so the left Haar measures μ_ω are bi-invariant. If G is also connected then even the left-invariant top-degree differential forms ω are bi-invariant.*

Warning 8.10. It can happen for G compact but disconnected that although μ_ω (and so really the "density" $|\omega|$) is bi-invariant, the differential form ω is *not* bi-invawriant. An example for which this happens is $G = \mathrm{O}(2) = \mathrm{SO}(2) \rtimes \mathbf{Z}/2\mathbf{Z}$, for which the non-trivial element of $\mathbf{Z}/2\mathbf{Z}$ conjugates on the circle $\mathrm{SO}(2)$ via inversion (which has the effect of negation on the invariant 1-form dt/t on the circle).

For some other disconnected compact groups, such as $\mathrm{SO}(r,s)$ with $r,s > 0$, the bi-invariance does hold at the level of differential forms. For those who know a bit about schemes: the secret conceptual distinction between the disconnected groups $\mathrm{SO}(r,s)$ and $\mathrm{O}(2)$ (or more generally $\mathrm{O}(n)$ for $n > 1$) that explains this difference in bi-invariance at the level of top-degree left-invariant differential forms is that $\mathrm{SO}(r,s)$ is "Zariski-connected" whereas $\mathrm{O}(n)$ is never "Zariski-connected".

Bi-invariance of μ_ω does fail in some non-compact cases. An example is the so-called "$ax+b$ group" that is $\mathbf{R}^\times \ltimes \mathbf{R}$ or more concretely the group of matrices

$$\left\{ M \in \mathrm{GL}_2(\mathbf{R}) : M = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}$$

In HW4, nonzero left-invariant and right-invariant 2-forms on this group will be worked out explicitly, and will thereby be seen to be rather different from each other.

Remark 8.11. The good news is that bi-invariance at the level of the top-degree differential form does hold for a large class of non-compact G: the so-called "Zariski-connected reductive" groups. We don't give a general definition here (I could talk about that topic for many hours, it is so wonderful), so we content ourselves with just mentioning some basic examples:

$$\mathrm{GL}_n(\mathbf{R}), \mathrm{Sp}_{2n}(\mathbf{R}), \mathrm{SL}_n(\mathbf{R}), \mathrm{GL}_n(\mathbf{C}), \mathrm{SO}(r,s), \mathrm{Sp}_{2n}(\mathbf{C}), \ldots$$

Remark 8.12. For compact (perhaps disconnected!) G there is a canonical Haar measure dg characterized by

$$\int_G dg = 1.$$

(Indeed, by regularity of the measure and compactness of G any Haar measure μ satisfies $0 < \int_G d\mu < \infty$, so we can uniquely scale μ by $\mathbf{R}_{>0}$ to arrange the total volume to be 1.) For finite G, this corresponds to the point mass $1/(\#G)$ at all $g \in G$, which gives rise to the averaging procedure used a lot in the representation theory of finite groups, such as to make a G-invariant positive-definite hermitian form on any (finite-dimensional, $\mathbf{C}$-linear!) G-representation V. Such a hermitian form h ensures that any G-subrepresentation $W \subset V$ has a G-stable linear complement, namely its h-orthogonal complement $W^\perp \subset V$. This yields complete reducibility for representations of finite groups G by dimension induction.

To establish complete reducibility for compact Lie groups by the exact same argument as for finite groups, we just need:

Proposition 8.13. *For compact Lie groups G and (continuous finite-dimensional!) representation*

$$\rho : G \to \mathrm{GL}(V),$$

there exists a G-invariant positive-definite hermitian form on V.

Proof. In Appendix D we use $\int_{\rho(G)}$ to show $\rho(G) \subset \mathrm{U}(h)$ for some positive-definite hermitian h on V. Such an inclusion is precisely the statement that G leaves h invariant:

$$h(gv, gv') = h(v, v') \text{ for all } v, v' \in V, g \in G.$$

$\square$

Remark 8.14. Positive-definiteness of the hermitian form (rather than non-degeneracy) is crucial to ensure that when we restrict to subspaces it remains nondegenerate and that orthogonal complements are linear complements. More importantly, in the averaging process used to construct the invariant h from an initial choice of h_0 on V, if h_0 were merely non-degenerate rather than positive-definite then we would have no reason to expect that the G-averaged h obtained from h_0 is even non-degenerate.

Corollary 8.15. *For a compact Lie group G and representation $\rho : G \to \mathrm{GL}(V)$, its decomposition as a direct sum of irreducible representations has the following uniqueness property: if*

$$V \simeq \bigoplus_{i \in I} V_i^{\oplus n_i}, \quad V \simeq \bigoplus_{j \in J} (V'_j)^{\oplus n'_j}$$

for $\{V_i\}$ pairwise non-isomorphic irreducible representations with $n_i > 0$ and $\{V'_j\}$ pairwise isomorphic irreducible representations with $n'_j > 0$, there is a bijection $\sigma : I \simeq J$ so that $V_i \simeq V'_{\sigma(i)}$ and moreover $n_i = n'_{\sigma(i)}$.

This is proved exactly as in the case of finite groups, at least if you learned finite group representations from me (over a general field). But you might have seen the proof for finite groups in a form that relies in Schur's Lemma, so let's now address that (in a way which will not lead to circularity):

Lemma 8.16 (Schur's lemma)**.** *For an irreducible finite-dimensional $\mathbf{C}$-linear representation $\rho : \Gamma \to \mathrm{GL}(V)$ of an arbitrary group Γ,*

$$\mathrm{End}_\Gamma(V) = \mathbf{C}.$$

Moreover, for a second such irreducible representation W of Γ,

$$\dim_{\mathbf{C}} \mathrm{Hom}_\Gamma(V, W) = \begin{cases} 0 & \text{if } V \not\simeq W, \\ 1 & \text{if } V \simeq W. \end{cases}$$

Proof. One uses the same eigenvalue argument (working over $\mathbf{C}$) as for finite groups. This is pure algebra, requiring nothing about the group Γ (no topology, etc.). □

Corollary 8.17. *For a commutative topological group G and (continuous finite-dimensional $\mathbf{C}$-linear) irreducible representation $\rho : G \to \mathrm{GL}(V)$, necessarily* $\dim V = 1$.

That is, any such (V, ρ) corresponds to a continuous character $\chi : G \to \mathbf{C}^\times$.

Proof. Since G is commutative, G acts on V through $\mathrm{End}_G(V)$, and this in turn consists of the scalar endomorphisms by Schur's Lemma. Therefore, G acts through $\mathrm{End}_G(V)^\times = \mathbf{C}^\times$, which is to say $\rho(g)$ is a scalar for all $g \in G$. Thus, all subspaces of V are G-stable but ρ is irreducible, so V must be 1-dimensional. □

Corollary 8.18. *For a compact commutative Lie groups G, every representation is a direct sum of* 1*-dimensional representations.*

Proof. This follows from complete reducibility and Corollary 8.17. □

Here is an example that will be extremely important for the rest of the course.

Example 8.19. For $G = (S^1)^r$ (i.e., a connected commutative compact Lie group), let's work out all possibilities for continuous homomorphisms

$$\chi : G = (S^1)^r \to \mathbf{C}^\times.$$

The image is a compact subgroup and so must lie inside $S^1 \subset \mathbf{C}^\times$, so

$$\chi(z_1, \ldots, z_r) = \chi_1(z_1) \cdots \chi_r(z_r)$$

for $\chi_j : S^1 \to S^1$ a continuous homomorphism (restriction of χ to the jth factor circle). What are the possibilities for such χ_j?

Proposition 8.20. *We have*

$$\mathrm{End}_{\mathrm{Lie}}(S^1) = \mathbf{Z},$$

with $n \in \mathbf{Z}$ corresponding to the power-endomorphism $z \mapsto z^n$ of S^1.

Once this is proved, we conclude that

$$\chi(z_1, \ldots, z_r) = \prod z_j^{a_j}$$

for a unique $(a_1, \ldots, a_r) \in \mathbf{Z}^r$. This gives a very explicit description of all (continuous) characters of tori.

Proof. For any $f : S^1 \to S^1$, the effect $\mathrm{Lie}(f) : \mathrm{Lie}(S^1) \to \mathrm{Lie}(S^1)$ on Lie algebras is an endomorphism of a line (i.e., $\mathrm{Lie}(S^1)$ is 1-dimensional over $\mathbf{R}$), so $\mathrm{Lie}(f)$ is multiplication by some $c \in \mathbf{R}$. Functoriality of the exponential map thereby gives a commutative diagram

$$\begin{tikzcd} \mathrm{Lie}(S^1) \arrow[r, "c"] \arrow[d, "\exp_{S^1}"'] & \mathrm{Lie}(S^1) \arrow[d, "\exp_{S^1}"] \\ S^1 \arrow[r, "f"'] & S^1 \end{tikzcd} \tag{8.1}$$

By commutativity, c-multiplication on the $\mathbf{R}$-line $\mathrm{Lie}(S^1)$ must carry the discrete subgroup $\ker \exp_{S^1}$ into itself. This discrete subgroup is non-trivial (as S^1 is compact), and more specifically it has the form $\mathbf{Z}\ell$ for some nonzero $\ell \in \mathrm{Lie}(S^1)$, so $c \cdot \ell = n \cdot \ell$ for some $n \in \mathbf{Z}$. Hence, $c = n \in \mathbf{Z}$. But $z \mapsto z^n$ has that effect on the Lie algebra (recall that the group law induces addition on the Lie algebra), so f coincides with that power map. $\square$

9. Character theory

9.1. **Isotypic decomposition.** Recall that for finite groups, the decomposition of a representation (always finite-dimensional, $\mathbf{C}$-linear) as a direct sum of irreducible representations is generally not unique in any intrinsic/functorial sense, but the multiplicity with which each irreducible appears in such a decomposition is intrinsic. Similarly, for any representation V of a compact Lie group G, the decomposition into irreducibles is not intrinsic. However, by exactly the same argument as for finite groups, the *isotypic decomposition*

$$V = \bigoplus_{\sigma \text{ irreducible}} V(\sigma)$$

for the span $V(\sigma)$ of all copies of σ inside V is both intrinsic *and* functorial. A slicker description of $V(\sigma)$ is this:

$$V(\sigma) = \mathrm{image}(\sigma \otimes_{\mathbf{C}} \mathrm{Hom}_G(\sigma, V) \to V)$$

(G-equivariant map via evaluation, injective by Schur's Lemma).

Definition 9.1. For V a representation of a compact Lie group G and σ an irreducible representation of G, $\mathrm{Hom}_G(\sigma, V)$ is called the *multiplicity space*. (The reason for the name is that by Schur's lemma, $\dim \mathrm{Hom}_G(\sigma, V)$ is the multiplicity of σ in a decomposition of V as a direct sum of irreducible representations.)

Definition 9.2. For a torus T, the characters $\chi : T \to \mathbf{C}^\times$ with $V(\chi) \neq 0$ are called the *weights* of V. For general χ, we call $V(\chi)$ the χ-weight space of V (it might vanish).

By functoriality of the isotypic decomposition, for a subrepresentation $W \subset V$ we have $W = \oplus W(\sigma)$ with $W(\sigma) = W \cap V(\sigma) \subset V(\sigma)$. Nothing like this is available in the language of the decomposition as a direct sum of irreducible representations; this is why the isotypic decomposition is sometimes more useful than mere complete reducibility in constructions and proofs.

9.2. **Application of the weight space decomposition for tori.** We'll next describe the irreducible representations of $\mathrm{SO}(3)$. The natural quotient map

$$\mathrm{SU}(2) \to \mathrm{SO}(3)$$

has central kernel $\{\pm 1\}$, so the irreducible representations of $\mathrm{SO}(3)$ are those irreducible representations of $\mathrm{SU}(2)$ on which -1 acts trivially. Note that on any irreducible representation of $\mathrm{SU}(2)$ the action of the central -1 must be via a scalar (by Schur's lemma), so since $(-1)^2 = 1$ in $\mathrm{SU}(2)$ we see that the scalar must be $\pm 1 \in \mathbf{C}^\times$.

For $n \geq 0$ consider

$$V_n := \mathrm{Sym}^n(\mathbf{C}^2)$$

as a representation of $\mathrm{SU}(2) \subset \mathrm{GL}_2(\mathbf{C})$, where $\mathbf{C}^2$ denotes the representation of $\mathrm{SU}(2)$ arising from its tautological inclusion into $\mathrm{GL}_2(\mathbf{C})$. Note that $\dim V_n = n + 1$. Concretely,

$$V_n = \{\text{degree-}n \text{ homogeneous polynomials } P(z_1, z_2)\}$$

with

$$(g \cdot P)(z_1, z_2) = P((z_1, z_2)\, g)$$

where

$$(z_1, z_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (az_1 + cz_2, bz_1 + dz_2).$$

Lemma 9.3. *The V_n's for which* $\mathrm{SU}(2)$ *factors through* $\mathrm{SO}(3)$ *are the cases of even n.*

Proof. By definition, the effect of $g = -1$ is

$$P(z) \mapsto P(-z) = (-1)^n P(z),$$

so -1 acts by multiplication by $(-1)^n$. Therefore, the V_n's for which $\mathrm{SU}(2)$ factors through $\mathrm{SO}(3)$ are the cases of even n. □

Proposition 9.4. *The representations V_n are irreducible for* $\mathrm{SU}(2)$.

Proof. The rough idea is to compute the weight space decomposition of V_n for some torus $T \subset \mathrm{SU}(2)$, then do the same for another torus $T' \subset \mathrm{SU}(2)$, and finally compare them. This is not the "professional" way of proceeding (which would be to use character theory that we haven't yet introduced), but it concretely illustrates the power of working with maximal tori in connected compact Lie groups G to get a handle on the representation theory of any such G; this will be an important theme later in the course.

To carry this out, consider the diagonal torus

$$T := \left\{ \lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in S^1 \right\} \subset \mathrm{SU}(2)$$

For $0 \leq j \leq n$, we have

$$\begin{aligned} \lambda(t).\left(z_1^j z_2^{n-j}\right) &= (tz_1)^j \left(t^{-1}z_2\right)^{n-j} \\ &= t^{2j-n}\left(z_1^j z_2^{n-j}\right). \end{aligned}$$

Therefore, each line $\mathbf{C}z_1^j z_2^{n-j}$ is a T-eigenline with weight $\chi_j(t) = t^{2j-n}$. That is, each of the lines are stable and given by a character. Further, these characters are pairwise distinct. Hence, these lines $\mathbf{C}z_1^j z_2^{n-j}$ are precisely the isotypic subspaces for the T-action on V_n.

Since the isotypic subspaces are 1-dimensional, it follows that any subrepresentation of V_n as a T-representation must be a direct sum among a subset of these lines. (The situation would not be as clean as this if some isotypic subspaces for the T-action on V_n had dimension > 1.) This applies in particular to any nonzero $\mathrm{SU}(2)$-subrepresentation W of V_n, and so *severely restricts* the possibilities for any such W. Let's check how another torus $T' \subset \mathrm{SU}(2)$ acts on these lines: consider

$$T' := \mathrm{SO}(2) = \left\{ r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in S^1 \right\} \subset \mathrm{SU}(2).$$

We have

$$\begin{aligned} r_\theta\left(z_1^j z_2^{n-j}\right) &= (\cos\theta z_1 + \sin\theta z_2)^j \left(-\sin\theta z_1 + \cos\theta z_2\right)^{n-j} \\ &= (\cos\theta)^j \left(-\sin\theta\right)^{n-j} z_1^n + (\text{other monomials}). \end{aligned}$$

Using generic θ to make the coefficient of z_1^n nonzero, we conclude that any nonzero $\mathrm{SU}(2)$-subrepresentation W (which we know has to be the span of *some* non-empty collection of the monomial basis lines, and hence contains $\mathbf{C}(z_1^j z_2^{n-j})$ for *some* $0 \leq j \leq n$) must contain $\mathbf{C}z_1^n$ because we have exhibited a vector in W with nonzero coefficient in the z_1^n-direction.

But now that W contains $\mathbf{C}z_1^n$ we conclude that W contains

$$r_\theta.z_1^n = (\cos\theta z_1 + \sin\theta z_2)^n,$$

which has nonzero coefficient on *all* monomials if θ is generic enough (to avoid the vanishing at θ of the various nonzero "trigonometric polynomial" coefficients of each $z_1^j z_2^{n-j}$ in $r_\theta.z_1^n$). Hence, $W = V_n$ as desired. $\square$

Theorem 9.5. *(1) The representations V_n are all the irreducible representations of* $\mathrm{SU}(2)$*. Likewise, the V_{2n} are all the irreducible representations of* $\mathrm{SO}(3)$*.*

(2) V_{2m} as an $\mathrm{SO}(3)$ *representation is the space of homogenous degree-m harmonic polynomials (i.e., those with $\Delta f = 0$) in $\{x_1, x_2, x_3\}$.*

The proof of this uses character theory. We now introduce the basic ideas in character theory (extending what is familiar from the case of finite groups), and the application of this to prove the above Theorem is developed in HW4. In all that we do for this topic, the only role of the "Lie" condition is that it makes the theory of Haar measure more tangible; granting the existence and uniqueness theorems on Haar measure (and continuity of the modulus character), all that we do applies without change to any compact Hausdorff topological group.

9.3. **Characters of compact Lie groups.** Let G be a compact Lie group and $\rho : G \to \mathrm{GL}(V)$ a representation. (Recall all representations are assumed to be $\mathbf{C}$-linear, finite-dimensional, and continuous.)

Definition 9.6. The *character* of (V, ρ) is

$$\chi_V(g) = \chi_\rho(g) := \mathrm{tr}\left(\rho(g)\right).$$

Remark 9.7. This is a class function and is C^∞ as a map $G \to \mathbf{C}$ because ρ is C^∞ (and hence the matrix entries of ρ are C^∞-functions on G). We will never need this; continuity of χ_V is entirely sufficient for our needs (to ensure various integrals below make sense without any complications).

Example 9.8. For V and V' two G representations, we get the representation $V \otimes V'$ via

$$g.\left(v \otimes v'\right) = \rho(g)(v) \otimes \rho'(g)(v').$$

Thus,

$$\begin{aligned} \chi_{V\otimes V'}(g) &= \mathrm{tr}\left(\rho(g) \otimes \rho'(g)\right) \\ &= \mathrm{tr}\left(\rho(g)\right) \cdot \mathrm{tr}\left(\rho'(g)\right) \\ &= \chi_V(g)\chi_{V'}(g), \end{aligned}$$

where the second equality uses that for $T : V \to V, T' : V' \to V'$,we have

$$\mathrm{tr}\left(T \otimes T'\right) = \mathrm{tr}\left(T\right) \cdot \mathrm{tr}(T')$$

(as can be seen by looking at eigenvalues upon upper-triangularizing the matrices for T and T').

Example 9.9. For a G-representation V, the dual V^* is a representation via

$$\left(g \cdot \ell\right)(v) = \ell(\rho(g^{-1})(v)).$$

(The use of g^{-1} ensures we have a left action.) Thus,

$$\chi_{V^*} = \mathrm{tr}(\rho(g^{-1})^*) = \mathrm{tr}(\rho(g^{-1})) = \overline{\mathrm{tr}(\rho(g))},$$

the last step using that the eigenvalues of $\rho(g)$ lie in S^1 (where $z^{-1} = \overline{z}$). To see this, we use that by compactness of $\rho(G)$ we have $\rho(G) \subset \mathrm{U}(h)$ for some positive-definite hermitian form h on V (and by definition of the hermitian condition, any eigenvector for an h-hermitian operator on V has eigenvalues in S^1). Alternatively, if the eigenvalue were not on the unit circle then its powers would become unbounded or approach 0, which would contradict compactness of $\rho(G) \subset \mathrm{GL}(V)$. Therefore, $\chi_{V^*} = \overline{\chi_V}$.

For two G-representations V and W, make $\mathrm{Hom}_{\mathbf{C}}\left(V, W\right)$ a G-representation via

$$g.T = \rho_W(g) \circ T \circ \rho_V(g)^{-1},$$

so $\mathrm{Hom}_{\mathbf{C}}\left(V, W\right)^G = \mathrm{Hom}_{\mathbf{C}[G]}(V, W)$. It is an instructive exercise to check (by computing on elementary tensors) that the natural isomorphism of $\mathbf{C}$-vector spaces

$$\mathrm{Hom}_{\mathbf{C}}\left(V, W\right) \simeq W \otimes V^*$$

is G-equivariant. Hence, $\chi_{\mathrm{Hom}_{\mathbf{C}}(V,W)} = \chi_W \cdot \overline{\chi_V}$. To exploit this, we'll average over G via the volume-1 measure dg.

The key observation is that for any G-representation V, the linear map

$$\begin{aligned}\Pi\colon V &\to V\\ v &\mapsto \int_G (\rho(g)(v))dg\end{aligned}$$

(where the vector-valued integral makes sense without complications since ρ is continuous, G is compact, and V is finite-dimensional) is a *projector* onto V^G: if $v \in V^G$ then $\Pi(v) = v$ because the integrand collapses to the constant function $g \mapsto v$ and $\int_G dg = 1$, and for any $v \in V$ we have $\Pi(v) \in V^G$ because for any $g_0 \in G$ we have

$$\begin{aligned}\rho(g_0)\left(\Pi(v)\right) &= \int_G \rho(g_0)\left(\rho(g)v\right) dg\\ &= \int_G \rho\left(g_0 g\right)(v)dg\\ &= \int_G \rho(g)(v)dg\\ &= \Pi(v),\end{aligned}$$

using left-invariance of dg for the third equality.

Since $\Pi : V \to V$ is given by the endomorphism-valued integral $\int_G \rho(g)dg$ and any linear operator passes through this integral (very easy to see upon choosing a $\mathbf{C}$-basis of V), applying the linear $\mathrm{tr} : \mathrm{End}(V) \to \mathbf{C}$ yields:

Corollary 9.10. *We have* $\int_G \chi_V(g)dg = \mathrm{tr}\,(\Pi) = \dim V^G$.

Proof. The first equality has been explained above, and the second is because Π is a projector onto V^G (compute its trace using a basis of V extending a basis of V^G). $\square$

Applying Corollary 9.10 to the G-representation $\mathrm{Hom}_{\mathbf{C}}(V,W)$ for G-representations V and W, we immediately obtain (using Schur's Lemma for the second assertion):

Proposition 9.11 (Orthogonality Relation)**.** *For any G-representations V and W,*

$$\int_G \chi_V(g)\overline{\chi_W}(g)dg = \dim \mathrm{Hom}_G\left(V,W\right).$$

In particular, for irreducible V and W this vanishes if $V \not\simeq W$ and is equal to 1 if $V \simeq W$.

The upshot is that the construction

$$\langle \chi_V, \chi_W\rangle := \int_G \chi_V(g)\overline{\chi_W}(g)dg$$

behaves like the analogous expression for finite groups using averaging (which the preceding literally recovers when a finite group is viewed as a compact Lie group!).

Corollary 9.12. *The character χ_V determines the isomorphism class of a G-representation V, and*

$$\langle \chi_V, \chi_V\rangle = 1$$

if and only if V is irreducible.

Proof. See Appendix I for the proof, which is literally the same as in the case of finite groups (due to the "orthogonality relation" established above). $\square$

10. Weyl's Unitary Trick

Today we will discuss Weyl's unitary trick. This is a method to use compact Lie groups to control the representation theory of non-compact Lie groups via relating their Lie algebras. As an example, we will be able to use $\mathrm{SU}(n)$ to study suitable representations of $\mathrm{SL}_n(\mathbf{C})$. Most of what we discuss today is being said just for general cultural awareness of what is "out there"; none of this will be used later in the course except for the discussion of the representation theory of $\mathfrak{sl}_2(\mathbf{C})$ which we'll come to near the end of today. In particular, apart from this final topic, we will say very little about the details of proofs, just some highlights.

10.1. **Warm-up to complex structure.** At the end of §H.4, we have:

Theorem 10.1. *If G, G' are Lie groups with G connected and $\pi_1(G) = 1$ (i.e., G is simply connected), then*
$$\mathrm{Hom}_{\mathrm{Lie}}(G, G') \simeq \mathrm{Hom}_{\mathbf{R}\text{-}\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}').$$

For example, the verification of the triviality of $\pi_1(G)$ for $G = \mathrm{SU}(n), \mathrm{Sp}(n)$ is treated in HW5.

Example 10.2. Take $G' = \mathrm{GL}_n(\mathbf{F})$ for $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. Then for G as in Theorem 10.1, a continuous homomorphism $\rho : G \to G'$ is the "same" as a map of Lie algebras $L\rho : \mathfrak{g} \to \mathfrak{g}' = \mathrm{Mat}_n(\mathbf{F})$ over $\mathbf{R}$. Concretely, the map $L\rho$ is given by
$$\begin{aligned} L\rho \colon \mathfrak{g} &\to \mathfrak{g}' \\ v &\mapsto \frac{d}{dt}|_{t=0} \left(\rho\left(\exp_G(tv)\right)\right). \end{aligned}$$

Corollary 10.3. *Suppose G is connected and simply connected. Passing to Lie algebras defines a bijection of sets of isomorphism classes:*

$$\begin{array}{c} \{n\text{-dimensional representations of } G \text{ over } \mathbf{C} \text{ up to isomorphism}\} \\ \big\downarrow \\ \{n\text{-dimensional representations of the real Lie algebra } \mathfrak{g} \text{ over } \mathbf{C}\} \end{array} \tag{10.1}$$

Note that n-dimensional representations of the real Lie algebra $\mathfrak{g}$ can be though of as suitable $\mathbf{R}$-bilinear pairings $\mathfrak{g} \times V \to V$ for an n-dimensional $\mathbf{C}$-vector space V (carrying the Lie bracket on $\mathfrak{g}$ over to the commutator of $\mathbf{C}$-linear endomorphisms of V).

This can also be thought of as V being given a structure of left module over the *universal enveloping algebra*
$$U(\mathfrak{g}) = T(\mathfrak{g}) / \langle X \otimes Y - Y \otimes X - [X, Y] \rangle$$
where $T(W)$ is the tensor algebra $\oplus_{m \geq 0} W^{\otimes m}$ for a vector space W and we form the quotient by the 2-sided ideal generated by the indicated relations. In the purely algebraic theory of Lie algebras, one can say quite a bit about the structure of $U(\mathfrak{g})$ when $\mathfrak{g}$ has favorable properties (such as being "semisimple" over a field of characteristic 0).

An important refinement of the preceding considerations is to consider G a complex Lie group and ρ a finite-dimensional holomorphic representation (i.e., $\rho : G \to \mathrm{GL}(V)$ is a map of complex Lie groups, a much stronger condition that being a map of C^∞ real manifolds). We can ask: is there a bijection like Theorem 10.1 in the holomorphic setting? Yes!

The key point is that Frobenius theorems still hold in the complex-analytic setting without redoing the proofs: the crux is that if M is a complex manifold and $N \subset M$ is a closed C^∞-submanifold then N is a complex submanifold if and only if for all $n \in N$ the $\mathbf{R}$-subspace

$$T_n(N) \subset T_n(M)$$

is a $\mathbf{C}$-subspace. Using this, one can show the following:

Fact 10.4. In the setup of Theorem 10.1 (so G is connected and simply connected) with G and G' both complex Lie groups, the natural map

$$\mathrm{Hom}_{\mathrm{hol}}(G, G') \to \mathrm{Hom}_{\mathbf{C}\text{-}\mathrm{Lie}}\left(\mathfrak{g}, \mathfrak{g}'\right)$$

is bijective.

10.2. **Algebraicity and holomorphicity.** We now take a brief digression to explain a fact which works over $\mathbf{C}$, but completely fails over $\mathbf{R}$ (roughly due to the additional subtleties of the links between analytic and algebraic geometry over $\mathbf{R}$ as opposed to over $\mathbf{C}$). We first define unipotence. There are many equivalent definitions. Here is one:

Definition 10.5. An $n \times n$ matrix over a field is *unipotent* if its eigenvalues all equal 1.

Miracle 10.6. Suppose G and G' are respectively Zariski-closed over $\mathbf{C}$ in $\mathrm{GL}(V)$ and $\mathrm{GL}(V')$ for finite-dimensional $\mathbf{C}$-vector spaces V, V' such that G is "reductive": has no non-trivial Zariski-closed normal subgroup all of whose elements are unipotent (there are more elegant ways to think about this which make it look less ad hoc).

The miracle is that all holomorphic homomorphisms $G \to G'$ are necessarily algebraic; i.e., they are given by everywhere-defined rational functions in the matrix entries. For example, the reductivity condition holds for $G = \mathrm{SL}_n(\mathbf{C}), \mathrm{Sp}_{2n}(\mathbf{C})$ (and many many more examples which we don't have time to discuss). This is a result in the spirit of Serre's GAGA theorem for projective varieties over $\mathbf{C}$, but it has nothing whatsoever to do with those results; it is very specific to the "affine group variety" setting over $\mathbf{C}$.

Remark 10.7. The avoidance of unipotent normal subgroups (i.e., the reductivity condition) in Miracle 10.6 is crucial. For example, it rules out the holomorphic homomorphism

$$\exp\colon \mathbf{C} \to \mathbf{C}^\times$$
$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mapsto e^z$$

that is visibly not "algebraic" as a map from the affine line to the affine curve GL_1.

Warning 10.8. Beware that Miracle 10.6 fails over $\mathbf{R}$! (The proof of Miracle 10.6 uses in a crucial way that algebraic geometry over $\mathbf{C}$ is in certain senses better-behaved than algebraic geometry over $\mathbf{R}$.) For example, one can exhibit "PGL_3" as an affine group variety over $\mathbf{R}$ yet the inverse to the real-analytic isomorphism $\mathrm{SL}_3(\mathbf{R}) \to \mathrm{PGL}_3(\mathbf{R})$ (the real-analyticity of cube root on $\mathbf{R}^\times$ has no analogue on the entirety of $\mathbf{C}^\times$) gives rise to a real-analytic representation $\mathrm{PGL}_3(\mathbf{R}) \simeq \mathrm{SL}_3(\mathbf{R}) \to \mathrm{GL}_3(\mathbf{R})$ that can be shown to *not* arise from an "algebraic" map $\mathrm{PGL}_3 \to \mathrm{GL}_3$ (in the sense of algebraic geometry over $\mathbf{R}$).

10.3. **Using Weyl's unitary trick to understand representations of $\mathrm{SL}_n(\mathbf{C})$.** By definition we have $\mathrm{SU}(n) \subset \mathrm{SL}_n(\mathbf{C})$, so naturally $\mathfrak{su}(n) \subset \mathfrak{sl}_n(\mathbf{C})$ as Lie algebras over $\mathbf{R}$. As shown in HW4 in the case $n = 2$ and will be shown later in general, the induced map

$$\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{su}(n) \to \mathfrak{sl}_n(\mathbf{C})$$

of Lie algebras over $\mathbf{C}$ is an isomorphism. (Concretely, an $\mathbf{R}$-basis of $\mathfrak{su}(n)$ is a $\mathbf{C}$-basis of $\mathfrak{sl}_n(\mathbf{C})$.) Therefore, at the level of categories of finite-dimensional $\mathbf{C}$-linear representations,

$$\begin{aligned} \mathrm{Rep}^{\mathbf{C}}(\mathrm{SU}(n)) &= \mathrm{Rep}^{\mathbf{C}}_{\mathbf{R}\text{-}\mathrm{Lie}}(\mathfrak{su}(n)) \\ &= \mathrm{Rep}^{\mathbf{C}}_{\mathbf{C}\text{-}\mathrm{Lie}}(\mathfrak{sl}_n(\mathbf{C})) \\ &= \mathrm{Rep}_{\mathrm{hol}}(\mathrm{SL}_n(\mathbf{C})) \end{aligned}$$

(using that $\pi_1(\mathrm{SL}_n(\mathbf{C})) = 1$ and $\pi_1(\mathrm{SU}(n)) = 1$). But $\mathrm{SU}(n)$ is *compact*, so the objects in the source category are completely reducible, and hence the finite-dimensional holomorphic representations of $\mathrm{SL}_n(\mathbf{C})$ are also completely irreducible.

Such representations of $\mathrm{SL}_n(\mathbf{C})$ can be classified by thinking in terms of the representation theory of $\mathfrak{sl}_n(\mathbf{C})$. The latter task is a special case of the general theory of semisimple Lie algebras (over a field of characteristic 0), for which the representation theory is best understood using root systems (a purely combinatorial structure that we will introduce later as a powerful tool in the structure theory of connected compact Lie groups). Let's now focus on understanding the representations of $\mathfrak{sl}_2(\mathbf{C})$, since this will be important in our subsequent work: remarkably, in the general structure theory of connected compact Lie groups the good properties of $\mathfrak{sl}_2(\mathbf{C}) = \mathfrak{su}(2)_{\mathbf{C}}$ play a crucial role.

10.4. **Representations of $\mathfrak{sl}_2(\mathbf{C})$ via algebra.** We'll only need the following discussion over $\mathbf{C}$, but we formulate it over a general field k of characteristic 0 for conceptual clarity. Let

$$\mathfrak{sl}_2(k) = \{M \in \mathrm{Mat}_2(k) : \mathrm{tr}(M) = 0\},$$

so we have the explicit k-basis

$$\mathfrak{sl}_2(k) = kX^- \oplus kH \oplus X^+$$

for

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

These satisfy the relations

$$[H, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = H.$$

We want to describe the finite-dimensional representations over k of $\mathfrak{sl}_2(k)$.

Warning 10.9. There are infinite-dimensional irreducible represenentations of $\mathfrak{sl}_2(k)$ over k. We will not discuss these.

In Appendix J, purely algebraic methods establish the following remarkable result:

Theorem 10.10. *Let V be a nonzero finite-dimensional $\mathfrak{sl}_2$ representation over k. The action of X^+ on V is nilpotent and $\ker(X^+)$ contains an H-eigenvector v_0. The H-eigenvalue on any such v_0 is a non-negative integer $m \geq 0$, and the subrepresentation $W := \mathfrak{sl}_2 \cdot v_0 \subset V$ satisfies the following conditions:*

$$
\begin{tikzcd}
kv_m \arrow[r, bend left, "X^+"] & kv_{m-1} \arrow[l, bend left, "X^-"] \arrow[r, bend left, "X^+"] & kv_{m-2} \arrow[l, bend left, "X^-"] \arrow[r, bend left, "X^+"] & \cdots \arrow[l, bend left, "X^-"] \arrow[r, bend left, "X^+"] & kv_0 \arrow[l, bend left, "X^-"]
\end{tikzcd}
\tag{10.2}
$$

FIGURE 1. The H-eigenspaces with the action of the corresponding raising and lowering operators

(1) *The H-action is semisimple (i.e., diagonalizable) on W, with $\dim W = m+1$ and with H-eigenspaces that are lines with respective eigenvalues*
$$m, m-2, m-4, \ldots, m-2j, \ldots, -m.$$
(2) $\ker(X^+|_W) = kv_0$. *Explicitly, the vector*
$$v_j := \frac{1}{j!}\left(X^-\right)^j(v_0)$$
spans the H-eigenline for eigenvalue $m-2j$ with $0 \leq j \leq m$. Moreover,
$$X^-(v_j) = (j+1)\,v_{j+1}, \;\; X^+(v_j) = (m-j+1)\,v_{j-1}, \;\; H(v_j) = (m-2j)\,v_j$$
(with the convention $v_{-1} = 0, v_{m+1} = 0$), so the isomorphism class of W is described entirely in terms of $m = \dim(W) - 1$ (no free parameters!).

(3) *The $\mathfrak{sl}_2$-representation W is absolutely irreducible, it is the unique one with dimension $m+1$, and for all integers $m \geq 0$ such a W exists: the mth symmetric power of the standard 2-dimensional representation of $\mathfrak{sl}_2(k)$. In particular, these are exactly the irreducible representations of $\mathfrak{sl}_2$ over k.*

Remark 10.11. Note that the eigenspace kv_0 for $X^+|_W$ as in the statement of Theorem 10.10 is 1-dimensional and has H-eigenvalue m strictly large than all others. We call m the *highest H-weight in W*, and call v_0 a "highest-weight vector" in W.

Remark 10.12. In Appendix J, the above list of *all* irreducible finite-dimensional $\mathfrak{sl}_2(k)$-representations is used to prove (via Ext's over a universal enveloping algebra) that the representation theory over k of $\mathfrak{sl}_2(k)$ is completely reducible! We know the complete reducibility for $k = \mathbf{C}$ by topological/analytic means via the unitary trick, but for later purposes it will be very important that we have found *all* the irreducible finite-dimensional representations of $\mathfrak{sl}_2(\mathbf{C})$; this in turn is also established via the unitary trick in HW4 by means of SU(2), but it is interesting that Appendix J gives the same result by purely algebraic means over any field of characteristic 0.

For our purposes the work in Appendix J should be regarded as an instructive warm-up in hands-on work with the Lie algebra $\mathfrak{sl}_2$. In this way one sees that the general development of the structure theory of compact Lie groups later in the course does not logically depend on the unitary trick.

11. WEYL'S CONJUGACY THEOREM: APPLICATIONS

We'll start by stating the Conjugacy Theorem and spend the rest of today deducing some applications which will have tremendous importance throughout the rest of the course.

These will also provide a compelling reason to care about the Conjugacy Theorem, whose proof will require a couple of lectures.

Theorem 11.1 (Weyl's Conjugacy Theorem)**.** *Let G be a connected compact Lie group.*

(1) All maximal tori in G (i.e., tori not contained in strictly bigger tori in G) are conjugate to each other.
(2) Every $g \in G$ lies in a maximal torus in G.

Remark 11.2. Here are some remarks on the theorem.

(1) Connectedness of G is crucial.
(2) For connectedness and dimension reasons, every torus in G is obviously contained in a maximal one.
(3) We showed the second part of the Conjugacy Theorem for the special case $G = \mathrm{SU}(n)$ in HW4.

Example 11.3. The torus

$$T := \left\{ \begin{pmatrix} e^{2\pi i\theta_1} & & \\ & \ddots & \\ & & e^{2\pi i\theta_n} \end{pmatrix} \right\} \subset \mathrm{SU}(n) \subset \mathrm{GL}_n(\mathbf{C})$$

in $G = \mathrm{SU}(n)$ is its own centralizer, as follows from a direct computation by considering preservation of T-weight spaces in $\mathbf{C}^n$ (i.e., any $g \in G$ that centralizes T must act on $\mathbf{C}^n$ in a way that preserves the isotypic decomposition under T, and those isotypic subspaces are the standard basis lines which have pairwise *distinct* T-weights). Thus, T is maximal as a torus in $\mathrm{U}(n)$ (since by commutativity of tori, T is centralized by any torus containing T).

The same works for the torus

$$T' = \left\{ \begin{pmatrix} r_{2\pi\theta_1} & & \\ & \ddots & \\ & & r^{2\pi\theta_n} \end{pmatrix} \right\} \subset \mathrm{SO}(2n) \subset \mathrm{GL}_{2n}(\mathbf{C})$$

in $H = \mathrm{SO}(2n)$ using that the evident 2-dimensional subspace $\mathbf{R}e_{2j-1} \oplus \mathbf{R}e_{2j}$ are the T'-isotypic subspaces of $\mathbf{R}^{2n}$. (We are using the complete reducibility of continuous T'-representations on finite-dimensional $\mathbf{R}$-vector spaces, which is proved via averaging just as over $\mathbf{C}$.) Here $r_{2\pi\theta}$ denotes the usual 2×2 rotation matrix through an angle $2\pi\theta$.

Remark 11.4. The proof of the Conjugacy Theorem will use G-invariant integration on the manifold G/T. Next time we'll discuss $\int_{G/H}$ for Lie groups G and closed subgroups H, which will be developed in HW5. Beware that for H disconnected or non-compact, G/H can be non-orientable and in fact can *fail* to have a G-invariant nonzero regular Borel measure (we'll see examples next time).

Fortunately, such difficulties will turn out not to occur for compact connected H (such as H a compact torus in the context of intended application for the proof of the Conjugacy Theorem). This will be ultimately because the only continuous homomorphism into $\mathbf{R}^\times$ from a connected compact group is the trivial homomorphism.

For the rest of today we take G to be a connected compact Lie group, accept the Conjugacy Theorem as true, and deduce some consequences. We saw above in some examples that a useful way to prove that a given torus in G is maximal is to show that the torus is its own centralizer. The first part of the first corollary below shows that this technique is not only sufficient but in some sense also necessary.

Corollary 11.5. *(1) A torus $T \subset G$ is maximal if and only if $T = Z_G(T)$.*
(2) $Z_G = \bigcap_{\text{max'l } T} T$.

Warning 11.6. Note that Z_G may be disconnected in Corollary 11.5(2). (For example, if $G = \mathrm{SU}(n)$ then Z_G is the diagonally embedded finite group μ_n of nth roots of unity.)

Proof of Corollary 11.5. For proving (2), choose $z \in Z_G$ and pick a maximal torus $T_0 \ni z_0$. Any maximal T has the form gT_0g^{-1} for some $g \in G$ by the Conjugacy Theorem, so $T \ni gzg^{-1} = z$.

For the reverse inclusion, pick any $g \in \bigcap_{\text{max'l } T} T$. For any $g_0 \in G$, we have $g_0 \in T_0$ for T_0 some maximal torus (by the Conjugacy Theorem). But $g \in T_0$ by the hypothesis on g and T_0 is commutative, so g and g_0 commute. Since $g_0 \in G$ was arbitrary, this implies $g \in Z_G$. This completes the proof of (2) in general.

For the proof of (1), one implication is obvious: if $T = Z_G(T)$ then T is a maximal torus because tori are commutative. To prove the converse, we assume T is maximal, pick $g \in Z_G(T)$, and aim to show that $g \in T$. Since T is commutative, $T \subset Z_G(g)$. But T is also *connected*, so $T \subset Z_G(g)^0$. Observe that $Z_G(g)^0$ is a connected compact Lie group! Further, T is a maximal torus in $Z_G(g)^0$ because T is maximal in G. If we knew $g \in Z_G(g)^0$, then by centrality of g in $Z_G(g)^0$, we could apply Corollary 11.5(2) to $Z_G(g)^0$ (!) to conclude $g \in T$ as desired.

So, it suffices to show $g \in Z_G(g)^0$. By the Conjugacy Theorem we know g lies in *some* maximal torus T'. But T' is commutative and connected, so $g \in T' \subset Z_G(g)^0$. □

Remark 11.7. In the proof of Corollary 11.5(1), the really key point was to show $g \in Z_G(g)^0$, and more specifically to find *some* connected closed subgroup containing g in its center (as such a subgroup lies in $Z_G(g)^0$). One might hope that the connected commutative closed subgroup $\overline{\langle g \rangle}^0$ (where $\langle g \rangle := g^{\mathbf{Z}}$) does the job, but it can happen that $g \notin \overline{\langle g \rangle}^0$, such as for g with finite order (which exist in abundance in any torus, and hence in any G by the Conjugacy Theorem). A more interesting example with $g \notin \overline{\langle g \rangle}^0$ is

$$g = (-1, \zeta) \in S^1 \times S^1$$

with ζ not of finite order; in this case $\overline{\langle g \rangle}^0 = \{1\} \times S^1$ clearly does not contain g.

Remark 11.8. It can really happen that $Z_G(g)$ is disconnected (so the trivial inclusion $g \in Z_G(g)$ does not give us "for free" that $g \in Z_G(g)^0$). For example, take $G = \mathrm{SO}(3)$ and

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In $\mathbf{R}^3$ the eigenspaces for g are $\mathbf{R}\,(e_1 + e_2)$ with eigenvalue 1 and

$$\mathbf{R}\,(e_1 - e_2) \oplus \mathbf{R}e_3$$

with eigenvalue -1. One can then check via direct consideration of preservation of such eigenspaces that

$$Z_G(g) \simeq \mathrm{O}(2)$$

for $\mathrm{O}(2)$ arising from the 2-dimensional vector space with basis $\{e_1 - e_2, e_3\}$ and acting on the 1-eigenline $\mathbf{R}(e_1 + e_2)$ via $\det^{\pm 1}$. That is, for the ordered basis $\{e_1 + e_2, e_1 - e_2, e_3\}$ we have

$$Z_G(g) = \begin{pmatrix} \det^{\pm 1} & 0 \\ 0 & \mathrm{O}(2) \end{pmatrix},$$

and this is visibly disconnected.

Remark 11.9. For $g \notin Z_G$ we know that $g \in Z_G(g)^0$ yet $Z_G(g)^0 \neq G$ (since g is assumed to be non-central in G), so $\dim Z_G(g)^0 < \dim G$. This is sometimes useful for dimension induction arguments with connected compact Lie groups (to pass to a situation where an element becomes central).

The next corollary has a level of importance that is impossible to over-estimate, and by the end of this course you will appreciate how awesomely powerful it is.

Corollary 11.10. *For any torus $S \subset G$, the group $Z_G(S)$ is connected. In particular, if $S \neq 1$ then $Z_G(S)/S$ is a connected compact Lie group with $\dim Z_G(S)/S < \dim G$.*

Proof. Pick $g \in Z_G(S)$. We know $g \in Z_G(g)^0$, and clearly $S \subset Z_G(g)$, so $S \subset Z_G(g)^0$ because S is connected. Pick a maximal torus $S' \subset Z_G(g)^0$ containing S. By Corollary 11.5 applied to $Z_G(g)^0$ and its central element g, we know that $g \in S'$! But $S' \subset Z_G(S)$ since S' is commutative and $S' \supset S$, so $S' \subset Z_G(S)^0$ because S' is connected. Thus, $g \in S' \subset Z_G(S)^0$. ☐

Remark 11.11. The real significance of Corollary 11.10 arises for non-central tori $S \subset G$, because then $Z_G(S)$ is a lower-dimensional *connected* compact Lie group. The Conjugacy Theorem provides lots of such S when G is non-commutative (which is equivalent to G not being a torus because G is compact and connected), since G contains tori through all elements. Many future dimension induction arguments will involve such passage to torus centralizers (without losing contact with the *connectedness* condition!).

Consider a surjective homomorphism of connected compact Lie groups $f : G \to \overline{G}$. For a torus $S \subset G$ and $\overline{S} := f(S) \subset \overline{G}$, note that $\overline{S}$ is a connected commutative compact subgroup of $\overline{G}$, so a Lie subgroup by closedness and thus a torus (as the only connected commutative compact Lie groups are tori). The following really fanstastic result has no analogue whatsoever in the setting of finite groups; it is a remarkable (and remarkably useful) result that hinges in an essential way on the connectedness of the groups under consideration.

Corollary 11.12. *In the above setup, the natural map*

$$\pi : Z_G(S) \to Z_{\overline{G}}(\overline{S})$$

is surjective.

Proof. We know that $\overline{S}$ is a torus, so by Corollary 11.10 the compact Lie groups $Z_G(S)$ and $Z_{\overline{G}}(\overline{S})$ are connected. Therefore, it is enough to show Lie π is surjective as a map between the corresponding Lie algebras. Indeed, if a map between Lie groups is surjective on Lie algebras then it is a submersion near the identity, hence open near the identity and so globally open (via translation arguments). The image is therefore open, yet an open subgroup of a topological group is always closed and so must be the entire group in the connected case. Thus a map of Lie groups having *connected* target is surjective when it is surjective between Lie algebras. It is therefore sufficient to show Lie π is surjective since $Z_{\overline{G}}(\overline{S})$ is connected.

By Corollary H.4.6, for any Lie group H and closed subgroup $H' \subset H$, we have

$$\mathrm{Lie}\left(Z_H(H')\right) = \mathfrak{h}^{H'}$$

inside $\mathfrak{h}$. Thus, $\mathrm{Lie}(\pi)$ is identified with the natural map

$$\mathfrak{g}^S \to \overline{\mathfrak{g}}^{\overline{S}}$$

induced by the map $\mathfrak{g} \to \overline{\mathfrak{g}}$. By Exercise 5(iii) in HW3 we know that a surjective homomorphism between Lie groups is always surjective between Lie algebras (the proof used Sard's Theorem), so $\mathfrak{g} \to \overline{\mathfrak{g}}$ is surjective. Viewing $\overline{\mathfrak{g}}$ as an S-representation (over $\mathbf{R}$) via $S \to \overline{S}$, it therefore suffices to show that for any *surjection* $V \to \overline{V}$ of (finite-dimensional continuous) S-representations over $\mathbf{R}$, the induced map

$$V^S \to \overline{V}^S$$

is surjective.

Applying $\mathbf{C} \otimes_{\mathbf{R}} (\cdot)$ and using that $(V^\Gamma)_{\mathbf{C}} = (V_{\mathbf{C}})^\Gamma$ for any group Γ acting linearly on any $\mathbf{R}$-vector space, it suffices to show the natural map

$$(V_{\mathbf{C}})^S \to (V'_{\mathbf{C}})^S$$

is surjective. But this follows by considering the weight-space decompositions for $V_{\mathbf{C}}$ and $V'_{\mathbf{C}}$ as $\mathbf{C}$-linear representations of S (continuous and finite-dimensional, as always); i.e., this follows from complete reducibility for torus representations over $\mathbf{C}$ (much as the same holds for isotypic components for any fixed irreducible representation for a finite group; the subspace of invariant vectors is the isotypic component for the trivial representation). □

There is actually no need to extend scalars to $\mathbf{C}$ at the end of the preceding proof since complete reducibility holds just as well over $\mathbf{R}$. However, the device of passing to the complexified Lie algebras will be an essential tool in many later considerations, so it is instructive to see it emerge here too (at the very least it simplifies the nature of the torus representations, even though that isn't strictly necessary in the present circumstances).

12. Integration on Coset Spaces

12.1. **Weyl Groups.** We'll start by filling out a loose end on the Conjugacy Theorem. Consider a connected compact Lie group G and a torus $S \subset G$.

Definition 12.1. The *Weyl group* of (G, S) is

$$W(G,S) := N_G(S)/Z_G(S) \hookrightarrow \mathrm{Aut}(S)$$
$$n \mapsto \left(s \mapsto nsn^{-1}\right)$$

Remark 12.2. Once we have proven the Conjugacy Theorem, for maximal T we'll have $W(G,T) = N_G(T)/T$.

Example 12.3. Choose $n \geq 2$. Consider $G = \mathrm{U}(n)$ and T the diagonal maximal torus

$$\left\{ \begin{pmatrix} e^{2\pi i\theta_1} & & \\ & \ddots & \\ & & e^{2\pi i\theta_n} \end{pmatrix} \right\}$$

Thinking about the T-weight spaces in $\mathbf{C}^n$, we see that

$$N_G(T) = \mathfrak{S}_n \rtimes T$$

where $\mathfrak{S}_n$ is the symmetric group of $n \times n$ permutation matrices. The reason is as follows. On the one hand, clearly $\mathfrak{S}_n \subset N_G(T)$ (it permutes the standard basis vectors). But any element of the normalizer *permutes* the T-weight spaces (being the isotypic subspaces for the T-action on $\mathbf{C}^n$) which are the standard basis lines and the $\mathfrak{S}_n$-subgroup achieves each such permutation exactly once. So on the other hand, for any $n \in N_G(T)$ there is a unique $\sigma \in \mathfrak{S}_n$ such that the product $\sigma^{-1}n \in N_G(T)$ has permutation effect that is trivial (and hence lies in the diagonal subgroup of $\mathrm{U}(n)$ that in turn is *exactly* T; check!). Thus,

$$W(G,T) = \mathfrak{S}_n.$$

Remark 12.4. In Appendix K, the following are proved:

(1) For $n \geq 2$ and $G' := \mathrm{SU}(n)$ with maximal diagonal torus T', once again $W(G', T') = \mathfrak{S}_n$ but it is not possible to lift this Weyl group isomorphically into $N_{G'}(T')$. For more details, see §K.2.

(2) Granting the Conjugacy Theorem, so $Z_G(S)$ is connected, $W(G,S)$ is finite if and only if $N_G(S)^0 = Z_G(S)$. In Proposition K.4.1, the finiteness of $W(G,S)$ is directly proved (not conditional on anything). This will be *used* for maximal S in the proof of the Conjugacy Theorem.

Remark 12.5. Why intuitively should the Weyl group be finite? The idea of the finitness of $W(G,S) = N_G(S)/Z_G(S) \subset \mathrm{Aut}(S)$ is as follows. There is an anti-equivalence of categories

$$\{\text{tori}\} \leftrightarrow \{\text{finite free } \mathbf{Z}\text{-modules}\}$$
$$S \mapsto X(S) = \mathrm{Hom}(S, S^1).$$

Since $\mathrm{Hom}(S, S^1) \simeq \mathbf{Z}^r$ for $S = \left(S^1\right)^r$, $\mathrm{Aut}(S) \simeq \mathrm{Aut}\left(X(S)\right)^{\mathrm{opp}} \simeq \mathrm{GL}_r(\mathbf{Z})$ is naturally "discrete". But its subgroup $N_G(S)/Z_G(S)$ is naturally compact, and a compact discrete space is finite!

Of course, this is not actually a proof. The issue is that we have not established a rigorous link between the compact topology of $N_G(S)$ and the ad-hoc discrete topology on $\mathrm{GL}_r(\mathbf{Z})$. The details for such a justification are given in the proof of finiteness of $W(G,S)$.

12.2. **Invariant integration on** G/H**.** We'll now develop the notion of invariant integration on coset spaces for Lie groups, which we will need for the proof of the Conjugacy Theorem.

Let G be a Lie group (we are not assuming G is compact) and let $H \subset G$ be a closed subgroup. Then, G/H is a C^∞ manifold (recall that G/H has a unique C^∞-structure making $G \to G/H$ a submersion yielding the expected tangent space at the identity coset). Let μ_G and μ_H be left Haar measures on G and H respectively.

Example 12.6. We can take

$$\mu_G(A) = \int_A |\omega|$$
$$\mu_H(B) = \int_B |\eta|$$

for Borel sets $A \subset G$ and $B \subset H$ where ω and η are top-degree nonzero left-invariant differential forms on G and H respectively.

Question 12.7. Does there exist a nonzero regular Borel measure $\overline{\mu}$ on G/H that is G-invariant (using the natural left-translation action of G on G/H)?

Even if this $\overline{\mu}$ exists, in order for it to be useful for computing integrals we need a "Fubini property", stated below in (12.1). For $f \in C_c(G)$, define the function

$$\begin{aligned} \overline{f} \colon G/H &\to \mathbf{C} \\ \overline{g} &\mapsto \int_H f(gh)\, d\mu_H(h), \end{aligned}$$

where g is a representative for $\overline{g}$ (the choice of which doesn't matter: if we change the choice of lift g of $\overline{g}$ then it amounts to right-multiplying g by some $h_0 \in H$, but μ_H is invariant under $h \mapsto h_0 h$). In HW5 it is shown that $\overline{f} \in C_c(G/H)$. The desired Fubini property is that for all such f we have:

$$\int_G f(g) d\mu_G(g) = \int_{G/H} \overline{f}(\overline{h}) d\overline{\mu}(\overline{h}). \tag{12.1}$$

In favorable situations such a Borel measure $\overline{\mu}$ exists with uniqueness properties similar to that of a Haar measure:

Fact 12.8.
1. By [Lang, Ch. XII, Theorem 4.3], if a G-invariant $\overline{\mu}$ exists then after unique $\mathbf{R}_{>0}$-scaling the Fubini relation (12.1) holds for all $f \in C_c(G)$. By [Lang, Ch. XII, Theorem 4.1], the map
$$\begin{aligned} C_c(G) &\to C_c(G/H) \\ f &\mapsto \overline{f} \end{aligned}$$
is surjective, so by the Riesz representation theorem the Fubini relation uniquely determines $\overline{\mu}$ given the choices of μ_G and μ_H (as μ_G controls the left side of (12.1) and μ_H controls the definition of $f \mapsto \overline{f}$). In particular, this gives a unique characterization of $\overline{\mu}$ without direct reference to the G-invariance property.
2. By [Nach, p. 138], $\overline{\mu}$ exists if and only if $\Delta_G|_H = \Delta_H$.

With more work, one can even prove an L^1 version of (12.1), going beyond the case of compactly-supported continuous functions. We won't need this in what follows, so we leave it as an exercise for the reader who wishes to revisit their experience of learning the proof of the L^1-version of the usual Fubini theorem for products of measure spaces.

We'll try to make a left G-invariant nonzero top-degree differential form $\overline{\omega}$ on G/H, and then $\int |\overline{\omega}|$ yields such a $\overline{\mu}$ (by HW5, Exercise 2(iv)). This is *stronger* than the existence of $\overline{\mu}$, since the measure involves slightly less information than the invariant differential form (roughly speaking, it discards orientation issues).

Sometimes there may not exist any $\overline{\mu}$:

Example 12.9. Take $G = \mathrm{GL}_2(\mathbf{R})$ and $H \subset G$ the "$ax + b$ group"

$$H = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}$$

Let $G' = \mathrm{SL}_2(\mathbf{R})$ and

$$H' = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} = H \cap G'.$$

The evident natural C^∞ map

$$G'/H' \to G/H$$

is easily seen to be bijective and even an isomorphism on tangent spaces at the identity points, hence (via translations) a diffeomorphism.

By Fact 12.8(2), a G-invariant $\overline{\mu}$ exists since $\Delta_{G'} = 1$ (HW4, Exercise 2(i)) and $\Delta_{H'} = 1$ (as H' is commutative). But $\Delta_G = 1$ (HW4, Exercise 2(i)) and $\Delta_H \neq 1$ (HW4, Exercise 2(ii)). Therefore, no $\overline{\mu}$ exists by Fact 12.8(2). Concretely, this amounts to checking if the left action of

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

preserves $\overline{\mu}'$, but a tangent space calculation at the identity coset shows that it scales $\overline{\mu}'$ by $1/|t|$.

Further, non-orientability of G/H obstructs the existence of $\overline{\omega}$:

Example 12.10. Let $n \geq 2$ and let

$$G = \mathrm{SO}(n) \supset H := \begin{pmatrix} \mathrm{O}(n-1) & 0 \\ 0 & \det^{-1} \end{pmatrix} \supset \begin{pmatrix} \mathrm{SO}(n-1) & \\ & 1 \end{pmatrix},$$

so $H \simeq \mathrm{O}(n-1)$ is disconnected. We find

$$\mathbf{Z}/2\mathbf{Z} = H/\mathrm{SO}(n-1) \subset \mathrm{SO}(n)/\mathrm{SO}(n-1) \simeq S^{n-1} \subset \mathbf{R}^n$$

where the isomorphism is given by $g \mapsto g(e_n)$. The left action of $\mathbf{Z}/2\mathbf{Z}$ on S^{n-1} is the antipodal map, so $\mathrm{SO}(n)/H \simeq \mathbf{RP}^{n-1}$. This is non-orientable for odd $n \geq 3$, so for such n there is no nonzero left-invariant top-degree form on G/H (since it would have to be everywhere non-vanishing, contradicting non-orientability of G/H).

Now, pick left-invariant nonzero top-degree ω on G and η on H. We want to find a preferred $\overline{\omega}(\overline{e}) \in \det\left(T_{\overline{e}}\left(G/H\right)^*\right)$ which consistently translates around to give a G-invariant $\overline{\omega}$ on G/H. We will use the elements

$$\omega(e) \in \det\left(T_e(G)^*\right), \eta(e) \in \det\left(T_e(H)^*\right)$$

to define $\overline{\omega}(\overline{e})$.

The natural map

$$\begin{aligned} G &\to G/H \\ e &\mapsto \overline{e} \end{aligned}$$

is a submersion whose $\overline{e}$-fiber is $H \subset G$. More generally, consider a submersion of manifolds

$$\begin{aligned} X &\to Y \\ x &\mapsto y \end{aligned}$$

so the fiber $X_y := \pi^{-1}(y)$ is a closed C^∞-submanifold of X. We have an exact sequence of vector spaces

$$0 \longrightarrow T_x(X_y) \longrightarrow T_x(X) \longrightarrow T_y(Y) \longrightarrow 0. \tag{12.2}$$

As shown on HW5, any short exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0 \tag{12.3}$$

yields a canonical isomorphism of *lines*

$$\det(V') \otimes \det(V'') \simeq \det(V)$$

using wedge products. Therefore, by dualizing the exact sequence of tangent spaces and passing to top exterior powers we have a canonical isomorphism of lines

$$\Omega_G^{\mathrm{top}}(e) \simeq \Omega_H^{\mathrm{top}}(e) \otimes \Omega_{G/H}^{\mathrm{top}}(\overline{e}).$$

Thus, we can uniquely define $\overline{\omega}(\overline{e})$ by the requirement on this isomorphism that

$$\omega(e) \mapsto \eta(e) \otimes \overline{\omega}(\overline{e}).$$

The action of H on G/H fixes $\overline{e}$, so to consistently translate $\overline{\omega}(\overline{e})$ around to define a G-invariant (necessarily C^∞; why?) global $\overline{\omega}$ we need exactly that the natural left H-action on $T_{\overline{e}}(G/H)$ has trivial determinant.

That is, we want the induced H-action on the dual line $\Omega_{G/H}^{\mathrm{top}}(\overline{e})$ to be trivial. The action on this line is given by the determinant of the "coadjoint" C^∞ homomorphism

$$\begin{aligned} \mathrm{coAd}_{G/H} \colon H &\to \mathrm{GL}\left(T_{\overline{e}}^*(G/H)\right) \\ h &\mapsto d\ell_{h^{-1}}(\overline{e})^* \end{aligned}$$

(the use of h^{-1} "cancels" the contravariance of duality to make the map a homomorphism rather than an anti-homomorphism). Note that such triviality holds automatically if H is compact and connected, as such groups have no non-trivial continuous homomorphism into $\mathbf{R}^\times$! This shows that the desired $\overline{\omega}$ exists on G/H whenever H has no non-trivial continuous homomorphisms to $\mathbf{R}^\times$, such as whenever H is both compact and connected.

13. Proof of Conjugacy Theorem I

Let G be a connected compact Lie group, $T \subset G$ a maximal torus. Consider the map (easily seen to be C^∞; check!):

$$q\colon\ (G/T) \times T \to G \qquad (13.1)$$
$$(\overline{g}, t) \mapsto gtg^{-1}.$$

This is a C^∞ map between connected compact smooth manifolds that are orientable (since we saw last time that for any Lie group G and closed subgroup H, the smooth manifold G/H admits a nowhere-vanishing even G-invariant top-degree smooth differential form when H is connected and compact) and clearly have the same dimension. Next time, we'll combine coset integration $\int_{G/T}$ with techniques developed today to show q is surjective. This is sufficient for our needs:

Proposition 13.1. *If q is surjective then the Conjugacy Theorem holds for G.*

Proof. The Conjugacy Theorem (Theorem 11.1) has two parts. Let's show that each part follows from q being surjective.

(2) For any $g_0 \in G$ there exist $g \in G, t \in T$ so that $g_0 = gtg^{-1} \in gTg^{-1}$. Since gTg^{-1} is a (maximal) torus, this proves any element of G lies in a (maximal) torus.

(1) Pick a maximal torus T'. We seek some $g \in G$ so that $gT'g^{-1} = T$ (which is equivalent to $T' = g^{-1}Tg$, and is the same as showing $T' \subset g^{-1}Tg$ since T' is maximal; this in turn is equivalent to showing $gT'g^{-1} \subset T$). To achieve this, we seek a sufficiently generic element $t' \in T'$ in the sense that that $\overline{\langle t' \rangle} = T'$. If we have such a t' then surjectivity of q provides $g \in G$ so that $gt'g^{-1} \in T$. Hence, $g\langle t' \rangle g^{-1} \subset T$ so passing to closures gives $gT'g^{-1} \subset T$, which is enough for our purposes. To find such a t', we use Lemma 13.2 below.

$\square$

Lemma 13.2 (Density Lemma)**.** *For any torus S, there exists $s \in S$ so that $\overline{\langle s \rangle} = S$.*

Example 13.3. Take $S = S^1$. Then any element not of finite order works.

Proof of Density Lemma. The idea is that for any $s \in S$, $\overline{\langle s \rangle}$ contains $\overline{\langle s \rangle}^0$ as a torus of finite index in $\overline{\langle s \rangle}$. Furthermore, we claim there are only countably many proper subtori in S. Indeed, because $X(S) \simeq \mathbf{Z}^r$ has only countably many torsion-free quotients (corresponding to saturated $\mathbf{Z}$-submodules), the countability claim follows from the categorical anti-equivalence between tori and finite free $\mathbf{Z}$-modules. Therefore, there are only countably many possibilities for $\overline{\langle s \rangle}$ if it is a proper subgroup of S, and those are all closed subgroups of S, hence nowhere-dense. The Baire Category Theorem thereby gives the desired s. See §K.5 for full details. $\square$

By Proposition 13.1, to prove the Conjugacy Theorem it suffices to show the map q in (13.1) is surjective.

Question 13.4. How to show $q : (G/T) \times T \to G$ is surjective?

Our method will be to consider more generally a C^∞ map $f : M' \to M$ between compact connected oriented manifolds of the same dimension $d > 0$. We'll define an invariant $\deg(f) \in \mathbf{Z}$ so that:

(1) if $\deg f \neq 0$ then f is surjective;
(2) this "degree" is computable in some useful sense.

Lemma 13.5. *For $f : M' \to M$ a C^∞ map of compact connected oriented smooth manifolds, there exists a unique number* $\deg(f) \in \mathbf{Z}$ *so that for all $\omega \in \Omega^{\mathrm{top}}(M)$,*

$$\int_{M'} f^*\omega = \deg(f) \cdot \int_M \omega$$

using integration with respect to the chosen orientations on M and M'.

Beware that $\deg(f)$ depends on the choices of orientations for M and M'; it changes by a sign if we change one of those orientations. Before proving Lemma 13.5, let's show that if $\deg f \neq 0$ then f is surjective.

Lemma 13.6. *For $f : M' \to M$ a C^∞ map of compact connected oriented smooth manifolds, if* $\deg f \neq 0$ *then f is surjective.*

Proof. Suppose f is not surjective, so $f(M') \subset M$ is a proper closed subset. Choose ω compactly supported inside a coordinate ball in $M - f(M')$, which is a *nonempty* open subset of M, with a coefficient function in oriented coordinates that is non-negative and not identically zero. Thus, $\int_M \omega \neq 0$. But, $f(M') \cap \mathrm{Supp}(\omega) = \emptyset$, so $f^*\omega = 0$ and hence $\int_{M'} f^*\omega = 0$. This integral is also equal to $\deg(f) \cdot \int_M \omega \neq 0$, a contradiction. $\square$

Proof of Lemma 13.5. Let d be the common dimension of M and M'. We have a commutative diagram

$$\begin{tikzcd}
& \mathrm{H}^d_{\mathrm{dR}}(M) \arrow[r, "f^*"] \arrow[dl, "\int_{M,or_M}"'] \arrow[d, "\mathrm{dR}"] & \mathrm{H}^d_{\mathrm{dR}}(M') \arrow[d, "\mathrm{dR}"] \arrow[dr, "\int_{M',or_{M'}}"] & \\
\mathbf{R} \arrow[r, "P.D."', "\simeq"] & \mathrm{H}^d(M,\mathbf{R}) \arrow[r, "f^*"] & \mathrm{H}^d(M,\mathbf{R}) & \mathbf{R} \arrow[l, "P.D."', "\simeq"] \\
\mathbf{Z} \arrow[u] \arrow[r, "P.D.", "\simeq"'] & \mathrm{H}^d(M,\mathbf{Z}) \arrow[u] \arrow[r, "f^*"] & \mathrm{H}^d(M,\mathbf{Z}) \arrow[u] & \mathbf{Z} \arrow[u] \arrow[l, "P.D."', "\simeq"]
\end{tikzcd} \tag{13.2}$$

where "$P.D.$" denotes Poincaré Duality (using the orientations on M or M'). The vertical maps labeled "dR" are the deRham comparison isomorphisms in top degree, as are the maps labeled $P.D.$ and $\int_{M,or_M}, \int_{M',or_{M'}}$ via the chosen orientations.

The composite map $\mathbf{Z} \to \mathbf{Z}$ along the bottom must be multiplication by some integer, which we call $\deg(f)$. Thus, the composite map $\mathbf{R} \to \mathbf{R}$ across the middle row is also multiplication by $\deg(f)$, and the commutativity of the top part of the diagram (13.2) shows that it satisfies the desired formula for all ω. (Note that all top-degree differential forms are closed, and integration of top-degree differential forms factors through top-degree deRham cohomology by Stokes' Theorem since these manifolds have empty boundary.) $\square$

Remark 13.7. For our needs, just $\deg(f) \in \mathbf{R}$ is good enough, so there is no need for the heavier algebraic topology input along the bottom of the diagram (13.2): all we need is that (by connectedness and compactness) the oriented integration maps are isomorphisms of top-degree deRham cohomology onto $\mathbf{R}$, as we can then define $\deg(f) \in \mathbf{R}$ to be the number yielding the composite $\mathbf{R}$-linear map $\mathbf{R} \to \mathbf{R}$ across the top part of the diagram.

(The fact that $\deg(f)$ defined in this way is actually an integer may then seem like a miracle. The preceding argument explains such integrality via algebraic topology, though it can be seen in other ways too; an alternative approach with other topological input is given in [BtD, Ch. I, 5.19].)

Now we'll use another kind of ω to get a formula for $\deg(f)$ in some cases.

Lemma 13.8. *Suppose $f : M' \to M$ is a smooth map between compact connected oriented smooth manifolds and $m_0 \in M$ is a point for which $f^{-1}(m_0) = \{m'_1, \ldots, m'_r\}$ is a finite set such that for all j we have*

$$df(m'_j) : T_{m'_j}(M') \simeq T_{m_0}(M).$$

That is (by the Inverse Function Theorem), we assume f is a C^∞ isomorphism near each m'_j. Then:

(1) There is an open connected $U \ni m_0$ so that

$$f^{-1}(U) = \coprod_{j=1}^{r} U'_j$$

for connected open U'_j so that $m'_j \in U'_j \xrightarrow{f} U$, with $f : U'_j \to U$ a diffeomorphism for each j (perhaps not orientation-preserving, using the unique orientations on each of the connected U'_j and U arising from the chosen orientations on M' and M respectively).

(2) We have

$$\deg(f) = \sum_{j=1}^{r} \varepsilon_j$$

where $\varepsilon_j = \pm 1$ records if $f : U'_j \simeq U$ preserves or reverses the orientations arising from M' and M respectively.

Remark 13.9. For $q : (G/T) \times T \to G$, we'll see next time that such an $m_0 \in G$ can be taken to be any "generic" $t \in T \subset G$. It will turn out that for such an m_0, calculations using $\int_{G/T}$ will yield that $q^{-1}(m_0)$ has size $\#(N_G(T)/T)$ (with finiteness of $N_G(T)/T$ proved directly in Proposition K.4.3 by analytic means via 1-parameter subgroups; this finiteness argument is a-priori stronger than finiteness of $N_G(T)/Z_G(T)$ since we don't know that $Z_G(T) = T$ until after the Conjugacy Theorem has been completely proved) *and* that all $\varepsilon_j = +1$, so $\deg(q) = \#(N_G(T)/T) \neq 0$ (yielding the desired surjectivity of q).

Proof of Lemma 13.8. We prove the two parts in turn.

(1) Since f is a local C^∞ isomorphism near each m'_j, we can pick pairwise disjoint open subsets $V'_j \ni m'_j$ in M' so that f carries each V'_j diffeomorphically onto an open neighborhood $V_j \ni m_0$ in M. Note that f is topologically proper (being a map between compact Hausdorff spaces) and $\coprod_j V'_j$ is an open subset of the source space M' that contains the entire fiber $f^{-1}(m_0)$.

Now recall that for a proper map $g : X \to Y$ between locally compact Hausdorff spaces, if $y \in Y$ is a point and $U \subset X$ is an open subset such that $g^{-1}(y) \subset U$ then there is an open set $W \subset Y$ around y such that $g^{-1}(W) \subset U$. (Informally, the open subsets $g^{-1}(W)$ for open $W \subset Y$ around y are cofinal among the open subsets of X

containing $g^{-1}(y)$.) This fact from point-set topology is an elementary exercise, the key being closedness of g. Applying this in our situation, there is an open $U \ni m_0$ so that $f^{-1}(U) \subset \coprod V'_j$. Now define $U'_j = f^{-1}(U) \cap V'_j$, but then shrink U a bit more to make sure U and all U'_j are connected. This proves the first part, and gives a bit more (which will be useful for the proof of the second part): we can arrange that all U'_j and U are each contained in coordinate charts in M' and M respectively.

(2) As mentioned above, we take U small enough so arrange that both U and each U'_j lie inside coordinate charts of M and M' respectively.

Now choose ω compactly supported in U with $\int_M \omega \neq 0$. Note that $f^*(\omega)$ is compactly supported in $f^{-1}(U) = \coprod U'_j$. Since $f : U'_j \simeq U$ is a diffeomorphism between connected open subsets of coordinate domains whose effect on orientation is governed by the sign ε_j, the usual Change of Variables Formula for integration of compactly supported functions on $\mathbf{R}^n$ yields $\int_{U'_j} f^*\omega = \varepsilon_j \int_U \omega$. Hence, by definition of $\deg(f)$ we have

$$\begin{aligned}
\deg(f) \cdot \int_M \omega &= \int_{M'} f^*\omega \\
&= \sum_j \int_{U'_j} f^*\omega \\
&= \sum_j \varepsilon_j \int_U \omega \\
&= \sum_j \varepsilon_j \int_M \omega \\
&= (\sum_j \varepsilon_j) \int_M \omega.
\end{aligned}$$

Now cancelling the nonzero $\int_M \omega$ from both sides does the job. □

14. Proof of the Conjugacy Theorem II

Today, we'll wrap up the proof of the Conjugacy Theorem. Let's begin by reviewing notation and results from last time. Let G be a connected compact Lie group, $T \subset G$ a maximal torus, W the group $N_G(T)/T$. We do not yet know $T = Z_G(T)$, so we don't know $W = W(G,T)$, but we see in Appendix K that W is finite. In particular, we know $N_G(T)^0 = T$; this will be very important near the end today.

Last time we defined the C^∞ map

$$\begin{aligned}
q\colon\ (G/T) \times T &\to G \\
(g,t) &\mapsto \overline{g}.t
\end{aligned}$$

with $\overline{g}.t$ just notation for gtg^{-1} for any representative g of $\overline{g}$. We reduced the proof of the Conjugacy Theorem to showing q is surjective. Further, we saw last time in Lemma 13.6 that it is enough to show $\deg(q) \neq 0$ for a choice of orientations on $G, T, G/T$. Now choose $t_0 \in T$ such that $\overline{\langle t_0 \rangle} = T$; we'll check that we can apply the formula "$\deg = \sum \varepsilon_j$" from Lemma 13.8 using the fiber $q^{-1}(t_0)$ over t_0.

Remark 14.1. Later it will turn out that the condition "$\overline{\langle t_0 \rangle} = T$" on t_0 for applicability of Lemma 13.8 to its fiber (to be justified below for t_0 as above) can be considerably weakened to just a "Zariski-open" condition on t_0 (corresponding to "distinct eigenvalues" for $G = \mathrm{U}(n)$), but we don't need such refinement at this time.

Lemma 14.2. *We have*

$$q^{-1}(t_0) = \left\{ \left(w, w^{-1}.t_0 \right) : w \in W \right\}.$$

In particular, $q^{-1}(t_0)$ is finite of size $\#W$.

Proof. Consider some $g \in G$ and $t \in T$ with $gtg^{-1} = t_0$. Then,

$$T = \overline{\langle t_0 \rangle} = g\overline{\langle t \rangle}g^{-1} \subset gTg^{-1}.$$

The torus gTg^{-1} is a torus of the same dimension as T that contains T. This implies $T = gTg^{-1}$, so $g \in N_G(T)$ and hence $\overline{g} \in W$. Likewise, clearly $w.t = t_0$, so $t = w^{-1}.t_0$. □

In order to prove the Conjugacy Theorem by the degree method from last time, it remains to show that for each

$$\xi := (\overline{g}, t) \in q^{-1}(t_0)$$

the tangent map

$$dq(\xi)\colon T_{\overline{g}}\left(G/T\right) \times T_t(T) = T_{\xi}\left(\left(G/T\right) \times T\right) \to T_{t_0}(G)$$

is an isomorphism and even orientation-preserving. That is, for oriented bases (to make sense of a "determinant") we just need to show $\det\left(dq(\xi)\right) > 0$. To proceed, we next have to specify orientations, by defining suitable nowhere-vanishing top-degree differental forms on G, T, and G/T.

We will see below that G/T is even-dimensional, so in the end it won't for matter if we switch the order of G/T and T when defining a product orientation on $(G/T) \times T$ (switching the factors in a product of oriented connected smooth manifolds with respective dimensions d and d' changes the product orientation by $(-1)^{dd'}$). But we will be precise about our orientation definitions during the argument anyway (since we don't yet know $\dim(G/T)$ is even).

14.1. **Finishing the proof of the Conjugacy Theorem.**

Fix top-degree differential forms dg on G and dt on T that are nonzero and left-invariant (hence they are bi-invariant, since the groups are compact and connected). We emphasize that these are differential forms, not measures.

We have seen earlier that a (preferred) left-invariant nonzero top-degree differential form $d\overline{g}$ on G/T is specified by the condition that the canonical isomorphism

$$\Omega^{\mathrm{top}}_{G/T}(\overline{e}) \otimes \Omega^{\mathrm{top}}_{T}(e) \to \Omega^{\mathrm{top}}_{G}(e)$$

satisfies $(d\overline{g})(\overline{e}) \otimes (dt)(e) \mapsto (dg)(e)$. Since $d\overline{g}$ is nonzero at $\overline{e}$ and left-invariant, it is nowhere-vanishing and so defines an orientation on G/T.

We now get top-degree C^∞ differential forms $q^*(dg)$, $d\overline{g} \wedge dt$ on $(G/T) \times T$, the second of which is nowhere-vanishing, so

$$q^*(dg) = J\left(d\overline{g} \wedge dt\right)$$

for some $J \in C^\infty\left(\left(G/T\right) \times T\right)$.

Unravelling definitions, our task is exactly to show $J(\xi) > 0$ for all $\xi \in q^{-1}(t_0)$. This positivity is immediate from Proposition 14.4 stated below, whose formulation requires us to first introduce some notation:

Notation 14.3. For a Lie group G and a closed subgroup $H \subset G$, define the map

$$\begin{aligned} \mathrm{Ad}_{G/H} \colon H &\to \mathrm{GL}\left(T_{\overline{e}}\left(G/H\right)\right) \\ h &\mapsto d\ell_h(\overline{e}) \end{aligned}$$

that is clearly a C^∞ homomorphism.

It is important to note that both maps $\ell_h, c_h : G/H \to G/H$ which preserve the point $\overline{e}$ since right-translation $r_{h^{-1}}$ on G induces the identity map on G/H. Hence, $d\ell_h(\overline{e}) = dc_h(\overline{e})$ on $\mathfrak{g}/\mathfrak{h}$ is induced by $dc_h(e) = \mathrm{Ad}_G(e)$ on $\mathfrak{g}$ for $c_h : G \to G$ that fixes e (whereas ℓ_h on G generally does not fix e). Therefore, the map $\mathrm{Ad}_{G/H}$ on $\mathfrak{g}/\mathfrak{h}$ is really induced by $\mathrm{Ad}_G|H$ on $\mathfrak{g}$, explaining the notation.

Now it remains to show:

Proposition 14.4. *We have*

(1) *for all* $(\overline{g}, t) \in (G/T) \times T$,

$$J\left(\overline{g}, t\right) = \det(\mathrm{Ad}_{G/T}(t^{-1}) - 1)$$

(2) *If* $t_0 \in T$ *satisfies* $\overline{\langle t_0 \rangle} = T$ *then*

$$\det\left(\mathrm{Ad}_{G/T}(t_0^{-1}) - 1\right) > 0.$$

Moreover, G/T *has even dimension.*

Remark 14.5. The generic choices for $t_0 \in T$ as in Proposition 14.4(2) exhaust a dense subset of T. (In fact, even the powers t_0^m for nonzero integers m constitute such a dense subset: that each power generates a dense subgroup is a nice exercise, which we'll explain a bit later.) Thus, by continuity, we see $J \geq 0$ everywhere.

Proof of Proposition 14.4. We prove the two parts, though the first part is mostly relagated to Appendix L.

(1) Using bi-invariance of dg as a differential form one can show

$$J(\overline{g}, t) = J\left(\overline{e}, t\right)$$

and we can use that $dm = +$ and $d(\mathrm{inv})$ is negation to eventually arrive at the desired formula (using the link of $\mathrm{Ad}_{G/T}$ to Ad_G). For details, see Appendix L.

(2) It is enough to show that the linear operator $\mathrm{Ad}_{G/T}(t_0^{-1}) - 1$ on the $\mathbf{R}$-vector space $\mathfrak{g}/\mathfrak{t}$ has no real eigenvalues (in particular, the eigenvalues are nonzero). Indeed, in this case, they must all come in non-real conjugate pairs, so their product will be a positive real number (and the "pairs" aspect will also show $\dim(G/T)$ is even!).

Now it is the same to show that $\mathrm{Ad}_{G/T}(t_0^{-1})$ has no real eigenvalues. But the subgroup $\mathrm{Ad}_{G/T}(T) \subset \mathrm{GL}(\mathfrak{g}/\mathfrak{t}) = \mathrm{GL}_n(\mathbf{R})$ is compact, so all eigenvalues of its elements lie in S^1. Therefore, the only possible eigenvalues in $\mathbf{R}$ are ± 1, so it is enough to show that 1 is not an eigenvalue of

$$\mathrm{Ad}_{G/H}\left(t_0^{-1}\right)^2.$$

But $\mathrm{Ad}_{G/T}(t_0^{-1})^2 = \mathrm{Ad}_{G/T}(t_0^{-2})$, and we further claim $\overline{\langle t_0^{-2}\rangle} = T$. Indeed, $\overline{\langle t_0^n\rangle} = T$ for any $n \in \mathbf{Z} - \{0\}$ because

$$\overline{\langle t_0^n\rangle} \subset T = \overline{\langle t_0\rangle}$$

with the cokernel a torus that is killed by n and hence is trivial.

Now we may and do rename t_0^{-2} as t_0, so it is enough to show $\mathrm{Ad}_{G/T}(t_0)$ doesn't have 1 as an eigenvalue. This operator is the effect of $\mathrm{Ad}_G(t_0)$ on $\mathfrak{g}/\mathfrak{t}$, so we want to show

$$(\mathfrak{g}/\mathfrak{t})^{\mathrm{Ad}_G(t_0)=1} = 0.$$

Since $\overline{\langle t_0\rangle} = T$, the space of fixed vectors under $\mathrm{Ad}_G(t_0) : \mathfrak{g}/\mathfrak{t} \to \mathfrak{g}/\mathfrak{t}$ agrees with the space of T-fixed vectors, by continuity of the action. Therefore, it is the same to show $(\mathfrak{g}/\mathfrak{t})^T = 0$ (using the Ad_G action of T). By complete reducibility of T-representations (over $\mathbf{R}$, or after tensoring up to $\mathbf{C}$) we have

$$(\mathfrak{g}/\mathfrak{t})^T = \mathfrak{g}^T/\mathfrak{t}^T = \mathfrak{g}^T/\mathfrak{t}$$

yet $\mathfrak{g}^T = \mathrm{Lie}(Z_G(T)) = \mathrm{Lie}(Z_G(T)^0)$ and

$$T \subset Z_G(T)^0 \subset N_G(T)^0 = T$$

(as we noted right at the start of today!). Thus, $\mathfrak{g}^T = \mathrm{Lie}(T) = \mathfrak{t}$, so we get the desired vanishing.

□

Here is an illustration of the Weyl Jacobian formula; we'll go into more detail on this next time.

Example 14.6. Let $G = \mathrm{U}(n)$ and T the maximal diagonal torus $\left(S^1\right)^n$. For

$$t = \begin{pmatrix} e^{2\pi i\theta_1} & & \\ & \ddots & \\ & & e^{2\pi i\theta_n} \end{pmatrix}$$

we claim

$$\det\left(\mathrm{Ad}_{G/T}(t^{-1}) - 1\right) = \prod_{1\le j<j'\le n} |e^{2\pi i\theta_j} - e^{2\pi i\theta_{j'}}|^2,$$

which is visibly positive when the eigenvalues are distinct (a *much* weaker condition on t than genericity).

15. Weyl Integration Formula and Roots

Last time we finished the proof of the Conjugacy Theorem. Let's record one more consequence of that and then move on. As is our typical setup, let G be a connected compact Lie group and $T \subset G$ a maximal torus. We know that $T = Z_G(T)$, and the resulting group

$$W := W(G,T) = N_G(T)/Z_G(T) = N_G(T)/T$$

is finite. Appendix M gives details of the following application.

Let $\mathrm{Conj}(G) := G/\sim$ with $x \sim y$ if there is $g \in G$ with $gxg^{-1} \sim y$; this is the set of conjugacy classes in G, and we equip it with the quotient topology from G. The continuous composite map $T \to G \to \mathrm{Conj}(G)$ is invariant under the W-action on T (as that is induced by $N_G(T)$-conjugation on G), so we obtain a continuous map of quotient spaces

$$T/W \xrightarrow{\alpha} \mathrm{Conj}(G).$$

In fact, this map α is a homeomorphism of compact Hausdorff spaces, so it induces an isomorphism of $\mathbf{C}$-algebras

$$\{\text{cont. } \mathbf{C}\text{-valued class functions on } G\} \simeq \{\text{cont. } T \to \mathbf{C} \text{ invariant under } W\} \qquad (15.1)$$
$$f \mapsto f|_T.$$

The proof of injectivity of the map α uses the Conjugacy Theorem for $Z_G(t)^0$ for any $t \in T$. (See Appendix M for more details on the preceding assertions.)

To appreciate the importance of (15.1), we introduce the following concept:

Definition 15.1. The *representation ring* $R(G)$ in the $\mathbf{Z}$-subalgebra of $C^0(\mathrm{Conj}\, G)$ generated by χ_V for irreducible representations V of G.

Since $\sum \chi_{V_j} = \chi_{\bigoplus_j V_j}$ and $\chi_V \cdot \chi_{V'} = \chi_{V \otimes V'}$ for representations of G, $R(G)$ consists of exactly the differences $\chi_W - \chi_{W'}$ for representations W and W' of G (so an element of $R(G)$ is a "virtual representation"). In particular,

$$R(G) = \bigoplus_{\chi \text{ irreducible}} \mathbf{Z} e_\chi$$

with the basis vectors e_χ satisfying the multiplication law $e_\chi e_{\chi'} = \sum_j n_j e_{\chi_j}$ when $V \otimes V' = \oplus V_j^{\oplus n_j}$ for irreducible representations V and V' with respective characters χ and χ' and the irreducible subrepresentations V_j occurring in $V \otimes V'$.

Example 15.2. For the torus $G = (S^1)^n$ the ring $R(G)$ is the Laurent polynomial ring

$$R(G) = \mathbf{Z}\left[\chi_1^{\pm 1}, \ldots, \chi_n^{\pm 1}\right]$$

for the projections

$$\chi_j = \mathrm{pr}_j : (S^1)^n \to S^1.$$

This expresses the fact that every irreducible representation of G is uniquely of the form $\prod_j \chi_j^{a_j}$ for $(a_1, \ldots, a_n) \in \mathbf{Z}^n$. More intrinsically, the representation ring of a torus T is exactly the group ring over $\mathbf{Z}$ of its character lattice $X(T)$.

Since the representation ring of a torus is so concrete, it is interesting that in general, for any G and a maximal torus $T \subset G$, we have an inclusion

$$R(G) \to R(T)^W$$
$$f \mapsto f|_T$$

induced by the equality $C^0(\mathrm{Conj}\, G) = C^0(T)^W$ from (15.1). For $G = \mathrm{U}(n)$ and T the diagonal torus, this inclusion at the level of representations rings is an isomorphism, as shown directly in [BtD, IV, 3.13]. The Weyl character formula, to be discussed near the end of the course, gives such an equality in general and more importantly it identifies the image of the set $\mathrm{Irred}(G)$ of (characters of) irreducible representations of G inside $R(T)^W$.

The starting point on the long road to the character formula is:

Theorem 15.3 (Weyl Integration Formula)**.** *Choose top-degree left-invariant (hence bi-invariant) differential forms dg, dt on G, T respectively whose associated measures are volume 1, and use these to orient G and T accordingly (so $\int_G dg = 1$ and $\int_T dt = 1$ as oriented integrations). Define the Weyl Jacobian $J : T \to \mathbf{R}_{\geq 0}$ by*

$$J(t) := \det\left(\mathrm{Ad}_{G/T}(t^{-1}) - 1\right).$$

Then,

$$\int_G f(g)dg = \frac{1}{\#W}\int_T f(t)J(t)dt$$

We've already done much of the work of the proof of the Weyl Integration Formula when we computed the degree of the map q in our proof of the Conjugacy Theorem. Before we give the proof of this formula (which will be quite short, given what has already been done), we illustrate it with a key basic class of examples:

Example 15.4. Let $G = \mathrm{U}(n)$ and T the diagonal maximal torus. Then

$$\int_{\mathrm{U}(n)} f(g)dg = \frac{1}{n!}\int_{(\mathbf{R}/\mathbf{Z})^n} f\left(\begin{pmatrix} e^{2\pi i\theta_1} & & \\ & \ddots & \\ & & e^{2\pi i\theta_n}\end{pmatrix}\right)\prod_{1\leq j<j'\leq n}|e^{2\pi i\theta_j} - e^{2\pi i\theta'_j}|^2 d\theta_1\ldots d\theta_n.$$

Here we're using that for

$$t = \begin{pmatrix} e^{2\pi i\theta_1} & & \\ & \ddots & \\ & & e^{2\pi i\theta_n}\end{pmatrix}$$

we have $J(t) = \prod_{1\leq j<j'\leq n}|e^{2\pi i\theta_j} - e^{2\pi i\theta'_j}|^2$, a fact that we asserted at the end of last time and will prove at the end of today.

Proof of Theorem 15.3. Recall

$$\begin{aligned} q\colon\ (G/T)\times T &\to G \\ (\overline{g}, t) &\mapsto gtg^{-1}\end{aligned}$$

of degree $\#W$. By definition of $\deg q$ applied to $\omega = f(g)dg$ on G we have

$$\begin{aligned}\int_G f(g)dg &= \int_G \omega \\ &= \frac{1}{\deg q}\int_{(G/T)\times T} f^*\omega \\ &= \frac{1}{\#W}\int_{(G/T)\times T} f(t)q^*(dg)\end{aligned}$$

since f is a class function. We saw in the proof of the Conjugacy Theorem that $q^*(dg) = J(t)d\overline{g}\wedge dt$ on $(G/T)\times T$ for the (nowhere-vanishing) left-invariant top-degree differential form $d\overline{g}$ on G/T associated to dg and dt. Orienting G/T using $d\overline{g}$, we know by the coset-Fubini property that $\int_{G/T} d\overline{g} = 1$ since $\int_G dg = 1$ and $\int_T dt = 1$ (see HW5, Exercise 2(iv) which gives an equivalent version in terms of the associated Haar measures without reference to orientations), so

$$\begin{aligned}\int_G f(g)dg &= \frac{1}{\#W}\left(\int_{G/T} d\overline{g}\right)\left(\int_T f(t)J(t)dt\right)\\ &= \frac{1}{\#W}\left(\int_T f(t)J(t)dt\right).\end{aligned}$$

□

To use the Weyl Integration Formula, we need to understand the function

$$J(t) = \det\left(\mathrm{Ad}_{G/T}(t^{-1}) - 1\right)$$

that is a determinant of the endomorphism of the $\mathbf{R}$-vector space $\mathfrak{g}/\mathfrak{t}$ or of its complexification $\mathfrak{g}_{\mathbf{C}}/\mathfrak{t}_{\mathbf{C}}$ arising from $\mathrm{Ad}_G(t^{-1}) - 1$. Therefore, we want to understand the T-action on $\mathfrak{g}_{\mathbf{C}}/\mathfrak{t}_{\mathbf{C}}$, or really on $\mathfrak{g}_{\mathbf{C}}$ through the adjoint representation.

As a T-representation over $\mathbf{C}$ we have,

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \bigoplus \left(\bigoplus_{a\in\Phi} (\mathfrak{g}_{\mathbf{C}})_a\right)$$

for some finite subset

$$\Phi = \Phi(G,T) \subset X(T) - \{0\}$$

and corresponding weight spaces $(\mathfrak{g}_{\mathbf{C}})_a$ for the non-trivial weights that appear; $\mathfrak{t}_{\mathbf{C}}$ is the weight space for the trivial character since

$$\mathfrak{t} = \mathrm{Lie}(T) = \mathrm{Lie}\left(Z_G(T)\right) = \mathfrak{g}^T.$$

Remark 15.5. The character lattice $X(T)$ is often denoted additively (even though its elements are homomorphisms into S^1), so the trivial character is denoted by 0. For that reason, we'll often denote characters in terms of exponentials, writing t^χ to mean $\chi(t)$ for $t \in T$ and $\chi \in X(T)$.

Remark 15.6. The W-action on T induces a W-action on the character lattice $X(T)$ that must permute the finite subset Φ (as is the case for the collection of non-trivial T-weights occuring in any representation of G over $\mathbf{C}$).

Example 15.7. Let $G = \mathrm{U}(n) \subset \mathrm{GL}_n(\mathbf{C})$. Let $T = \left(S^1\right)^n$ be the diagonal maximal torus. We have the center

$$Z_G = S^1 \xrightarrow{\Delta} \left(S^1\right)^n = T$$

of scalar matrices in $\mathrm{U}(n)$, and

$$\mathfrak{u}(n)_{\mathbf{C}} \simeq \mathfrak{gl}_n(\mathbf{C})$$

for $\mathbf{R}$-dimension reasons since

$$\mathfrak{u}(n) \cap i\left(\mathfrak{u}(n)\right) = \{0\}.$$

Further, Ad_G on $\mathfrak{g}_{\mathbf{C}} = \mathfrak{gl}_n(\mathbf{C})$ is just induced by

$$\mathrm{Ad}_{\mathrm{GL}_n(\mathbf{C})}$$

which is conjugation on $\mathfrak{gl}_n(\mathbf{C}) = \mathrm{Mat}_n(\mathbf{C})$, by functoriality of Ad_H in H.

We know in this case

$$N_G(T) = T \rtimes S_n$$

for S_n identified with $n \times n$ permutation matrices. Letting $E_{ij} \in \mathfrak{gl}_n(\mathbf{C})$ denote the matrix with a 1 in the ijth entry and 0's elsewhere for $i \neq j$, for any

$$t := \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \in T$$

a straightforward calculation yields that

$$tE_{ij}t^{-1} = \frac{t_i}{t_j} E_{ij}.$$

We have the identification

$$\begin{aligned} \mathbf{Z}^n &\to X(T) \\ \vec{z} &\mapsto \chi_{\vec{z}} \end{aligned}$$

where

$$\chi_{\vec{z}}(t) = \prod_j t_j^{z_j} =: t^{\chi_{\vec{z}}}.$$

Therefore, $\mathbf{C}E_{ij}$ is a T-eigenline with eigencharacter $t^{a_i - a_j}$ where $a_i : t \mapsto t_i$ is the ith projection $T = (S^1)^n \to S^1$. Therefore,

$$\Phi(G, T) = \left\{a_i - a_j : 1 \leq i, j \leq n, i \neq j\right\} \subset \{\vec{x} \in \mathbf{Z}^n : \sum_i x_i = 0\} = X\left(T/Z\right) \subset X(T)$$

with $\left(\mathfrak{g}_{\mathbf{C}}\right)_a$ a line for each $a \in \Phi$.

Further, observe $-\Phi = \Phi$ and $\Phi \cap (\mathbf{Q}a) = \{\pm a\}$ for each $a \in \Phi$ (i.e., the only $\mathbf{Q}$-linear dependence for distinct elements of Φ is for the pairs $\{a, -a\}$).

Example 15.8. Inside the Euclidean hyperplane $X\left(T/Z\right)_{\mathbf{R}} \subset X(T)_{\mathbf{R}} = \mathbf{R}^n$ we have

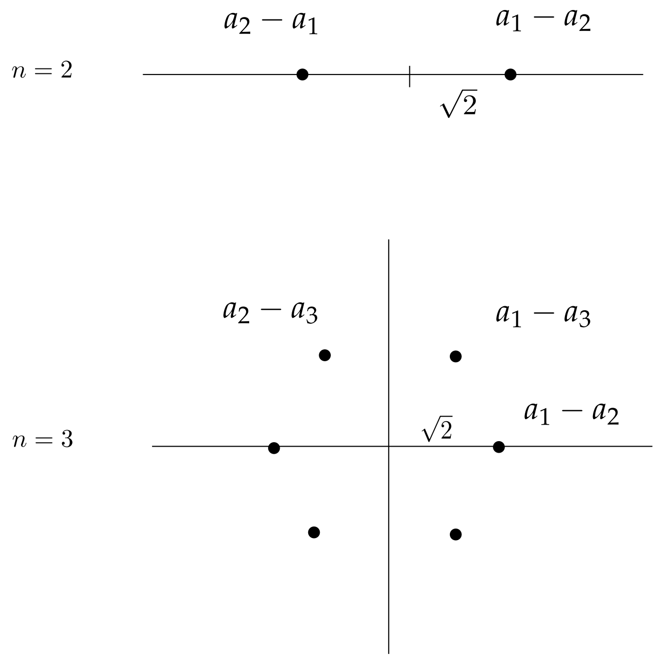


FIGURE 2. Elements of $\Phi(\mathrm{U}(n), T)$ for $n = 2, 3$ (lengths and angles verified via dot products in $X(T)_{\mathbf{R}} = \mathbf{R}^n$).

See Appendix N for Weyl group computations for $\mathrm{SU}(n)$ ($n \geq 2$), $\mathrm{SO}(n)$ ($n \geq 3$), and $\mathrm{Sp}(n)$ ($n \geq 1$). These are also worked out in [BtD, Ch. IV, (3.3)–(3.8)]. The pair $(X(T/Z)_{\mathbf{Q}}, \Phi)$ will be an instance of a root system (a concept we will define later).

Example 15.9. For $G = \mathrm{U}(n)$, and T the diagonal maximal torus, let's calculate the Weyl Jacobian from the Weyl Integration Formula: since $z^{-1} = \overline{z}$ for $z \in S^1$, we have

$$\begin{aligned}
\deg\left(\mathrm{Ad}_{G/T}(t^{-1}) - 1\right) &= \prod_{a \in \Phi} \left(t^{-a} - 1\right) \\
&= \prod_{i \neq j} \left(t^{a_i - a_j} - 1\right) \\
&= \prod_{i < j} \left|t^{a_i - a_j} - 1\right|^2 \\
&= \prod_{i < j} |t^{a_i} - t^{a_j}|^2 \\
&= \prod_{i < j} \left|t_i - t_j\right|^2,
\end{aligned}$$

where the second to last equality uses that $|t^{a_j}| = 1$ for all j.

16. ROOT SYSTEMS AND CLIFFORD ALGEBRAS

Let's start by recalling what we obtained last time. As usual, consder a connected compact Lie group G and a maximal torus T. Let $W = W(G, T) = N_G(T)/T$. We made a W-*stable* finite subset $\Phi \subset X(T) - \{0\}$ giving the nontrivial T-weights on $\mathfrak{g}_{\mathbf{C}}$ with its Ad_G-action of T. Also, $\Phi \subset X(T/Z_G)$ because the center Z_G of G acts trivially under the adjoint representation (since Ad_G is given by differentiating the conjugation action of G on itself, under which Z_G acts trivially). Note that the torus T is its own centralizer on G (by maximality), so it contains Z_G (and hence T/Z_G makes sense).

Keep in mind that although elements of

$$X(T) := \mathrm{Hom}(T, S^1)$$

are of "multiplicative" nature, we will often use the notation $t^\chi := \chi(t)$ and thereby write the "multiplication" in $X(T)$ additively. One reason for the additive notation is that historically people worked much more on the Lie algebra level than the Lie group level, and that is akin to working with logarithms (i.e., going backwards through the exponential map). But there are other reasons that additive notation will be convenient, as we shall see.

Let's review the outcome of calculations from last time:

Example 16.1. For $G = \mathrm{U}(n)$ and T the diagonal torus $(S^1)^n$ we saw $\Phi = -\Phi$. We also saw that $\dim(\mathfrak{g}_{\mathbf{C}})_a = 1$ for all $a \in \Phi$.

Explicitly, for

$$Z_G =: Z = S^1 \xrightarrow{\Delta} (S^1)^n = T \subset \mathrm{U}(n) = G,$$

we have

$$\bigoplus \mathbf{Z} a_i = \mathbf{Z}^n = X(T) \supset X(T/Z) = \{\sum_j x_j = 0\} \supset \Phi = \{a_i - a_j : i \neq j\}$$

for the projections $a_i : T = (S^1)^n \to S^1$, so $t^{a_i - a_j} = t_i/t_j$ for all $t \in T = (S^1)^n$. Observe also that $\mathbf{Z}\Phi = X(T/Z)$.

The natural action of $W = \mathfrak{S}_n$ on $X(T) = \mathbf{Z}^n$ is the usual coordinate permutation (this is immediate from the definition of how W acts on $X(T)$ and the identification of W with the group of $n \times n$ permutation matrices in $\mathrm{U}(n)$ (whose conjugation effect on a diagonal torus is to permute the diagonal entries in accordance with the corresponding element of $\mathfrak{S}_n$). The general fact that the W-action on $X(T)$ preserves Φ (which holds just by unwinding definitions) is thereby seen quite vividly in the case of $\mathrm{U}(n)$.

Another nice feature of the case of $\mathrm{U}(n)$ is that for all $a \in \Phi$,

$$(\mathbf{Q}a) \cap \Phi = \{\pm a\}$$

inside $X(T)_{\mathbf{Q}}$, as we see by inspection. Later we'll see that this holds in complete generality; it is not at all obvious (e.g., in general why couldn't there be $a, b \in \Phi$ so that $2a = 3b$, or in other words two distinct non-trivial T-weights χ, χ' on $\mathfrak{g}_{\mathbf{C}}$ such that $\chi^2 = \chi'^3$?). Further, we computed the weight space explicitly for each element of $\Phi(\mathrm{U}(n), T)$, getting

$$(\mathfrak{g}_{\mathbf{C}})_{a_i - a_j} = \mathbf{C}E_{ij}$$

where E_{ij} is the matrix with a 1 in the ij-entry and 0's elsewhere via the identification

$$\mathfrak{u}(n)_{\mathbf{C}} \simeq \mathfrak{gl}_n(\mathbf{C}) = \mathrm{Mat}_n(\mathbf{C}).$$

We'll next work out the analogue of Example 16.1 for $\mathrm{SU}(n)$, not from scratch but rather by explaining conceptually why much of the output of the $\mathrm{U}(n)$-calculation must agree with analogues for $\mathrm{SU}(n)$ (the latter case being a bit less explicit since the Weyl group for $\mathrm{SU}(n)$ is not liftable to a subgroup of the normalizer of the maximal torus, in contrast with the case of $\mathrm{U}(n)$ via permutation matrices). The reason for working out the relationship between the outcomes for $\mathrm{SU}(n)$ and $\mathrm{U}(n)$ is that it is a warm-up to a general procedure we'll have to go through to bypass headaches caused by the presence of a non-trivial central torus.

Example 16.2. Let

$$G' = \mathrm{SU}(n), T' = T \cap \mathrm{SU}(n),$$

so T' is a maximal torus of $\mathrm{SU}(n)$ (as can be proved directly with a weight-space argument for the T'-action on $\mathbf{C}^n$ to infer that $T' = Z_{G'}(T')$, similarly to how we proved maximality of T in $G := \mathrm{U}(n)$). Later, we'll see $\mathrm{SU}(n)$ coincides the the commutator subgroup $[\mathrm{U}(n), \mathrm{U}(n)]$, and the special case $n = 2$ will be done in HW7, where it will be seen that every element of $\mathrm{SU}(2)$ is actually a single commutator on the nose.

Let's show that (G', T') encodes the same information as (G, T) for W, Φ. The real lesson of what we will now see is that central tori are a necessary but manageable nuisance. Letting $G = \mathrm{U}(n)$ and $Z = Z_G = S^1$, we have an exact sequence

$$1 \longrightarrow \mu_n \longrightarrow Z \times \mathrm{SU}(n) \longrightarrow \mathrm{U}(n) \longrightarrow 1 \tag{16.1}$$

where $\mu_n = Z_{G'} =: Z'$ inside $Z_G = S^1$ via the "anti-diagonal" map

$$\begin{aligned} \mu_n &\to Z \times \mathrm{SU}(n) \\ \zeta &\mapsto \left(\zeta, \zeta^{-1}\right). \end{aligned}$$

In particular, the multiplication map $Z \times \mathrm{SU}(n) \to \mathrm{U}(n)$ is an instance of:

Definition 16.3. A map $f : H \to G$ of connected Lie groups is called an *isogeny* if $\dim G = \dim H$ and f is surjective.

Remark 16.4. Note that if a map is an isogeny, the kernel is necessarily discrete (for Lie algebra reasons), so the kernel is finite in the case of compact Lie groups.

Thus, in our setting with $G = \mathrm{U}(n)$ and $G' = \mathrm{SU}(n)$ we have

$$\begin{tikzcd} T'/Z' \arrow[r, "\simeq"] \arrow[d] & T/Z \arrow[d] \\ G'/Z' \arrow[r, "\simeq"] & G/Z \end{tikzcd} \tag{16.2}$$

and an exact sequence

$$1 \longrightarrow T' \longrightarrow T = \left(S^1\right)^n \xrightarrow{\det} S^1 \longrightarrow 1. \tag{16.3}$$

This short exact sequence of tori yields an exact sequence on character lattices

$$\begin{tikzcd} 1 \arrow[r] & X(S^1) \arrow[r] \arrow[d, "\simeq"'] & X(T) \arrow[r] \arrow[d, "\simeq"] & X(T') \arrow[r] & 1 \\ & \mathbf{Z} \arrow[r, "\mathrm{diag}"] & \mathbf{Z}^n & & \end{tikzcd} \tag{16.4}$$

so $X(T') = \mathbf{Z}^n/\Delta(\mathbf{Z})$ as a quotient of $X(T) = \mathbf{Z}^n$.

But $Z \times T' \to T$ is an isogeny, so $X(T) \to X(Z \times T') = X(Z) \oplus X(T')$ is an injection between finite free $\mathbf{Z}$-modules of the *same* rank, hence it is a finite-index inclusion. Explicitly we have

$$\begin{tikzcd} X(Z) \oplus X(T') \arrow[d, "="] & X(T) \arrow[l] \arrow[d, "="] & \\ \mathbf{Z} \oplus (\mathbf{Z}^n/\Delta) & \mathbf{Z}^n \arrow[l] & X(T/Z) = X(T'/Z') = \left\{\sum_j x_j = 0\right\} \arrow[l, "\supset"'] \\ \left(\sum_j x_j, \vec{x} \bmod \Delta\right) & \vec{x} \arrow[l, mapsto] & \end{tikzcd} \tag{16.5}$$

The identification $G'/Z' = G/Z$ carrying T'/Z' over to T/Z yields $G'/T' = G/T$ as coset spaces. Passing to tangent spaces at the identity yields that $\mathfrak{g}'/\mathfrak{t}' = \mathfrak{g}/\mathfrak{t}$ respecting the actions of $T' \hookrightarrow T$, so by chasing weights under $T = Z\dot{T}'$ on the complexifications we obtain $\Phi' = \Phi$ inside $X(T'/Z') = X(T/Z)$.

Rationalizing, we obtain a *canonical* isomorphism

$$X(Z)_{\mathbf{Q}} \oplus X(T')_{\mathbf{Q}} \xleftarrow{(\dagger)} X(T)_{\mathbf{Q}}.$$

(In general passing to rationalizations of character lattices gets rid of discrepancies caused by isogenies between tori.) Since $T' = T \cap G'$ with G' normal in G, and $T = T' \cdot Z$ with $Z = Z_G$ central in G, we obtain

$$W' := N_{G'}(T')/T' = N_G(T)/T =: W.$$

Explicitly, this equality $W' = W$ is compatible with their actions on the two sides of $(\dagger)$, with the W-action on $X(Z)_{\mathbf{Q}}$ being trivial (since Z is central in G). Compatibly with $(\dagger)$ we also have $\{0\} \times \Phi' = \Phi$.

Remark 16.5. In the general case, the role of $\mathrm{SU}(n)$ will be replaced by the commutator subgroup $G' := [G, G]$ which turns out (not at all obviously!) to be closed, and G' will turn out to satisfy $G' = [G', G']$ (so it is "very non-commutative"!). Likewise, Z above will be replaced with either Z_G or Z_G^0 (the maximal central torus), depending on the situation.

Since $T/Z_G^0 \to T/Z_G$ is an isogeny of tori, we have

$$X(T/Z_G)_{\mathbf{Q}} \simeq X(T/Z_G^0)_{\mathbf{Q}}.$$

We'll later show in general for any (G, T), the following properties hold:

- the pair $(X(T/Z_G)_{\mathbf{Q}}, \Phi)$ is a *reduced root system* (a concept to be defined later, but purely combinatorial and admitting a development on its own without reference to connected compact Lie groups),
- the subgroup $W \hookrightarrow \mathrm{GL}(X(T/Z_G)_{\mathbf{Q}}) = \mathrm{GL}(X(T')_{\mathbf{Q}})$ (injective because $W = W'$ and $Z_{G'}(T') = T'$!) is generated by certain special "reflections" of the $\mathbf{Q}$-vector space $X(T/Z_G)_{\mathbf{Q}} = X(T/Z_G^0)_{\mathbf{Q}} = X(T')_{\mathbf{Q}}$.

Warning 16.6. Many textbook treatments of root systems look at $W \hookrightarrow \mathrm{GL}\left(X(T/Z_G)_{\mathbf{R}}\right)$ instead of $W \hookrightarrow \mathrm{GL}\left(X(T/Z_G)_{\mathbf{Q}}\right)$ and give $X(T/Z_G)_{\mathbf{R}}$ a Euclidean structure at the outset.

This is very bad! One should (as in [Bou2]) keep the vector space rational until the real numbers are needed for a serious reason, and definitely *not* bake a Euclidean structure into the initial setting. We will come back to this later.

Remark 16.7. The viewpoint of root systems $(X(T/Z_G)_{\mathbf{Q}}, \Phi)$ will be very convienient for analyzing the reprsentation theory of G later.

We've talked so far about $\mathrm{SU}(n), \mathrm{SO}(2m+1), \mathrm{Sp}(n), \mathrm{SO}(2m)$ (which are called type A,B,C,D respectively, due to a classification theorem for irreducible root systems to be discussed later). In HW5 Exercise 4 we saw that $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ are simply connected, whereas for $n \geq 3$ we saw that $\#\pi_1(\mathrm{SO}(n)) \leq 2$. For $\mathrm{SO}(n)$ it turns out that there is a *connected* degree-2 covering space, known as the *Spin group* $\mathrm{Spin}(n)$ (so $\pi_1(\mathrm{SO}(n)) = \mathbf{Z}/2\mathbf{Z}$), as we now discuss.

Here is some preliminary motivation (so don't worry too much about omitted details). For connected Lie groups G and discrete normal subgroups $\Gamma \subset G$ (necessarily central by HW1), HW5 provided a surjection

$$\pi_1(G/\Gamma) \twoheadrightarrow \Gamma.$$

In fact, in Appendix O something more refined is done: there is a natural short exact sequence

$$1 \longrightarrow \pi_1(G) \longrightarrow \pi_1(G/\Gamma) \longrightarrow \Gamma \longrightarrow 1. \tag{16.6}$$

Thus, using that the universal cover of a connected Lie group is naturally a Lie group (upon fixing a point in the fiber over the identity, to serve as an identity for the group law), the equality $\pi_1(\mathrm{SO}(n)) = \mathbf{Z}/2\mathbf{Z}$ for $n \geq 3$ is exactly the statement that there is an exact sequence

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1. \tag{16.7}$$

for some *connected* Lie group $\mathrm{Spin}(n)$ (necessarily compact since it is a finite-degree covering space of the compact $\mathrm{SO}(n)$). We now aim to sketch the idea of how such a connected double cover is built.

Example 16.8. If $n = 3$, we have an explicit connected double cover

$$\mathrm{SU}(2) \to \mathrm{SO}(3)$$

(so $\pi_1(\mathrm{SO}(3)) = \mathbf{Z}/2\mathbf{Z}$).

Remark 16.9. In class, Brian performed the "belt trick." Essentially, he took off his belt, had a tall(er) volunteer hold one end of the belt rigidly in space in their outstretched hand, he held the other end to make the belt essentially horizontal to the ground, and he walked in a loop around the other person while lifting the belt over their head and bringing his end of the belt back to its beginning position and orientation. This left the belt with a twist, and he then walked in another loop in the same direction except now going under the volunteer's outstretched arm (walking in front of them), after which the twisting disappeared!

How exactly does this illustrate that $\pi_1(\mathrm{SO}(3))$ contains a non-trivial 2-torsion element? Visualize an element of $\mathrm{SO}(3)$ as an orthonormal frame, and think about one such frame assigned uniformly at all points along a line segment on the entire top of the belt. Thinking of a path in $\mathrm{SO}(3)$ as a continuous family of orthonormal frames, the end of the belt in Brian's hand is sweeping out the same *closed* loop σ in $\mathrm{SO}(3)$ both times he walks around in a circle (a closed loop since Brian's end of the belt returns to its original position in space at the end of both circular trips). But at any fixed point ξ of the belt away from the ends the first walk-around is not a *closed* loop in $\mathrm{SO}(3)$ due to the twist (which prevents the orthonormal frame based at ξ from returning to its original position in space). However, after going around twice, the motion of each point ξ of the belt *is* a closed loop in $\mathrm{SO}(3)$ (the corresponding orthonormal frame at ξ returning to its initial position) precisely because the belt ends up untwisted after the double walk-around.

Tracing out what is happening to the orthonormal frame at each point ξ of the belt during the entire process thereby gives a "continuous family" of closed loops linking the closed loop $\sigma * \sigma$ in $\mathrm{SO}(3)$ at Brian's endpoint of the belt to the identity loop in $\mathrm{SO}(3)$ at the other end of the belt (the rigid hand of the volunteer). In other words, this entire process visualizes the homotopy in $\mathrm{SO}(3)$ between $\sigma * \sigma$ and the identity. (The fact that $\sigma \not\simeq 1$ in $\pi_1(\mathrm{SO}(3))$ is not rigorously justified by this visualization.)

There are a lot of videos of the "belt trick" on the Internet, generally done in a very different way which completely misses the point of the mathematical content (i.e., visualizing the homotopy to the identity of the square of the generator of $\pi_1(\mathrm{SO}(3))$). Some day Tadashi Tokieda will do a proper job of this in an Internet video, perhaps on Numberphile, and in the meantime here are his illustrations of the 3 main steps of the process in reverse (corresponding to doing the walk-around under the arm of the volunteer first, and then going around the volunteer with the belt passing over their head, with the stool serving in the role of the rigid hand of the volunteer):

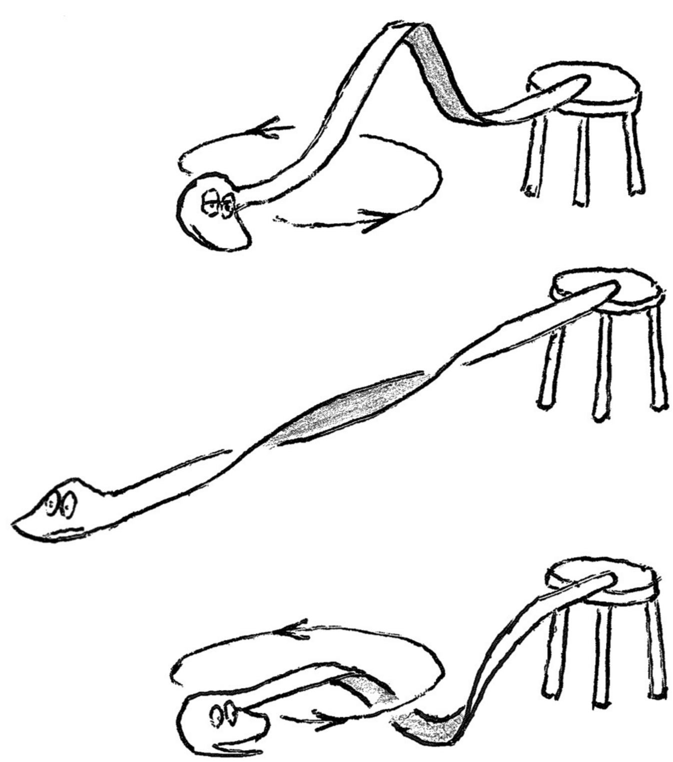


FIGURE 3. T. Tokieda's illustration of the belt trick from class in reverse

To build compact spin groups in general, we now define Clifford algebras. (See Appendix P for the full details on what follows.) Let (V, q) be a finite-dimensional quadratic space over a field k; a key example will be $k = \mathbf{R}$, $V = \mathbf{R}^n$, $q = q_n := -\sum_{i=1}^n x_i^2$. The idea of Clifford algebras is to try to extract a square root of q. More precisely, consider k-linear maps $V \xrightarrow{f} A$ to associative k-algebras A with identity such that $f(v)^2 = q(v) \cdot 1_A$. The universal such (A, f) is a quotient of the tensor algebra $T(V) = \oplus_{n \geq 0} V^{\otimes n}$, namely

$$C(q) = C(V, q) := T(V)/\langle v \otimes v - q(v) \cdot 1 \rangle =: C^+ \oplus C^-$$

for C^+ the even-graded part and C^- the odd-graded part.

Example 16.10. Taking $q = 0$ (but allowing $V \neq 0$!), $C(q)$ is the exterior algebra $\wedge^\bullet(V)$.

Fact 16.11. The map $V \to C(q)$ defined by $v \mapsto v$ is injective, because one shows that $C(q)$ has as a basis the products

$$\{e_{i_1} \cdots e_{i_r}\}_{i_1 < \cdots < i_r}$$

for an ordered basis $\{e_1, \ldots, e_n\}$ of V (recovering what we know for the exterior algebra).

Consider

$$\Gamma(q)' = \left\{ u \in C(q)^\times : uVu^{-1} = V,\ u \text{ homogeneous} \right\}$$

For $u \in \Gamma(q)'$ and any $v \in V$ we have $uvu^{-1} \in V$, so it makes sense to plug this into q:

$$q(uvu^{-1}) = uvu^{-1}uvu^{-1} = q(v).$$

Hence, $\Gamma(q)'$ acts on V *preserving* q. This gives a homomorphism $\Gamma(q)' \to \mathrm{O}(q)$.

Miracle 16.12. One can introduce a "norm" $\Gamma(q)' \to k^\times$ to remove an excess of scalars in $\Gamma(q)'$ (akin to using a determinant condition to get rid of the non-trivial central torus in $\mathrm{U}(n)$). Its kernel is a subgroup $\Gamma(q) \subset \Gamma(q)'$, and one shows $\Gamma(q) \to \mathrm{O}(q)$ is *surjective* in the special case $k = \mathbf{R}$, $V = \mathbf{R}^n$, $q = q_n := -\sum_i x_i^2$. It turns out that the preimage of $\mathrm{SO}(q_n) = \mathrm{SO}(n)$ in $\Gamma(q_n)$ is *connected* of degree 2 over $\mathrm{SO}(n)$; it is called $\mathrm{Spin}(n)$. The full details on this are discussed in Appendix P.

17. Torus centralizers and SU(2)

Today we will find out why SU(2) is the secret to life! We'll discuss central tori in connected compact Lie groups and explain why SU(2) is extremely ubiquitous beyond the commutative case.

For today, let G be a connected compact Lie group, $T \subset G$ a maximal torus, $Z := Z_G^0$ the maximal central torus (since a torus is the same as a connected compact commutative Lie group). Defining $G' := [G, G]$ to be the commutator subgroup (in the purely algebraic sense), later we'll show G' is a closed subgroup with $(G')' = G'$ (i.e., G' is "perfect") and that the natural homomorphism $Z \times G' \xrightarrow{\text{mult}} G$ is an isogeny (or equivalently $G' \to G/Z$ is an isogeny).

Example 17.1. In HW7, we'll see that every $g \in \mathrm{SU}(2)$ is a commutator, and this will turn out to imply $\mathrm{SU}(n)' = \mathrm{SU}(n)$. Thus, the isogeny $Z_{\mathrm{U}(n)} \times \mathrm{SU}(n) \to \mathrm{U}(n)$ considered last time is a special case of the general isogeny mentioned above.

One might ask when we would encounter G with nontrivial Z. So far, many of the examples we have seen, save $\mathrm{U}(n)$, have trivial central tori. However, groups with nontrivial central tori are ubiquitous. How do we supply these connected compact H with $Z_H^0 \neq 1$?

Let G be non-commutative. If we take any noncentral torus $S \subset G$, the torus centralizer $Z_G(S)$ is connected (as a consequence of the Conjugacy Theorem). Note that $S \subset G$ is contained in a maximal torus T', so $Z_G(S) \supset T'$. However, when S is chosen in a way related to $\Phi(G,T)$ that we will discuss soon, $Z_G(S)$ will often be non-commutative (and it has S as a central torus by design). Such centralizers will be a ubiquitous tool in the structural study of connected compact Lie groups.

Before continuing with torus centralizers, we digress briefly to discuss the link between subtori and $\mathbf{Q}$-vector spaces. Suppose T is a torus and let $V = X(T)_{\mathbf{Q}}$. We have a bijection

$$\begin{array}{c} \{V \subset X(T)_{\mathbf{Q}}\} \\ \updownarrow \\ \{\pi : X(T) \twoheadrightarrow \Lambda \text{ torsion-free }\} \\ \updownarrow \\ \{\text{ subtori } S \subset T\} \\ \updownarrow \\ \{\Lambda' \subset X(T) \text{ saturated submodule (i.e., } X(T)/\Lambda' \text{ is torsion free)}\} \end{array} \tag{17.1}$$

The maps are described by

$$\begin{aligned} \{V \subset X(T)_{\mathbf{Q}}\} &\to \{\Lambda' \subset X(T) \text{ saturated submodule}\} \\ V &\mapsto V \cap X(T), \end{aligned}$$

$$\begin{aligned} \{V \subset X(T)_{\mathbf{Q}}\} &\to \{\pi : X(T) \twoheadrightarrow \Lambda \text{ torsion-free }\} \\ V &\mapsto \text{ image of } X(T) \text{ in } X(T)_{\mathbf{Q}}/V, \end{aligned}$$

$$\begin{aligned} \{\Lambda' \subset X(T) \text{ saturated submodule}\} &\to \{\pi : X(T) \twoheadrightarrow \Lambda \text{ torsion-free}\} \\ S &\mapsto X(T/S) \subset X(T) \end{aligned}$$

where the restriction $X(T) \to X(S)$ is surjective (the short exact sequence of tori $1 \to S \to T \to T/S \to 1$ induces a short exact sequence of character lattices, as explained in Appendix K).

Here is a key example.

Example 17.2. Take $V = \mathbf{Q}a \subset X(T)_{\mathbf{Q}}$ for $a \in X(T) - \{0\}$, so $a : T \twoheadrightarrow S^1$. The corresponding subtorus $T_a \subset T$ is exactly $(\ker a)^0$, a codimension-1 subtorus (the unique one killed by a). Disconnectedness of $\ker(a)$ expresses that $\mathbf{Z}a \subset X(T)$ may not be saturated.

Returning to general noncommutative G, so $T \subset G$ is not central (by the Conjugacy Theorem), $\mathrm{Ad}_G|_T : \mathfrak{g} \to \mathfrak{g}$ is nontrivial (recall $\mathfrak{g}^T = \mathfrak{t}$), so $\Phi := \Phi(G,T)$ is *non-empty*. Pick $a \in \Phi$ and let $T_a \subset T$ be the unique codimension 1 subtorus killed by a. That is, $T_a := (\ker a)^0$. Consider $Z_G(T_a) \subset T$. In particular, $T \subset Z_G(T_a)$ is maximal and $T/T_a \subset Z_G(T_a)/T_a$ is a 1-dimensional maximal torus.

Lemma 17.3. *With notation as above, T_a is the maximal central torus in $Z_G(T_a)$.*

Remark 17.4. This lemma is very specific to codimension 1 subtori. It's also very important that a is a root. If a were not a root, $Z_G(T_a)$ would just equal T.

Proof. If T_a were not the maximal central torus in $Z_G(T_a)$ then the maximal central torus of $Z_G(T_a)$ would be a maximal torus, since the common dimension of the maximal tori of $Z_G(T_a)$ is $\dim T = 1 + \dim(T_a)$. Using that all maximal tori are conjugate, if a connected compact Lie group has a central maximal torus then it is commutative (again, we are using the Conjugacy Theorem). So then $Z_G(T_a) = T$.

So to get a contradiction it suffices to show $Z_G(T_a)$ is not commutative, which we will do via the Lie algebra. We shall now compute $\mathrm{Lie}(Z_G(T_a))_{\mathbf{C}}$ inside $\mathfrak{g}_{\mathbf{C}}$ for *any* $a \in X(T)$, and then get a contradiction when $a \in \Phi$. Since $T_a \subset T$, we have:

$$\mathrm{Lie}\,(Z_G(T_a))_{\mathbf{C}} = (\mathfrak{g}_{\mathbf{C}})^{T_a} = \mathfrak{t}_{\mathbf{C}} \oplus \left(\bigoplus_{b \in \Phi} (\mathfrak{g}_{\mathbf{C}})_b^{T_a} \right).$$

We want to understand when $(\mathfrak{g}_{\mathbf{C}})_b^{T_a} \neq 0$.

On $(\mathfrak{g}_{\mathbf{C}})_b$, the T-action is exactly via the character $b : T \twoheadrightarrow S^1$, by definition. Therefore, by definition, the T_a-invariants are

$$(\mathfrak{g}_{\mathbf{C}})_b^{T_a} = \begin{cases} 0 & \text{if } b|_{T_a} \neq 1 \\ (\mathfrak{g}_{\mathbf{C}})_b & \text{if } b|_{T_a} = 1. \end{cases}$$

We leave the following as an exercise:

Exercise 17.5. Let $T_a = (\ker a)^0$. Show $b|_{T_a} = 1 \iff b \in \mathbf{Q}a$. Hint: as a warm-up $b|_{\ker a} = 1$ if and only if $b \in \mathbf{Z}a$ because $b|_{\ker a} = 1$ says that b factors through a, and the endomorphisms of S^1 are power maps with an integer exponent:

(17.2)
$$\begin{tikzcd} T \arrow[rr, "a"] \arrow[dr, "b"'] & & S^1 \arrow[dl, "z^{\mathbf{Z}}"] \\ & S^1 & \end{tikzcd}$$

One then checks that being a rational multiple of a relates to being trivial just on the identity component of $\ker a$.

Therefore,

$$\mathrm{Lie}\,(Z_G(T_a))_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus (\bigoplus_{b \in \Phi \cap \mathbf{Q}a} (\mathfrak{g}_{\mathbf{C}})_b).$$

But now we recall that a is a root, so $a \in \Phi \cap \mathbf{Q}a$. Hence,

$$\mathrm{Lie}\,(Z_G(T_a)) \supsetneq \mathfrak{t},$$

so $Z_G(T_a) \supsetneq T$. ∎

Remark 17.6. The preceding proof shows that for $a \in X(T)$ such that $a \notin \mathbf{Q}b$ for all $b \in \Phi$ then $Z_G(T_a) = T$. Hence, it is rather special that when a is a root we have that T_a is the maximal central torus in $Z_G(T_a)$!

For $a \in \Phi(G, T)$, $Z_G(T_a)/T_a$ is noncommutative with maximal torus T/T_a and the roots of $Z_G(T_a)/T_a$ are precisely $\Phi \cap \mathbf{Q}a$ with the same weight spaces as for $\mathfrak{g}_{\mathbf{C}}$.

Therefore, for $a \in \Phi(G, T)$, if we want to compute the dimension of the a-weight space (Is it 1?), and determine which roots are rational multiples a (Is it $\{\pm a\}$? Is $-a$ actually a root?), then we can focus our attention on the non-commutative connected compact Lie group $Z_G(T_a)/T_a$ whose maximal tori are 1-dimensional.

Before we see why this is real progress, let's first work out what is going on in the usual running example:

Example 17.7. Let $G = \mathrm{U}(n)$ for some $n \geq 2$, and let T be the diagonal maximal torus, so $X(T) = \bigoplus_{i=1}^n \mathbf{Z}a_i$ for $a_i = \mathrm{pr}_i : T = \left(S^1\right)^n \to S^1$. We saw

$$\Phi = \left\{a_i - a_j : i \neq j\right\}$$

with $t^{a_i - a_j} = t_i/t_j$ for $t \in T$ (with t_i the ith entry of the diagonal unitary matrix t).

Fix $a = a_i - a_j$ for $i < j$ (corresponding to $E_{ij} \in \mathfrak{g}_{\mathbf{C}} = \mathfrak{gl}_n(\mathbf{C})$ that is upper triangular). Clearly $T_a = \left\{t \in T : t_i = t_j\right\}$. We have $T_a \subset Z_G(T_a)$, but what is $Z_G(T_a)$? Note that there is an "SU(2)" living on the span of the ith and jth basis vectors, $\mathrm{SU}(\mathbf{C}e_i \oplus \mathbf{C}e_j)$, and this visibly commutes with T_a.

As a sanity check (this isn't logically necessary), let's see explicitly that $T_a \cdot \mathrm{SU}(\mathbf{C}e_i \oplus \mathbf{C}e_j)$ contains T. The subgroup $\mathrm{SU}(\mathbf{C}e_i \oplus \mathbf{C}e_j)$ meets T in exactly the elements

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

for $z \in S^1$. Combing this torus in $\mathrm{SU}(\mathbf{C}e_i \oplus \mathbf{C}e_j)$ with T_a, we visibly get the entire maximal torus T (this is really an assertion in $\mathrm{U}(\mathbf{C}e_i \oplus \mathbf{C}e_j)$).

By inspection the commuting subgroups T_a and $\mathrm{SU}\left(\mathbf{C}e_i \oplus \mathbf{C}e_j\right)$ have *finite* intersection (equal to the center $\{\pm 1\}$ of SU(2)). Further, since $\Phi \cap \mathbf{Q}a = \{a, -a\}$, we have

$$\mathrm{Lie}\left(Z_G(T_a)/T_a\right)_{\mathbf{C}} = (\mathfrak{t}/\mathfrak{t}_a)_{\mathbf{C}} \oplus (\mathfrak{g}_{\mathbf{C}})_a \oplus (\mathfrak{g}_{\mathbf{C}})_{-a}$$

which is 3-dimensional over $\mathbf{C}$. Thus, the containment

$$Z_G(T_a) \supset T_a \cdot \mathrm{SU}\left(\mathbf{C}e_i \oplus \mathbf{C}e_j\right)$$

is an equality for dimension reasons!

Hence,

$$Z_G(T_a)/T_a = \mathrm{SU}(2)/\left\{\pm 1\right\} = \mathrm{SO}(3)$$

and (by centrality of T_a)

$$Z_G(T_a)' = \mathrm{SU}(\mathbf{C}e_i \oplus \mathbf{C}e_j)' = \mathrm{SU}(2)' = \mathrm{SU}(2).$$

We have the following big theorem, which is a vast generalization of what we saw above for $Z_G(T_a)'$ and $Z_G(T_a)/T_a$ for $G = \mathrm{U}(n)$:

Theorem 17.8. *If G is a non-commutative connected compact Lie group and* $\dim T = 1$ *then* $G = \mathrm{SU}(2)$ *or* $\mathrm{SO}(3)$.

Remark 17.9. This applies to $Z_H(T_a)/T_a$ for any non-commutative connected compact Lie group H, maximal torus T, and $a \in \Phi(H,T)$.

Here is a nice corollary.

Corollary 17.10. *In general* $\Phi \cap \mathbf{Q}a = \{\pm a\}$ *and* $\dim (\mathfrak{g}_{\mathbf{C}})_a = 1$.

Proof. Check this directly for $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$, and then apply Theorem 17.8 to $Z_G(T_a)/T_a$. □

Let's begin the proof of Theorem 17.8; we'll finish it next time. Note that $\dim (G/T) > 0$ and is even, so $\dim G/T \geq 2$, so $\dim G \geq 3$.

Lemma 17.11. *Theorem 17.8 holds in the case* $\dim G = 3$.

Proof. Fix a G-invariant positive-definite quadratic form q on $\mathfrak{g}$ (acted upon by the compact Lie group G via Ad_G). Thus, by connectedness of G we have

$$G \xrightarrow{\mathrm{Ad}_G} \mathrm{SO}(q) \subset \mathrm{GL}(\mathfrak{g})$$

where $\mathrm{SO}(3) = \mathrm{SO}(q) = \mathrm{O}(q)^0$ is 3-dimensional. But

$$(\ker \mathrm{Ad}_G)^0 = Z_G^0 \subset T = S^1$$

is a proper closed subgroup since T is not central (as G is non-commutative), so $Z_G^0 = 1$. Thus, $\ker \mathrm{Ad}_G$ is *finite*.

But $\dim G = 3 = \dim \mathrm{SO}(q)$, so for dimension reasons the inclusion $G/(\ker \mathrm{Ad}_G) \subset \mathrm{SO}(q) = \mathrm{SO}(3)$ is an equality. Hence, $G \to \mathrm{SO}(3)$ is an isogeny and more specifically a connected finite-degree covering space. Now form a pullback diagram of covering maps as compact (perhaps disconnected!) Lie groups:

$$\begin{array}{ccc} H & \longrightarrow & \mathrm{SU}(2) = S^3 \\ \downarrow & & \downarrow \\ G & \longrightarrow & \mathrm{SO}(3) \end{array} \tag{17.3}$$

with the vertical maps of degree 2. Suppose the bottom horizontal map is not an isomorphism (i.e., it is a finite-degree covering space with degree > 1). Then the top horizontal map is also a finite-degree covering map with degree > 1. But $\mathrm{SU}(2) = S^3$ is *simply connected*, so its covering space H must be *disconnected*.

Hence, for the degree-2 covering map $H \to G$, it follows that the open and closed subset $H^0 \subset H$ must be a lower-degree covering space over G and so has to be degree 1! This gives $H^0 \simeq G$. But $H^0 \to \mathrm{SU}(2)$ is a *connected* finite-degree covering space of $\mathrm{SU}(2) = S^3$, so it must also have degree 1 and hence is an isomorphism. We conclude that if $\mathrm{Ad}_G : G \to \mathrm{SO}(3)$ is not an isomorphism then $G \simeq H^0 \simeq \mathrm{SU}(2)$. □

18. Commutators and rank-1 subgroups

Last time we began the proof of the following crucial result.

Theorem 18.1. *Let G be a connected compact Lie group with rank* 1 *that is non-commutative. Then* $G \simeq \mathrm{SU}(2)$ *or* $G \simeq \mathrm{SO}(3)$.

Recall that the *rank* of a compact connected Lie group is the dimension of its maximal tori.

Proof. Note that $\dim(G/T) > 0$ and is even, so $\dim G/T \geq 2$, so $\dim G \geq 3$. Last time, in Lemma 17.11, we settled the case $\dim G = 3$. So, we just have to show $\dim G = 3$.

To this end, choose $T \subset G$ a maximal torus (with $\dim T = 1$ by assumption). Consider the weight space decomposition

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \left(\oplus_{\alpha \in \Phi} (\mathfrak{g}_{\mathbf{C}})_a\right).$$

with $\Phi := \Phi(G, T) \neq \emptyset$. We want to show $\dim_{\mathbf{C}}(\mathfrak{g}_{\mathbf{C}}) = 3$. It is enough to show that $\Phi = \{\pm a_0\}$ for some $a_0 \in X(T) - \{0\}$ with $\dim_{\mathbf{C}}(\mathfrak{g}_{\mathbf{C}})_a = 1$ for $a \in \{\pm a_0\}$.

We'll now highlight the key points. For full details, see Appendix Q. We'll connect our task to the representation theory of $\mathfrak{sl}_2(\mathbf{C})$ that has been discussed earlier.

To start, we give a direct argument for why the roots show up in opposite pairs (though we're not yet claiming that opposite roots have weight spaces of the same dimension, let alone of dimension 1).

Lemma 18.2. *We have $\Phi = -\Phi$.*

Proof. Note that $T = S^1$ acts on $\mathfrak{g}/\mathfrak{t}$. This is a (continuous) representation over $\mathbf{R}$, and as such is completely reducible (as for any compact group, due to averaging against Haar measure). When this representation is tensored up to $\mathbf{C}$ it has only nontrivial weights (since $\mathfrak{g}_{\mathbf{C}}^T = \mathfrak{t}_{\mathbf{C}}$ and the T-action on $\mathfrak{g}_{\mathbf{C}}$ is completely reducible). Hence, we want to understand the irreducible nontrivial representations of S^1 over $\mathbf{R}$.

The irreducible nontrivial representations ρ of S^1 over $\mathbf{R}$ are precisely the standard 2-dimensional representation (acting by rotations on $\mathbf{R}^2$) precomposed with some power map $z \mapsto z^n$ for $n \in \mathbf{Z}_{>0}$. To see this, one uses knowledge of the complex representation theory and then examines which representations have a character preserved under complex conjugation (and then one has to build a suitable descent over $\mathbf{R}$, namely from the asserted list of possibilities). This is carried out in HW7, Exercise 5(ii).

By inspection, $\rho_{\mathbf{C}} = \chi \oplus \chi^{-1}$ for some $\chi \in X(S^1)$; note that this consists of a character and its reciprocal. Thus, the same holds for $(\rho \circ z^n)_{\mathbf{C}}$ for any $n \in \mathbf{Z}$. Therefore, by complete reducibility of S^1-representations over $\mathbf{R}$, we are done. $\square$

Now, pick the highest weight $a_0 \in \Phi \subset X(T) = X(S^1) = \mathbf{Z}$. (Here we are fixing an isomorphism $T \simeq S^1$.) Since T acts on each weight space through a $\mathbf{C}^\times$-valued character, for any $a, b \in X(T)$ we have

$$[(\mathfrak{g}_{\mathbf{C}})_a, (\mathfrak{g}_{\mathbf{C}})_b] \subset (\mathfrak{g}_{\mathbf{C}})_{a+b}$$

(interesting only for $a, b \in \Phi$, but note that it can happen that $a + b = 0$, in which case the right side is $\mathfrak{t}_{\mathbf{C}}$). Using that $[\cdot,\cdot]_{\mathfrak{g}} \neq 0$ and that G has rank 1, one can show (using 1-parameter subgroups) that the subspace

$$\left[(\mathfrak{g}_{\mathbf{C}})_{a_0}, (\mathfrak{g}_{\mathbf{C}})_{-a_0}\right] \subset \mathfrak{t}_{\mathbf{C}}$$

is nonzero and so exhausts $\mathfrak{t}_{\mathbf{C}}$.

Then via suitable $\mathbf{C}^\times$-scaling, we can find $X_\pm \in (\mathfrak{g}_{\mathbf{C}})_{\pm a_0}$ and $H \in \mathfrak{t}_{\mathbf{C}}$ so that $\{X_+, X_-, H\}$ is an $\mathfrak{sl}_2$-triple (i.e., their span is $\mathfrak{sl}_2(\mathbf{C})$ with the usual bracket relations).

Remark 18.3. We are doing linear algebra at the complexified level $\mathfrak{g}_{\mathbf{C}}$! The elements $X_{\pm}$ and H we have made have no reason whatsoever to lie in $\mathfrak{g}$, so this has no direct meaning in terms of G. In particular, we're not claiming $H \in \mathfrak{t}$ (only in $\mathfrak{t}_{\mathbf{C}}$). Working with such elements is like an out-of-body experience.

Now we have a Lie subalgebra $\mathfrak{sl}_2(\mathbf{C}) \hookrightarrow \mathfrak{g}_{\mathbf{C}}$ whose diagonal is the line $\mathfrak{t}_{\mathbf{C}}$. This has the special feature that the 0-weight space for $\mathfrak{t}_{\mathbf{C}}$ on $\mathfrak{g}_{\mathbf{C}}$ (in the sense of Lie-algebra representations) is just $\mathfrak{t}_{\mathbf{C}}$ since connectedness of G and T forces $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) = \mathrm{Lie}(Z_G(T)) = \mathfrak{t}$, and $\mathfrak{t}_{\mathbf{C}}$ is a *line*. This is a very small weight-0 space. Moreover, we can view $\mathfrak{g}_{\mathbf{C}}$ as an $\mathfrak{sl}_2(\mathbf{C})$-representation, and by design $\mathfrak{g}_{\mathbf{C}}$ has a_0 as its highest weight. Therefore, by design of the $\mathfrak{sl}_2(\mathbf{C})$-subalgebra, all weights for H on $\mathfrak{g}_{\mathbf{C}}$ lie in $\{-2, -1, 0, 1, 2\}$.

Since the representation theory of $\mathfrak{sl}_2(\mathbf{C})$ is completely reducible, and the copy of $\mathfrak{sl}_2(\mathbf{C})$ inside $\mathfrak{g}_{\mathbf{C}}$ already accounts for the entire weight-0 space for H, there is no room for the occurrence of the trivial representation inside $\mathfrak{g}_{\mathbf{C}}$. We now look at the list of nontrivial irreducible representations of $\mathfrak{sl}_2(\mathbf{C})$. For weight reasons, the only options that can occur inside $\mathfrak{g}_{\mathbf{C}}$ are $\mathfrak{sl}_2(\mathbf{C})$ and the standard 2-dimensional representation std_2. The former can only occur once since the weight-0 space for H in $\mathfrak{g}_{\mathbf{C}}$ is a line, so

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}_2(\mathbf{C}) \oplus (\mathrm{std}_2)^{\oplus N}$$

as $\mathfrak{sl}_2(\mathbf{C})$-representations for some $N \geq 0$. In §Q.1 it is shown (again using that G has rank 1) that $N = 0$. This completes the proof. □

In general, for $a \in \Phi$, we have a short exact sequence

$$1 \longrightarrow T_a \longrightarrow Z_G(T_a) \longrightarrow Z_G(T_a)/T_a \longrightarrow 1, \tag{18.1}$$

the inclusion $T_a \to Z_G(T_a)$ is central and the quotient is either SO(3) or SU(2). We'd like to lift the right side back into $Z_G(T_a)$, at least up to isogeny. This is a special case of a general result:

Proposition 18.4. *Consider a central extension of connected compact Lie groups*

$$1 \longrightarrow S \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1 \tag{18.2}$$

for S a torus and $\overline{G} = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$. Then, $G' := [G, G]$ is closed in G and the multiplication map $S \times G' \to G$ (a homomorphism since S commutes with G') is an isogeny. In particular, $G' \to \overline{G}$ is an isogeny and G' is either SO(3) *or* SU(2).

Moreover,

(1) *if $\overline{G} = \mathrm{SU}(2)$ or $G' = \mathrm{SO}(3)$ then we have a unique splitting (i.e., $S \times G' = G$ via multiplication), so $G' = \overline{G}$;*
(2) *if $\overline{G} = \mathrm{SO}(3)$ and $G' = \mathrm{SU}(2)$ then $(S \times G')/\mu \simeq G$, where $\mu := S \cap G' = \{\pm 1\} \subset \mathrm{SU}(2) = G'$ is anti-diagonally embedded in $S \times G'$.*

Remark 18.5. To give some intuition for this result, because SU(2) and so also its quotient SO(3) coincides with its own commutator subgroup we may hope that the commutator $G' = [G, G]$ should give a "section up to isogeny" $\overline{G} \to G$. A general principle is to use commutators to build an isogeny complement to a central subgroup (a principle that will acquire more substance when we build on the current work to tackle commutator subgroups in higher-rank situations later).

Sketch of proof. The full details are in §2 of the handdount on SU(2). The idea of the proof is to build an "isogeny-section" to the exact sequence by using Lie algebras. Let's first focus on the Lie algebras. We have

$$0 \longrightarrow \mathfrak{s} \longrightarrow \mathfrak{g} \longrightarrow \overline{\mathfrak{g}} \simeq \mathfrak{su}(2) \longrightarrow 0 \tag{18.3}$$

in which $\mathfrak{s}$ is central in $\mathfrak{g}$ (in the Lie algebra sense!) because S is central in G. We claim that this exact sequence uniquely splits.

Note that $\mathfrak{su}(2)$ is its own commutator subalgebra (i.e., brackets generate all of $\mathfrak{su}(2)$) since it suffices to check this at the complexified level and $\mathfrak{su}(2)_{\mathbf{C}} = \mathfrak{sl}_2(\mathbf{C})$ (and one can check explicitly that usual basis vectors for weight spaces in $\mathfrak{sl}_2(\mathbf{C})$ are brackets among $\mathbf{Q}^{\times}$-multiplies of the standard basis vectors for weight spaces). Therefore, checking that the above sequence of Lie algebras splits is *equivalent* to checking that the natural map $[\mathfrak{g}, \mathfrak{g}] \to \overline{\mathfrak{g}}$ is an isomorphism. This also shows that a splitting is unique if it exists.

This isomorphism criterion for splitting is equivalent to the same after complexifying, and $[\mathfrak{g}, \mathfrak{g}]_{\mathbf{C}} = [\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}}]$. Complexifying the exact sequence gives an exact sequence

$$0 \longrightarrow \mathfrak{s}_{\mathbf{C}} \longrightarrow \mathfrak{g}_{\mathbf{C}} \longrightarrow \overline{\mathfrak{g}}_{\mathbf{C}} \longrightarrow 0 \tag{18.4}$$

in which $\mathfrak{s}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$ is central and $\overline{\mathfrak{g}}_{\mathbf{C}} = \mathfrak{sl}_2(\mathbf{C})$. This makes $\mathfrak{g}_{\mathbf{C}}$ a representation of $\mathfrak{sl}_2(\mathbf{C})$ (using that $\mathfrak{s}_{\mathbf{C}}$ is central: the Lie bracket always factors through the quotient by any central Lie subalgebra), so it makes $\mathfrak{g}_{\mathbf{C}}$ into a Lie-algebra representation of $\mathfrak{sl}_2(\mathbf{C})$.

Now by *complete reducibility* of the representation theory of $\mathfrak{sl}_2(\mathbf{C})$, we get the unique splitting (since there is only room in $\mathfrak{g}_{\mathbf{C}}$ for one copy of $\mathfrak{sl}_2(\mathbf{C})$)! Hence, (18.3) splits as a sequence over $\mathbf{R}$. This gives a section $\mathfrak{su}(2) \to \mathfrak{g}$. Since the Lie group SU(2) is *simply connected*, this section integrates to a Lie group map $h : \mathrm{SU}(2) \to G$ that in turn is an inclusion at the Lie algebra level. Therefore, this map has kernel which is a 0-dimensional normal discrete subgroup, hence central and finite. The center of SU(2) is $\{\pm 1\}$. Therefore, the kernel of h is either 1 or $\{\pm 1\}$, so either SU(2) or $\mathrm{SU}(2)/\{\pm 1\} = \mathrm{SO}(3)$ is thereby found as a closed subgroup of G.

We have identified SU(2) or SO(3) with a closed subgroup $H \subset G$ for which $S \cap H$ is finite and central in H, making H an isogeny complement to S! In particular, $H = [H, H] = [G, G]$, the final equality because S is central (the commutator map $\Gamma \times \Gamma \to \Gamma$ for any group Γ factors through a map $(\Gamma/Z) \times (\Gamma/Z) \to \Gamma$ for any central subgroup $Z \subset \Gamma$). Hence, the commutator subgroup $[G, G]$ is closed! The rest of the argument is now some bookkeeping, and is given in Appendix Q. □

Remark 18.6. Did you know that "bookkeeping" is one of the few words in the English language with three consecutive pairs of double letters? Another one is "bookkeeper". Yet another is "bookkeepers"! There are no others, but I hereby patent the word "ibookkeeper" so that if Apple invents an app to destroy the accounting industry then I can blackmail them.

As an important application, we'll build a "reflection" in $W(G, T)$ attached to each root $a \in \Phi(G, T)$. Consider Proposition 18.4 applied to

$$1 \longrightarrow T_a \longrightarrow Z_G(T_a) \longrightarrow Z_G(T_a)/T_a \longrightarrow 1. \tag{18.5}$$

We get a central isogeny

$$q_a : T_a \times Z_G(T_a)' \to Z_G(T_a).$$

As is worked out in HW7, under a central quotient map between connected compact Lie groups, the formation of image and preimage give bijections between the sets of maximal tori in each. (This is immediate from the Conjugacy Theorem, and uses that maximal tori always contain the center and that the image of a maximal torus under a quotient map is a maximal torus.) Note in particular that the preimage of a maximal torus under a central quotient map between connected compact Lie groups is always connected!

Applying this to the maximal torus $T \subset Z_G(T_a)$ that contains T_a, its q_a-preimage is $T_a \times T'_a$ for $T'_a := T \cap Z_G(T_a)$. Since a maximal torus in a direct product of connected compact Lie groups is exactly a direct product of maximal tori of the factors (again, obvious from the Conjugacy Theorem), it follows that T'_a is a maximal torus in the group $Z_G(T_a)'$ that is either $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. In particular, $\dim T'_a = 1$. (Alternatively, since T_a is codimension-1 in T and $T_a \times T'_a \twoheadrightarrow T$ is an isogeny, necessarily T'_a has dimension 1.)

Observe that

$$W(G,T) = N_G(T)/T \supset N_{Z_G(T_a)}(T)/T = W(Z_G(T_a),T),$$

and by the centrality of T_a in $Z_G(T_a)$ we clearly have

$$W(Z_G(T_a),T) = W\left(Z_G(T_a)', T'_a\right) =: W_a$$

with $Z_G(T_a)'$ equal to either $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. The Weyl group W_a embeds into $\mathrm{GL}(X(T'_a)) = \mathbf{Z}^\times$ (since the Weyl group of any connected compact Lie group acts faithfully on the character lattice of the maximal torus), so it has size *at most* 2. In fact, its size is exactly 2:

Lemma 18.7. *We have $W_a = \{1, r_a\}$ for an element r_a, necessarily of order* 2.

Proof. We just need to produce a nontrivial element in the Weyl group W_a for $(Z_G(T_a)', T'_a)$, and by the Conjugacy Theorem all maximal tori in $Z_G(T_a)'$ are "created equal" for this purpose. Since $Z_G(T_a)'$ is equal to $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$, it suffices to do a direct check for each of $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ with *some* choice of maximal torus.

For $\mathrm{SU}(2)$ we consider the maximal torus

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} : z \in S^1 \right\} \subset \mathrm{SU}(2)$$

for which

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

works (i.e., it normalizes the maximal torus but doesn't centralize it). For $\mathrm{SO}(3)$, we consider the maximal torus

$$\left\{ \begin{pmatrix} [z]_{\mathbf{R}^2} & 0 \\ 0 & 1 \end{pmatrix} : z \in S^1 \right\} \subset \mathrm{SO}(3)$$

for which

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

works. $\square$

19. Reflections in Weyl groups

Let's review the construction from last time. For any (G, T) and $a \in \Phi = \Phi(G, T)$, define $T_a := (\ker a)^0 \subset T$ to be the unique codimension-1 subtorus killed by $a : T \to S^1$. So, we get $Z_G(T_a) \supset T \supset T_a$. There is an isogeny

$$T_a \times Z_G(T_a)' \xrightarrow{\text{mult}} Z_G(T_a)$$

with the (closed!) commutator subgroup $Z_G(T_a)'$ that is isomorphic to SU(2) or SO(3) and has as a 1-dimensional maximal torus

$$T_a' := T \cap Z_G(T_a)'$$

for which the multiplication map

$$T_a \times T_a' \xrightarrow{\text{mult}} T$$

is an isogeny of tori. This isogeny between tori of the same dimension induces an inclusion between character lattices

$$X(T_a) \times X(T_a') \hookleftarrow X(T)$$

that (for $\mathbf{Z}$-rank reasons) must be a finite-index inclusion. In particular, $X(T_a)_{\mathbf{Q}} \oplus X(T_a')_{\mathbf{Q}} \simeq X(T)_{\mathbf{Q}}$.

The Weyl group

$$W(G, T) = N_G(T)/T = N_G(T)/Z_G(T) \hookrightarrow \mathrm{GL}(X(T)) \subset \mathrm{GL}(X(T)_{\mathbf{Q}})$$

clearly contains as a subgroup $W(Z_G(T_a), T) = N_{Z_G(T_a)}(T)/T$. This subgroup has another description:

Lemma 19.1. *The natural map $W(Z_G(T_a)', T_a') \to W(Z_G(T_a), T)$ arising from the description T as $T_a \cdot T_a'$ is an isomorphism.*

Proof. Since $T_a' = T \cap Z_G(T_a)'$ and $Z_G(T_a)$ is the "almost direct product" of the subgroups $Z_G(T_a)'$ and T_a with T_a central in $Z_G(T_a)$ and contained in T, our task reduces to checking that an element $g \in Z_G(T_a)$ normalizes T if and only if it normalizes $T \cap Z_G(T_a)' = T_a'$.

If $g \in Z_G(T_a)$ normalizes T then since everything in $Z_G(T_a)$ normalizes the normal subgroup $Z_G(T_a)'$ we see that g certainly normalizes the intersection $T_a' = T \cap Z_G(T_a)'$. Conversely, if g normalizes T_a' then since it even centralizes T_a it certainly normalizes $T_a' \cdot T_a = T$. $\square$

To simplify notation (and conserve the valuable chalk being used in class), we introduce the following shorthand: we write G_a to denote the commutatator subgroup $Z_G(T_a)'$ (with maximal torus T_a'). This is either an SU(2) or SO(3), $\Phi(G_a, T_a') = \{\pm a|_{T_a'}\}$ (note that $a|_{T_a} = 1$ and $T = T_a \cdot T_a'$), and we have the direct sum of lines

$$\mathrm{Lie}(G_a)_{\mathbf{C}} = \mathrm{Lie}(T_a')_{\mathbf{C}} \oplus (\mathfrak{g}_{\mathbf{C}})_a \oplus (\mathfrak{g}_{\mathbf{C}})_{-a}$$

as the weight-space decomposition for the action of the maximal torus $T_a' \subset G_a$.

We have

$$W(G, T) \supset W(Z_G(T_a), T) = W(G_a, T_a') \hookrightarrow \mathrm{GL}\left(X(T_a')\right) = \mathbf{Z}^{\times} = \{\pm 1\},$$

so $W(G_a, T_a')$ has size at most 2. In fact its size is exactly 2, since we can exhibit a non-trivial coset class by checking both possibilities SU(2) and SO(3) for G_a directly (using a "standard" choice of maximal torus for the calculation, the choice of which doesn't matter thanks to the Conjugacy Theorem):

$$W\left(G_a, T_a'\right) = \begin{cases} W\left(\mathrm{SU}(2), \{\text{diagonal torus}\}\right) = \left\{\mathrm{id}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \right\} \\ W\left(\mathrm{SO}(3), \begin{pmatrix} r_\theta & \\ & 1 \end{pmatrix}\right) = \left\{\mathrm{id}, \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix}\right\}. \end{cases}$$

We let r_a denote the unique nontrivial element in $W(G_a, T_a')$. Since $W(G, T)$ is faithfully represented in its action on $X(T)_{\mathbf{Q}}$, the element r_a is determined by its action on $X(T)_{\mathbf{Q}} \simeq X(T_a)_{\mathbf{Q}} \oplus X(T_a')_{\mathbf{Q}}$. We now describe that action:

Lemma 19.2. *The effect of r_a is the identity on the hyperplane $X(T_a)_{\mathbf{Q}}$ and is negation on the line $X(T_a')_{\mathbf{Q}}$.*

Proof. The effect on $X(T_a)_{\mathbf{Q}}$ is 1 because r_a comes from $Z_G(T_a)' - T_a' \subset Z_G(T_a)$. Its effect on $X(T_a')_{\mathbf{Q}}$ is negation because by design r_a is a nontrivial element of $N_{Z_G(T_a)'}(T_a')/Z_{Z_G(T_a)'}(T_a')$ and $\mathrm{GL}(X(T_a')) = \mathbf{Z}^\times = \{1, -1\}$. $\square$

Warning 19.3. There is no natural notion of orthogonality in $X(T)_{\mathbf{Q}}$ or $X(T)_{\mathbf{R}}$ (yet!), so we cannot say that the element r_a is a "reflection" about a hyperplane in the sense of Euclidean geometry. Its eigenspace decomposition on $X(T)_{\mathbf{Q}}$ does pick out both a hyperplane and a linear complementary line, but it is not determined (yet) by that hyperplane or line.

Example 19.4. Let's now work out what all of the above is saying in the first nontrivial case: $G = \mathrm{SU}(3)$, T the diagonal maximal torus, and a choice of root $a \in \Phi = \Phi(G, T)$. Explicitly we can write T as

$$T = \left\{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & 1/(t_1 t_2) \end{pmatrix} : t_1, t_2 \in S^1 \right\} \simeq S^1 \times S^1.$$

This is an asymmetric description of $X(T)$ that we have previously described in more symmetric terms as $\mathbf{Z}^3/\Delta$ (the quotient of the character lattice $\mathbf{Z}^3$ of the diagonal maximal torus of $\mathrm{U}(3)$ modulo the diagonal). Here, we choose the noncanonical isomorphism $\mathbf{Z}^3/\Delta \simeq \mathbf{Z} \oplus \mathbf{Z}$ by picking out the first two basis vectors of $\mathbf{Z}^3$.

Let $\overline{a}_i \in \mathbf{Z}^3/\Delta$ denote the residue class of the ith standard basis element $a_i \in \mathbf{Z}^3$ (corresponding to projection to the ith entry on the diagonal torus of $\mathrm{U}(3)$), so the root system for $(\mathrm{SU}(3), T)$ is

$$\Phi = \left\{\overline{a}_i - \overline{a}_j : i \neq j\right\} \subset \mathbf{Z}^3/\Delta = (\oplus \mathbf{Z} a_i)/\Delta.$$

Let's now focus on the root $a = \overline{a}_1 - \overline{a}_2 : t \mapsto t_1/t_2 \in S^1$. In this case we easily compute that

$$T_a := (\ker a)^0 = \left\{ \begin{pmatrix} \tau & & \\ & \tau & \\ & & \tau^{-2} \end{pmatrix} : \tau \in S^1 \right\},$$

so preservation of the T_a-weight space decomposition of $\mathbf{C}^3$ yields that

$$Z_G(T_a) = \left\{ \begin{pmatrix} g & 0 \\ 0 & 1/\det(g) \end{pmatrix} : g \in \mathrm{U}(\mathbf{C}e_1 \oplus \mathbf{C}e_2) \right\} \simeq \mathrm{U}(2).$$

Joke 19.5. Here is a true story: a famous mathematician (who shall remain nameless) once overheard some students milling around before the start of class discussing their excitement about an upcoming concert. One of the students said "my favorite group is U2." The mathematician was puzzled by this and asked: "why not U(3) or U(4)?"

Continuing with our example, we see from the description of $Z_G(T_a)$ that

$$G_a := Z_G(T_a)' = \begin{pmatrix} \mathrm{SU}(\mathbf{C}e_1 \oplus \mathbf{C}e_2) & \\ & 1 \end{pmatrix} \simeq \mathrm{SU}(2)$$

with associated maximal torus

$$T_a' := T \cap G_a = \left\{ \begin{pmatrix} z & 0 & \\ 0 & 1/z & \\ & & 1 \end{pmatrix} : z \in S^1 \right\}.$$

Observe that

$$T_a \cap T_a' = \left\{ \begin{pmatrix} \varepsilon & & \\ & \varepsilon & \\ & & 1 \end{pmatrix} : \varepsilon = \pm 1 \right\} \simeq \mu_2$$

is finite (as we know it must be, since the multiplication map $T_a \times T_a' \to T$ is an isogeny), so we get an exact sequence

$$1 \longrightarrow \mu_2 \longrightarrow T_a \times T_a' \xrightarrow{\text{mult}} T \longrightarrow 1. \tag{19.1}$$

Hence, the natural finite-index inclusion $X(T) \to X(T_a) \oplus X(T_a')$ must have index 2.

We can see this inclusion explicitly upon making the following identifications: $X(T) \simeq \mathbf{Z}^2$ via the basis $\{\overline{a}_1, \overline{a}_2\}$, $X(T_a) \simeq \mathbf{Z}$ via projection to the upper left matrix entry (in effect, $\overline{a}_1|_{T_a}$ as a basis), and $X(T_a') \simeq \mathbf{Z}$ via the same projection (in effect, z as a basis). Under this identification of $X(T)$ with $\mathbf{Z}^2$, the pair $(n, m) \in X(T)$ corresponds to the character $t \mapsto t_1^n t_2^m$ in $X(T)$. Restricting that formula to each of T_a and T_a' that we have parameterized as above, we thereby see that the inclusion

$$X(T) \hookrightarrow X(T_a) \oplus X(T_a')$$

is identified with the map $\mathbf{Z}^2 \to \mathbf{Z}^2$ defined by

$$(n, m) \mapsto (n + m, n - m)$$

(since evaluating $t_1^n t_2^m$ with $t_1 = t_2 = \tau$ yields τ^{n+m} and with $t_1 = z, t_2 = 1/z$ yields z^{n-m}). This is visibly an index-2 inclusion, as we knew it had to be (since math is consistent).

From the abstract setting we know by design of r_a as an element of both $W(G_a, T_a') = W(Z_G(T_a), T) \subset W(G, T)$ that it acts compatibly on $X(T)$ and $X(T_a) \oplus X(T_a')$ respecting the direct sum decomposition. More specifically, we know that its effect on $X(T_a)$ is the identity and on $X(T_a')$ is negation. Let's see this directly in our situation with SU(3) and a as above. For later purposes we'll push this a bit further by computing how Φ sits inside $X(T) = \mathbf{Z}^2$ and analyzing how r_a acts on Φ under this description.

Since $\overline{a}_3 = -(\overline{a}_1 + \overline{a}_2)$ (why?), we have in $X(T) = \mathbf{Z}^2$ (using the basis $\{\overline{a}_1, \overline{a}_2\}$) that

$$\overline{a}_1 - \overline{a}_2 = (1, -1)$$
$$\overline{a}_2 - \overline{a}_3 = (1, 2)$$
$$\overline{a}_1 - \overline{a}_3 = (2, 1)$$

But the effect of r_a on $X(T_a) \oplus X(T'_a)$ is the identity on the first factor and negation on the second. Hence, on $(n, m) \in \mathbf{Z}^2 = X(T)$ that corresponds to $(n + m, n - m) \in X(T_a) \oplus X(T'_a)$ its yields the output $(n + m, m - n) \in X(T_a) \oplus X(T'_a)$ that comes from $(m, n) \in \mathbf{Z}^2 = X(T)$.

Therefore, for $a = \overline{a}_1 - \overline{a}_2$ the action of r_a on $X(T) = \mathbf{Z}^2$ has the following effect on some roots:

$$\begin{cases} \overline{a}_1 - \overline{a}_2 = (1, -1) \mapsto (-1, 1) = -\overline{a}_1 + \overline{a}_2 = -a \\ \overline{a}_2 - \overline{a}_3 = (1, 2) \mapsto (2, 1) = \overline{a}_1 - \overline{a}_3 \\ \overline{a}_1 - \overline{a}_3 = (2, 1) \mapsto (1, 2) = \overline{a}_2 - \overline{a}_3. \end{cases}$$

In terms of the Euclidian visualization of $X(T) = \mathbf{Z}^3/\Delta$ we have seen earlier, the effect of r_a is reflecting about the y-axis in Figure 4

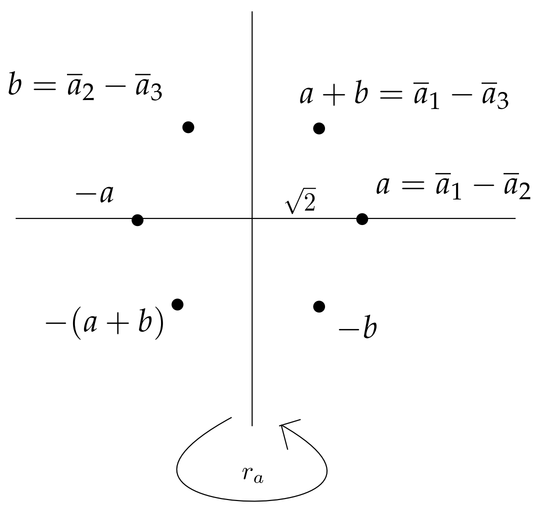


FIGURE 4. Roots for U(3) with Weyl group element r_a

It turns out that $W = S_3$ acts as D_3 (the dihedral group of order 6) on that regular hexagon whose vertices are the roots. (This calculation of the Weyl group, and for all special unitary groups more generally, will be taken up later.) Although there is no Euclidean structure instrinsic to our situation with $X(T)_{\mathbf{Q}}$, there is a precise sense in which r_a deserves to be called a reflection:

Definition 19.6. Let V be a finite-dimensional vector space over a field k of characteristic 0 (we will often care about $k = \mathbf{Q}, \mathbf{R}, \mathbf{C}$). A *reflection* on V is an isomorphism $r : V \simeq V$ so that $r^2 = \mathrm{id}$ with -1-eigenspace a line (so the 1-eigenspace is complementary hyperplane).

We emphasize that the -1-eigenline alone doesn't determine the complemtary hyperplane that is the 1-eigenspace of r since there is no "Euclidean structure" (non-degenerate quadratic form, say) specified that is preserved by r.

Example 19.7. Take $V = X(T)_{\mathbf{Q}}$ with r_a acting as -1 on the eigenline $X(T'_a)_{\mathbf{Q}}$ and as 1 on the hyperplane $X(T_a)_{\mathbf{Q}}$.

Note that $r_a \in W \subset \mathrm{GL}\left(X(T)_{\mathbf{Q}}\right)$ with W finite. This allows us to apply the following result to show that from the specification of W the element r_a is determined by the line on which it acts as negation ("as if" there were a Euclidean structure, which there is not), and likewise is determined by the hyperplane on which it acts as the identity.

Lemma 19.8. *Let $\Gamma \subset \mathrm{GL}(V)$ be a finite subgroup and $r, r' \in \Gamma$ two reflections. If r and r' negate the same line, or alternatively act as 1 on the same hyperplane, then $r = r'$.*

Proof. Suppose they act as -1 on the same line L. Consider the exact sequence

$$0 \longrightarrow L \longrightarrow V \longrightarrow V/L \longrightarrow 0 \tag{19.2}$$

that is compatible with each of r and r' acting on V, so we can consider the effect of $r'r^{-1} \in \Gamma$ on this situation. The effect on L in $(-1)^2 = 1$, and its effect on V/L is also 1, so $r'r^{-1}$ is unipotent, necessarily of finite order (since Γ is finite). By the same type of argument, if r and r' act as the identity on the same line then again $r'r^{-1}$ is unipotent, necessarily of finite order.

But over a field of characteristic 0, a unipotent matrix of finite order is trivial (exercise, using $\log(1+N)$ for nilpotent matrices N)! Hence, $r'r^{-1} = \mathrm{id}$, so $r' = r$. $\square$

20. Root systems

For (G, T, a) as usual, last time we constructed $r_a \in W(G_a, T'_a) = W(Z_G(T_a), T) \subset W(G, T)$ whose action on $X(T)_{\mathbf{Q}} = X(T_a)_{\mathbf{Q}} \oplus X(T'_a)_{\mathbf{Q}}$ is 1 on $X(T_a)_{\mathbf{Q}}$ and -1 on $X(T'_a)_{\mathbf{Q}}$; in particular, it is a reflection. The -1-eigenline has a rather direct description:

Lemma 20.1. *The line $X(T'_a)_{\mathbf{Q}} \subset X(T)_{\mathbf{Q}}$ is exactly $\mathbf{Q} \cdot a$.*

Proof. By definition, $T_a = (\ker a)^0$, so $a|_{T_a} = 1$. This means the projection to $X(T_a)_{\mathbf{Q}}$ is 0. Then, using that the inclusion on character lattices is given by

$$\begin{aligned} X(T) &\to X(T_a) \oplus X(T'_a) \\ \chi &\mapsto \left(\chi|_{T_a}, \chi|_{T'_a}\right), \end{aligned}$$

we're done. $\square$

Corollary 20.2. *We have $r_a(a) = -a$, and this characterizes the element $r_a \in W(G, T)$ as a reflection on $X(T)_{\mathbf{Q}}$.*

The point of this corollary is that it characterizes r_a in terms of a rather directly, without reference to the specific construction involving the pair (G'_a, T'_a).

Proof. This is immediate from Lemma 20.1 and Lemma 19.8 (with $\Gamma = W(G, T)$). $\square$

Here are some more consequences.

Corollary 20.3. *We have:*

1. *for $a, b \in \Phi$, $r_a = r_b$ if and only if $b = \pm a$;*
2. *on $X(T)_{\mathbf{Q}}$, $r_a(x) = x - \ell_a(x)a$ for a unique linear form $\ell_a \in X(T)^*_{\mathbf{Q}}$ (so $\ell_a(a) = 2$ since $r_a(a) = -a$).*

Proof. We prove these in order.

(1) The implication "$\Leftarrow$" holds because r_a and r_{-a} are reflections in the finite group $W(G,T)$ that negate the same line (so Lemma 19.8 forces $r_a = r_{-a}$); this could also seen from the construction using (G'_a, T'_a), but it is much more elegant to argue via Lemma 19.8. For the converse implication, if $r_a = r_b$ then the lines negated by these reflections coincide, hence $\mathbf{Q}a = \mathbf{Q}b$, so $b \in \Phi \cap \mathbf{Q}a = \{\pm a\}$.
(2) Via the decomposition $X(T)_{\mathbf{Q}} = X(T_a)_{\mathbf{Q}} \oplus X(T'_a)_{\mathbf{Q}}$ for which r_a acts as the identity on the hyperplane $X(T_a)_{\mathbf{Q}}$, we see that $r_a(x) - x \in X(T'_a)_{\mathbf{Q}} = \mathbf{Q}a$ for all x. Expressing the coefficient of a as $\ell_a(x)a$ for a function $\ell_a : X(T)_{\mathbf{Q}} \to \mathbf{Q}$, clearly ℓ_a is linear and is unique.

$\square$

Miracle 20.4. It will turn out that $\ell_a(\Phi) \subset \mathbf{Z}$ and even better: $\ell_a(X(T)) \subset \mathbf{Z}$. We'll give a Lie group interpretation of this via co-characters $X_*(T) = \mathrm{Hom}(S^1, T)$ later today, in terms of which this important integrality property will be established.

Remark 20.5. In HW7 you'll show that the finite subgroup $Z_G/Z_G^0 \subset G/Z_G^0$ is the center of G/Z_G^0 (which is perhaps quite surprising, since in general passing to a quotient of a group can lead to the center of the quotient being bigger than the image of the original center; think about a non-abelian solvable group). Moreover, in HW7 you'll show the following are equivalent:

(1) Z_G is finite (which is to say $Z_G^0 = 1$, or in other words G has no nontrivial central torus);
(2) $\mathbf{Q}\Phi = X(T)_{\mathbf{Q}}$ (i.e., Φ spans $X(T)_{\mathbf{Q}}$ over $\mathbf{Q}$);
(3) the $\mathbf{Z}$-span $\mathbf{Z}\Phi$ (called the "root lattice") inside the "character lattice" $X(T)$ has finite index (then necessarily equal to $\#Z_G$).

Further, it will be shown in HW8 that these conditions imply $G = G'$ (i.e., G is its own commutator subgroup, so it is "very non-commutative"). The converse is also true: if $G = G'$ then the above conditions all hold (this will be shown in HW9, where it will also be proved that G' is always *closed* in G).

Example 20.6. Here is a chart depicting the centers of several classical matrix groups.

G	Z_G
$\mathrm{SU}(n)$	μ_n
$\mathrm{Sp}(n)$	μ_2
$\mathrm{SO}(2m), m \geq 2$	μ_2
$\mathrm{SO}(2m-1), m \geq 2$	1

TABLE 1. Centers of classical groups

Recall that we always have that the faithful action of W on $X(T)$ preserving $\Phi \subset X(T) - \{0\}$, so W acts faithfully on $X(T)_{\mathbf{Q}}$ also preserving the image of Φ in $X(T)_{\mathbf{Q}}$. This entire discussion finally brings us to an important general concept:

Definition 20.7. Let k be a field of characteristic 0. A *root system* over k is a pair (V, Φ) for a finite-dimensional k-vector space V and a finite set $\Phi \subset V - \{0\}$ (called *roots*) so that:

RS1 the k-span $k\Phi$ is equal to V (so $V = 0 \iff \Phi = \emptyset$)
RS3 for all $a \in \Phi$, there is a reflection $r_a : V \simeq V$ satisfying
  (a) $r_a(a) = -a$ (so $-\Phi = \Phi$);
  (b) $r_a(\Phi) = \Phi$;
  (c) writing $r_a(x) = x - \ell_a(x)a$ for a unique $\ell_a \in V^*$, we have $\ell_a(\Phi) \subset \mathbf{Z}$ (this is called the Integrality Axiom).

A root system is called *reduced* if it additionally satisfies

RS2 $ka \cap \Phi = \{\pm a\}$.

We will always work with reduced root systems because the root systems that arise from connected compact Lie groups will turn out to be reduced. But there are good reasons to care about non-reduced root systems. For example, when classifying (or analyzing the structure of) non-compact connected Lie groups with a "semisimple" Lie algebra or connected "semisimple" linear algebraic groups over general fields (a notion that is most appropriately defined using ideas from algebraic geometry), there is always an associated root system (this is a rather hard fact!) but it can be non-reduced (when the group is "non-split", a basic example being special unitary groups over $\mathbf{R}$ with mixed signature).

It may seem in the definition of a root system that the reflection r_a is somehow ambiguous or is extra data to be specified. Let's see that such a reflection (if it exists) is uniquely determined by (V, Φ) and the required properties. The key point is that

$$r_a \in \Gamma := \{T \in \mathrm{GL}(V) : T(\Phi) = \Phi\}$$

with Φ a finite spanning set of V, and Γ is a *finite* group (since a linear automorphism is determined by its effect on a spanning set, and there are only finitely many permutations of the finite set Φ). Hence, r_a is uniquely determined by the further requirement that it negates the line ka (here we are again using Lemma 19.8). Further, as we have seen in the Lie group setting, for any root system (V, Φ) we have $r_a = r_b \iff b = \{\pm a\}$, using RS2 for "$\Rightarrow$".

We shall see that there is a very rich theory of root systems, and that in turn is a powerful tool in the the study of connected compact Lie groups (with finite center). To make the link, we need to verify the Integrality Axiom for $(X(T)_{\mathbf{Q}}, \Phi(G, T))$ when Z_G is finite. In fact, we shall do much more: we will show $\ell_a(X(T)) \subset \mathbf{Z}$ *without* any restriction on Z_G.

Definition 20.8. For (V, Φ) a root system, its **Weyl group** is the subgroup

$$W(\Phi) = \langle r_c \rangle_{c \in \Phi} \subset \mathrm{GL}(V).$$

This is *finite* since it consists of linear automorphisms that permute the finite spanning set Φ of V.

Remark 20.9. Later, we'll show that naturally

$$W(\Phi(G, T)) = W(G, T)$$

for any G with finite center. In fact, we'll do much better: the general assertion we will later establish (without hypotheses on Z_G) is that $W(G, T)$ is generated by the reflections r_a for $a \in \Phi(G, T)$; this is not at all evident.

20.1. **Geometric meaning of the integrality axiom, in the Lie theory setting.** Let's now briefly discuss the geometric meaning of the Integrality Axiom. Choose a $W(G,T)$-invariant inner product, $\langle\cdot|\cdot\rangle$ on $X(T)_{\mathbf{R}}$ (which happens to be $\mathfrak{t}^*$, which we won't need right now). Thus, all r_a act as isometries on $X(T)_{\mathbf{R}}$, so they r_a are actual reflections about $\mathbf{R}a$ in the sense of Euclidean geometry. Therefore,

$$r_a(x) = x - 2\left\langle x|\frac{a}{\|a\|}\right\rangle \frac{a}{\|a\|} = x - 2\frac{\langle x|a\rangle}{\langle a|a\rangle}a = x - \left\langle x|\frac{2a}{\langle a|a\rangle}\right\rangle a.$$

Thus,

$$a^{\vee} := \frac{2a}{\langle a|a\rangle} \in X(T)_{\mathbf{R}}$$

computes ℓ_a via $\langle\cdot|\cdot\rangle$. (Of course, $a^{\vee}$ viewed inside $X(T)_{\mathbf{R}}$ in this way depends on the choice of inner product. The more intrinsic object is the linear form ℓ_a, which involves no Euclidean structure. Of course, identifying a finite-dimensional $\mathbf{R}$-vector space with its dual via an inner product depends on the inner product.)

The Integrality Axiom says

$$n_{a,b} := \ell_a(b) = \frac{2\langle a|b\rangle}{\langle a|a\rangle} \in \mathbf{Z}$$

for all $a, b \in \Phi$. These $n_{a,b}$ are called the *Cartan numbers*. Note from the definition that $n_{a,b}$ has no reason to be symmetric in (a,b) nor any reason to be additive in a (and rank-2 examples as we will see below illustrate those features).

20.2. **Examples of root systems.** Our examples will be for 2-dimensional $\mathbf{R}$-vector spaces.

Example 20.10 ($\mathrm{A}_1 \times \mathrm{A}_1$)**.** Here we have $W = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, $c^{\vee} = c$ for all $c \in \Phi$, and $n_{a,b} = 0$.

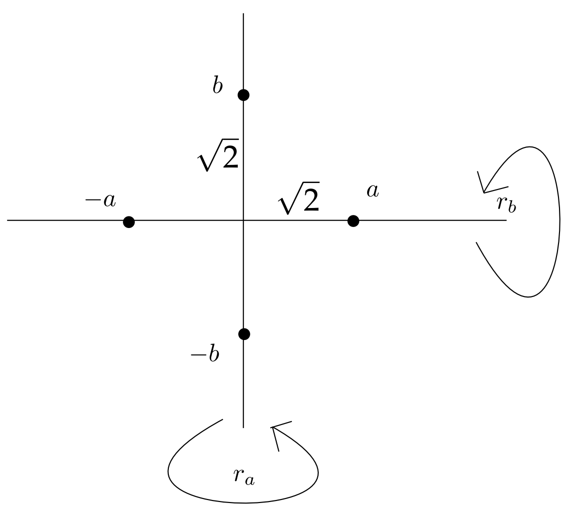


FIGURE 5. $\mathrm{A}_1 \times \mathrm{A}_1$ root system

Example 20.11 (A_2)**.** Here we have $c^{\vee} = c$ for all $c \in \Phi$, $W(\Phi) = S_3 = D_3$, and $n_{a,b} = -1 = n_{b,a}$.

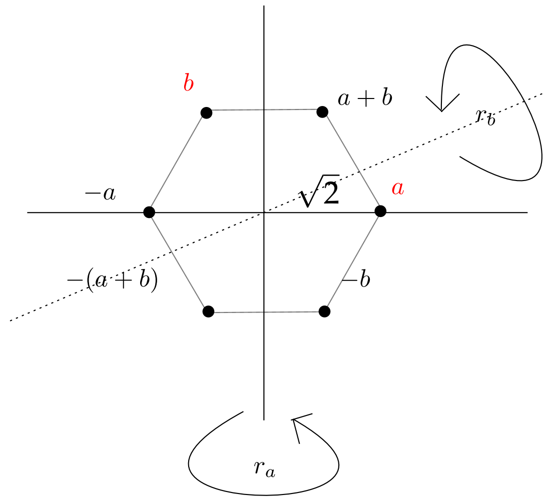


FIGURE 6. A_2 root system

Example 20.12 (B_2). Here we have $n_{a,b} = -2$, $n_{b,a} = -1$, and $W(\Phi) = S_2 \ltimes (\mathbf{Z}/2\mathbf{Z})^2$. (We write S_2 here because in the higher-rank root system to be called B_n the Weyl group is the semi-direct product $S_n \ltimes (\mathbf{Z}/2\mathbf{Z})^n$.)

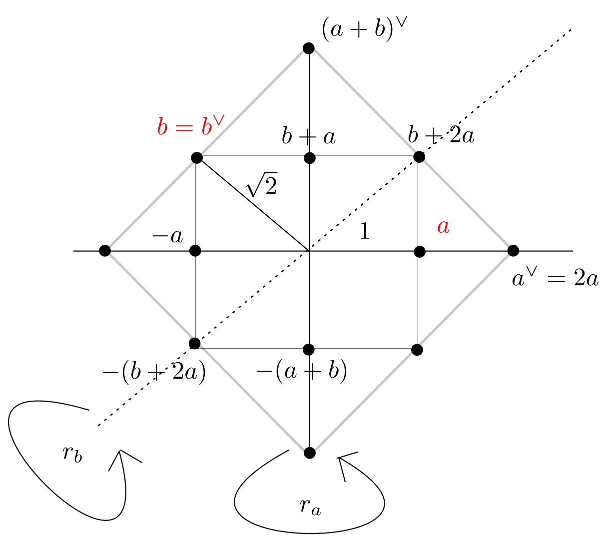


FIGURE 7. B_2 root system and the cocharacters $c^\vee$

Example 20.13 (G_2). Here $a^\vee = a$, $b^\vee = b/3$, $W(\Phi) = D_6$, and $n_{b,a} = -1, n_{a,b} = -3$.

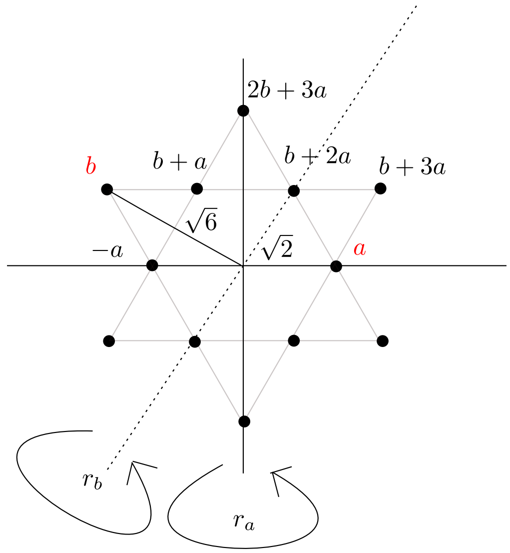


FIGURE 8. G_2 root system

Coming back to the verification of the Integrality Axiom for the data arising from Lie theory, we need to address the Lie group meaning of the integrality condition on $\ell_a \in X(T)^*_{\mathbf{Q}}$? Our goal is to show $\ell_a(X(T)) \subset \mathbf{Z}$ (with *no* finiteness hypothesis on Z_G), so in particular $\ell_a(\Phi(G,T)) \subset \mathbf{Z}$ as desired.

It is easy to see that the bilinear pairing

$$\begin{aligned} X(T) \times X_*(T) &\to \mathrm{End}(S^1) = \mathbf{Z} \\ (\chi, \lambda) &\mapsto \chi \circ \lambda = \langle \chi, \lambda \rangle \end{aligned}$$

is a perfect pairing of finite free $\mathbf{Z}$-modules. Thus, we seek a homomorphism of Lie groups

$$a^\vee : S^1 \to T$$

so that

$$r_a(x) = x - \langle x, a^\vee \rangle a$$

for all $x \in X(T)$. Since

$$r_a(a) = -a,$$

a necessary condition $\langle a, a^\vee = 2 \rangle$, and this is sufficient when $\dim T = 1$. In such "rank-1" cases, the condition is then that the isogeny $a : T \to S^1$ of 1-dimensional tori factors through the squaring endomorphism of S^1, which is exactly the condition that $\#(\ker a)$ divides 2. For SU(2) this is fine (the kernel has order 2) and for SO(3) the root viewed as a character is even an isomorphism. In general, one can check the construction for (G_a, T'_a) and $a|_{T'_a}$ works: we use the composite cocharacter $S^1 \to T'_a \subset T$; see Appendix R for details.

21. Weyl groups and coroots for root systems

Let (G, T, a) be the usual data: a compact connected Lie group G, a maximal torus T, and a root $a \in \Phi := \Phi(G, T)$. Last time we built a cocharacter

$$a^\vee : S^1 \to T$$

so that

$$r_a(x) = x - \langle x, a^\vee \rangle a$$

for all $x \in X(T)_{\mathbf{Q}}$; here, $\langle x, a^\vee \rangle := x \circ a^\vee \in \mathrm{End}(S^1) = \mathbf{Z}$. We built it by passing to the rank-1 case $(G_a = Z_G(T_a)', T'_a, a|_{T'_a})$ and defining a suitable $a^\vee : S^1 \to T'_a \subset T$. Let's make this explicit for SU(2) and SO(3) with specific maximal tori.

Example 21.1. Take

$$\begin{aligned} \mathrm{SL}_2(\mathbf{C}) \supset G = \mathrm{SU}(2) = \left(\mathbf{H}^\times\right)^1 &= \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\} \\ &\supset \left\{ t(z) = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} : z \in S^1 \right\} \\ &= T. \end{aligned}$$

Then, T acts on $\mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}_2(\mathbf{C})$ with weight spaces

$$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$$

with weights $z^2 =: a_+(t(z))$ and $z^{-2} =: a_-(t(z))$ respectively.

We use

$$\begin{aligned} a_+^\vee : S^1 &\to T \\ z &\mapsto \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} a_-^\vee : S^1 &\to T \\ z &\mapsto \begin{pmatrix} 1/z & 0 \\ 0 & z \end{pmatrix}, \end{aligned}$$

so $a_\pm \circ a_\pm^\vee = 2$.

Example 21.2. Take $G = \mathrm{SO}(3) = \mathrm{SU}(2)/\langle \pm 1 \rangle$. Let $(\mathbf{H}_0, N)$ denote the 3-dimensional space of pure imaginary quaternions equipped with the usual norm form N (which is a positive-definite quadratic form). Then, $\left(\mathbf{H}^\times\right)^1$ acts on the quadratic space $(\mathbf{H}_0, N)$ via conjugation, realizing the degree-2 isogeny $\mathrm{SU}(2) \to \mathrm{O}(3)^0 = \mathrm{SO}(3)$. We thereby compute that $\mathrm{SU}(2) \to \mathrm{SO}(3) = G$ satisfies

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mapsto \begin{pmatrix} r_{2\theta} & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking the usual maximal torus

$$T = \left\{ \begin{pmatrix} r_\alpha & 0 \\ 0 & 1 \end{pmatrix} = a_+(e^{i\alpha}) : \alpha \in \mathbf{Z}/2\pi\mathbf{Z} \right\} \subset \mathrm{SO}(3)$$

(with $a_+ : S^1 \to T$ an isomorphism) yields the coroot

$$\begin{aligned} a_+^\vee &: S^1 \to T \\ z = e^{i\theta} &\mapsto \begin{pmatrix} r_{2\theta} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

for the upper-triangular weight space in $\mathfrak{g}_{\mathbf{C}} \simeq \mathfrak{su}(2)_{\mathbf{C}} = \mathfrak{sl}_2(\mathbf{C})$. This map $a_+^\vee$ kills ± 1 (as it must do to satisfy $a_+ \circ a_+^\vee = 2$ with a_+ an isomorphism).

Since

$$\begin{aligned} x - \langle x, (-a)^\vee \rangle (-a) &= r_{-a} \\ &= r_a \\ &= x - \langle x, a^\vee \rangle a \end{aligned}$$

clearly

$$(-a)^\vee = -a^\vee;$$

this expresses the identity $\ell_{-a} = -\ell_a$.

The preceding discussion yields:

Proposition 21.3. *For general (G, T, a), the cocharacter*

$$a^\vee : S^1 \to T'_a \subset T$$

is either an isomorphism or a degree-2 isogeny onto $T'_a = T \cap G_a$, depending on whether $G_a =$ $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. Further, passing to $-a$ negates this map; i.e., $(-a)^\vee = -\left(a^\vee\right)$.

Remark 21.4. For groups G with finite center, it is an important fact that the root system will characterize G up to isogeny. We will come back to this later in the course (but won't get into its proof, which requires finer structural work with semisimple Lie algebras).

The notion of root datum (which is more general than that of a root system, and includes data associated to coroots) refines this to characterize G up to isomorphism with no restriction on Z_G. We will also return to this matter later, after we have developed some structure theory for root systems and seek to apply it to get structural results for compact connected Lie groups.

Warning 21.5. People usually write Φ to denote a root system, rather than the more accurate notation (V, Φ).

For any root system (V, Φ), we saw the reflection $r_a : V \simeq V$ negating $a \in \Phi$ is uniquely determined by (V, Φ), and deduced that $r_a = r_b$ if and only if $b = \pm a$. (Recall that in this course root systems are always reduced.) But $r_a(x) = x - \ell_a(x) a$, so if we define

$$a^\vee := \ell_a \in V^* - \{0\}$$

(and call this the *coroot* attached to $a \in \Phi$) then the fact that $r_a = r_{-a}$ forces $\ell_{-a} = -\ell_a$. Hence, we obtain in general for any root system that

$$(-a)^\vee = -(a^\vee).$$

Whereas the reflection r_a determines a only up to a sign, the coroot determines the root on the nose:

Lemma 21.6. *For $a, b \in \Phi$, if $a^\vee = b^\vee$ then $a = b$.*

Proof. Note that $\ker(a^\vee) =: H_a \subset V$ is the hyperplane where $r_a = 1$. Therefore, $V^{r_a=1} = V^{r_b=1}$. This implies $r_a = r_b$ (by using Lemma 19.8), so $b = \pm a$. But $(-a)^\vee = -(a^\vee) \neq a^\vee$ (as $a^\vee \neq 0$), so $b \neq -a$ and hence $b = a$. □

Definition 21.7. Define $\Phi^\vee \subset V^* - \{0\}$ as the image of

$$\Phi \hookrightarrow V^*$$
$$a \mapsto a^\vee.$$

For $a^\vee \in \Phi^\vee$ a coroot, define $r_{a^\vee} := (r_a)^* : V^* \simeq V^*$. Observe that $r_{a^\vee}$ is a reflection on V^* (since it has order 2 and -1 as an eigenvalue with multiplicity 1, inherited from r_a since passing to the dual endomorphism has no effect on the characteristic polynomial).

Lemma 21.8. *For $x^* \in V^*$, we have*

$$r_{a^\vee}(x^*) = x^* - \langle a, x^* \rangle a^\vee. \tag{21.1}$$

This is proved by expanding out the definitions, using that $r_{a^\vee}(x^*) = x^* \circ r_a$.

Theorem 21.9. *The datum $(V^*, \Phi^\vee)$ and $\{r_{a^\vee}\}_{a^\vee \in \Phi^\vee}$ is a root system. Further,*

$$r_{a^\vee}(b^\vee) = (r_a(b))^\vee.$$

Before proving this result, we make some observations. The data $(V^*, \Phi^\vee)$ is called the *dual root system*. By using (21.1), we see that $(a^\vee)^\vee = a$.

Further, when $k = \mathbf{Q}$ or $k = \mathbf{R}$ and we choose a $W(\Phi)$-invariant inner product $\langle \cdot | \cdot \rangle$ to identify V^* with V, we get

$$a^\vee = \frac{2a}{\langle a | a \rangle}.$$

This formula is visibly *not* generally additive in a (as we see in some of the explicit rank-2 examples earlier for some pairs of roots whose sum is a root).

In the purely combinatorial setting, we defined $W(\Phi)$ as a group generated by reflections in the roots, and this was easily seen to be finite. But in the Lie theory setting, we cooked up reflections in the finite group $W(G, T)$, and it is not at all evident if those generate $W(G, T)$. That is, for Z_G finite (so $\Phi(G, T)$ is a root system) we have the inclusion $W(\Phi(G, T)) \subset W(G, T)$ but equality isn't obvious.

Over the course of the next two lectures we will use honest Euclidean geometry and some new concepts (such as Weyl chambers) to prove that $W(G, T)$ is generated by the reflections in the roots (without any finiteness restriction on Z_G).

Sketch of Theorem 21.9. The full proof of this theorem is in Appendix S. The two key things to prove are:

(i) $\Phi^\vee$ spans V^*;

(ii) $(r_{a^\vee})(b^\vee) = (r_a(b))^\vee$ (so the dual reflections really do preserve $\Phi^\vee$).

Here are the main ideas for these two parts:

(i) If $k = \mathbf{Q}$ or $\mathbf{R}$ (or in general any ordered field,) one can build a $W(\Phi)$-invariant positive-definite quadratic form q via averaging. Consider the associated (non-degenerate) symmetric bilinear form $\langle\cdot|\cdot\rangle_q$ on V. Using this to identify V^* with V, the coroot $a^\vee \in V^*$ is identified with $(2/q(a))a$ which is a $k^\times$-multiple of a. Thus, the spanning is clear because Φ spans V.

For general k a miracle happens. Define the $\mathbf{Q}$-subspace $V_0 = \mathbf{Q}\Phi \subset V$, which is stable by all r_a (using the Integrality Axiom!). The miracle is that the natural surjection $k \otimes_{\mathbf{Q}} V_0 \twoheadrightarrow V$ is an isomorphism, so one can reduce to the settled case $k = \mathbf{Q}$!

(ii) To show $r_{a^\vee}(b^\vee) = b^\vee \circ r_a = r_a(b)^\vee$, one does not just blindly crank out $r_{(r_a(b))}$ and hope for the best. Instead, the key is that one needs a formula for $r_{(r_a(b))}$. In fact, $r_{(r_a(b))} = r_a r_b r_a^{-1}$. This holds because of the uniqueness of the reflection associated to a root and the fact that r_a acts on (V,Φ) (more generally: any automorphism of a root system (V,Φ) must be compatible with the associated reflections!).

□

Remark 21.10. In [Bou2], the Euclidean geometry aspects of reflection groups are developed in the setting of Hilbert spaces without assuming finite-dimensionality. There are no reasons to care about this beyond the finite-dimensional case, so assume finite-dimensionality without any concern when you read [Bou2].

Example 21.11. For $n \geq 2$, we have seen already that $\Phi(\mathrm{SU}(n))$ is $\mathbf{Q}^n/\Delta \supset \{\bar{e}_i - \bar{e}_j\}_{i\neq j}$. This is naturally identified with

$$\mathbf{Q}^n/\Delta \simeq \left\{\sum x_i = 0\right\} \supset \left\{e_i - e_j\right\}_{i\neq j}.$$

The dual root system is

$$\left\{\sum x_i^* = 0\right\} \supset \left\{a_{ij}^\vee = e_i^* - e_j^*\right\}$$

via the natural duality between $\mathbf{Q}^n/\Delta$ and the hyperplane $\sum x_i^* = 0$. Hence, this root system is self-dual. The root system $\Phi(\mathrm{SU}(n))$ is called A_{n-1} (the subscript since the underlying vector space has dimension $n-1$).

Next time, we'll pass to the dual root system over $\mathbf{R}$ to prove the equality

$$W(G,T) = \langle r_a\rangle_{a\in\Phi} \subset \mathrm{GL}(X(T)_{\mathbf{Q}}).$$

This will use genuine Euclidean geometry in the dual space $X(T)^*_{\mathbf{R}} = X_*(T)_{\mathbf{R}} \simeq \mathfrak{t}$ (an isomorphism we will explain next time).

The crucial concept needed in this proof will be *Weyl chambers*, defined in terms of connected components of complements of a (locally) finite collection of hyperplanes in a finite-dimensional $\mathbf{R}$-vector space. It is *exactly at this step* that we really have to finally use the ground field $k = \mathbf{R}$ in our work on root systems, and not just use pure algebra: defining such connected components is very painful purely in terms of algebra since being on certain sides of some of the hyperplanes, as can be described algebraically, might force one to be on a specific side of another hyperplane (that is, we can't just specify which side of each hyperplane we want to be on and expect the resulting condition to be non-empty). This is where topology over $\mathbf{R}$ is a very useful tool.

22. EQUALITY OF TWO NOTIONS OF WEYL GROUP I

Last time, we defined the *dual* $(V^*, \Phi^\vee)$ of a root system (V, Φ) and saw that for $n \geq 2$, $\mathrm{A}_{n-1} := \Phi\left(\mathrm{SU}(n)\right)$ is self-dual. In terms of the classification of irreducible roots systems, $\mathrm{D}_n := \Phi\left(\mathrm{SO}(2n)\right)$ is also self-dual, whereas $\mathrm{C}_n := \Phi\left(\mathrm{Sp}(n)\right)$ turns out to be dual to $\mathrm{B}_n := \Phi\left(\mathrm{SO}(2n+1)\right)$ for $n \geq 1$. We'll come back to the computation of these next time.

There is an important link between root systems over $\mathbf{R}$ and Lie algebras, as follows. For (G, T) with Z_G finite, the dual root system over $\mathbf{R}$ has underlying vector space

$$\left(X\left(T\right)_{\mathbf{R}}\right)^* = \left(X\left(T\right)^*\right)_{\mathbf{R}} = X_*\left(T\right)_{\mathbf{R}} \simeq \mathfrak{t} =: \mathrm{Lie}(T)$$

by assigning to each cocharacter $(S^1 \xrightarrow{f} T) \in X_*(T)$ the image under

$$df(e) : \mathrm{Lie}(S^1) \to \mathfrak{t}$$

of a chosen "standard basis" of $\mathrm{Lie}(S^1)$ arising from a fixed choice of uniformization of S^1 (say as $\mathbf{R}/\mathbf{Z}$ or $\mathbf{R}/2\pi\mathbf{Z}$). This defines the $\mathbf{R}$-linear map $X_*(T)_{\mathbf{R}} \to \mathfrak{t}$, and to show it is an isomorphism we first observe that this construction is compatible with direct products in T and is functorial in T. Thus, using an isomorphism $T \simeq (S^1)^r$ reduces the verification to the case $T = S^1$ that is elementary. This isomorphism $X_*\left(T\right)_{\mathbf{R}} \simeq \mathfrak{t}$ works without any hypotheses of Z_G; we only mentioned finiteness of Z_G to make a link with root systems.

Remark 22.1. For any root system (V, Φ), last time we also defined the Cartan numbers

$$n_{a,b} = \ell_a(b) = a^\vee(b).$$

Likewise, we have for the dual root system

$$n_{a^\vee, b^\vee} = \left(a^\vee\right)^\vee\left(b^\vee\right) = a\left(b^\vee\right) := b^\vee(a) = n_{b,a}$$

using $V \simeq \left(V^*\right)^*$.

In other words, the *Cartan matrix* for Φ (whose rows and columns are labeled by a common fixed enumeration of the set of roots, or really a specific subset called a *basis* that we'll discuss later, and whose ab-entry is $n_{a,b}$ for $a, b \in \Phi$). is transpose to the Cartan matrix of $\Phi^\vee$ (when we enumerate the coroots in accordance with how we do it for the roots).

22.1. **Weyl Chambers.**

We now draw various examples of rank-2 root systems over $\mathbf{R}$ and Weyl chambers (to be defined below) for each. For each root system (V, Φ) over $\mathbf{R}$ and root c, we defined the "root hyperplane"

$$H_{-c} = H_c := \ker(c^\vee) = V^{r_c = 1}.$$

The connected components K of $V - \left(\cup_{c\in\Phi} H_c\right)$ are called the *Weyl chambers*.

Example 22.2 ($\mathrm{A}_1 \times \mathrm{A}_1$)**.**

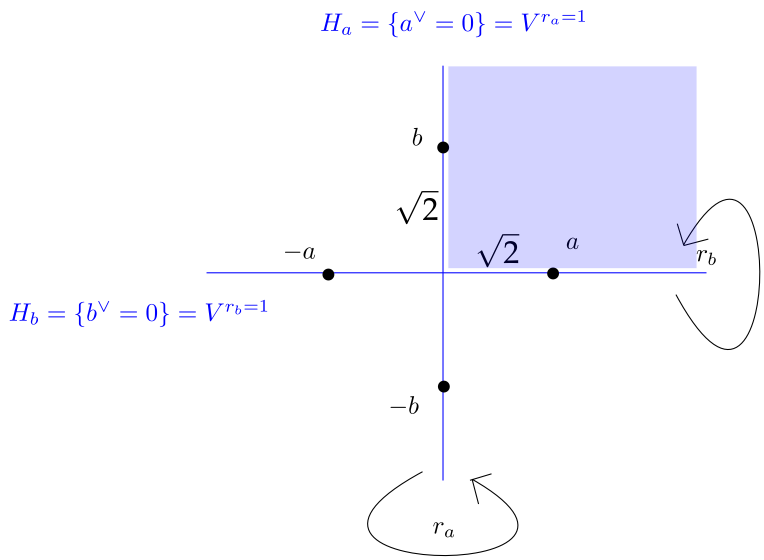


FIGURE 9. $A_1 \times A_1$ Weyl chambers

Example 22.3 (A_2).

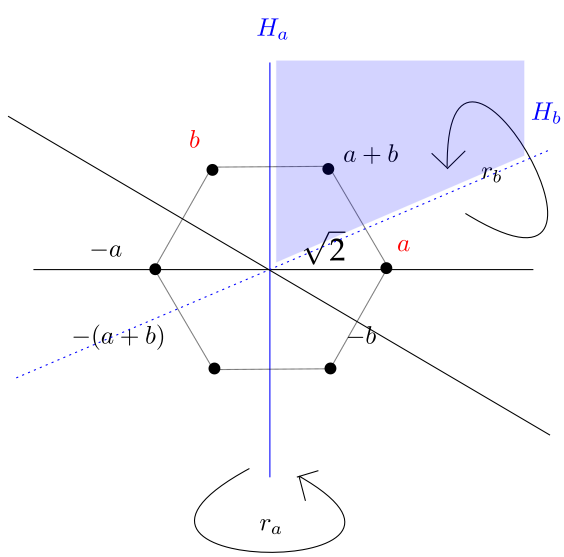


FIGURE 10. A_2 Weyl chambers

Example 22.4 (B_2).

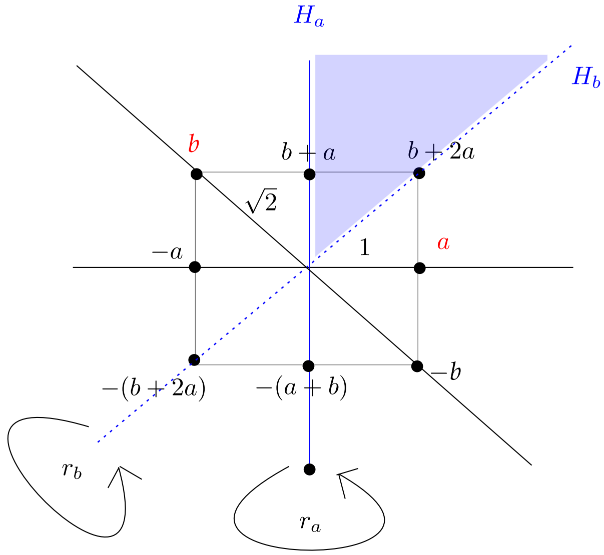


FIGURE 11. B_2 Weyl chambers

Example 22.5 (G_2).

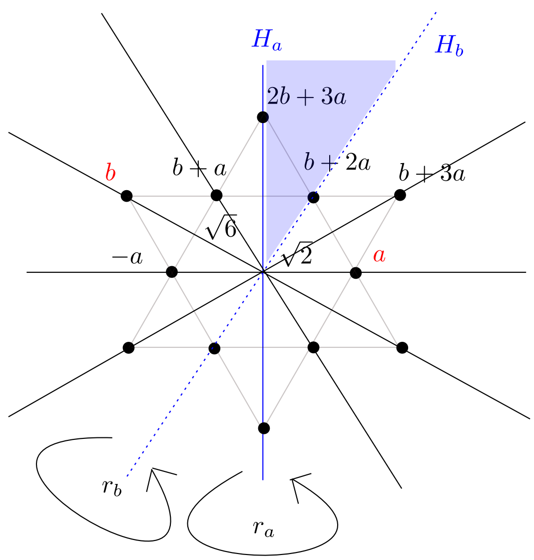


FIGURE 12. G_2 Weyl chambers

In each of the above cases, we see by inspection $\#K = \#W(\Phi)$. This is no accident: the action of $W = W(\Phi)$ on V certainly permutes the set of Weyl chambers (since it permutes the collection of the root hyperplanes, as those are intrinsic to the root system), and we'll see soon that the action is simply transitive.

In the Lie theory setting, $W(G,T)$ also acts on $X(T)_{\mathbf{R}}$ and permutes the collection of root hyperplanes (again, since these are intrinsically determined by the roots), so $W(G,T)$ permutes the set of Weyl chambers (with no assumptions on Z_G, such as finiteness).

We want to prove $W(G,T) = \langle r_a \rangle_{a \in \Phi}$. This would then imply $W(G,T) = W(\Phi(G,T))$ when Z_G is finite (so $(X(T)_{\mathbf{Q}}, \Phi(G,T))$ is actually a root system). In fact, we'll show that $W(G,T)$ is generated by a specific set of reflections.

Definition 22.6. For a Weyl chamber K of a root system, a *wall* of K is a hyperplane $H = H_c$ so that $\mathrm{int}_H(\overline{K} \cap H) \neq \emptyset$.

Theorem 22.7. *For Z_G finite, $W(G,T) = \langle r_c \rangle_{H_c \in \mathrm{wall}(K)}$ for a fixed choice of Weyl chamber K. In particular, $W(G,T) = \langle r_c \rangle_{c \in \Phi} =: W(\Phi)$.*

To prove Theorem 22.7, we need the following result that is part of HW8 (with hints):

Lemma 22.8. *For a root system (V, Φ) over $\mathbf{R}$ and a Weyl chamber K, the following hold:*

(1) $\partial_V(\overline{K}) = \bigcup_{H \in \mathrm{wall}(K)} \left(H \cap \overline{K}\right)$
(2) $\mathrm{int}_V(\overline{K}) = K$
(3) $K = \bigcap_{H \in \mathrm{wall}(K)} (H^{+_K})$*, where H^{+_K} denotes the connected component of $V - H$ containing K.*

Remark 22.9. The theory of Coxeter groups can be used to infer a presentation of $W(\Phi)$ using Theorem 22.7. For example, this yields for $\mathfrak{S}_n = W(\Phi(\mathrm{SU}(n)))$ a presentation as a group generated by the adjacent transpositions $(12), (23), \ldots, (n-1, n)$, subject to certain braid relations.

Proof of Theorem 22.7. To start, we pass to the dual root system, as we may do without any harm. (This will be important when proving Lemma 22.11 below.) Thus, now the underlying $\mathbf{R}$-vector space is $\mathfrak{t}$ on which $W = N_G(T)/T$ acts through the adjoint representation of G on $\mathfrak{g}$. Let $W' := \langle r_c \rangle_{H_c \in \mathrm{wall}(K)} \subset W(G,T)$.

We'll show W' acts transitively on the set of Weyl chambers and $W(G,T)$ has only trivial isotropy on any chamber (or equivalently on one chamber, once W'-transitivity for the action on the set of chambers is proved). The first of these says that W' is big enough and the second (which means $w.K = K \Rightarrow w = 1$ for $w \in W(G,T)$) says that the ambient group $W(G,T)$ is small enough. Putting these together clearly implies $W' = W(G,T)$. Thus, the proof follows from Lemma 22.10 and Lemma 22.11 below. □

Lemma 22.10. *W' acts transitively on the Weyl chambers.*

Proof. Choose Weyl chambers $K' \neq K$ and pick $y \in K', x \in K$. Pick $w' \in W'$ so that the distance $|w'.y - x|$is minimal. We'll show $w'.y \in K$ which implies $w'(K') = K$ (since if two connected components meet they must be the same).

Assume, for the sake of contradiction that $w'.y \notin K$ (so $w'.K' \neq K$). Fix a $W(G,T)$-invariant inner product on V. By renaming $w'.y$ as y and $w'.K' \neq K$ as K', we can assume $|y - x| \leq |w'.y - x|$ for all $w' \in W'$, yet with $y \in K' \neq K$. Consider the line segment $[y, x]$. It is an exercise to use Lemma 22.8 to deduce that $[y, x]$ crosses some wall H_a of K, so x and y lie on opposite sides of H (neither lies in H since they don't lie in any wall of any Weyl chamber). Thus, $r_a \in W'$. We will look at $r_a(y)$.

Using the orthogonal decomposition $V = H \oplus H^{\perp}$ to decompose y and x into their components along H and along the line $H^{\perp}$, applying $r_a = r_H$ to y has no effect on the H-component but negates the $H^{\perp}$-component. By considering $H^{\perp}$-components, we claim that $r_a(y)$ is closer to x than y, which would be a contradiction to the minimizing property we have arranged for y (as $r_a \in W'$). Working along the line $H^{\perp}$ in which the components of x and y are nonzero (as these don't live in any of the walls of any of the Weyl chambers) and lie on opposite sides of the origin (as x and y are on opposite sides of H), this is essentially saying that if we consider a negative number and a positive number, and we negate the negative number, then it becomes closer to the positive number. This completes the proof of transitivity of W' on the set of Weyl chambers.

$\square$

To prove Theorem 22.7, it only remains to prove the following lemma.

Lemma 22.11. *For $w \in W(G,T) = N_G(T)/T$ such that $w.K = K$, we necessarily have $w = 1$.*

Proof. Consider $K \subset X_*(T)_{\mathbf{R}} = \mathfrak{t}$. Pick $x \in K$. By openness of K in $\mathfrak{t}$ and density of $\mathbf{Q}$ in $\mathbf{R}$, we can approximate x so that $x \in X_*(T)_{\mathbf{Q}}$. Letting Γ be the finite cyclic group generated by w, consider $\sum_{\gamma\in\Gamma} \gamma.x \in K$, membership since K is a cone (really use Lemma 22.8(3)). This sum is fixed by w and lies in $X_*(T)_{\mathbf{Q}}$. We can scale by $\mathbf{Z}^+$ so $x \in X_*(T)$. This yields a map $x : S^1 \to T$ which is centralized by a representative $g \in N_G(T)$ of w. Next time we'll use an argument with roots to show that the centralizer of x is equal to T and use this to deduce $g \in Z_G(T) = T$, so the element $w \in W(G,T)$ represented by g is trivial, as desired. $\square$

23. Equality of two notions of Weyl group II

Last time we were confronted with the situation of a cocharacter $x : S^1 \to T$ correspondng to a point in $X_*(T) \subset X_*(T)_{\mathbf{R}} = \mathfrak{t}$ lying in some Weyl chamber (and hence *outside* all of the (co)root hyperplanes) and $g \in N_G(T)$ such that $gxg^{-1} = x$ in $X_*(T)$, and we wanted to show that $g \in T$ (so the class $w \in W(G,T)$ of g is trivial, which was our real goal at the end of last time, when showing the simple transitivity of the action of $W(G,T)$ on the set of Weyl chambers in the dual root system). Note that the action of $W(G,T)$ on $\mathfrak{t}$ via the surjection $N_G(T) \twoheadrightarrow W(G,T)$ is induced by the Ad_G-action on $\mathfrak{g}$ (differentiating at the identity for the effect of conjugation, much as $X_*(T) \hookrightarrow \mathfrak{t}$ is defined by differentiation of cocharacters at the identity).

We'll now show that the x-centralizer $Z_G(x) = Z_G\left(x(S^1)\right)$ in G (which clearly contains g) is just T, so $g \in T$ as desired. Since $x(S^1)$ is a torus, the centralizer $Z_G(x) = Z_G(x(S^1))$ is a *connected* compact Lie group that contains T (as $x(S^1) \subset T$), and T is visibly maximal in here since it is even maximal in G. We assume $Z_G(x) \neq T$ and seek a contradiction.

By *connectedness*, if the containment $T \subset Z_G(x)$ is not an equality then the subspace $\mathrm{Lie}(Z_G(x)) \subset \mathfrak{g}$ is strictly larger than $\mathfrak{t}$, forcing its complexification to have a nontrivial T-weight. But the T-action on $\mathfrak{g}_{\mathbf{C}}$ is completely reducible with $\mathfrak{t}_{\mathbf{C}}$ as the weight space for the trivial weight and the other weights being the roots, each with a 1-dimensional weight space. Thus, $\mathrm{Lie}\,(Z_G(x))_{\mathbf{C}}$ must contain the "root line" $(\mathfrak{g}_{\mathbf{C}})_a$ for some $a \in \Phi(G,T)$.

But $\mathrm{Lie}\,(Z_G(x))_{\mathbf{C}} = (\mathfrak{g}_{\mathbf{C}})^{x(S^1)=1}$, so $(\mathfrak{g}_{\mathbf{C}})_a \subset (\mathfrak{g}_{\mathbf{C}})^{x(S^1)=1}$. Since T acts on the a-root line through scaling against $a : T \to S^1$, and $x(S^1) \subset T$, it follows that $x(S^1)$ acts on $(\mathfrak{g}_{\mathbf{C}})_a$ through scalar multiplication against its image

$$a(x(S^1)) = (a \circ x)(S^1) \subset S^1 \subset \mathbf{C}^\times$$

under a. Hence, the containment in $(\mathfrak{g}_{\mathbf{C}})^{x(S^1)=1}$ forces $a \circ x \in \mathrm{End}(S^1) = \mathbf{Z}$ to vanish, which is to say $a \circ x : S^1 \to S^1$ is trivial.

Thus, $x(S^1) \subset (\ker a)^0 =: T_a$, so the cocharacter $x \in X_*(T) \subset \mathfrak{t}$ lies in $\mathrm{Lie}(T_a) = \ker(da(e))$ that is exactly the kernel of a viewed as a linear functional on $X_*(T)_{\mathbf{R}}$. This is exactly the (co)root hyperplane $H_{a^\vee}$ in the dual root system via the identitifcation $a = (a^\vee)^\vee$. But $x \in K$, so x does not lie in any (co)root hyperplanes in the dual root system! This is a contradiction, so indeed $Z_G(x) = T$ as desired, completing the proof.

Passing back to the original root system, we have now finally proved

$$W(G,T) = \langle r_a \rangle_{a \in \Phi} =: W(\Phi(G,T)) \tag{23.1}$$

inside $\mathrm{GL}(X(T)_{\mathbf{Q}})$ when Z_G is finite. We next want to show that *without restriction on* Z_G (e.g., no finiteness assumption), $W(G,T)$ is generated by the reflections r_a in the root hyperplanes:

Proposition 23.1. *For G any compact connected Lie group (no assumptions on Z_G) and T a maximal torus, the subgroup $W(G,T) \subset \mathrm{GL}(X(T)_{\mathbf{Q}})$ is generated by the reflections $\langle r_a \rangle_{a \in \Phi(G,T)}$.*

Proof. Let $\overline{G} := G/Z_G^0$ and $\overline{T} = T/Z_G^0$, so we know that $\overline{T}$ is a maximal torus of $\overline{G}$ and that $\overline{G}$ has center Z_G/Z_G^0 which is visibly finite. We also have $\overline{\mathfrak{g}}_{\mathbf{C}} = \mathfrak{g}_{\mathbf{C}}/\,\mathrm{Lie}(Z_G^0)$ with its natural action by $\overline{G} = G/Z_G^0$, and by considering weight lines for the action of the quotient $\overline{T}$ of T we see that this identifies $\Phi(\overline{G},\overline{T}) = \Phi(G,T)$ via the inclusion $X(\overline{T}) \subset X(T)$. (Each root is trivial on Z_G since it encodes the effect of T through the Ad_G.)

We also have the evident isomorphism

$$\overline{W} := W(\overline{G},\overline{T}) = N_{\overline{G}}(\overline{T})/\overline{T} \leftarrow (N_G(T)/Z_G^0)/(T/Z_G^0) = N_G(T)/T = W(G,T) =: W. \tag{23.2}$$

Via this identification (note the direction in which the isomorphism is defined!), the actions of $\overline{W}$ on $X(\overline{T})$ and W on $X(T)$ are compatible with the inclusion $X(\overline{T}) \subset X(T)$ that carries $\overline{\Phi}$ bijectively onto Φ.

It suffices to check that for $a \in \Phi$ and the corresponding $\overline{a} \in \overline{\Phi}$, we have

$$r_a|_{X(\overline{T})_{\mathbf{Q}}} = r_{\overline{a}},$$

because then the isomorphism $W \simeq \overline{W}$ defined in (23.2) satisfies $r_a \mapsto r_{\overline{a}}$, with the latter reflections generating $\overline{W}$ (forcing the former reflections to generate W, as we wanted to show).

Since at least $r_a|_{X(\overline{T})_{\mathbf{Q}}} \in \overline{W}$, with $\overline{W}$ a finite subgroup of $\mathrm{GL}(X(\overline{T})_{\mathbf{Q}})$, it's enough to check this is a reflection negating $\mathbf{Q}\overline{a}$ (because the finite group $\overline{W}$ has only has one reflection negating a given line in $X(\overline{T})_{\mathbf{Q}}$). Therefore, it is enough to show that the line $\mathbf{Q}\overline{a} \subset X\left(\overline{T}\right)_{\mathbf{Q}}$ is the same as $\mathbf{Q}a \subset X(T)_{\mathbf{Q}}$. However, this exactly expresses that $\overline{a} = a$ under the equality $\overline{\Phi} = \Phi$ induced by $X(\overline{T}) \subset X(T)$. $\square$

Without passing to the dual root system, when Z_G is finite we can ask if the Weyl group $W(G,T) = W(\Phi(G,T))$ generated by the reflections in the walls of one Weyl chamber of $\Phi(G,T)$. This question has *not* already been solved because it takes place in the initial root system rather than its dual and we haven't set up a theory of "dual cones" to define a concept of "dual Weyl chamber" (so as to functorially transport a result about Weyl chambers on the dual side back to a result about Weyl chambers in the original root system). The answer is affirmative:

Theorem 23.2. *Let (V, Φ) be any root system over $\mathbf{R}$. Let $K \subset V$ be a Weyl chamber. Then, $W(\Phi) = \langle r_a \rangle_{H_a \in \mathrm{wall}(K)}$ and $W(\Phi)$ acts transitively on the set of Weyl chambers.*

Proof. Let $W' = \langle r_a \rangle_{H_a \in \mathrm{wall}(K)}$. The earlier Euclidean geometry argument (in the proof of Lemma 22.10) shows that W' acts transitively on the set of chambers. So it remains to show $W' = W(\Phi)$. (The earlier proof of simple transitivity (for $W(G, T)$ acting on the set of Weyl chambers in the dual root system when G has finite center) used (G, T). In the present setting *there is no Lie group*, so we don't (yet) know that the transitive W'-action is simply transitive.)

It remains to show that any $w \in W(\Phi)$ lies in W'. By definition of $W(\Phi)$, it is enough to show $r_c \in W'$ for each $c \in \Phi$. Choose a $W(\Phi)$-invariant inner product on V, as we may do by averaging since $W(\Phi)$ is finite, so we can regard the action of every element of $W(\Phi)$ as an isometry. (This isn't strictly needed in what follows, but it simplifies the exposition.) Now r_c is orthogonal reflection in H_c.

First, we claim that the root hyperplane H_c is the wall of *some* Weyl chamber K'. This is "geometrically obvious" (especially if one stares at the pictures in the rank-2 case). The idea to prove this rigorously is to look in $V - H_c$ near a point $x \in H_c - \cup_{a \neq \pm c}(H_c \cap H_a)$ and to prove that near such x there is exactly one Weyl chamber on each side of H_c, and so to deduce that the closure of each of those meets H_c in a *neighborhood* of x in H_c (so H_c is a wall of each of those, by our initial definition of "wall" for a Weyl chamber).

By the established transitivity, there exists $w' \in W'$ so that $w'.K' = K$ so $w'(H_c) \in \mathrm{wall}(K)$ (since $H_c \in \mathrm{wall}(K')$ by design of K'). This says $w'(H_c) = H_a$ for some $a \in \Phi$ with H_a a wall of K, so $r_a \in W'$ by definition of W'. It follows that conjugation by the isometry w' carries orthogonal reflection in H_c to orthogonal reflection in H_a, which is to say $w' r_c (w')^{-1} = r_a$. But $r_a \in W'$, so $r_c = (w')^{-1} r_a w' \in W'$ as desired! □

The final topic in our development of the relationship between Weyl groups and reflections is the simple-transitivity upgrade of Theorem 23.2. In view of the transitivity just established, this comes down to proving:

Theorem 23.3. *For any root system (V, Φ) over $\mathbf{R}$, $w \in W(\Phi)$, and Weyl chamber $K \subset V$, if $w.K = K$ then $w = 1$.*

The proof of this needs a new method compared to what we did earlier for Weyl groups of connected compact Lie groups (with finite center) because there is no (G, T) here. It is a fact that every (reduced) (V, Φ) is the root system of some such Lie group, but that is a deep existence result far far beyond the level of the result we are presently aiming to prove (so it would be completely absurd to invoke it here, though awareness of this fact helps us to realize that the axioms of a root system are not missing any extra features of those arising from the theory of connected compact Lie groups).

Proof. We'll write w as a product of a finite number of reflections r_a in the walls H_a of K (as we may do!) and induct on the number of reflections that show up. Writing $w = r_{a_1} \cdots r_{a_n}$, if $n > 0$ then we'll build such an expression with fewer such reflections (so $w = 1$). The case $n = 1$ cannot occur, since r_{a_1} swaps the two open sides of H_{a_1} in V (i.e. swaps the two connected components of $V - H_{a_1}$) yet by definition K is on one of those sides (so the Weyl chamber $r_{a_1}.K$ is on the other side than K is and so cannot be equal to K). Hence, $n \geq 2$

and since $w.K = K$ whereas $r_{a_1}.K \neq K$, there is a minimal $i > 0$ such that $(r_{a_1} \cdots r_{a_i}).K$ is on the same side of H_{a_1} that K is. Note that $i \geq 2$ (for the same reason that $n \geq 2$).

Define $w' = r_{a_1} \cdots r_{a_{i-1}}$, so $w'.K$ and $w' r_{a_i}.K$ lie on opposite sides of H_{a_1} due to the minimality of i. Hence, applying w'^{-1} everywhere, the Weyl chambers K and $r_{a_i}.K$ are on opposite sides of the hyperplane $w'^{-1}(H_{a_1})$. The closures of these chambers therefore lie in such "opposite" closed half-spaces, so the intersection

$$\overline{K} \cap (r_{a_i}.\overline{K})$$

lies inside the hyperplane $w'^{-1}(H_{a_1})$.

But H_{a_i} is wall of K (by design of W'), so obviously the hyperplane $r_{a_i}(H_{a_i}) = H_{a_i}$ is a wall of $r_{a_i}(K)$. Clearly this common wall $H_{a_i} = V^{r_{a_i}=1}$ meets the closure of each of K and $r_{a_i}(K)$ in exactly the same region in H_{a_i} with non-empty interior. In particular, $\overline{K} \cap r_{a_i}.(\overline{K})$ lies inside H_{a_i} with *non-empty interior* in H_{a_i}. But we saw that this intersection also lies in the hyperplane $w'^{-1}(H_{a_1})$, and the only way two hyperplanes $H, H' \subset V$ can have H' containing a subset of H with non-empty interior in H is when $H' = H$. Hence, $w'^{-1}(H_{a_1}) = H_{a_i}$. But orthogonal reflection in $w'^{-1}(H_{a_1})$ is $w'^{-1} r_{a_1} w'$, yet orthogonal reflection in H_{a_i} is r_{a_i}, so

$$w'^{-1} r_{a_1} w' = r_{a_i}.$$

We conclude that

$$w' r_{a_i} = r_{a_1} w' = r_{a_2} \cdots r_{a_{i-1}},$$

where the second equality uses the definition of w' and the fact that $r_{a_1}^2 = 1$. Now going back to the definition of w, we obtain

$$w = w' r_{a_i} \cdots r_{a_n} = r_{a_2} \cdots r_{a_{i-1}} r_{a_{i+1}} \cdots r_{a_n}.$$

This expresses the original w as a product of $n-2$ reflections in the walls of K, so by induction on n we are done. $\square$

24. Bases of root systems

Let (V, Φ) be a root system over $\mathbf{R}$. Using a $W(\Phi)$-invariant inner product on V and our earlier results on root systems, the following is shown via the top half of [BtD, p. 199] and [BtD, p. 204-206] (ignoring [BtD, Cor 4.6]):

Theorem 24.1 (Basis theorem)**.** *Fix a Weyl chamber $K \subset V$ for Φ. For each wall $H_a = \ker(a^\vee)$ of K adjust a (and hence $a^\vee$) by a sign if necessary so that $a^\vee|_K > 0$. Let $B(K)$ be the set of such roots with $a^\vee|_K > 0$. Then:*

(1) The set $B(K)$ is a basis of V (so the walls of K are "independent" as lines in V^*), and

$$\Phi \subset (\mathbf{Z}_{\geq 0} B(K)) \cup (\mathbf{Z}_{\leq 0} B(K)).$$

(We call $B(K)$ a **basis** *for K and $\Phi^+(K) := \Phi \cap (\mathbf{Z}_{\geq 0} B(K))$ the* **positive system of roots** *with respect to $B(K)$. Since $W(\Phi) = \langle r_a \rangle_{a \in B(K)}$ by Theorem 23.3, we call such r_a the* **simple reflections** *with respect to $B(K)$.)*

(2) We have

$$\Phi^+(K) = \left\{ c \in \Phi : \langle v, c^\vee \rangle > 0 \text{ for all } v \in K \right\}.$$

Geometrically, this is saying that c makes an acute angle against all points in K.

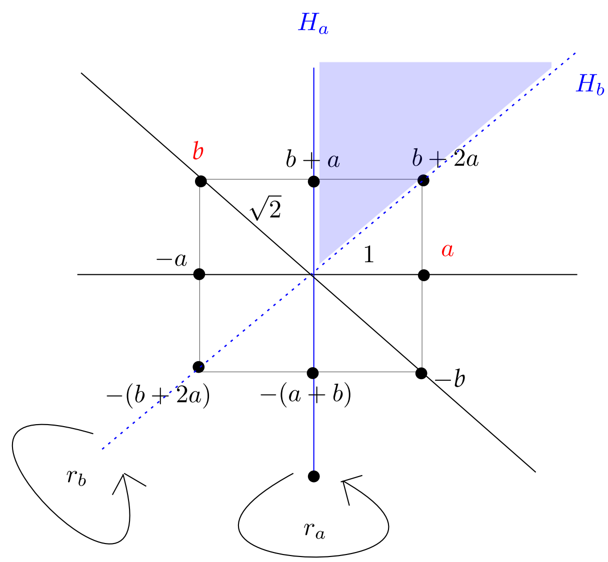
(3) The map

$$\{\text{Weyl chambers}\} \to \{\text{bases}\}$$
$$K \mapsto B(K)$$

is a bijection, with inverse given by

$$B \mapsto \left\{v \in V : \langle v, c^\vee \rangle > 0 \text{ for all } c \in B\right\} =: K(B).$$

Example 24.2. The property in (1) that all $B(K)$-coefficients of roots are integers which share the same sign (or are 0) is rather striking but has been seen in our rank-2 examples. For our Weyl chamber picture for B_2,



we have $a^\vee > 0$ to the right of H_a and $b^\vee > 0$ above H_b encoding the sign preference on each determined by the shaded Weyl chamber (whose walls are these two hyperplanes). The Weyl chamber K is the intersection of the regions above H_b and right of H_a. Here we can see that the points of the Weyl chamber K is exactly the set of points whose rays to the origin makes acute angles with both a and b.

Remark 24.3. More advanced work on root systems shows that subsets of Φ of the form $\Phi^+(K)$ as in Theorem 24.1 are exactly the subsets $\Phi \cap \{\ell > 0\}$ for $\ell \in V^*$ a linear form that is non-zero on Φ (i.e., these are the subsets obtained by picking a hyperplane missing all the roots and choosing those roots on one side of the hyperplane).

Further work in the parts of [BtD] indicated above also yields:

Corollary 24.4. *The Weyl action on the set of bases is simply transitive, and every root lies in some basis. In particular,* $\Phi = W(\Phi).B$ *for any basis* B*.*

Remark 24.5. Since $W(\Phi) = \langle r_a \rangle_{a \in B}$ for a basis B, one can recover the full Weyl group from just the reflections in the basis, and hence can recover the set of all roots from just a basis of roots (together with a Euclidean structure to define the reflection in each root without reference to $W(\Phi)$ at the outset).

By using the preceding corollary, [BtD, Prop. 4.13, Ch. IV] combined with our earlier formula $r_a(b)^\vee = r_{a^\vee}(b^\vee)$ for any $a, b \in \Phi$ gives a nice proof of:

Proposition 24.6. *If B is a basis of Φ then*

$$B^\vee := \{a^\vee : a \in B\}$$

is a basis of $\Phi^\vee$.

24.1. **Rank-2 root systems.** In Appendix T, the root systems $(\Phi, \Phi^\vee, W)$ are worked out in the various "classical" cases:

(1) $\mathrm{SU}(n)$ ($n \geq 2$), called A_{n-1},
(2) $\mathrm{SO}(2m+1)$ ($m \geq 2$), called B_m,
(3) $\mathrm{Sp}(n)$ ($n \geq 2$), called C_n,
(4) $\mathrm{SO}(2m)$ ($m \geq 3$), called D_m.

As a warm up for finding all rank-2 reduced root systems, note that there is only one reduced rank-1 root system (up to isomorphism), namely a pair of opposite nonzero vectors in a line; this is the root system A_1 for $\mathrm{SU}(2)$. To work out the possible rank-2 root systems, we'll work over $\mathbf{R}$ (which is harmless, since root systems have a unique $\mathbf{Q}$-structure) and choose a Weyl-invariant Euclidean structure at the outset to make a calculation (and near the end we'll have to explain why this choice of Euclidean structure is harmless for classification purposes).

Lemma 24.7. *Let c and c' be two roots. Then,*

$$n_{c,c'} n_{c',c} = 4\cos\left(c, c'\right)^2$$

where $\cos(c, c')$ *denotes the angle in* $[0, \pi]$ *between c and c'.*

Proof. In terms of the Euclidean structure to identify $a^\vee$ with $2a/(a \cdot a)$ for all roots a, the Cartan integer $n_{c,c'}$ is given by the formula

$$\begin{aligned} n_{c,c'} &:= \langle c', c^\vee \rangle \\ &= c' \cdot \frac{2c}{c \cdot c} \\ &= \frac{2\left(c' \cdot c\right)}{c \cdot c} \\ &= 2\frac{\|c'\|}{\|c\|} \cos\left(c, c'\right). \end{aligned}$$

Thus, $n_{c,c'} n_{c',c} = 4\cos(c, c')^2$. □

Now let $B = \{a, b\}$ be a basis for a rank-2 root system. Since a and b linearly independent (so $\cos(a, b)^2 < 1$), we have $0 \leq n_{a,b} n_{b,a} < 4$ by Lemma 24.7. Therefore, the integer $n_{a,b} n_{b,a}$ is either $0, 1, 2$, or 3. Using this, the angle between a and b lies in $(0, \pi)$ and is one of

$$\frac{\pi}{2}, \ \frac{\pi}{3} \text{ or } \frac{2\pi}{3}, \ \frac{\pi}{4} \text{ or } \frac{3\pi}{4}, \ \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

respectively.

Lemma 24.8. *We have* $\cos(a, b) \leq 0$, *which is to say the angle between a and b lies in $[\frac{\pi}{2}, \pi)$. In other words, this angle is one of the four possibilities $\pi/2, 2\pi/3, 3\pi/4, 5\pi/6$.*

Proof. Suppose not, which is to say the angle between a and b is acute so $a \cdot b > 0$. Then, $n_{a,b}$ and $n_{b,a}$ are both positive, but $n_{a,b}n_{b,a} < 4$. Thus, either $n_{a,b}$ or $n_{b,a} = 1$. Without loss of generality, say $n_{a,b} = 1$. Then, $r_b(a) = a - b$, but clearly $a - b \notin \mathbf{Z}_{\geq 0}B \cup \mathbf{Z}_{\leq 0}B$ since $B = \{a, b\}$, yet $r_b(a) \in \Phi$, so we have a contradiction. ☐

Since the integers $n_{a,b}, n_{b,a}$ are non-positive with product at most 3, and either both are 0 or both are nonzero, in the nonzero case at least one of them is equal to -1. By symmetry, we can swap a and b if necessary so that $n_{b,a} = -1$ in the nonzero cases. Since $W(\Phi) = \langle r_a, r_b \rangle$ and $\Phi = W(\Phi) \cdot \{a, b\}$, we arrive at:

Corollary 24.9. *With $n_{b,a} = -1$ when nonzero, the angle between a and b is one of the following:*

(1) *$\pi/2$, corresponding to $n_{a,b} = 0$, and the root system $\mathrm{A}_1 \times \mathrm{A}_1$,*
(2) *$2\pi/3$, corresponding to $n_{a,b} = -1$ and root system A_2,*
(3) *$3\pi/4$, corresponding to $n_{a,b} = -2$ and root system B_2,*
(4) *$5\pi/6$, corresponding to $n_{a,b} = -3$ and root system G_2.*

Proof. We'll whittle ourselves down to only one possibility for the root system (up to isomorphism) for each of the possible angles we have available, and since we have already constructed rank-2 root systems realized each option we will then be done (it is clear that the four rank-2 root systems we have already seen are pairwise non-isomorphic as root systems). In each case, we will work out geometric possibilities for a and b, and then determine the entire root system via two crucial theoretical facts discussed earlier: $W(\Phi) = \langle r_a, r_b \rangle$ and $\Phi = W(\Phi) \cdot \{a, b\}$.

Recall that $n_{a,b} = 0$ if and only if $n_{b,a} = 0$, and in the nonzero cases we have arranged that $n_{b,a} = -1$ and so $n_{a,b} \in \{-1, -2, -3\}$. If $n_{a,b} = 0$ then a is orthogonal to b and their lengths are unrelated (it is an artifact of the choice of inner product). Since $W(\Phi) = \langle r_a, r_b \rangle$ and $\Phi = W(\Phi) \cdot \{a, b\}$, the root system is clearly $\mathrm{A}_1 \times \mathrm{A}_1$.

Now we may and do assume $n_{a,b} \in \{-1, -2, -3\}$ and $n_{b,a} = -1$. We have already determined the angle between a and b in these respective cases (as $2\pi/3, 3\pi/4, 5\pi/6$ respectively). In the case $n_{a,b} = -1$, we find

$$2\frac{\|b\|}{\|a\|}(-1/2) = n_{a,b} = -1$$

so $\|a\| = \|b\|$ with an angle of $2\pi/3$ between them. By repeatedly applying the orthogonal reflections in these two roots (i.e., reflecting across H_a and H_b), one gets the known root system A_2. (This final step again uses that $W(\Phi) = \langle r_a, r_b \rangle$ and $\Phi = W(\Phi) \cdot \{a, b\}$. We will implicitly use this in each of the remaining two cases as well.)

In the case $n_{a,b} = -2$ we have

$$2\frac{\|b\|}{\|a\|}\left(\frac{-1}{\sqrt{2}}\right) = n_{a,b} = -2.$$

We obtain $\|b\| = \sqrt{2} \cdot \|a\|$, and the angle between these is $3\pi/4$. In this case, the same procedure of repeatedly reflecting through H_a and H_b yields the known root system B_2.

Finally, in the case $n_{a,b} = -3$ we have

$$2\frac{\|b\|}{\|a\|}\left(\frac{-\sqrt{3}}{2}\right) = n_{a,b} = -3,$$

which implies $\|b\| = \sqrt{3}\|a\|$. The angle between these is $5\pi/6$, and this time repeatedly reflecting through H_a and H_b involves some more effort, but eventually yields the known root system G_2. $\square$

Remark 24.10. By inspection, in each non-orthogonal case (i.e., the rank-2 root systems other than $A_1 \times A_1$), one can check without too much difficulty that there is no single line in $V_{\mathbf{C}}$ stable under the W-action, so the W-action on V is absolutely irreducible. But a W-invariant inner product is a "symmetric" isomorphism of W-representations $V \simeq V^*$, and the isomorphism is unique up to $\mathbf{R}^\times$-scaling due to Schur's Lemma over $\mathbf{C}$. Hence, the choice of W-invariant inner product is unique up to scaling by an element of $\mathbf{R}^\times$, and even an element of $\mathbf{R}_{>0}$ (by positive-definiteness of inner products). Thus, the Euclidean structure we chose in such cases is *unique up to scaling*, so the *ratio* between the lengths of two roots is *intrinsic* to the root system.

By inspection of the rank-2 cases, we see that in the non-orthogonal cases there are exactly two root-lengths. The value of these lengths is not intrinsic (but it is a common convention to uniquely scale it to make the shorter root-length be $\sqrt{2}$, so $c^\vee = c$ via the Euclidean structure for roots c of the shortest length in such cases). However, we have noted that the ratio of root lengths is intrinsic, so it is well-posed in the non-orthogonal cases to speak of one root having length at most that of another root. In this sense, one sees by inspection that for a basis $\{a, b\}$ we have:

$$n_{a,b}n_{b,a} = \left(\frac{\text{longer root length}}{\text{shorter root length}}\right)^2$$

(when nonzero).

Corollary 24.11. *Let $\{a, b\}$ be a basis, which we assume to be non-orthogonal (equivalently $r_a(b) \neq b$, or in other words $n_{a,b} \neq 0$), with labels chosen to make the intrinsic ratio $\|b\|/\|a\|$ be at least 1. This ratio determines the isomorphism class of the root system, and hence determines $n_{a,b}$.*

Proof. This is simple checking of the ratio in each of the non-orthogonal rank-2 cases that exist (i.e., A_2, B_2, G_2). $\square$

25. Dynkin diagrams

Last time, we discussed the classification of root systems in rank 2. We'll now discuss the classification of root systems (V, Φ) for arbitrary rank.

Remark 25.1. For the rank-2 case, the value $n_{ab} \in \mathbf{Z}$ for some basis $\{a, b\}$ determines (V, Φ) uniquely, provided that we "break the symmetry" in a and b by swapping the labels if necessary to arrange that $n_{ba} = -1$ for the non-orthogonal case. In the non-orthogonal case, the value of n_{ab} is determined by and determines the ratio $\|b\|/\|a\| \geq 1$.

Theorem 25.2. *The root system (V, Φ) is determined up to isomorphism by the Cartan matrix $(n_{a,a'})_{(a,a') \in B \times B}$ for a basis B. Moreover, if (V', Φ') is a root system with a basis B' and $f : B \xrightarrow{\sim} B'$ is a bijection such that*

$$n_{f(a),f(a')} = n_{aa'},$$

then the linear isomorphism

$$f : V = \bigoplus kB \simeq \bigoplus kB' = V'$$

carries Φ bijectively onto Φ'.

Proof. See [BtD, Prop. 5.2, Ch. V] for a full proof. The proof uses two key facts: $W(\Phi) = \langle r_b \rangle_{b \in B}$ and $\Phi = W(\Phi).B$, at which point it becomes a nice argument in linear algebra. $\square$

Definition 25.3. For two root systems (V_1, Φ_1) and (V_2, Φ_2), their *direct sum* is the root system $(V_1 \oplus V_2, \Phi_1 \times \{0\} \sqcup \{0\} \times \Phi_2)$, where for $a \in \Phi_1$ we use the reflection $r_{(a,0)}$ on $V_1 \oplus V_2$ defined to be $r_a \oplus \mathrm{id}$, and likewise for roots in Φ_2.

A root system (V, Φ) is *reducible* if it is a direct sum of two nonzero root systems.

Remark 25.4. In the reducible setting, with (V, Φ) a direct sum of (V_1, Φ_1) and (V_2, Φ_2), we have $n_{a_1,a_2} = 0$ for $a_i \in \Phi_i$, or equivalently, $r_{a_1}(a_2) = a_2$. This is immediate from how the reflections of the direct sum are built.

For $k = \mathbf{R}$ and $K_i \subset V_i$ a Weyl chamber for Φ_i, $K_1 \times K_2 \subset V_1 \oplus V_2$ is easily checked to be a Weyl chamber for the direct sum, with walls indexed by $B_1 \sqcup B_2$. Thus for bases B_1, B_2 of (V_1, Φ_1) and (V_2, Φ_2) respectively, $B := B_1 \sqcup B_2$ is a basis of the direct sum. The Cartan matrix written in the ordered basis B (putting B_1 before B_2) is given by

$$C(\Phi) = \begin{pmatrix} C(\Phi_1) & 0 \\ 0 & C(\Phi_2) \end{pmatrix}.$$

We then have

$$W(\Phi) = \langle r_b \rangle_{b \in B_1 \sqcup B_2} = \langle r_{b_1} \rangle_{b_1 \in B_1} \times \langle r_{b_2} \rangle_{b_2 \in B_2} \subset \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \subset \mathrm{GL}(V_1 \oplus V_2).$$

Thus $W(\Phi) = W(\Phi_1) \times W(\Phi_2)$. But then *all* bases of Φ are a disjoint union of bases of Φ_1 and Φ_2, since $W(\Phi)$ acts transitively on the set of bases (or one can argue more directly in terms of Weyl chambers, again, on the set of which $W(\Phi)$ acts transitively).

Definition 25.5. A root system (V, Φ) is *irreducible* if it is not reducible and not $(0, \emptyset)$.

Definition 25.6. Two roots $a, b \in \Phi$ are *orthogonal* if any of the following equivalent conditions hold:

- $n_{ab} = 0$
- $n_{ba} = 0$
- $\langle a, b^\vee \rangle = 0$
- $\langle b, a^\vee \rangle = 0$
- The roots a and b are orthogonal under some (or equivalently, any) $W(\Phi)$-invariant inner product.

Remark 25.7. For a $(V, \Phi) = (V_1, \Phi_1) \oplus (V_2, \Phi_2)$, since $W(\Phi) = W(\Phi_1) \times W(\Phi_2)$, elements of Φ_1 are orthogonal to elements of Φ_2.

It follows that any nonzero root system is a direct sum of irreducibles, which are necessarily pairwise orthogonal. This decomposition satisfies a strong uniqueness property.

Proposition 25.8. *If $(V, \Phi) \simeq \oplus (V_i, \Phi_i)$, the decomposition of the set $\Phi = \bigsqcup \Phi_i$ is unique. In particular, since V_i is the span of Φ_i, so the decomposition of Φ fully determines the decomposition of (V, Φ).*

See Appendix U for the proof. Note that this uniqueness is *much stronger* than the uniqueness of decomposing a finite group representation (in characteristic 0) into a direct sum of irreducibles: for the latter there is a lot of ambiguity about the underlying subspaces of such irreducible constituents in the direct sum in case of multiplicities, but for the root system decomposition there is *no* such ambiguity, even for a direct sum of several copies of the same irreducible root system (since the actual partition of the set of roots is uniquely determined!).

As a result, the key case in the classification is the irreducible case. From now on, we will assume that (V, Φ) is an irreducible root system.

Consider two distinct roots $a, a' \in \Phi$ that are elements of a basis B of Φ and that are not orthogonal. Then we can recover the Cartan integer $n_{aa'}$ from a rank-2 root system $\Phi_{a,a'} = (\mathbf{R}a \oplus \mathbf{R}a', \Phi \cap \mathrm{span}_{\mathbf{Z}}(a, a'))$. More precisely, HW8 Exercise 3 shows that $\Phi_{aa'}$ is a root system, and since a and a' are not orthogonal this is not $\mathrm{A}_1 \times \mathrm{A}_1$. Thus, it is A_2, B_2, or G_2. By using that $a, a' \in B$ and thinking about $\Phi^+(K(B))$ we see (exercise!) that $\{a, a'\}$ is a basis of $\Phi_{a,a'}$. But we observed in these rank-2 non-orthogonal cases the ratio $\|a\|/\|a'\| \geq 1$ (swap a, a' to make this ≥ 1) of root-lengths for a basis is intrinsic to the root system $\Phi_{a,a'}$, and that this ratio determines and is determined by $n_{a,a'}$.

Fact 25.9. It is an important fact in general, which we have noted by inspection in the non-orthogonal rank-2 cases (exactly the irreducible rank-2 cases!), that for irreducible root systems (V, Φ), the group action $W(\Phi) \curvearrowright V$ is absolutely irreducible. For a proof, see [Bou2, Ch. VI, §1.2, Cor. to Prop. 5].

We will not use this fact, but it significance is that for $k = \mathbf{R}$, the $W(\Phi)$-invariant inner product on V is unique up to scaling, so ratio of root-lengths for any two roots in Φ is intrinsic. In particular, the "Euclidean geometry" way of thinking about root systems over $\mathbf{R}$ in the irreducible case is equivalent to the purely linear-algebraic approach (over $\mathbf{R}$), since the Euclidean structure is uniquely determined up to a positive scaling factor (often normalized by setting the shortest root length to be $\sqrt{2}$).

Remark 25.10. The Cartan matrix is determined by the data of the integers $n_{a,a'}$ for distinct $a, a' \in B$, since $n_{aa} = 2$.

The conclusion of this discussion is:

Proposition 25.11. *The root system (V, Φ) is determined up to isomorphism by the ratios $\|a\|/\|a'\|$ for distinct non-orthogonal $a, a' \in B$, where B is a chosen basis. This ratio is either $1, (\sqrt{2})^{\pm 1}$, or $(\sqrt{3})^{\pm 1}$.*

The data in the preceding result is precisely the information captured in the following concept:

Definition 25.12. The *Dynkin diagram* $\Delta(\Phi)$ of (V, Φ) is a decorated ("weighted") directed graph without loops (i.e., no edge whose vertices coincide) defined as follows. The vertices of $\Delta(\Phi)$ are the elements of a chosen basis B, and for distinct $a, b \in B$ arranged with $\|a\|/\|b\| \geq 1$ in the non-orthogonal case the edges joining a and b are given by:

Orthogonal case:	$\overset{a}{\bullet}\quad\overset{b}{\bullet}$	
Non-orthogonal case with $\|a\|/\|b\| = 1$:	$\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$	A_2
Non-orthogonal case with $\|a\|/\|b\| = \sqrt{2}$:	$\bullet\!\Rightarrow\!\bullet$	B_2
Non-orthogonal case with $\|a\|/\|b\| = \sqrt{3}$:	$\bullet\!\Rrightarrow\!\bullet$	G_2

In these pictures, the arrow points from the longer root to the short one (when the roots a, b are non-orthogonal).

Classifying irreducible root systems (V, Φ) is now equivalent to classifying the possibilities for their connected Dynkin diagrams, which is a problem in Euclidean geometry (this might seem to require Fact 25.9, but it doesn't because once we have a complete list of diagrams we just need to exhibit *some* root system with that diagram and it doesn't matter if we do that via Euclidean methods or not). The classification of connected Dynkin diagrams arising from (reduced) irreducible root systems is discussed in [Hum]. In [Hum, §11], a list of all *possible* connected Dynkin diagrams arising from (reduced) root systems is found, and in [Hum, §12.1] it is shown that everything on this list of possibilities really occurs; the arguments are a mixture of elementary graph theory considerations and Euclidean geometry (the existence argument requites explicit constructions in the Euclidean spaces $\mathbf{R}^n$).

Remark 25.13. It is an unexpected fact coming out of the classification that for an irreducible (and reduced!) root system there are at most 2 root lengths (this is a well-posed statement due to Fact 25.9; we will never use it). I don't know a proof which doesn't amount to staring at the possibilities for Dynkin diagrams.

Remark 25.14. An important fact lying rather beyond the course is that every (reduced!) root system arises as the root system of a connected compact Lie group. The proof goes through complex semisimple Lie algebras and complex Lie groups. One reassuring feature of this fact is that it guarantees that there are no "missing axioms" for the notion of root system that arises in the study of compact Lie groups.

Where are we at? Let's now summarize the structural results that we have achieved so far (bringing in results from homework as needed). Let G be a connected compact Lie group, and T a maximal torus.

(1) If Z_G is finite then we have several consequences: (i) $G = G'$ (i.e., G is its own commutator subgroup), by HW8 Exercise 2(iii), (ii) $(X(T)_{\mathbf{Q}}, \Phi = \Phi(G,T))$ is a root system with Weyl group $W(\Phi)$ equal to $W_G(T)$, (iii) $X(T)/\mathbf{Z}\Phi \simeq X(Z_G)$ is finite with size $\#Z_G$, by HW7 Exercise 2(iii).

(2) If Z_G is finite and $\pi : G \to \overline{G}$ is an isogeny to another connected compact Lie group (so π is the quotient modulo a finite central subgroup and $Z_{\overline{G}} = Z_G/(\ker \pi)$ is finite) then $\overline{T} := \pi(T)$ is a maximal torus of $\overline{G}$ and composition with π defines an isomorphism of root systems

$$(X(\overline{T})_{\mathbf{Q}}, \overline{\Phi}) \simeq (X(T)_{\mathbf{Q}}, \Phi)$$

(see HW8 Exercise 4(i), resting on the fact that all roots of (G,T) are trivial on Z_G, and hence trivial on $\ker \pi \subset Z_G$, so as characters of T they all factor through the quotient $T/(\ker \pi) = \overline{T}$).

Moreover, the dual isomorphism $X_*(T)_{\mathbf{Q}} \simeq X_*(\overline{T})_{\mathbf{Q}}$ composing cocharacters with $\pi : T \to \overline{T}$ carries $a^\vee$ to $\overline{a}^\vee$ for $a \in \Phi$ corresponding to $\overline{a} \in \overline{\Phi}$.

(3) On HW9, the following are proved. The commutator subgroup G' is *always* closed, with $(G')' = G'$, and for the maximal central torus Z_G^0 the multiplication map

$$\mu : Z_G^0 \times G' \to G$$

is an isogeny (so $G = G'$ if and only if Z_G is finite!). In particular, since μ corresponds to a connected finite-degree covering space of G and $\pi_1(S^1) = \mathbf{Z}$, it follows that if $\pi_1(G)$ is finite then the maximal central torus Z_G^0 is trivial, which is to say Z_G is finite.

Conversely, by HW9 Exercise 3(v) (which rests on HW8 Exercise 4(iv) that in turn involves serious input from the theory of root systems), if Z_G is finite then the universal cover $\widetilde{G} \to G$ that has abelian kernel $\pi_1(G)$ is a *finite-degree* covering space, so $\pi_1(G)$ is finite.

The upshot of this is that the following three conditions are *equivalent*:

- Z_G is finite,
- $\pi_1(G)$ is finite,
- $G = G'$.

We call such G *semisimple*. It can be shown with more knowledge about the structure of finite-dimensional Lie algebras in characteristic 0 that for compact connected G, being semisimple is equivalent to the condition on the Lie algebra $\mathfrak{g}$ that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. (Warning: there is a general notion of "semisimple" for finite-dimensional Lie algebras $\mathfrak{h}$ in characteristic 0, and it implies the property of $\mathfrak{h}$ being its own commutator subalgebra but the converse is generally not true beyond the setting of $\mathfrak{h}$ over $\mathbf{R}$ that arise as the Lie algebra of a compact Lie group.)

For semisimple connected compact G with maximal torus T and root system Φ, the identification of coroots with cocharacters of T identifies $X(T)$ with a sublattice of the $\mathbf{Z}$-dual

$$P := (\mathbf{Z}\Phi^\vee)' \subset X(T)^*_{\mathbf{Q}} = X_*(T)_{\mathbf{Q}}$$

of the "coroot lattice" $\mathbf{Z}\Phi^\vee \subset X_*(T)$. The notation P comes from the French word *poid* for weight. Since all roots are characters of T, and the finite-degree universal cover $\pi : \widetilde{G} \to G$ has the *same* root system (using its maximal torus $\widetilde{T} = \pi^{-1}(T)$) since π is an isogeny, we have containments

$$Q := \mathbf{Z}\Phi \subset X(T) \subset X(\widetilde{T}) \subset (\mathbf{Z}\Phi^\vee)' =: P$$

of full-rank lattices in the $\mathbf{Q}$-vector space $X(T)_{\mathbf{Q}}$. In particular, these are finite-index inclusions (Q has as a $\mathbf{Z}$-basis the elements of a basis B of the root system, and P has as a $\mathbf{Z}$-basis the dual basis to the basis $B^\vee$ of the dual root system, with the inclusion $Q \hookrightarrow P$ having its matrix relative to these bases given by the Cartan matrix or its transpose, depending on the time of day).

In particular, the finite kernel $\pi_1(G) = \ker(\widetilde{G} \to G) = \ker(\widetilde{T} \to T)$ is dual to the cokernel of the finite-index inclusion $X(\widetilde{T}) \hookrightarrow X(T)$, so

$$\#\pi_1(G) = [X(\widetilde{T}) : X(T)] \,|\, [P : X(T)].$$

Appendix V proves the fundamental fact that

$$X(\widetilde{T}) = P$$

(related to the fact that in the simply connected case there is no obstruction to integrating a Lie algebra representation to a Lie group representation). Combining this with (1)(iii) above, we then obtain the equality

$$(\#\pi_1(G))(\#Z_G) = [X(\widetilde{T}) : X(T)][X(T) : X(T/Z_G)] = [P : X(T)][X(T) : Q] = [P : Q]$$

whose right side has *nothing to do* with where G sits within its "isogeny class". The lesson is that there is a trade-off between $\#\pi_1(G)$ and $\#Z_G$ as G wanders around between $\widetilde{G}$ and the maximal central quotient $\widetilde{G}/Z_{\widetilde{G}}$: as $\pi_1(G)$ gets smaller (i.e., G becomes "closer" to $\widetilde{G}$) the center Z_G gets bigger and vice-versa.

Roughly speaking, the position of the lattice $X(T)$ between Q and P keeps track of where G sits as a quotient of the simply connected compact $\widetilde{G}$; the inclusion of the finite subgroup $\ker(\widetilde{G} \to G) \subset Z_{\widetilde{G}} \subset \widetilde{T}$ into $\widetilde{T}$ is dual to the quotient map from P onto the finite-order quotient $P/X(T) = X(\widetilde{T})/X(T)$ that dominates P/Q. This is the first step towards formulating a combinatorial classification of G under the isomorphism rather than just up to central isogeny (improving on the root system, which is only an isogeny invariant).

26. Normal subgroup structure

Recall from HW6 Exercise 6 that the commuting left and right multiplications on $\mathbf{H} \simeq \mathbf{R}^4$ by the group $\mathrm{SU}(2)$ of norm-1 quaternions preserve the positive-definite quadratic form given by the norm, and thereby yields a homomorphism of connected compact Lie groups

$$(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2 \simeq \mathrm{SO}(4)$$

which is an isomorphism.

Loosely speaking, $\mathrm{SO}(4)$ is built by pasting two $\mathrm{SU}(2)$'s along their common central μ_2. The interesting thing is that the constraint of *normality* in $\mathrm{SO}(4)$ really distinguishes these two $\mathrm{SU}(2)$ subgroups from others. For example, the diagonally embedded $\mathrm{SU}(2)/\mu_2$ in the left side is *not* normal (precisely because $\mathrm{SU}(2)$ is very non-commutative)! So this "isogeny decomposition" of $\mathrm{SO}(4)$ is much more rigid than decomposing a finite abelian group into a product of cyclic groups or an abelian variety into an isogenous product of simple abelian varieties.

Today we will discuss a far-reaching generalization for all connected compact Lie groups that are semisimple (meaning any of the equivalent conditions discussed in the summary at the end of last time: finite center, is its own commutator subgroup, has semisimple Lie algebra, or has finite fundamental group)

Goal 26.1. A general nontrivial semisimple connected compact Lie group is built canonically from finitely many nontrivial connected closed normal subgroups G_i that are "almost simple" (meaning: has no nontrivial connected closed normal proper subgroup) and which commute with each other (hence have central pairwise intersections) in the same way that $\mathrm{SO}(4)$ "decomposes" as two canonical copies of $\mathrm{SU}(2)$ above. These G_i's will be *uniquely* determined as a collection of subgroups of G, and correspond to the irreducible components of the root system $\Phi(G, T)$ upon choosing a maximal torus T (but the G_i's will be *independent* of T; keep in mind that all T are conjugate to each other inside G and the G_i's are going to be normal in G).

For a maximal torus T in a semisimple G, if $B \subset \Phi(G, T)$ is a basis, then

$$X(T) \supset \mathbf{Z}\Phi = \bigoplus_{b \in B} \mathbf{Z}b = X(T/Z_G),$$

where the containment $X(T) \supset \mathbf{Z}\Phi =: Q$ of the root lattice inside the character lattice has finite index (and we recall that the simultaneous kernel inside T of the roots is Z_G by consideration of the T-action on $\mathfrak{g}_{\mathbf{C}}$). Thus $T/Z_G \simeq \prod_{b \in B} S^1$ via $\bar{t} \mapsto (\bar{t}^b)$. This is a concrete description of the "adjoint torus" $T/Z_G \subset G^{\mathrm{ad}} := G/Z_G \subset \mathrm{GL}(\mathfrak{g})$.

At the other extreme, consider the universal cover $\widetilde{G} \to G$ that we know from HW9 is an isogeny; in particular, $\widetilde{G}$ is compact connected and semisimple. The preimage of

T is the corresponding maximal torus $\widetilde{T} \subset \widetilde{G}$ and we have $\Phi(\widetilde{G}, \widetilde{T}) = \Phi(G, T)$ inside $X(T)_{\mathbf{Q}} = X(\widetilde{T})_{\mathbf{Q}}$ (as the root system is invariant under isogenies of semisimple connected compact Lie groups). We have the finite-index inclusion

$$X(T) \hookrightarrow X(\widetilde{T}) \subset (\mathbf{Z}\Phi^\vee)',$$

into the dual lattice $(\mathbf{Z}\Phi^\vee)' \subset X(T)_{\mathbf{Q}} = X_*(T)_{\mathbf{Q}}^\vee$ to the coroot lattice $\mathbf{Z}\Phi^\vee \subset X_*(T)_{\mathbf{Q}}$. The dual lattice $(\mathbf{Z}\Phi^\vee)'$ is called the *weight lattice* of the root system and is denoted P, from the French word *poid*, for weight.

Miracle 26.2. The containment $X(\widetilde{T}) \subset (\mathbf{Z}\Phi^\vee)' = P$ (which expresses the fact that coroots arises as cocharacters for any semisimple connected compact Lie group with a given root system, such as $(\widetilde{G}, \widetilde{T})$) is an equality; equivalently, the containment $\mathbf{Z}\Phi^\vee \subset X_*(\widetilde{T})$ is an equality. This is very important: it tells us that there are *no further conditions* on $X(T)$ beyond that it lies between Q and P. In particular, all elements of P really do occur as characters for the simply connected member of the isogeny class with the given root system. See Appendix V for the proof that $P = X(\widetilde{T})$; it is a very serious theorem, involving real input from topological considerations.

For $B^\vee := \{b^\vee \mid b \in B\}$ the corresponding basis of $\Phi^\vee$, we have

$$\bigoplus_{b \in B} \mathbf{Z}b^\vee = \mathbf{Z}\Phi^\vee = X_*(\widetilde{T}).$$

Thus we have an isomorphism

$$\prod_{b \in B} S' \simeq \widetilde{T}$$

given by

$$(x_b) \mapsto \prod_{b \in B} b^\vee(x_b).$$

So we have a very concrete description for two tori, T/Z_G and $\widetilde{T}$, whose character lattices are directly described in terms of the root system. The tori "in between" such as T remain somewhat mysterious: all we can say is $Q \subset X(T) \subset P$ (and all intermediate lattices between P and Q arise this way from some central isogenous quotient of $\widetilde{G}$ since G^{ad} as an isogenous quotient of $\widetilde{G}$ *with trivial center* must necessarily coincide with $\widetilde{G}^{\mathrm{ad}}$ (why?), forcing P/Q to be dual to $Z_{\widetilde{G}}$).

Remark 26.3. The dual basis to $B^\vee$ in the lattice $X(\widetilde{T})$ dual to $X_*(\widetilde{T})$ is usually denoted as $\{\varpi_b\}_{b \in B}$ and is called the set of *fundamental weights*. We'll encounter this in our discussion of representation theory in the final two lectures.

Here are some basic examples.

Example 26.4. For $m \geq 3$, $\mathrm{SO}(m)$ has $X(T) \subset P = X(\widetilde{T})$ of index 2 (see Appendix T). This corresponds to the fact that $\widetilde{\mathrm{SO}(m)} \to \mathrm{SO}(m)$ is a degree-2 cover, called $\mathrm{Spin}(m)$.

Example 26.5. The groups $\mathrm{SU}(n)$ for $n \geq 2$ and $\mathrm{Sp}(n)$ for $n \geq 1$ have $X_*(T) = \mathbf{Z}\Phi^\vee$, so $G = \widetilde{G}$ and these groups are simply connected. Explicitly $\mathrm{SU}(n)$, we have seen that $X(T) = \mathbf{Q}^n/\Delta$, so

$$X_*(T) = \left\{\sum x_j = 0\right\} \subset \mathbf{Q}^n,$$

and the coroots are exactly $e_i^* - e_j^*$ for $i \neq j$. These coroots visibly span $X_*(T)$, as promised (and $\{e_i^* - e_{i+1}^*\}_{1 \leq i < n}$ is $B^\vee$ for one choice of B).

Remark 26.6. Since $P = X(\widetilde{T})$ and $\ker(\widetilde{G} \to G) = \ker(\widetilde{T} \to T)$,

$$[P : X(T)] = \deg(\widetilde{T} \to T) = \deg(\widetilde{G} \to G) = \#\pi_1(G).$$

As we discussed previously, paradoxically it is the contribution from central tori (i.e., the commutative case) which is the most unpleasant part of the description of connected compact Lie groups, or in other words it is the semisimple cases which admit the most elegant structure theorem. Here is the main result, which is fully proved in Appendix X:

Theorem 26.7. *For a nontrivial semisimple connected compact Lie group G, the irreducible decomposition of $\Phi(G, T)$ determines the normal subgroup structure as follows.*

The minimal *nontrivial closed connnected normal subgroups $\{G_i \subset G\}_{i \in I}$ satisfy the following:*

1. *there are finitely many G_i,*
2. *the G_i's pairwise commute,*
3. *the G_i's are all semisimple,*
4. *$\prod G_i \xrightarrow{\pi} G$ given by multiplication is an isogeny,*
5. *for a maximal torus $T \subset G$ and the corresponding maximal torus $\pi^{-1}(T) = \prod T_i$ in $\prod G_i$ (so $T_i = T \cap G_i$ is a maximal torus in G_i), the equality $X(T)_{\mathbf{Q}} = \bigoplus X(T_i)_{\mathbf{Q}}$ arising from the finite-index inclusion $X(T) \hookrightarrow \prod X(T_i)$ identifies $\Phi := \Phi(G, T)$ with the subset $\sqcup \Phi_i$ for $\Phi_i := \Phi(G_i, T_i) \subset X(T_i)$, and $\{(X(T_i)_{\mathbf{Q}}, \Phi_i)\}$ is the set of irreducible components of $(X(T)_{\mathbf{Q}}, \Phi)$,*
6. *the connected closed normal subgroups $N \subset G$ are exactly the subgroups $G_J = \langle G_i \rangle_{i \in J}$ for uniquely determined $J \subset I$.*

Remark 26.8. The final part shows that in contrast with the situation for general groups, in the setting of semisimple connected compact Lie groups and connected closed subgroups thereof, normality is transitive! (Tossing in a central torus is harmless: the canonical isogeny decomposition as a direct product of the maximal central torus and the semisimple commutator subgroup immediately implies that the same conclusion then holds without assuming the ambient group to be semisimple.)

Remark 26.9. As an important special case, the nontrivial semisimple G is "almost simple," i.e. the only normal connected closed subgroups are 1 and G, if and only if the root system Φ is irreducible.

Comments on the proof. The above theorem is proved (in Appendix X) essentially in the opposite order of its formulation: the G_i's are first made using Φ_i's, and are then eventually seen after quite a bit of work to be the same as the G_i's defined near the start of the theorem.

Let's address two key aspects of the proof: why irreducibility of the root system forces a connected closed normal proper subgroup $N \subset G$ to be trivial, and how the G_i's are built using the structure of the root system.

Consider a connected closed normal subgroup N of G. First we claim that N must be semisimple. In view of the natural isogeny $Z_N^0 \times N' \to N$ with N' semisimple, we have to show that $Z_N^0 = 1$. By normality of N in G, clearly Z_N^0 is normal in G. Since normal tori in connected compact Lie groups are automatically central, so $Z_N^0 \subset Z_G^0 = 1$. Thus $N = N'$, so N is semisimple as claimed.

To show $N = 1$ if $N \neq G$, pick a maximal torus $S \subset N$ and consider $\Phi(N, S)$. Certainly S is not maximal in G, since otherwise by normality of N and the Conjugacy Theorem we'd have $N = G$. Let $T \subset G$ be a maximal torus containing S, so $\dim S < \dim T$. Note that $T \cap N \subset Z_N(S) = S$, so the action of $N_G(T)$ on N preserves $T \cap N = S$. The action of the Weyl group $W(\Phi) = W(G, T)$ on $X(T)_{\mathbf{Q}}$ therefore preserves the *proper* subspace $X(S)_{\mathbf{Q}}$. But since Φ is irreducible, the action of $W(\Phi)$ on $X(T)_{\mathbf{Q}}$ is irreducible (see HW9, Exercise 1(ii)), so $X(S)_{\mathbf{Q}} = 0$ and thus $N = 1$!

To build the subgroups G_i, consider the irreducible components (V_i, Φ_i) of $(X(T)_{\mathbf{Q}}, \Phi)$. By working with root systems at the level of *rational* vector spaces (not $\mathbf{R}$-vector spaces!), we can now exploit the following elementary fact (exercise, using saturation of lattices in subspaces of finite-dimensional $\mathbf{Q}$-vector spaces): quotients of $X(T)_{\mathbf{Q}}$ as a $\mathbf{Q}$-vector spaces correspond naturally to subtori of T. This implies (check!) that for any direct sum decomposition $W \oplus W'$ of $V := X(T)_{\mathbf{Q}}$ there are unique subtori $S, S' \subset T$ such that $S \times S' \to T$ is an isogeny and the resulting isomorphism $X(S)_{\mathbf{Q}} \oplus X(S')_{\mathbf{Q}} = X(T)_{\mathbf{Q}}$ identifies $X(S)_{\mathbf{Q}}$ with W and $X(S')_{\mathbf{Q}}$ with W'.

Hence, the decomposition $\bigoplus V_i$ of $X(T)_{\mathbf{Q}}$ from the irreducible components of the root system define subtori $T_i \subset T$ for which $\prod T_i \to T$ is an isogeny identifying $X(T_i)_{\mathbf{Q}}$ with V_i as a direct summand of $X(T)_{\mathbf{Q}}$. Now comes the great idea: if the desired descriptions in (3) and (4) were to hold, then $Z_G(\prod_{j \neq i} T_j)$ should be the subgroup generated by G_i and the tori $T_j = Z_{G_j}(T_j)$ which commute with it, so passing to the commutator subgroup $Z_G(\prod_{j \neq i} T_j)'$ should yield G_i! This motivates taking as a *definition*

$$G_i := (Z_G(\prod_{j \neq i} T_j))',$$

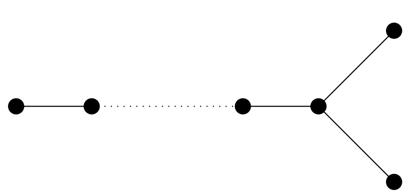
and then one has to do work to *prove* that these work as desired in the theorem (in particular, these are independent of T and in fact normal and intrinsically characterized via minimality conditions).

The moral of the story is that formation of torus centralizers and commutator subgroups is a very powerful tool for digging out normal connected closed subgroups of connected compact Lie groups. $\square$

Example 26.10. Let's illustrate the basic flavor of the structure theorem by making a couple of new semisimple connected compact Lie groups from the basic building blocks corresponding to the irreducible root systems. The root system for E_6 shows that the simply connected version, which we'll denote as E_6, has center μ_3. The group $\mathrm{SU}(12)$ has center $\mu_{12} \supset \mu_3$, so we can glue to make a new group:

$$G = (\mathrm{SU}(12) \times \mathrm{E}_6)/\mu_3.$$

For another such construction, we will again work with $\mathrm{SU}(12)$, but now use its central μ_4. In is a general fact that the simply connected compact groups $\mathrm{Spin}(2n)$ of type D_n for *odd* $n \geq 3$ have center μ_4 (whereas for even n the center is $\mu_2 \times \mu_2$). In terms of the Dynkin diagram



and the description of the torus in the simply connected case as $\prod_{b \in B} b^\vee(S^1)$ via the coroot basis $B^\vee$, the central μ_4 is "supported" in the coroot groups $b^\vee(S^1)$ for b corresponding to the vertices at the end of the short legs and in alternating vertices along the long arm (but which of the two senses of "alternating" depends on the parity of n). Be that as it may, we can form the gluing

$$(\mathrm{SU}(12) \times \mathrm{Spin}(10))/\mu_4.$$

The upshot of the structure theorem is that a general semisimple connected compact Lie group is always built by variations on this construction: an isogenous (central!) quotient of a direct product $\prod \widetilde{G}_i$ of several semisimple connected compact Lie groups that are simply connected. In this sense, much of the serious structural work with semisimple connected compact Lie groups can focus on the cases with an irreducible root systems. This is very useful in general proofs.

Since it turns out (with hard work beyond the level of this course, to prove the so-called Isomorphism Theorem) that the simply connected compact connected Lie groups are determined up to isomorphism by the root system, and the classical groups exhaust all the root systems of types A, B, C, D, apart from the 5 exceptional root systems in the classification we have a rather concrete grip on what a general semisimple connected compact Lie group can look like! This isn't to say that one should usually prove general theorems via case-analysis of each of the classical and 5 exceptional types (far from it: using the classification to prove a *general* result should be avoided whenever possible), but nonetheless this isogeny classification is a really wonderful gift from the heavens for testing good examples and occasionally for proving general results. For instance, the only known proof of the Hasse Principle for simply connected groups in the arithmetic theory of linear algebraic groups over number fields requires extensive case-analysis via an analogous but deeper classification result for connected semisimple groups over number fields in terms of root systems and Galois cohomology.

27. Return to representation theory

We start with some comments on commutators.

(1) For a general compact connected Lie group G, we know $G' \subset G$ is closed, $(G')' = G'$ (i.e., G' is semisimple), and

$$Z_G^0 \times G' \to G$$

is an isogeny; this was all seen in HW9, Exercise 4(ii).

For semisimple G, we have isogenies

$$\prod \widetilde{G}_i \to \prod G_i \to G,$$

where $\{G_i\}$ is the (finite!) set of the minimal nontrivial connected normal closed subgroups of G (which are necessarily semisimple), and $\widetilde{G}_i \to G_i$ is the universal cover. The composite map has finite kernel $Z \subset \prod Z_{\widetilde{G}_i}$. Further, the $Z_{\widetilde{G}_i}$ are explicitly described for each irreducible root system Φ_i, as shown in Appendix Y.

(2) For any connected Lie group G and connected normal Lie subgroups $H_1, H_2 \subset G$, the commutator $(H_1, H_2) \subset G$ is the connected Lie subgroup associated to $[\mathfrak{h}_1, \mathfrak{h}_2]$ (see Appendix W for a proof).

But beware that if $H_i \subset G$ are closed, then (H_1, H_2) can fail to be closed beyond the case of compact G. This can even fail for $G = \mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{R})$; examples are given in Appendix W.

The upshot is that up to killing a finite central subgroup, describing the representation theory of a compact connected G (for finite-dimensional continuous $\mathbf{C}$-linear representations) reduces to the case that $G = T \times \prod_i G_i$ where T is a torus and G_i are simply connected semisimple with irreducible root system. Indeed, by the above comments, we know every compact connected Lie group H admits a central isogeny from such a group, and so representations of of H are just representations of G killing a certain finite central subgroup. This motivates us to address how irreducible representations arise for a direct product of compact Lie groups, which works out exactly as for finite groups:

Proposition 27.1. *Let G_1 and G_2 be compact Lie groups.*

(1) If (V_i, ρ_i) are irreducible representations of G_i then $V_1 \otimes V_2$ is irreducible as a $G_1 \times G_2$ representation.

(2) Conversely, if V is an irreducible representation of $G_1 \times G_2$ then $V \simeq V_1 \otimes V_2$ for irreducible representations V_i of G_i.

Remark 27.2. This result reduces the general study of irreducible representations for connected compact Lie groups G to the case of simply connected semisimple G with irreducible Φ (since we understand the irreducible representations of tori).

Proof. We prove the two parts in order.

(1) Let $\chi_i : G \to \mathbf{C}$ be the character of G_i and let dg_i be the volume-1 Haar measure on G_i. Then, $V_1 \otimes V_2$ has character $\chi(g_1 g_2) = \chi_1(g_1)\chi_2(g_2)$ since

$$\operatorname{tr}\left(\rho(g_1) \otimes \rho_2(g_2)\right) = \operatorname{tr}\left(\rho_1(g_1)\right) \operatorname{tr}\left(\rho_2(g_2)\right).$$

Therefore,

$$\begin{aligned} \langle \chi, \chi \rangle_{G_1 \times G_2} &= \int_{G_1 \times G_2} \chi_1(g_1)\chi_2(g_2)\overline{\chi_1(g_1)\chi_2(g_2)} dg_1 dg_2 \\ &= \langle \chi_1, \chi_1 \rangle_{G_1} \langle \chi_2, \chi_2 \rangle_{G_2} \\ &= 1 \cdot 1 \\ &= 1. \end{aligned}$$

(2) Since V is irreducible, hence nonzero, V has an irreducible subrepresentation V_1 as a G_1-representation. Then, the V_1-isotypic subspace W of V as a G_1-representation is G_2-stable because (i) G_2 commutes with the action of G_1, (ii) isotypic subspaces are intrinsic. Hence, W is $G_1 \times G_2$ stable (and nonzero), so $W = V$ by irreducibility for $G_1 \times G_2$.

Consider the canonical evaluation isomorphism of G_1-representations

$$V_1 \otimes_{\mathbf{C}} \operatorname{Hom}_{G_1}(V_1, V) \simeq W = V.$$

This is an isomorphism due to complete reducibility for G_1 and Schur's lemma. By inspection, it is also G_2-equivariant (using the G_2 action on $V_2 := \operatorname{Hom}_{G_1}(V_1, V)$t through the G_2-action on V). Thus, V_2 must be irreducible, because it is nonzero and tensoring V_1 against a nonzero proper G_2-subrepresentation of V_2 would produce a nonzero proper $G_1 \times G_2$-subrepresentation of V (contradicting the irreducibility of V). It follows that $V = V_1 \otimes V_2$ for irreducible V_1 and V_2.

□

There are now two key issues to be addressed:

(1) We want to describe in terms of Φ (a root system) the T-restriction of characters of irreducible representations of the unique simply connected semisimple compact connected Lie group G with root system Φ.
(2) Some time ago, we introduced a coarser notation: the representation ring $R(G) \subset R(T)^W$ (concretely, $R(G)$ consists of differences of characters of representations of G; this is a space of class functions that is a ring under pointwise operations, and is a natural object of interest for general representation-theoretic questions). Since $W = W(\Phi)$ and $R(T)$ is the group ring of $X(T)$, the ring $R(T)^W$ is very accessible via the root system. Is the inclusion of $R(G)$ into $R(T)^W$ an equality?

Warning 27.3. Let $R_{\text{eff}}(G) \subset R(G)$ the subset corresponding to characters of actual representations (as opposed to virtual representations, which may involve formal inverses of representations). It is generally not true that $R_{\text{eff}}(G) = R_{\text{eff}}(T)^W$ (this already fails for $G = \text{SU}(2)$).

Remark 27.4. The answer to (2) above is affirmative, but the proof, which is given in Appendix Z, uses the answer to (1) via the Theorem on the Highest Weight which will be discussed next time.

Remark 27.5. In [BtD, Ch. VI, §5-§7], $R(G) = R(T)^W$ is calculated for the classical types (A through D). Which representations are defined over $\mathbf{R}$ is also discussed there (this property is more than the character being $\mathbf{R}$-valued; there are "quaternionic" obstructions).

Now we prepare to discuss the nature of the answer to (1) above. This is a multi-part result called the *Weyl character formula* that we will discuss next time. To state it, we need a new construction, called the "half-sum of positive roots":

Construction 27.6. Fix Φ^+ (corresponding to a Weyl chamber K, or equivalently a basis B). Define

$$\rho := \frac{1}{2} \sum_{c \in \Phi^+} c.$$

For simply connected semisimple (G, T) with root system Φ, recall $X(T) = P := (\mathbf{Z}\Phi^\vee)'$ (where $'$ means "dual lattice"). Clearly $\rho \in (1/2)X(T)$. But the situation is much better!

Example 27.7. For $G = \text{SU}(2)$ of type A_1, we have $\Phi^+ = \{a\}$ is a choice among the two roots, and by inspection a is twice a basis for $X(T)$ (since the conjugation action of $\text{diag}(z, 1/z)$ on the upper triangular nilpotent matrices in $\mathfrak{sl}_2(\mathbf{C})$ is multiplication by z^2). Hence, $\rho = (1/2)a$ is a basis for $X(T)$.

Example 27.8. Consider $\text{Sp}(2)$ of type C_2, with basis $\{a, b\}$ for short a and long b. Choose the Euclidean structure to make a have length $\sqrt{2}$, so $a^\vee = a$ and $b^\vee = 2b/4 = b/2$. One can then check that

$$X(T) = \left(\mathbf{Z}\Phi^\vee\right)' = \mathbf{Z}a \oplus \mathbf{Z}(b/2).$$

Since $\Phi^+ = \{a, b, b + a, b + 2a\}$, we see that $\rho = (1/2)(4a + 3b) = 2a + (3/2)b \in X(T)$.

The two preceding examples illustrate an important general fact:

Proposition 27.9. *We have $\rho \in P$. In fact, for a basis $B = \{b_1, \dots, b_n\}$ and the corresponding fundamental weights $\{\varpi_1, \dots, \varpi_n\} \subset X(T) = P$ dual to the basis $B^\vee$ for $\mathbf{Z}\Phi^\vee$, we have $\rho = \sum \varpi_i$. Equivalently, $\langle \rho, b_i^\vee \rangle = 1$ for all i.*

Remark 27.10. See [BtD, Ch. V §6] for the computation of ρ for all the classical types (i.e., all root systems of types A through D).

Proof. Fix a choice of $b \in B$. By definition

$$r_b(\rho) = \rho - \langle \rho, b^\vee \rangle b,$$

so it is equivalent to show $r_b(\rho) = \rho - b$ for each such $b \in B$. By definition

$$\rho = (1/2)(b + \sum_{c \in \Phi^+ - \{b\}} c),$$

and we claim that the action of r_b on Φ *permutes* $\Phi^+ - \{b\}$. Once this is shown, it follows that $r_b(\rho - b/2) = \rho - b/2$ since $\rho - b/2$ is the half-sum of the elements of $\Phi^+ - \{b\}$, so

$$r_b(\rho) = r_b((\rho - b/2) + b/2) = (\rho - b/2) + r_b(b/2) = (\rho - b/2) - b/2 = \rho - b$$

as desired.

It remains to prove the permutation claim. Since the general effect $r_b(x) = x - \langle x, b^\vee \rangle b$ on the ambient vector space $\bigoplus_{b' \in B} \mathbf{Q} b'$ only changes x in the b-direction yet anything in $\Phi^+ - \{b\}$ has $\mathbf{Z}_{>0}$-coefficients somewhere away from b (our root systems are reduced!), the effect of r_b on anything in $\Phi^+ - \{b\}$ *still* has a positive B-coefficient somewhere away from b. But everything in $\Phi = \Phi^+ \coprod -\Phi^+$ has all its nonzero B-coefficients of the *same* sign (!), so indeed $r_b(\Phi^+ - \{b\}) \subset \Phi^+$ and hence

$$r_b(\Phi^+ - \{b\}) = \Phi^+ - \{b\}$$

(since $b = r_b(-b)$ with $-b \notin \Phi^+$). This establishes the permuation claim. $\square$

28. Examples and Fundamental Representations

Let G be a simply connected semisimple connected compact Lie group (such as $\mathrm{SU}(n)$). Let $T \subset G$ be a maximal torus. Choose $\Phi^+ \subset \Phi(G, T)$ a positive system of roots. Recall this is equivalent to specifying a Weyl chamber $K \subset \mathbf{R}\Phi = X(T)_{\mathbf{R}} = \mathfrak{t}^*$, or equivalently a basis $B = B(K)$ for Φ (where B is obtained from the walls of K).

Since G is simply connected, we have

$$X(T) = P := \left(\mathbf{Z}\Phi^\vee\right)' \supset Q = \mathbf{Z}\Phi,$$

where $\mathbf{Z}\Phi^\vee$ is the coroot lattice and Q is the root lattice (and P is called the weight lattice). Last time we defined

$$\rho = \frac{1}{2} \sum_{c \in \Phi^+} c$$

and showed that $\rho \in P$; more explicitly, we showed $\rho = \sum \varpi_i$, with ϖ_i the set of fundamental weights (the basis of P that is $\mathbf{Z}$-dual to the basis $B^\vee$ of the coroot lattice).

Since

$$\overline{K} = \left\{ x \in X(T)_{\mathbf{R}} : \langle x, b^\vee \rangle \geq 0 \text{ for all } b \in B \right\}$$

and

$$K = \left\{ x \in X(T)_{\mathbf{R}} : \langle x, b^{\vee} \rangle > 0 \text{ for all } b \in B \right\} \subset \overline{K},$$

the result from last time that

$$\langle \rho, b^{\vee} \rangle = 1 > 0 \text{ for all } b \in B$$

gives that

$$\rho \in K \cap X(T) = K \cap P.$$

Example 28.1. Let's write down what the above is saying in the case $G = \mathrm{SU}(3)$ with T the diagonal torus. Recall that $\Phi(G, T)$ is of type A_2. Let $\Delta \subset \mathbf{Z}^{\oplus 3}$ denote the diagonally embedded copy of $\mathbf{Z}$, so

$$X(T) = \mathbf{Z}^{\oplus 3}/\Delta$$

(where $\mathbf{Z}^{\oplus 3}$ is the character lattice of the diagonal torus of $\mathrm{U}(3)$). Let $\overline{e}_i$ be the image in $X(T)$ of the ith standard basis vector of $\mathbf{Z}^{\oplus 3}$, so

$$X(T) = \mathbf{Z}\overline{e}_1 \oplus \mathbf{Z}\overline{e}_2 \hookleftarrow \mathbf{Z}\Phi = \mathbf{Z}a \oplus \mathbf{Z}b$$

for

$$\begin{aligned} a &:= \overline{e}_1 - \overline{e}_2, \\ b &:= \overline{e}_2 - \overline{e}_3 \end{aligned}$$

a basis of Φ.

Since

$$\begin{aligned} \langle a, b^{\vee} \rangle &= -1 \\ \langle b, a^{\vee} \rangle &= -1, \end{aligned}$$

the weight lattice

$$P = \left\{ xa + yb \in \mathbf{Q}a \oplus \mathbf{Q}b : \langle xa + yb, a^{\vee} \rangle \in bz \in \mathbf{Z}, \langle xa + yb, b^{\vee} \rangle \in \mathbf{Z} \right\}$$

is seen by direct computation to be

$$\mathbf{Z}\left(\frac{2a+b}{3}\right) \oplus \mathbf{Z}\left(\frac{a+2b}{3}\right) \supset \mathbf{Z}a \oplus \mathbf{Z}b = \mathbf{Z}\Phi = Q.$$

This exhibits by inspection that the inclusion $Q \subset P$ has index 3, which we know it must since P/Q is S^1-dual to $Z_G = Z_{\mathrm{SU}(3)} = \mu_3$ of size 3.

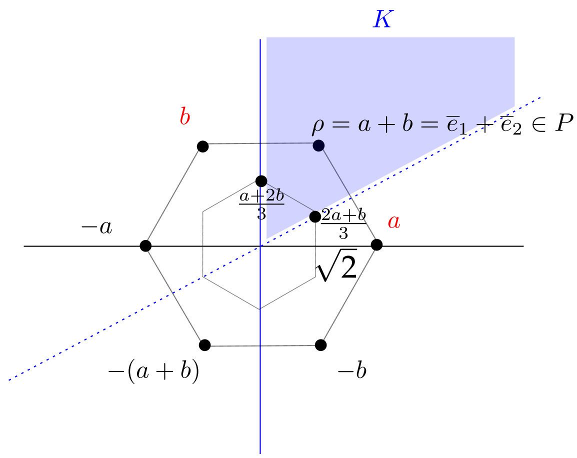
To express the fact $P = X(T)$, we compute

$$3e_1 = 2(e_1 - e_2) + (e_2 - e_3) + (e_1 + e_2 + e_3),$$

so in $X(T) = \mathbf{Z}^{\oplus 3}/\Delta$ we have

$$3\overline{e}_1 = 2a + b.$$

Hence, $(2a + b)/3 = \overline{e}_1$ in $X(T)_{\mathbf{Q}}$. Similarly $(a + 2b)/3 = \overline{e}_2$. These lattices P and Q can be pictured as follows:



The following three fundamental theorems governing irreducible G-representations are all due to Weyl and all express key features of the representation theory of G over $\mathbf{C}$ in essentially combinatorial terms. They can also be expressed in the language of Lie algebras, since by simple connectedness the finite-dimensional continuous $\mathbf{C}$-linear representations of G are "the same" as those of $\mathfrak{g}$ and hence of $\mathfrak{g}_{\mathbf{C}}$ which is in fact a "semisimple" Lie algebra. Books on Lie algebras such as [Hum] develop a theory of root systems for semisimple Lie algebras over algebraically closed fields of characteristic 0, and establish analogues of all three theorems below based on purely algebraic tools such as infinite-dimensional Verma modules.

Theorem 28.2 (Theorem on the Highest Weight)**.** *For a simply connected semisimple connected compact Lie group G as above, we have a bijection*

$$\overline{K} \cap P \to \mathrm{Irred}(G)$$
$$\lambda \mapsto V_\lambda$$

where V_λ is uniquely characterized as having λ as its unique highest T-weight: if μ is any T-weight occurring in V_λ then $\lambda \geq \mu$ in the sense of coefficient-wise comparison for the basis $\{\varpi_i\}$ of fundamental weights for $P = X(T)$.

Furthermore, the λ-weight space in V_λ is 1-dimensional.

In the preceding theorem, one calls V_λ the "highest-weight representation" attached to λ; by uniqueness, $\lambda = 0$ corresponds to the trivial representation. The weights in $\overline{K} \cap P$ are often referred to as "dominant weights"; this all depends on the choice of K (or equivalently of Φ^+).

Theorem 28.3 (Weyl Character Formula)**.** *Consider the subset*

$$T^{\mathrm{reg}} := \{t \in T : t^a \neq 1 \text{ for all } a \in \Phi\}$$

of "regular" elements in T (the complement of finitely many proper closed subtori). For $\varepsilon : W \to \{\pm 1\}$ the determinant of the W-action on $X(T)_{\mathbf{Q}} = P_{\mathbf{Q}}$ and $t \in T^{\mathrm{reg}}$ we have

$$\chi_{V_\lambda}(t) = \frac{\sum_{w \in W} \varepsilon(w) t^{w.(\lambda+\rho)}}{\sum_{w \in W} \varepsilon(w) t^{w.\rho}}.$$

(Implicit in this is that the "Weyl denominator" $\sum_{w \in W} \varepsilon(w) t^{w.\rho}$ is nonzero for $t \in T^{\mathrm{reg}}$.)

The character formula is describing the element χ_{V_λ} in the Laurent polynomial ring $\mathbf{Z}[X(T)]$ as a rational function. We will exhibit this explicitly for $\mathrm{SU}(2)$ shortly.

Theorem 28.4 (Weyl Dimension Formula)**.** *We have*

$$\dim V_\lambda = \prod_{a \in \Phi^+} \frac{\langle \chi + \rho, a^\vee \rangle}{\langle \rho, a^\vee \rangle}.$$

The above three results, Theorem 28.2, Theorem 28.3, and Theorem 28.4, are proven in [BtD, Ch. VI, 1.7, 2.6(ii)] and rest on the Weyl Integration Formula that we have discussed and used a lot as well as the Peter-Weyl theorem from [BtD, Ch. III] which rests on a lot of functional analysis and says that the Hilbert space $L^2(G, \mathbf{C})$ with its left regular representation of G is a Hilbert direct sum of the finite-dimensional irreducible continuous G-representations, each occurring with finite multiplicity equal to its dimension. There is no connectedness hypothesis on G, so one could use this to prove the above statement for finite groups, where $L^2(G)$ is the group ring for finite G, but that would be overkill.

Let's now see some examples.

Example 28.5. Suppose $G = \mathrm{SU}(2)$, and take T to be the diagonal torus

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} : z \in S^1 \right\}.$$

We have $\Phi = \{\pm a\} \subset X(T) = \mathbf{Z}(a/2)$ where a corresponds to squaring (± 2 in the lattice). Choosing Φ^+ to consist of the single root

$$a : \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \mapsto z^2$$

gives $\rho = a/2$ and $X(T) = \mathbf{Z} \ni 1 = \rho$. Then,

$$\Phi^\vee = \left\{ \pm a^\vee : S^1 \to T \right\}$$

where

$$a^\vee : z \mapsto \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}.$$

Thus,

$$\Phi^\vee = \{\pm 1\} \subset P = \mathbf{Z}$$

and

$$K = \mathbf{R}_{>0} \subset X(T)_{\mathbf{R}} \simeq \mathbf{R}.$$

We now have

$$\overline{K} \cap P = \mathbf{Z}_{\geq 0},$$

so

$$\mathrm{Irred}\,(\mathrm{SU}(2)) = \{V_n\}_{n \geq 0}$$

for V_n the unique irreducible representation with highest weight n, where "n" corresponds to the map

$$\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \mapsto z^n.$$

Lemma 28.6. *The representation V_n above is given by* $\mathrm{Sym}^n(\mathbf{C}^2)$.

Proof. By the Weyl dimension formula, we can easily check $\dim V_n = (n+1)/1 = n+1$, and we also can check directly that $\dim \mathrm{Sym}^n(\mathbf{C}^2) = n+1$. But, writing $\mathbf{C}^2 = \mathbf{C}e_1 \oplus \mathbf{C}e_2$, we see

$$\mathrm{Sym}^n(\mathbf{C}^2) = \bigoplus_{j=0}^{n} \mathbf{C}e_1^j e_2^{n-j}$$

with $e_1^j e_2^{n-j}$ a T-eigenvector having weight

$$\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \mapsto z^j (1/z)^{n-j} = z^{2j-n}$$

for $0 \leq j \leq n$. Thus, $\mathrm{Sym}^n(\mathbf{C}^2)$ has n as the unique highest weight, moreover with multiplicity 1.

By complete reducibility we know that $\mathrm{Sym}^n(\mathbf{C}^2)$ is a direct sum of irreducible representations, so since each irreducible has a unique highest weight it follows that $V_n \subset \mathrm{Sym}^n(\mathbf{C}^2)$. (Here we are using that if W is a representation of G with unique highest T-weight λ, by complete reducibility and the Theorem on the Highest Weight it is immediate that necessarily $V_\lambda \subset W$. In particular, if moreover $\dim W$ coincides with $\dim V_\lambda$ given by the Weyl Dimension Formula then necessarily $W = V_\lambda$. Observe that this style of argument doesn't require directly proving W is irreducible at the outset; such irreducibility emerges at the very end from the conclusion!)

But $\dim \mathrm{Sym}^n(\mathbf{C}^2) = n+1 = \dim V_n$, so the (abstract) inclusion $V_n \subset \mathrm{Sym}^n(\mathbf{C}^2)$ is an equality. $\square$

Let's see what the Weyl Character Formula says for $V_n = \mathrm{Sym}^n(\mathbf{C}^2)$. By using the basis of vectors $e_1^j e_2^{n-j}$, we see that upon identifying T with S^1 via $z \mapsto \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}$ we have

$$\chi_{V_n}(z) = z^n + z^{n-2} + \cdots + z^{-n}.$$

As long as $z \neq \pm 1$, which is exactly the condition $z^{\pm 2} \neq 1$, which is to say $z \in T^{\mathrm{reg}}$, this Laurent polynomial can be rewritten as

$$\frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}},$$

and this rational expression is exactly the ratio in the Weyl Character Formula (with $n+1$ playing the role of $\lambda + \rho$).

Remark 28.7. HW10 Exercise 4 gives more such examples of identifying V_λ for specific λ, for $\mathrm{SU}(3)$ and $\mathrm{Sp}(2)$.

Remark 28.8. In practice the Weyl Dimension Formula can be hard to use, and there are two more useful variants that are proved in [BtD, Ch. VI] as corollaries of the Dimension Formula: Konstant's Theorem in [BtD, Ch. VI, Thm. 3.2] that computes in root-system terms the multiplicity of a character $\mu \in X(T) = P$ as a T-weight in V_λ, and Steinberg's Theorem in [BtD, Ch. VI, Prop. 3.4] that computes the multiplicity of a given irreducible representation $V_{\lambda''}$ in the tensor product $V_\lambda \otimes V_{\lambda'}$ of two irreducible representations, all labeled by their highest weights relative to the choice of Φ^+. (Kostant's Theorem is used in the proof of Steinberg's Theorem.)

As an application of the Theorem on the Highest Weight, we can make the following definition.

Definition 28.9. The *fundamental representations* of G (relative to the choice of Φ^+) are the highest-weight representations V_{ϖ_i} associated to the fundamental weights ϖ_i.

By design, the number of fundamental representations is the dimension of the maximal tori. In [BtD, Ch. VI, §5] the fundamental representations are worked out for all of the classical groups.

Example 28.10. There are $n-1$ fundamental representations of $\mathrm{SU}(n)$, given by the ith exterior powers of the standard n-dimensional representation for $1 \le i \le n-1$ (proved by reading off the highest weight from the determination of the T-weights of the standard representation, and then using the Weyl Dimension Formula).

In the special case $n = 3$ this standard representation is 3-dimensional, so its second exterior power is the same as its dual; i.e., the standard representation and its dual are the fundamental representations for $\mathrm{SU}(3)$.

Any dominant weight $\lambda \in \overline{K} \cap P$ has its basis expansion in the fundamental weights in the form

$$\lambda = \sum n_i \varpi_i$$

for integers $n_i = \langle \lambda, b_i^\vee \rangle \in \mathbf{Z}_{\ge 0}$ (with $\{b_i\}$ the enumeration of B corresponding to the enumeration of the fundamental weights). Hence, the tensor product

$$\otimes V_{\varpi_i}^{\otimes n_i}$$

has $\sum n_i \varpi_i = \lambda$ as its unique highest weight, moreover with multiplicity 1. It then follows from complete reducibility and the Theorem on the Highest Weight that this tensor product contains V_λ as a subrepresentation, moreover with multiplicity 1. So, if one knows the fundamental representations, by tensoring them together and decomposing one obtains *all* irreducible representations of G! There's a lot of interesting combinatorics associated with finding these tensor product decompositions. In Appendix Z, this relation of V_λ to the fundamental representations is used to prove that the inclusion $R(G) \subset R(T)^W$ is an equality.

Remark 28.11. We end by recording a further remarkable application of the Peter-Weyl Theorem (proved in [BtD, Ch. III] and discussed briefly at the end of Appendix Z), called *Tannaka-Krein Duality*. Define the subspace

$$A(G) = \{G\text{-finite vectors}\} \subset C^0(G, \mathbf{R});$$

this is the space of continuous $\mathbf{R}$-valued functions f on G whose translates under pre-composition with left translation by elements $g \in G$ is finite-dimensional (i.e., the functions $f(g(\cdot))$ on G lie in a finite-dimensional space of functions).

It is not hard to prove that that $A(G)$ is an $\mathbf{R}$-subalgebra of $C^0(G)$, but much less evident is that $A(G)$ is a finitely generated $\mathbf{R}$-algebra (which ultimately rests on showing, via the Peter-Weyl Theorem, that G admits a faithful finite-dimensional representation over $\mathbf{C}$). Hence, it makes sense to form the affine algebraic scheme $G^{\mathrm{alg}} := \mathrm{Spec}(A(G))$ over $\mathbf{R}$.

It turns out that G^{alg} has a natural structure of (smooth) affine algebraic group over $\mathbf{R}$, and the natural map of sets $G \to G^{\mathrm{alg}}(\mathbf{R})$ (sending $g \in G$ to evaluation at g) is an isomorphism of Lie groups! The construction $G \rightsquigarrow G^{\mathrm{alg}}$ thereby defines a faithful functor from the category of compact (not necessarily connected) Lie groups into the category of affine algebraic groups over $\mathbf{R}$.

Amazingly, this functor turns out to be an equivalence onto a very specific full subcategory of such algebraic groups (with G^0 going over to $(G^{\mathrm{alg}})^0$, the latter defined in terms of Zariski-topology connectedness, and G a maximal compact subgroup of $G^{\mathrm{alg}}(\mathbf{C})$). This explains conceptually the fact of experience that all examples of compact Lie groups *and* all continuous maps between them that one ever encounters are described in terms of matrix groups and polynomial functions in matrix entries. Further details on this, building on [BtD, Ch. III], are given in [C, D.2.4-D.2.6, D.3] (assuming some familiarity with the purely algebraic theory of linear algebraic groups in characteristic 0).

Appendix A. Quaternions

A.1. **Introduction.** Inside the $\mathbf{C}$-algebra $\mathrm{Mat}_n(\mathbf{C})$ there is the $\mathbf{R}$-subalgebra $\mathrm{Mat}_n(\mathbf{R})$ with the property that the natural map of $\mathbf{C}$-algebras

$$\mathbf{C} \otimes_{\mathbf{R}} \mathrm{Mat}_n(\mathbf{R}) \to \mathrm{Mat}_n(\mathbf{C})$$

(satisfying $c \otimes M \mapsto cM$) is an isomorphism. (Proof: compare $\mathbf{C}$-bases on both sides, using the standard $\mathbf{R}$-basis of $\mathrm{Mat}_n(\mathbf{R})$ and the analogous $\mathbf{C}$-basis for $\mathrm{Mat}_n(\mathbf{C})$.) There are many other $\mathbf{R}$-subalgebras $A \subset \mathrm{Mat}_n(\mathbf{C})$ with the property that the natural map of $\mathbf{C}$-algebras $\mathbf{C} \otimes_{\mathbf{R}} A \to \mathrm{Mat}_n(\mathbf{C})$ is an isomorphism: $A = g\mathrm{Mat}_n(\mathbf{R})g^{-1}$ for any $g \in \mathrm{GL}_n(\mathbf{C})$. This is a bit "fake" since such an $\mathbf{R}$-subalgebra A is just $\mathrm{Mat}_n(\mathbf{R})$ embedded into $\mathrm{Mat}_n(\mathbf{C})$ via applying an automorphism of $\mathrm{Mat}_n(\mathbf{C})$ (namely, g-conjugation) to the usual copy of $\mathrm{Mat}_n(\mathbf{R})$ inside $\mathrm{Mat}_n(\mathbf{C})$.

But are there any fundamentally different A, such as one that is *not isomorphic* to $\mathrm{Mat}_n(\mathbf{R})$ as an $\mathbf{R}$-algebra? Any such A would have to have $\mathbf{R}$-dimension equal to n^2. In the mid-19th century, Hamilton made the important discovery that for $n = 2$ there *is* a very different choice for A. This exotic 4-dimensional $\mathbf{R}$-algebra is denoted $\mathbf{H}$ in his honor, called the *quaternions*.

A.2. **Basic construction.** Define $\mathbf{H} \subset \mathrm{Mat}_2(\mathbf{C})$ to be the $\mathbf{R}$-span of the elements

$$\mathbf{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{i} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{j} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{k} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Explicitly, for $a, b, c, d \in \mathbf{R}$ we have

$$a \cdot \mathbf{1} + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

for $\alpha = a + bi, \beta = c + di \in \mathbf{C}$. It follows that such an $\mathbf{R}$-linear combination vanishes if and only if $\alpha, \beta = 0$, which is to say $a, b, c, d = 0$, so $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is $\mathbf{R}$-linearly independent; we call it the *standard basis* for $\mathbf{H}$. These calculations also show that $\mathbf{H}$ can be alternatively described as the set of elements of $\mathrm{Mat}_2(\mathbf{C})$ admitting the form

$$M = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

for $\alpha, \beta \in \mathbf{C}$.

It is easy to verify by direct calculation (do it!) that the following relations are satisfied:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \ \mathbf{ij} = \mathbf{k} = -\mathbf{ji}$$

and likewise

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

For any two quaternions

$$h = a \cdot \mathbf{1} + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}, \ h' = a' \cdot \mathbf{1} + b' \cdot \mathbf{i} + c' \cdot \mathbf{j} + d' \cdot \mathbf{k}$$

with $a, b, c, d, a', b', c', d' \in \mathbf{R}$, the product $hh' \in \mathrm{Mat}_2(\mathbf{C})$ expands out as an $\mathbf{R}$-linear combination in the products $\mathbf{ee}'$ for $\mathbf{e}, \mathbf{e}'$ in the standard basis. But we just saw that all products among pairs from the standard basis are in $\mathbf{H}$, establishing the first assertion in:

Proposition A.2.1. *The quaternions are an* $\mathbf{R}$*-subalgebra of* $\mathrm{Mat}_2(\mathbf{C})$*, and the natural map of* $\mathbf{C}$*-algebras* $\mu : \mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \to \mathrm{Mat}_2(\mathbf{C})$ *is an isomorphism.*

The stability of $\mathbf{H}$ under multiplication could also be checked using the description as matrices of the form $\left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$ with $\alpha, \beta \in \mathbf{C}$.

Proof. The source and target of μ are 4-dimensional over $\mathbf{C}$, so for the isomorphism assertion it suffices to check injectivity. More specifically, this is the assertion that the standard basis of $\mathbf{H}$ (viewed inside $\mathrm{Mat}_2(\mathbf{C})$) is even linearly independent over $\mathbf{C}$ (not just over $\mathbf{R}$).

Taking a, b, c, d from $\mathbf{C}$, we have

$$a \cdot \mathbf{1} + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k} = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix},$$

so if this vanishes then we have $a \pm bi = 0 = \pm c + di$ with $a, b, c, d \in \mathbf{C}$ (not necessarily in $\mathbf{R}$!). It is then clear that $a = 0$, so $b = 0$, and likewise that $di = 0$, so d and c vanish too. $\square$

The *center* of an associative ring with identity is the subset of elements commuting with everything under multiplication. This is a commutative subring (with the same identity).

Corollary A.2.2. *The center of* $\mathbf{H}$ *coincides with* $\mathbf{R} = \mathbf{R} \cdot \mathbf{1}$.

Proof. Let $Z \subset \mathbf{H}$ be the center, so $\mathbf{R} \subset Z$. To prove equality it suffices to show $\dim_{\mathbf{R}} Z \leq 1$. But $\mathbf{C} \otimes_{\mathbf{R}} Z$ is certainly contained in the center of $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \simeq \mathrm{Mat}_2(\mathbf{C})$, and the center of the latter is just the evident copy of $\mathbf{C}$. This shows that $\dim_{\mathbf{R}} Z = \dim_{\mathbf{C}}(\mathbf{C} \otimes_{\mathbf{R}} Z) \leq 1$. $\square$

A.3. **Conjugation and norm.** For $h = a \cdot \mathbf{1} + b \cdot \mathbf{i} + c \cdot \mathbf{j} + d \cdot \mathbf{k}$, define its *conjugate* to be

$$\overline{h} = a \cdot \mathbf{1} - b \cdot \mathbf{i} - c \cdot \mathbf{j} - d \cdot \mathbf{k},$$

so clearly $\overline{\overline{h}} = h$. We call h a *pure quaternion* if $a = 0$, or equivalently $\overline{h} = -h$. Although multiplication in $\mathbf{H}$ is not commutative, in a special case commutativity holds:

Proposition A.3.1. *The products* $h\overline{h}$ *and* $\overline{h}h$ *coincide and are equal to* $a^2 + b^2 + c^2 + d^2$*. This is also equal to* $\det(h)$ *viewing* h *inside* $\mathrm{Mat}_2(\mathbf{C})$.

There is also a much easier identity: $h + \overline{h} = 2a = \mathrm{Tr}(h)$.

Proof. The expression $a^2 + b^2 + c^2 + d^2$ is unaffected by replacing h with $\overline{h}$, so if we can prove $h\overline{h}$ is equal to this expression in general then applying that to $\overline{h}$ gives the same for $\overline{h} \cdot \overline{\overline{h}} = \overline{h}h$. Hence, we focus on $h\overline{h}$. Writing h as a 2×2 matrix, we have

$$h = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

for $\alpha = a + bi$ and $\beta = c + di$. Since $a - bi = \overline{\alpha}$ and $-c - di = -\beta$, $\overline{h}$ corresponds to the analogous matrix using $\overline{\alpha}$ in place of α and $-\beta$ in place of β. Hence,

$$h\overline{h} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix} = \begin{pmatrix} \alpha\overline{\alpha} + \beta\overline{\beta} & 0 \\ 0 & \overline{\beta}\beta + \alpha\overline{\alpha} \end{pmatrix}.$$

This is $\mathbf{1}$ multiplied against the real scalar $|\alpha|^2 + |\beta|^2 = a^2 + b^2 + c^2 + d^2$. $\square$

We call $a^2 + b^2 + c^2 + d^2$ the *norm* of h, and denote it as $\mathrm{N}(h)$; in other words,

$$\mathrm{N}(h) = h\overline{h} = \overline{h}h$$

by viewing $\mathbf{R}$ as a subring of $\mathbf{H}$ via $c \mapsto c\mathbf{1}$.

It is clear by inspection of the formula that if $h \neq 0$ then $\mathrm{N}(h) \in \mathbf{R}^\times$, so in such cases $\overline{h}/\mathrm{N}(h)$ is a multiplicative inverse to h! Hence,

$$\mathbf{H}^\times = \mathbf{H} - \{0\};$$

we say $\mathbf{H}$ is a *division algebra* (akin to a field, but without assuming multiplication is commutative; multiplicative inverses do work the same on both sides). The $\mathbf{R}$-algebra $\mathbf{H}$ is very different from $\mathrm{Mat}_2(\mathbf{R})$ since the former is a division algebra whereas the latter has lots of zero-divisors!

Remark A.3.2. The $\mathbf{R}$-linear operation $h \mapsto \overline{h}$ is an "anti-automorphism" of $\mathbf{H}$: it satisfies $\overline{hh'} = \overline{h'} \cdot \overline{h}$ for any $h, h' \in \mathbf{H}$. One way to see this quickly is to note that the cases $h = 0$ or $h' = 0$ are easy, and otherwise it suffices to check equality after multiplying on the left against $hh' \in \mathbf{H}^\times$. But

$$\begin{aligned}(hh')\overline{hh'} = \mathrm{N}(hh') = \det(hh') = \det(h)\det(h') &= \mathrm{N}(h)\mathrm{N}(h') \\ &= h\overline{h}\mathrm{N}(h') \\ &= h\mathrm{N}(h')\overline{h} \\ &= hh'\overline{h'} \cdot \overline{h}\end{aligned}$$

(the second to last equality using that $\mathbf{R}$ is central in $\mathbf{H}$, and the final equality using associativity).

Appendix B. Smoothness of inversion

B.1. **Main goal.** The aim of this appendix is to prove that smoothness of inversion can be dropped from the definition of a Lie group (i.e.., it is a consequence of the other conditions in the definition).

Theorem B.1.1. *Let G be a C^∞-manifold and suppose it is equipped with a group structure such that the composition law $m : G \times G \to G$ is C^∞. Then inversion $G \to G$ is C^∞ (in particular, continuous!).*

This will amount to an exercise in the Chain Rule and the behavior of tangent spaces on product manifolds. The most important lesson from the proof is that the tangent map $\mathrm{d}m(e,e) : \mathrm{T}_e(G) \oplus \mathrm{T}_e(G) \to \mathrm{T}_e(G)$ is $(v, w) \mapsto v + w$ (so near the identity point, to first order *all* C^∞ group laws look like vector addition).

The proof will freely use that in general with product manifolds, the *linear* identification

$$\mathrm{T}_{(x_0,y_0)}(X \times Y) \simeq \mathrm{T}_{x_0}(X) \oplus \mathrm{T}_{y_0}(Y)$$

is defined in both directions as follows. Going from left to right, the first component $\mathrm{T}_{(x_0,y_0)}(X \times Y) \to \mathrm{T}_{x_0}(X)$ of this identification is $\mathrm{d}(\mathrm{pr}_1)(x_0, y_0)$ where $\mathrm{pr}_1 : X \times Y \to X$ is the projection $(x, y) \mapsto x$, and similarly for the second component (using $\mathrm{pr}_2 : X \times Y \to Y$). Going from right to left, the first factor inclusion

$$\mathrm{T}_{x_0}(X) \hookrightarrow \mathrm{T}_{x_0}(X) \oplus \mathrm{T}_{y_0}(Y) \simeq \mathrm{T}_{(x_0,y_0)}(X \times Y)$$

(using $v \mapsto (v, 0)$ for the initial step) is $\mathrm{d}(i_{y_0})(x_0, y_0)$ where $i_{y_0} : X \to X \times Y$ is the "right slice" map defined by $x \mapsto (x, y_0)$, and the second factor inclusion of $\mathrm{T}_{y_0}(Y)$ into $\mathrm{T}_{(x_0,y_0)}(X \times Y)$ is similar using the "left slice" map $j_{x_0} : Y \to X \times Y$ defined by $y \mapsto (x_0, y)$.

B.2. **Proof of Theorem B.1.1.** Consider the "shearing transformation"

$$\Sigma : G \times G \to G \times G$$

defined by $\Sigma(g, h) = (g, gh)$. This is bijective since we are using a group law, and it is C^∞ since the composition law m is assumed to be C^∞. (Recall that if M, M', M'' are C^∞ manifolds, a map $M \to M' \times M''$ is C^∞ if and only if its component maps $M \to M'$ and $M \to M''$ are C^∞, due to the nature of product manifold structures.)

We claim that Σ is a diffeomorphism (i.e., its inverse is C^∞). Granting this,

$$G = \{e\} \times G \to G \times G \overset{\Sigma^{-1}}{\to} G \times G$$

is C^∞, but explicitly this composite map is $g \mapsto (g, g^{-1})$, so its second component $g \mapsto g^{-1}$ is C^∞ as desired. Since Σ is a C^∞ bijection, the C^∞ property for its inverse is equivalent to Σ being a *local isomorphism* (i.e., each point in its source has an open neighborhood carried diffeomorphically onto an open neighborhood in the target). By the Inverse Function Theorem, this is equivalent to the isomorphism property for the tangent map

$$\mathrm{d}\Sigma(g, h) : \mathrm{T}_g(G) \oplus \mathrm{T}_h(G) = \mathrm{T}_{(g,h)}(G \times G) \to \mathrm{T}_{(g,gh)}(G \times G) = \mathrm{T}_g(G) \oplus \mathrm{T}_{gh}(G)$$

for all $g, h \in G$ (where $\mathrm{d}f(x) : \mathrm{T}_x(X) \to \mathrm{T}_{f(x)}(Y)$ is the tangent map for $f : X \to Y$).

We shall now use left and right translations to reduce this latter "linear" problem to the special case $g = h = e$, and in that special case we will be able to compute the tangent map explicitly and see the isomorphism property by inspection. For any $g \in G$, let $\ell_g : G \to G$ be the left translation map $x \mapsto gx$; this is C^∞ with inverse $\ell_{g^{-1}}$. Likewise, let $r_g : G \to G$ be the right translation map $x \mapsto xg$, which is also a diffeomorphism (with inverse $r_{g^{-1}}$). Note that $\ell_g \circ r_h = r_h \circ \ell_g$ (check!), and for any $g, h \in G$,

$$(\ell_g \times (\ell_g \circ r_h)) \circ \Sigma = \Sigma \circ (\ell_g \times r_h)$$

since evaluating both sides at any $(g', h') \in G \times G$ yields $(gg', g(g'h')h) = (gg', (gg')(h'h))$. But $\ell_g \times r_h$ is a diffeomorphism carrying (e, e) to (g, h), so the Chain Rule applied to the tangent maps induced by both sides of this identity at (e, e) yields that $\mathrm{d}\Sigma(e, e)$ is an isomorphism if and only if $\mathrm{d}\Sigma(g, h)$ is an isomorphism. Hence, indeed it suffices to treat the tangential isomorphism property only at (e, e).

Let $\mathfrak{g} := \mathrm{T}_e(G)$, so we wish to describe the map

$$\mathrm{d}\Sigma(e, e) : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}.$$

It suffices to show that this map is $(v, w) \mapsto (v, v + w)$, the "linear" version of the shearing map construction (since it is clearly an isomorphism). This says that the map $\mathrm{d}\Sigma(e, e)$ has as its first and second components the respective maps $(v, w) \mapsto v$ and $(v, w) \mapsto v + w$.

By the Chain Rule and the *definition* of the direct sum decomposition for tangent spaces on product manifolds (reviewed at the start), these components are the respective differentials at (e, e) of the components $\mathrm{pr}_1 \circ \Sigma = \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ \Sigma = m$ of Σ. Since tangent maps are *linear* and $(v, w) = (v, 0) + (0, w)$, our problem is to show that the compositions of $\mathrm{d}(\mathrm{pr}_1)(e, e)$ with the inclusions of $\mathfrak{g}$ into the respective first and second factors of $\mathfrak{g} \oplus \mathfrak{g}$ are $v \mapsto v$ and $w \mapsto 0$ and the compositions of $\mathrm{d}m(e, e)$ with those inclusions are $v \mapsto v$ and $w \mapsto w$ respectively.

Since factor inclusions at the level of tangent spaces are differentials of the associated slice maps at the level of manifolds, by the Chain Rule we want to show that the maps

$$\mathrm{pr}_1 \circ i_e, \mathrm{pr}_1 \circ j_e, m \circ i_e, m \circ j_e$$

from G to G carrying e to e have associated differentials $\mathfrak{g} \to \mathfrak{g}$ at e respectively equal to

$$v \mapsto v, w \mapsto 0, v \mapsto v, w \mapsto w.$$

But clearly $\mathrm{pr}_1 \circ i_e : g \mapsto \mathrm{pr}_1(g, e) = g$ is the identity map of G (so its differential on every tangent space is the identity map), and by the group law axioms $m \circ i_e$ and $m \circ j_e$ are also the identity map of G (and hence also have differential equal to the identity map on every tangent space). Finally, the map $\mathrm{pr}_1 \circ j_e$ is the map $g \mapsto \mathrm{pr}_1(e, g) = e$ that is the *constant map* to e, so its differential at every point vanishes. This completes the proof of Theorem B.1.1.

Since $m \circ (\mathrm{id}_G, \mathrm{inv}) : G \to G \times G \to G$ is the *constant map* to e (i.e., $m(g, g^{-1}) = e$), its differential vanishes on all tangent spaces. Computing at the point e on the source, it follows via the Chain Rule and our computation of $\mathrm{d}m(e, e)$ as ordinary addition that the map

$$\mathrm{d}(\mathrm{id}_G)(e) + \mathrm{d}(\mathrm{inv})(e) : \mathfrak{g} \to \mathfrak{g}$$

vanishes. But $\mathrm{d}(\mathrm{id}_G)(g) : \mathrm{T}_g(G) \to \mathrm{T}_g(G)$ is the identity map for any $g \in G$, so by setting $g = e$ we conclude that $v + (\mathrm{d}(\mathrm{inv})(e))(v) = 0$ for all $v \in \mathfrak{g}$. In other words:

Corollary B.2.1. *The tangent map* $\mathrm{d}(\mathrm{inv})(e) : \mathfrak{g} \to \mathfrak{g}$ *is negation.*

This says that near the identity point, to first order the inversion in every Lie group looks like vector negation.

Appendix C. The adjoint representation

Let G be a Lie group. One of the most basic tools in the investigation of the structure of G is the conjugation action of G on itself: for $g \in G$ we define $c_g : G \to G$ to be the C^∞ automorphism $x \mapsto gxg^{-1}$. (This is not interesting when G is commutative, but we will see later that *connected* commutative Lie groups have a rather simple form in general.)

The *adjoint representation* of G on its tangent space $\mathfrak{g} = \mathrm{T}_e(G)$ at the identity is the homomorphism

$$\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$$

defined by $\mathrm{Ad}_G(g) = \mathrm{d}c_g(e)$. This is a homomorphism due to the Chain Rule: since $c_{g'} \circ c_g = c_{g'g}$ and $c_g(e) = e$, we have

$$\mathrm{Ad}_G(g'g) = \mathrm{d}c_{g'}(e) \circ \mathrm{d}c_g(e) = \mathrm{Ad}_G(g') \circ \mathrm{Ad}_G(g).$$

In this appendix we prove the smoothness of Ad_G (which [BtD] seems to have overlooked), compute the derivative

$$\mathrm{d}(\mathrm{Ad}_G)(e) : \mathfrak{g} \to \mathrm{T}_1(\mathrm{GL}(\mathfrak{g})) = \mathrm{End}(\mathfrak{g})$$

at the identity, and use Ad_G to establish a very useful formula relating the Lie bracket rather directly to the group law on G near e.

C.1. **Smoothness and examples.** To get a feeling for the adjoint representation, let's consider the case $G = \mathrm{GL}_n(\mathbf{F})$ for $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$. For any $X \in \mathfrak{g} = \mathrm{Mat}_n(\mathbf{F})$, a parametric curve in G through the identity with velocity vector X at $t = 0$ is $\alpha_X(t) := \exp(tX)$. Thus, the differential $\mathrm{Ad}_G(g) = \mathrm{d}c_g(e)$ sends $X = \alpha'_X(0)$ to the velocity at $t = 0$ of the parametric curve

$$c_g \circ \alpha_X : t \mapsto g \exp(tX) g^{-1} = 1 + gtXg^{-1} + \sum_{j \geq 2} \frac{t^j}{j!} g X^j g^{-1},$$

so clearly this has velocity gXg^{-1} at $t = 0$. In other words, $\mathrm{Ad}_G(g)$ is g-conjugation on $\mathrm{Mat}_n(\mathbf{F})$. This is visibly smooth in g.

We can use a similar parametric curve method to compute $\mathrm{d}(\mathrm{Ad}_G)(e)$ for $G = \mathrm{GL}_n(\mathbf{F})$, as follows. Choose $X \in \mathfrak{g}$, so $\alpha_X(t) := \exp(tX)$ is a parametric curve in G with $\alpha'_X(0) = X$. Hence, $\mathrm{d}(\mathrm{Ad}_G)(e)(X)$ is the velocity at $t = 0$ of the parametric curve $\mathrm{Ad}_G(\exp(tX)) \in \mathrm{GL}(\mathfrak{g})$. In other words, it is the derivative at $t = 0$ of the parametric curve $c_{\exp(tX)} \in \mathrm{GL}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$. For $Y \in \mathfrak{g}$,

$$\begin{aligned} \exp(tX) \circ Y \circ \exp(-tX) &= (1 + tX + t^2(\cdot)) \circ Y \circ (1 - tX + t^2(\cdot)) \\ &= (Y + tXY + t^2(\cdot)) \circ (1 - tX + t^2(\cdot)), \end{aligned}$$

and this is equal to $Y + t(XY - YX) + t^2(\cdots)$, so its $\mathrm{End}(\mathfrak{g})$-valued velocity vector at $t = 0$ is the usual commutator $XY - YX$ that we know to be the Lie bracket on $\mathfrak{g}$.

Next we take up the proof of smoothness in general. First, we localize the problem near the identity using the elementary:

Lemma C.1.1. *Let G and H be Lie groups. A homomorphism of groups $f : G \to H$ is continuous if it is continuous at the identity, and it is C^∞ if it is C^∞ near the identity.*

Proof. The left-translation $\ell_g : G \to G$ is a homeomorphism carrying e to g, and likewise $\ell_{f(g)} : H \to H$ is a homeomorphism. Since

$$f \circ \ell_g = \ell_{f(g)} \circ f \tag{C.1.1}$$

(as f is a homomorphism), continuity of f at g is equivalent to continuity of f at e. This settles the continuity aspect. In a similar manner, since left translations are diffeomorphisms and ℓ_g carries an open neighborhood of e onto one around g (and similarly for $\ell_{f(g)}$ on H), if f is C^∞ on an open U around e then f is also C^∞ on the open $\ell_g(U)$ around g due to (C.1.1). Since the C^∞-property is local on G, it holds for f if it does so on an open set around every point. ∎

Finally, we prove smoothness of Ad_G. Since the conjugation-action map $c : G \times G \to G$ defined by $(g, g') \mapsto gg'g^{-1}$ is C^∞ and $c(e, e) = e$, we can choose a open coordinate domains $U \subset U' \subset G$ around e so that $c(U \times U) \subset U'$. Let $\{x_1, \dots, x_n\}$ be a coordinate system on U' with $x_i(e) = 0$, and define $f_i = x_i \circ c : U \times U \to \mathbf{R}$ as a function on $U \times U \subset \mathbf{R}^{2n}$. Let $\{y_1, \dots, y_n, z_1, \dots, z_n\}$ denote the resulting product coordinate system on $U \times U$.

Each f_i is smooth and $c_g : U \to U'$ has ith component function $f_i(g, z_1, \dots, z_n)$ with $g \in U$. Thus, the matrix $\mathrm{Ad}_G(g) = \mathrm{d}(c_g)(e) \in \mathrm{Mat}_n(\mathbf{R})$ has ij-entry equal to $(\partial f_i / \partial z_j)(0)$. Hence, smoothness of Ad_G on U reduces to the evident smoothness of each $\partial f_i / \partial z_j$ in the first n coordinates $y_1, \dots, y_n$ on $U \times U$ (after specializing the second factor U to e). By the preceding Lemma, this smoothness on U propagates to smoothness for Ad_G on the entirety of G since Ad_G is a homomorphism.

C.2. **Key formula for the Lie bracket.** For our Lie group G, choose $X, Y \in \mathfrak{g}$. In class we mentioned the fact (to be proved next time) that there is a unique Lie group homomorphism $\alpha_X : \mathbf{R} \to G$ satisfying $\alpha_X'(0) = X$. The automorphism $\mathrm{Ad}_G(\alpha_X(t))$ of $\mathfrak{g}$ therefore makes sense and as a point in the open subset $\mathrm{GL}(\mathfrak{g})$ of $\mathrm{End}(\mathfrak{g})$ it depends smoothly on t (since Ad_G is smooth). Evaluation on Y for this matrix-valued path defines a smooth path

$$t \mapsto \mathrm{Ad}_G(\alpha_X(t))(Y)$$

valued in $\mathfrak{g}$. We claim that the velocity of this latter path at $t = 0$ is $[X, Y]$. In other words:

Theorem C.2.1. *For any $X, Y \in \mathfrak{g}$,*

$$[X, Y] = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(\mathrm{Ad}_G(\alpha_X(t))(Y)).$$

Observe that the left side uses the construction of *global* left-invariant differential operators whereas the right side is defined in a much more localized manner near e. The "usual" proof of this theorem uses the notion of Lie derivative, but the approach we use avoids that.

Proof. Since Y is the velocity at $s = 0$ of the parametric curve $\alpha_Y(s)$, for any $g \in G$ the vector $\mathrm{Ad}_G(g)(Y) = \mathrm{d}(c_g)(e)(Y) \in \mathfrak{g}$ is the velocity at $s = 0$ of the parametric curve $c_g(\alpha_Y(s))$. Thus, for any t, $\mathrm{Ad}_G(\alpha_X(t))(Y)$ is the velocity at $s = 0$ of

$$a(t, s) := c_{\alpha_X(t)}(\alpha_Y(s)) = \alpha_X(t)\alpha_Y(s)\alpha_X(-t).$$

Note that $a(t, 0) = e$ for all t, so for each t the velocity to $s \mapsto a(t, s) \in G$ lies in $\mathrm{T}_e(G) = \mathfrak{g}$; this velocity is nothing other than $\mathrm{Ad}_G(\alpha_X(t))(Y)$, but we shall suggestively denote it as $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}a(t, s)$. Since this is a parametric curve valued in $\mathfrak{g}$, we can recast our problem as proving the identity

$$[X, Y] \stackrel{?}{=} \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}a(t, s)$$

where $a(t, s) = \alpha_X(t)\alpha_Y(s)\alpha_X(-t)$. We shall compute each side as a point-derivation at e on a smooth function φ on G and get the same result.

For the right side, an exercise to appear in HW3 (Exercise 9 in I.2) shows that its value on φ is the ordinary 2nd-order multivariable calculus derivative

$$\frac{\partial^2}{\partial_t\partial_s}|_{(0,0)}\varphi(a(t, s))$$

of the smooth function $\varphi \circ a : \mathbf{R}^2 \to \mathbf{R}$. By a clever application of the Chain Rule, it is shown in [BtD] (on page 19, up to swapping the roles of the letters s and t) that this 2nd-order partial derivative is equal to the difference

$$\frac{\partial^2}{\partial t\partial s}|_{(0,0)}\varphi(\alpha_X(t)\alpha_Y(s)) - \frac{\partial^2}{\partial t\partial s}|_{(0,0)}\varphi(\alpha_Y(s)\alpha_X(t)).$$

Letting $\widetilde{X}$ and $\widetilde{Y}$ respectively denote the left-invariant vector fields extending X and Y at e, we want this difference of 2nd-order partial derivatives to equal the value $[X, Y](\varphi) =$

$[\widetilde{X},\widetilde{Y}](\varphi)(e)$ at e of $\widetilde{X}(\widetilde{Y}(\varphi))-\widetilde{Y}(\widetilde{X}(\varphi))$, so it suffices to prove in general that

$$\frac{\partial^2}{\partial t\partial s}|_{(0,0)}\varphi(\alpha_X(t)\alpha_Y(s)) = X(\widetilde{Y}\varphi)$$

(and then apply this with the roles of X and Y swapped).

In our study next time of the construction of 1-parameter subgroups we will see that for any $g\in G$, $(\widetilde{Y}\varphi)(g)=(\partial_s|_{s=0})(\varphi(g\alpha_Y(s)))$. Thus, setting $g=\alpha_X(t)$, for any t the s-partial at $s=0$ of $\varphi(\alpha_X(t)\alpha_Y(s))$ is equal to $(\widetilde{Y}\varphi)(\alpha_X(t))$. By the same reasoning now applied to X instead of Y, passing to the t-derivative at $t=0$ yields $(\widetilde{X}(\widetilde{Y}\varphi))(e)=X(\widetilde{Y}(\varphi))$. ☐

C.3. **Differential of adjoint.** Finally, we connect the Lie bracket to the adjoint representation of G:

Theorem C.3.1. *Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. Then $\mathrm{d}(\mathrm{Ad}_G)(e)\in\mathrm{End}(\mathfrak{g})$ is equal to $\mathrm{ad}_{\mathfrak{g}}$. In other words, for $X\in\mathfrak{g}$, $\mathrm{d}(\mathrm{Ad}_G)(e)(X)=[X,\cdot]$.*

Proof. Choose $X\in\mathfrak{g}$, so $\alpha_X(t)$ is a parametric curve in G with velocity X at $t=0$. Consequently, $\mathrm{d}(\mathrm{Ad}_G)(e)(X)$ is the velocity vector at $t=0$ to the parametric curve $\mathrm{Ad}_G(\alpha_X(t))$ valued in the open subset $\mathrm{GL}(\mathfrak{g})$ of $\mathrm{End}(\mathfrak{g})$.

Rather generally, if $L:(-\epsilon,\epsilon)\to\mathrm{End}(V)$ is a parametric curve whose value at $t=0$ is the identity then for any $v\in V$ the velocity to $t\mapsto L(t)(v)$ at $t=0$ is $L'(0)(v)$. Indeed, the second-order Taylor expression $L(t)=1+tA+t^2B(t)$ for a smooth parametric curve $B(t)$ valued in $\mathrm{End}(V)$ implies that $L(t)(v)=v+tA(v)+t^2B(t)(v)$, so this latter curve valued in V has velocity $A(v)$. But clearly $A=L'(0)$, so our general velocity identity is proved.

Setting $L=\mathrm{Ad}_G\circ\alpha_X$ and $v=Y$, we conclude that $\mathrm{Ad}_G(\alpha_X(t))(Y)$ has velocity at $t=0$ equal to the evaluation at Y of the velocity at $t=0$ of the parametric curve $\mathrm{Ad}_G\circ\alpha_X$ valued in $\mathrm{End}(\mathfrak{g})$. But by the Chain Rule this latter velocity is equal to

$$\mathrm{d}(\mathrm{Ad}_G)(\alpha_X(0))\circ\alpha_X'(0)=\mathrm{d}(\mathrm{Ad}_G(e))(X),$$

so $\mathrm{d}(\mathrm{Ad}_G(e))(X)$ carries Y to the velocity at $t=0$ that equals $[X,Y]$ in Theorem C.2.1. ☐

Appendix D. Maximal compact subgroups

D.1. **Introduction.** It is a general fact (beyond the scope of this course) that if G is a Lie group with finitely many connected components then: every compact subgroup of G is contained in a maximal one (i.e., one not strictly contained in a larger compact subgroup), all maximal compact subgroups $K\subset G$ are G-conjugate to each other, and K meets every connected component of G with $G^0\cap K$ connected and itself a maximal compact subgroup of G^0. Nearly all treatments of the story of maximal compact subgroups of Lie groups in textbooks only address the connected case, but [Hoch, Ch. XV, Thm. 3.1]) does treat the wider case with $\pi_0(G)$ merely finite (possibly not trivial); I don't think one can deduce the case of finite $\pi_0(G)$ from the case of trivial $\pi_0(G)$. Cases with $\pi_0(G)$ finite but possibly non-trivial do arise very naturally as the group of $\mathbf{R}$-points of affine group varieties over $\mathbf{R}$.

In this appendix, we address a few classes of examples for which such K and their conjugacy can be verified directly. Some aspects of the technique used below actually play an essential role in the treatment of the general case (but we don't have time to get into that, so we refer the interested reader to Hochschild's book for such further details).

D.2. **The definite cases.** The basic building blocks for everything below emerge from two cases: $\mathrm{GL}(V)$ for a finite-dimensional nonzero vector space V over $\mathbf{R}$ and $\mathrm{GL}(W)$ for a finite-dimensional nonzero vector space W over $\mathbf{C}$. We know by Gram-Schmidt that $\mathrm{GL}(V)$ acts transitively on the set of all positive-definite (non-degenerate) quadratic forms q on V (this just expresses that all such q become "the same" in suitable linear coordinates), so the compact subgroups $\mathrm{O}(q)$ of $\mathrm{GL}(V)$ constitute a single conjugacy class. Likewise, $\mathrm{GL}(W)$ acts transitively on the set of all positive-definite (non-degenerate) hermitian forms h on W (this just expresses that all such h become "the same" in suitable $\mathbf{C}$-linear coordinates), so the compact subgroups $\mathrm{U}(h)$ of $\mathrm{GL}(W)$ constitute a single conjugacy class.

We claim that every compact subgroup of $\mathrm{GL}(V)$ lies in some $\mathrm{O}(q)$, and every compact subgroup of $\mathrm{GL}(W)$ lies in some $\mathrm{U}(h)$. It is elementary to check that a compact Lie group (so finite π_0; why?) has no proper closed C^∞-submanifold of the same dimension with the same number of connected components (why not?). Hence, it would follow that the compact subgroups $\mathrm{O}(q) \subset \mathrm{GL}(V)$ are all maximal and likewise for the compact subgroups $\mathrm{U}(h) \subset \mathrm{GL}(W)$, since we know from HW1 that such inclusions are closed C^∞-submanifolds.

Remark D.2.1. As an exercise, the interested reader can then deduce similar results for $\mathrm{SL}(V)$ and $\mathrm{SL}(W)$ using $\mathrm{SO}(q)$ and $\mathrm{SU}(h)$ (retaining conjugacy properties because $\mathrm{O}(q) \cdot \mathbf{R}_{>0} \to \mathrm{GL}(V)/\mathrm{SL}(V)$ and $\mathrm{U}(h) \cdot \mathbf{R}_{>0} \to \mathrm{GL}(W)/\mathrm{SL}(W)$ are surjective, where $\mathbf{R}_{>0}$ is the evident central subgroup of such scalars in each case).

So let K be a compact subgroup of $\mathrm{GL}(V)$. We seek a positive-definite $q : V \to \mathbf{R}$ such that $K \subset \mathrm{O}(q)$. Choose a positive-definite inner product $\langle \cdot, \cdot \rangle_0$ on V. We want to make a new one that is K-invariant by averaging. If K is a finite group then this can be done as a genuine average: make a new bilinear form

$$\langle v, w \rangle = \frac{1}{\#K} \sum_{k \in K} \langle kv, kw \rangle_0.$$

(also works if we omit the scaling factor $1/\#K$). This is manifestly K-invariant by design, and positive-definite. In the more meaty case that K is not finite (but compact), one has to use a *Haar measure* on K. So let us briefly digress to record the basic existence/uniqueness results on Haar measures (which for Lie groups we will later build via differential forms).

If G is any locally compact Hausdorff group, a *left Haar measure* is a regular Borel measure μ on the topological space G with the invariance property $\mu(gA) = \mu(A)$ for all Borel sets $A \subset G$ and all $g \in G$. (If we use Ag then we speak of a "right Haar measure".) For example, if $G = \mathbf{R}^n$ then the Lebesgue measure is a left Haar measure. As another example, if G is a discrete group (e.g., a finite group with the discrete topology) then *counting measure* m_G (i.e., $m_G(A) = \#A \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$) is a left Haar measure. The "regularity" condition in the definition of a Haar measure is a technical property which avoids some pathologies, and imposes in particular that $\mu(U) > 0$ for non-empty open subsets $U \subset G$ and $\mu(C) < \infty$ for compact subsets $C \subset G$. (For example, counting measure m_G on any G is a translation-invariant Borel measure but if G is non-discrete then there exist infinite compact $C \subset G$ and $m_G(C) = \infty$, so m_G is not regular and thus not a Haar measure if G is non-discrete.)

The basic result about left Haar measures μ is that they exist and are unique up to an $\mathbf{R}_{>0}$-scaling factor (and likewise for right Haar measures). For Lie groups we will construct them using differential forms. For many interesting groups the left Haar measures are also right Haar measures, in which case we call G *unimodular*. We'll later show that compact groups are unimodular (as are important non-compact Lie groups such as $\mathrm{SL}_n(\mathbf{R})$, $\mathrm{Sp}_{2n}(\mathbf{C})$, and so on, but we will not need this). In case G is *compact*, regularity implies $\mu(G)$ is both finite and positive, so we can scale μ by $1/\mu(G)$ to arrive at μ satisfying $\mu(G) = 1$. This "normalized" property removes all scaling ambiguity and so pins down a canonical (left) Haar measure in the compact case, denoted μ_G. For example, if G is finite then the normalized (left) Haar measure is $\mu_G(A) = \#A/\#G$; i.e., μ_G assigns mass $1/\#G$ to each element of G.

Coming back to a compact subgroup K of $\mathrm{GL}(V)$, we use Haar measures to prove:

Lemma D.2.2. *There exists a K-invariant positive-definite inner product $\langle \cdot, \cdot \rangle$ on V.*

Proof. Pick a positive-definite inner product $\langle \cdot, \cdot \rangle_0$. We will make a K-invariant one by averaging that initial choice in the sense of integration over K against a right Haar measure μ_K (which happens to also be a left Haar measure, though we don't need that):

$$\langle v, w \rangle = \int_K \langle kv, kw \rangle_0 \, \mathrm{d}\mu_K.$$

(The integrand is a continuous function of k, so the integral makes sense and converges since K is compact.) This new pairing is certainly bilinear and *positive-definite* (why?), and it is K-invariant precisely because μ_K is a right Haar measure (replacing v and w with $k_0 v$ and $k_0 w$ respectively for some $k_0 \in K$ amounts to translating the integrand through *right-translation* against k_0, so the right-invariance of μ_K ensures the integral is unaffected by this intervention of k_0). $\square$

Remark D.2.3. The preceding argument would run into difficulties if we tried to build a K-invariant non-degenerate symmetric bilinear form with an indefinite signature: the problem is that the integration construction would not have positive or negative-definiteness properties, and so we would not have a way to ensure the end result is non-degenerate (or even to control its signature).

This K-invariant inner product $\langle \cdot, \cdot \rangle$ corresponds to a K-invariant positive-definite quadratic form q on V, so $K \subset \mathrm{O}(q)$. That does the job for $\mathrm{GL}(V)$. In the case of $\mathrm{GL}(W)$, we can likewise build a K-invariant positive-definite hermitian form h on W (so $K \subset \mathrm{U}(h)$) by exactly the same K-averaging method beginning with an initial choice of positive-definite hermitian form on W. In particular, as with definite orthogonal groups, the compact subgroups $\mathrm{U}(h) \subset \mathrm{GL}(W)$ are maximal.

As as we noted in Remark D.2.1, it follows immediately from our results for $\mathrm{GL}(V)$ and $\mathrm{GL}(W)$ that we get analogous results for $\mathrm{SL}(V)$ and $\mathrm{SL}(W)$ using $\mathrm{SO}(q)$ and $\mathrm{SU}(h)$ (with positive-definite q on V and h on W).

D.3. **Indefinite orthogonal groups.** Now we fix a quadratic form $q : V \to \mathbf{R}$ with signature (r, s) with $r, s > 0$, so $\mathrm{O}(q) \simeq \mathrm{O}(r, s)$ is non-compact. Let $n = r + s$. The technique from HW1 in the positive-definite case adapts with only minor changes in the indefinite case to yield that $\mathrm{O}(q)$ is a closed C^∞-submanifold of $\mathrm{GL}(V)$. Can we describe its maximal compact subgroups and see the conjugacy of all of them by a direct method? The answer is "yes", and the key input will be the spectral theorem, combined with the analysis of compact subgroups inside $\mathrm{GL}_m(\mathbf{R})$ in the previous section for any $m > 0$ (e.g., $m = r, s$). First we build a conjugacy class of compact subgroups of $\mathrm{O}(q)$, and then check that they are maximal and every compact subgroup lies in one of these.

Consider a direct-sum decomposition

$$V = V^+ \oplus V^-$$

for which $q|_{V^+}$ is positive-definite, $q|_{V^-}$ is negative-definite, and V^+ is B_q-orthogonal to V^-. Such decompositions exist precisely by the classification of non-degenerate quadratic spaces over $\mathbf{R}$: we know that in suitable linear coordinates q becomes $\sum_{j=1}^{r} x_j^2 - \sum_{j=r+1}^{n} x_j^2$, so we can take V^+ to be the span of the first r such basis vectors and V^- to be the span of the others in that basis.

For any such decomposition, choices of orthonormal bases for the positive-definite spaces (V^+, q) and $(V^-, -q)$ give a description of q as $\sum_{j=1}^{d} y_j^2 - \sum_{j=d+1}^{n} y_j^2$ for linear coordinates dual to the union of those two bases, with $d = \dim V^+$. Hence, $d = r$ by the well-definedness of signature, so $\dim V^+ = r$ and $\dim V^- = n - r = s$. We thereby get a compact subgroup

$$\mathrm{O}(V^+) \times \mathrm{O}(V^-) \subset \mathrm{O}(q)$$

that is a copy of $\mathrm{O}(r) \times \mathrm{O}(s)$. (Here we write $\mathrm{O}(V^\pm)$ to denote $\mathrm{O}(\pm q|_{V^\pm})$.) By using the closed C^∞-submanifold $\mathrm{GL}(V^+) \times \mathrm{GL}(V^-) \subset \mathrm{GL}(V)$ (visualized via block matrices), we see that $\mathrm{O}(V^+) \times \mathrm{O}(V^-)$ is a closed C^∞-submanifold of $\mathrm{GL}(V)$ which is contained in the closed C^∞-submanifold $\mathrm{O}(q) \subset \mathrm{GL}(V)$, so $\mathrm{O}(V^+) \times \mathrm{O}(V^-)$ is a closed C^∞-submanifold of $\mathrm{O}(q)$.

We claim that these subgroups constitute a single $\mathrm{O}(q)$-conjugacy class, and that these are all maximal, with every compact subgroup of $\mathrm{O}(q)$ contained in one of these. First we address conjugacy:

Lemma D.3.1. *If (V^+, V^-) and (U^+, U^-) are two such pairs for (V, q) then each is carried to the other by an element of* $\mathrm{O}(q)$.

Proof. We have isomorphisms of quadratic spaces

$$(V^+, q|_{V^+}) \perp (V^-, q|_{V^-}) \simeq (V, q) \simeq (U^+, q|_{U^+}) \perp (U^-, q|_{U^-})$$

(where $(W', q') \perp (W'', q'')$ means $W' \oplus W''$ equipped with the quadratic form $Q(w' + w'') = q'(w') + q''(w'')$; it is easy to check that W' and W'' are B_Q-orthogonal with $Q|_{W'} = q'$ and $Q|_{W''} = q''$). But $(V^+, q|_{V^+})$ and $(U^+, q|_{U^+})$ are positive-definite quadratic spaces with the *same* dimension r, and likewise $(V^-, -q|_{V^-})$ and $(U^-, -q|_{U^-})$ are positive-definite quadratic spaces with the *same* dimension $n - r = s$.

By the Gram-Schmidt process, any two positive-definite quadratic spaces over $\mathbf{R}$ with the same finite dimension are isomorphic (as quadratic spaces), so there exist linear isomorphisms $T^\pm : V^\pm \simeq U^\pm$ that carry $\pm q|_{V^\pm}$ over $\pm q|_{U^\pm}$. Hence, the linear automorphism

$$V = V^+ \oplus V^- \overset{T^+ \oplus T^-}{\simeq} U^+ \oplus U^- = V$$

preserves q (why?) and carries $V^\pm$ over to $U^\pm$. This is exactly an element of $\mathrm{O}(q) \subset \mathrm{GL}(V)$ carrying the pair (V^+, V^-) over to (U^+, U^-). $\square$

We conclude from the Lemma that the collection of compact subgroups

$$\mathrm{O}(V^+) \times \mathrm{O}(V^-) \subset \mathrm{O}(q)$$

(that are also compact Lie groups) constitute a single $\mathrm{O}(q)$-conjugacy class. Thus, exactly as in the treatment of the definite case, considerations of dimension and finiteness of π_0 imply that such subgroups are maximal provided that we can show *every* compact subgroup $K \subset \mathrm{O}(q)$ is contained in one of these.

In other words, for a given K, our task reduces to finding an ordered pair (V^+, V^-) as above that is stable under the K-action on V. To achieve this, we will use the spectral theorem over $\mathbf{R}$. We first choose a K-invariant positive-definite inner product $\langle \cdot, \cdot \rangle$ on V, as we have already seen can be done (in effect, this is just applying our knowledge about the maximal compact subgroups of $\mathrm{GL}(V)$ and that every compact subgroup of $\mathrm{GL}(V)$ is contained in one of those). Let's use this inner product to identify V with its own dual. For our indefinite non-degenerate quadratic form $q : V \to \mathbf{R}$, consider the associated symmetric bilinear form $B_q : V \times V \to \mathbf{R}$ that is a perfect pairing (by non-degeneracy of q). This gives rise to a linear isomorphism $T_q : V \simeq V^*$ via $v \mapsto B_q(v, \cdot) = B_q(\cdot, v)$ which is self-dual (i.e., equal to its own dual map via double-duality on V) due to the symmetry of B_q.

Composing T_q with the self-duality $\iota : V^* \simeq V$ defined by the choice of $\langle \cdot, \cdot \rangle$, we get a composite linear isomorphism

$$f : V \overset{T_q}{\simeq} V^* \simeq V;$$

explicitly, $f(v)$ is the unique element of V such that $B_q(v, \cdot) = \langle f(v), \cdot \rangle$ in V^*. The crucial observation is that f is *self-adjoint* with respect to $\langle \cdot, \cdot \rangle$ (i.e., f is self-dual relative to ι); this is left to the interested reader to check as an exercise (do check it!). Thus, by the spectral theorem, f is diagonalizable. The eigenvalues of f are nonzero since f is an isomorphism. (Typically f will have very few eigenvalues, with multiplicity that is large.) But $\langle \cdot, \cdot \rangle$ is K-invariant by design, and B_q is K-invariant since $K \subset \mathrm{O}(q)$, so f commutes with the K-action (i.e., $k.f(v) = f(k.v)$ for all $k \in K$ and $v \in V$). Hence, K must preserve each eigenspace of f. For v in the eigenspace V_λ for a given eigenvalue λ of f, we have

$$2q(v) = B_q(v, v) = \langle f(v), v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v, \rangle = \lambda \|v\|^2$$

where $\| \cdot \|$ is the norm associated to the positive-definite $\langle \cdot, \cdot \rangle$. Hence, $q|_{V_\lambda}$ is *definite* with sign equal to that of λ.

If V^+ denotes the span of the eigenspaces for the positive eigenvalues of f and V^- denotes the span of the eigenspaces for the negative eigenvalues of f then $q|_{V^\pm}$ is definite with sign $\pm$, each of $V^\pm$ are K-stable, and $V^+ \oplus V^- = V$. Hence, (V^+, V^-) is exactly an ordered pair of the desired type which is K-stable!

It follows similarly to the cases of $\mathrm{SL}(V)$ that for indefinite q on V with signature (r, s) that $\mathrm{SO}(q)$ has as its maximal compact subgroups exactly the disconnected compact groups

$$\{(g, g') \in \mathrm{O}(V^+) \times \mathrm{O}(V^-)) \mid \det(g) = \det(g')\}$$

that is isomorphic to the more concretely-described group

$$\{(T, T') \in \mathrm{O}(r) \times \mathrm{O}(s) \mid \det(T) = \det(T')\}$$

(and that every compact subgroup of $\mathrm{SO}(q)$ lies in one of these).

D.4. **Indefinite unitary groups.** Now we consider a complex vector space W with positive finite dimension n and an indefinite non-degenerate hermitian form h on W of type (r, s) with $0 < r < n$ and $s = n - r$. As in the indefinite orthogonal case, the technique from HW1 for positive-definite hermitian forms adapts to show that $\mathrm{U}(h)$ is a closed C^∞-submanifold of $\mathrm{GL}(W)$.

Inside $\mathrm{U}(h)$ there are compact subgroups $\mathrm{U}(W^+) \times \mathrm{U}(W^-) \simeq \mathrm{U}(r) \times \mathrm{U}(s)$ for $W^\pm \subset W$ complementary subspaces on which h is definite with sign $\pm$ and that are h-orthogonal to each other. Similarly to the orthogonal case, these are closed C^∞-submanifolds of $\mathrm{U}(h)$ and necessarily $\dim_{\mathbf{C}} W^+ = r$ and $\dim_{\mathbf{C}} W^- = s$ with orthonormal bases of $(W^\pm, \pm h|_{W^\pm})$ yielding a description of h as

$$h(w, w') = \sum_{j=1}^{n} \varepsilon_j w_j \overline{w}'_j$$

where $\varepsilon_j = 1$ for $1 \leq j \leq r$ and $\varepsilon_j = -1$ for $r+1 \leq j \leq n$.

An argument similar to the orthogonal case shows that $\mathrm{U}(h)$ acts transitively on the set of such ordered pairs (W^+, W^-) (using there is only one isomorphism class of positive-definite hermitian spaces of a given finite dimension). The spectral theorem for self-adjoint operators over $\mathbf{C}$ (rather than over $\mathbf{R}$) then enables us to adapt the technique in the orthogonal case (exerting some extra attention to the intervention of conjugate-linearity) to deduce that any compact subgroup $K \subset \mathrm{U}(h)$ is contained in $\mathrm{U}(W^+) \times \mathrm{U}(W^-)$ for some such ordered pair (W^+, W^-). We have directly proved the expected results for compact and maximal compact subgroups of $\mathrm{U}(h)$ in the indefinite case akin to the case of orthogonal groups.

It follows similarly to the case of $\mathrm{SL}(W)$ that for indefinite h on W with type (r, s) that $\mathrm{SU}(h)$ has as its maximal compact subgroups exactly the connected compact groups

$$\{(g, g') \in \mathrm{U}(W^+) \times \mathrm{O}(W^-)) \mid \det(g) = \det(g')\}$$

that is isomorphic to the more concretely-described group

$$\{(T, T') \in \mathrm{U}(r) \times \mathrm{U}(s) \mid \det(T) = \det(T')\}$$

(and that every compact subgroup of $\mathrm{SU}(h)$ lies in one of these).

Appendix E. ODE

As applications of connectivity arguments and the contraction mapping theorem in complete metric spaces (in fact, an infinite-dimensional complete normed vector space: continuous functions on a compact interval, endowed with the sup-norm), in this appendix we prove and illustrate the classical local existence and uniqueness theorem for a wide class of first-order ordinary differential equations (ODE).

As a special case this includes linear ordinary differential equations of *any* order, and in this linear case we prove a remarkable *global* existence theorem (and give counterexamples in the non-linear case). We end by illustrating the surprisingly subtle problems associated with variation of auxiliary parameters and initial conditions in an ODE (a fundamental issue in applications to physical problems, where parameters and initial conditions must be allowed to have error); this effect of variation is a rather non-trivial problem that underlies the theory of integral curves which we address in Appendix F (and is absolutely basic in differential geometry).

You should at least skim over this appendix (reading the examples, if not the proofs) in order to appreciate the elegance and power of the basic theory of ODE in terms of its applicability to rather interesting problems.

E.1. **Motivation.** In classical analysis and physics, a fundamental topic is the study of systems of linear ordinary differential equations (ODE). The most basic example is as follows. Let $I \subseteq \mathbf{R}$ be a non-trivial interval (i.e., not a point but possibly half-open/closed, perhaps unbounded, etc.) and choose smooth functions $f, a_0, \ldots, a_{n-1}$ on I as well as constants $c_0, \ldots, c_{n-1} \in \mathbf{R}$, with $n \geq 1$. We seek to find a smooth function $u : I \to \mathbf{R}$ satisfying

$$u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_1 u' + a_0 u = f$$

on I, subject to the "initial conditions" $u^{(j)}(t_0) = c_j$ for $0 \leq j \leq n-1$ at $t_0 \in I$. (For example, in physics one typically meets 2nd-order equations, in which case the initial conditions on the value and first derivative are akin to specifying position and velocity at some time.) We call this differential equation *linear* because the left side depends $\mathbf{R}$-linearly on u (as the operations of differentiation and multiplication by a smooth function are $\mathbf{R}$-linear self-maps of $C^\infty(I)$).

Actually, for more realistic examples we want more: we should let u be vector-valued rather than just $\mathbf{R}$-valued. Let V be a finite-dimensional vector space over $\mathbf{R}$ (classically, $\mathbf{R}^N$). The *derivative* $u' : I \to V$ of a map $u : I \to V$ is defined in one of three equivalent ways: use the habitual "difference-quotient" definition (which only requires the function to have values in a normed $\mathbf{R}$-vector space), differentiate componentwise with respect to a choice of basis (the choice of which does not impact the derivative being formed; check!), or take $u'(t) = Du(t)(\partial|_t) \in T_{u(t)}(V) \simeq V$ for ∂ the standard "unit vector field" on the interval $I \subseteq \mathbf{R}$.

For C^∞ mappings $f : I \to V$ and $A_j : I \to \mathrm{Hom}(V, V)$ $(0 \leq j \leq n-1)$, as well as vectors $C_j \in V$ $(0 \leq j \leq n-1)$, we seek a C^∞ mapping $u : I \to V$ such that

$$u^{(n)}(t) + (A_{n-1}(t))(u^{(n-1)}(t)) + \cdots + (A_1(t))(u'(t)) + (A_0(t))(u(t)) = f(t) \tag{E.1.1}$$

for all $t \in I$, subject to the initial conditions $u^{(j)}(t_0) = C_j$ for $0 \leq j \leq n-1$. If $V = \mathbf{R}^N$ and we write $u(t) = (u_1(t), \ldots, u_N(t))$ then (E.1.1) is really a system of N linked nth-order linear ODE's in the functions $u_i : I \to \mathbf{R}$, and the n initial conditions $u^{(j)}(t_0) = C_j$ in $V = \mathbf{R}^N$ are Nn conditions $u_i^{(j)}(t_0) = c_{ij}$ in $\mathbf{R}$, where $C_j = (c_{1j}, \ldots, c_{Nj}) \in \mathbf{R}^N = V$. For most problems in physics and engineering we have $n \leq 2$ with $\dim V$ large.

The "ideal theorem" is that (E.1.1) with its initial conditions should exist and be unique, but we want more! Indeed, once one has an existence and uniqueness theorem in hand, it is very natural to ask: how does the solution depend on the initial conditions $u^{(j)}(t_0) = C_j$? That is, as we vary the C_j's, does the solution exhibit smooth dependence on these initial conditions? And what if the A_j's depend continuously (or smoothly) on some auxiliary parameters not present in the initial conditions (such as some friction constants or other input from the surrounding physical setup)? That is, if $A_j = A_j(t, z)$ for z in an auxiliary space, does the solution $t \mapsto u(t, z)$ for each z depend "as nicely" on z as do the A_j's? How about dependence on the *combined* data $(t, z, C_1, \ldots, C_n)$?

It has been long recognized that it is hopeless to try to study differential equations by explicitly exhibiting solutions. Though clever tricks (such as integrating factors and separation of variables) do find some solutions in special situations, one usually needs a general theory to predict the "dimension" of the space of solutions and so to tell us whether there may be more solutions remaining to be found. Moreover, we really want to understand *properties* of solutions: under what conditions are they unique, and if so do they exist for all time? If so, what can be said about the long-term behavior (growth, decay, oscillation, etc.) of the solution? Just as it is unwise to study the properties of solutions to polynomial equations by means of explicit formulas, in the study of solutions to differential equations we cannot expect to get very far with explicit formulas. (Though in the rare cases that one can find an explicit formula for some or all solutions it can be helpful.)

The first order of business in analyzing these questions is to bring the situation into more manageable form by expressing the given nth-order problem (E.1.1) as a first-order differential equation. This immensely simplifies the notational complexity and thereby helps us to focus more clearly on the essential structure at hand. Here is an illustration of the classical trick known to all engineers for reducing linear initial-value problems to the first-order case; the idea is to introduce an *auxiliary vector space*.

Example E.1.1. Consider the equation

$$y'' + P(x)y' + Q(x)y = R(x) \tag{E.1.2}$$

on an interval I, with $P, Q, R \in C^\infty(I)$. Say we impose the conditions $y(t_0) = c_0$ and $y'(t_0) = c_1$ for some $c_0, c_1 \in \mathbf{R}$. We shall rephrase this as a first-order equation with values in $\mathbf{R}^2$. The idea is to study the derivative of the vector $u(x) = (y(x), y'(x)) \in \mathbf{R}^2$. Define

$$A(x) = \begin{pmatrix} 0 & 1 \\ -Q(x) & -P(x) \end{pmatrix},\ \ f(x) = \begin{pmatrix} 0 \\ R(x) \end{pmatrix},\ \ c = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}.$$

For a mapping $u : I \to V = \mathbf{R}^2$ given by $u(x) = (u_0(x), u_1(x))$, consider the first-order initial-value problem in $\mathbf{R}^2$:

$$u'(x) = (A(x))(u(x)) + f(x),\ \ u(t_0) = c. \tag{E.1.3}$$

Unwinding what this says on components, one gets the equivalent statements $u_0' = u_1$ and u_0 satisfies the original differential equation (E.1.2) with $u_0(t_0) = c_0$ and $u_0'(t_0) = c_1$. Hence, our second-order problem for an $\mathbf{R}$-valued function u_0 with 2 initial conditions (on u_0 and u_0' at t_0) has been reformulated as a first-order problem (E.1.3) for an $\mathbf{R}^2$-valued mapping u with one initial condition (at t_0).

To apply this trick in the general setup of (E.1.1), let W be the n-fold direct sum V^n, and let $A : I \to \mathrm{Hom}(W, W)$ be defined by

$$\begin{aligned} A(t) : (v_0, \dots, v_{n-1}) &\mapsto (v_1, v_2, \dots, v_{n-1}, -((A_{n-1}(t))(v_{n-1}) + \cdots + (A_0(t))(v_0))) \\ &\in V^n \\ &= W, \end{aligned}$$

so A is smooth. Let $F : I \to W = V^n$ be the smooth mapping $F(t) = (0, \dots, 0, f(t))$, and let $C = (C_0, \dots, C_{n-1}) \in W$. A mapping $u : I \to W$ is given by

$$u(t) = (u_0(t), u_1(t), \dots, u_{n-1}(t))$$

for mappings $u_j : I \to V$, and $u : I \to W$ is smooth if and only if all $u_j : I \to V$ are smooth (why?). By direct calculation, the first-order equation

$$u'(t) = (A(t))(u(t)) + F(t) \tag{E.1.4}$$

with initial condition $u(t_0) = C$ is the *same* as requiring two things to hold:

- $u_j = u_0^{(j)}$ for $1 \leq j \leq n-1$ (so $u_1, \dots, u_{n-1}$ are redundant data when we know u_0),
- u_0 satisfies the initial-value problem

$$u_0^{(n)}(t) + (A_{n-1}(t))(u_0^{(n-1)}(t)) + \cdots + (A_0(t))(u_0(t)) = f(t) \tag{E.1.5}$$

with initial conditions $u_0^{(j)}(t_0) = C_j$ for $0 \leq j \leq n-1$; this is (E.1.1) by another name.

In other words, our original nth-order linear V-valued problem (E.1.5) with n initial conditions in V is equivalent to the *first-order* linear W-valued problem (E.1.4) with *one* initial condition. This is particularly striking in the classical case $V = \mathbf{R}$: an nth-order linear ODE for an $\mathbf{R}$-valued function can be recast as first-order linear ODE provided we admit *vector-valued* problems (for $V = \mathbf{R}$ we have $W = \mathbf{R}^n$).

For these reasons, in the theory of linear ODE's the "most general" problem is the vector-valued problem

$$u'(t) = (A(t))(u(t)) + f(t), \quad u(t_0) = v_0$$

for smooth $f : I \to V$ and $A : I \to \mathrm{Hom}(V, V)$ and a point $v_0 \in V$, where V is permitted to be an *arbitrary* finite-dimensional vector space. (We could write $V = \mathbf{R}^N$, but things are cleaner if we leave V as an abstract finite-dimensional vector space.) Note in particular that the classical notion of "system of nth-order linear ODE's" (expressing derivatives of each of several $\mathbf{R}$-valued functions as linear combinations of all of the functions, with coefficients that depend smoothly on t) is just a single first-order vector-valued ODE in disguise. Hence, we shall work exclusively in the language of *first-order vector-valued ODE's*. Observe also that we may replace "smooth" with C^p in everything that went before (for $0 \leq p \leq \infty$), with the caveat that we cannot expect the solution to have class of differentiability any better than C^{p+1}.

E.2. **The local existence and uniqueness theorem.** For applications in differential geometry and beyond, it is important to allow for the possibility of non-linear ODE's (e.g, non-linear expressions such as u^2 or $(u')^2$ showing up in the equation for $\mathbf{R}$-valued u). The engineer's trick of introducing auxiliary functions whose first derivatives are set equal to other functions still reduces problems to the first-order case (via an auxiliary vector space).

What do we really mean by "non-linear first-order ODE"? Returning to the linear case considered above, we can rewrite the equation $u'(t) = (A(t))(u(t)) + f(t)$ as follows:

$$u'(t) = \phi(t, u(t))$$

where $\phi : I \times V \to V$ is the smooth mapping $\phi(t, v) = (A(t))(v) + f(t)$. (This is a C^p mapping if A and f are C^p in t.)

Generalizing, we are interested in C^p solutions to the "initial-value problem"

$$u'(t) = \phi(t, u(t)), \; u(t_0) = v_0 \tag{E.2.1}$$

where $\phi : I \times U \to V$ is a C^p mapping for an open set $U \subseteq V$ and $v_0 \in U$ is a point. Of course, it is implicit that a solution $u : I \to V$ has image contained in U so that the expression $\phi(t, u(t))$ makes sense for all $t \in I$. Such a solution u is certainly of class C^0, and in general if it is of class C^r with $r \leq p$ then by (E.2.1) we see u' is a composite of C^r mappings and hence is C^r. That is, u is C^{r+1}. Thus, by induction from the case $r = 0$ we see that a solution u is necessarily of class C^{p+1}.

The key point of the above generalization is that $\phi(t,\cdot) : U \to V$ need not be the restriction of an affine-linear map $V \to V$ (depending on t). Such non-linearity is crucial in the study of vector flow on manifolds, as we shall see.

The first serious theorem about differential equations is the *local* existence and uniqueness theorem for equations of the form (E.2.1):

Theorem E.2.1. *Let $\phi : I \times U \to V$ be a C^p mapping with $p \geq 1$, and choose $v_0 \in U$ and $t_0 \in I$. There exists a connected open subset $J \subseteq I$ around t_0 and a differentiable mapping $u : J \to U$ satisfying* (E.2.1) *(so u is C^{p+1}). Moreover, such a solution is uniquely determined on any J where it exists.*

Before we prove the theorem, we make some observations and look at some examples. The existence aspect of the theorem is local at t_0, but the uniqueness aspect is more global: on any J around t_0 where a solution to the initial-value problem exists, it is uniquely determined. In particular, if $J_1, J_2 \subseteq I$ are two connected open neighborhoods of t_0 in I on which a solution exists, the solutions must agree on the interval neighborhood $J_1 \cap J_2$ (by uniqueness!) and hence they "glue" to give a solution on $J_1 \cup J_2$. In this way, it follows from the uniqueness aspect that there exists a maximal connected open subset $J_{\max} \subseteq I$ around t_0 (depending on u, ϕ, and v_0) on which a solution exists (containing all other such connected opens). However, in practice it can be hard to determine $J_{\max}$! (We give some examples to illustrate this below.) Also, even once the local existence and uniqueness theorem is proved, for applications in geometry we need to know more: does the solution u exhibit C^p-dependence on the initial condition v_0, and if ϕ depends "nicely" (continuously, or better) on "auxiliary parameters" then does u exhibit just as nice dependence on these parameters? (See Example E.3.6.) Affirmative answers will be given at a later time.

Remark E.2.2. Although consideration of $\phi(t, u(t))$ permits equations with rather general "non-linear" dependence on u, we are still requiring that u' only intervene linearly in the ODE. Allowing non-linear dependence on higher derivatives is an important topic in advanced differential geometry, but it is not one we shall need to confront in our study of elementary differential geometry.

A fundamental dichotomy between the first-order *linear* ODE's (i.e., the case when $\phi(t,\cdot)$ is the restriction of an affine-linear self-map $v \mapsto A(t)v + f(t)$ of V for each $t \in V$) and the general case is that in Theorem E.3.1 we will prove a *global* existence theorem in the linear case (for which we may and do always take $U = V$): the initial-value problem will have a solution across the *entire* interval I (i.e., $J_{\max} = I$ in such cases). Nothing of the sort holds in the general non-linear case, even when $U = V$:

Example E.2.3. If we allow non-linear intervention of derivatives then uniqueness fails. Consider the $\mathbf{R}$-valued problem $(u')^2 = t^3$ on $(-1,1)$. This has exactly two solutions (not just one!) on $(0,1)$ with any prescribed initial condition at $t_0 = 1/2$. The solutions involve $\pm\sqrt{t}$ and so continuously extend to $t = 0$ without differentiability there. In other words, even without "blow-up" in finite time, solutions to initial-value ODE's that are non-linear in the derivative may not exist across the entire interval even though they admit limiting values at the "bad" point. The non-uniqueness of this example shows that the linearity in u' is crucial in the uniqueness aspect of Theorem E.2.1.

Since this example does not fit the paradigm $u'(t) = \phi(t, u(t))$ in the local existence and uniqueness theorem, you may find this sort of non-linear example to be unsatisfying. The next example avoids this objection.

Example E.2.4. Even for first-order initial-value problems of the form $u'(t) = \phi(t, u(t))$, for which there is uniqueness (given $u(t_0)$), non-linearity in the second variable of ϕ can lead to the possibility that the solution does not exist across the entire interval (in contrast with what we have said earlier in the linear case, and will prove in Theorem E.3.1). For example, consider

$$u' = 1 + u^2, \;\; u(0) = 0.$$

The unique local solution near $t_0 = 0$ is $u(t) = \tan(t)$, and as a solution it extends to $(-\pi/2, \pi/2)$ with blow-up (in absolute value) as $t \to \pm\pi/2$. It is not at all obvious from the shape of this particular initial-value problem that the solution fails to propagate for all time. Hence, we see that the problem of determining $J_{\max} \subseteq I$ in any particular case can be tricky. The blow-up aspect of this example is not a quirk: we will prove shortly that failure of the image of u near an endpoint τ of J in I to be contained in a compact subset of U as $t \to \tau$ is the only obstruction to having $J_{\max} = I$ for initial-value problems of the form in Theorem E.2.1.

Let us now turn to the proof of Theorem E.2.1.

Proof. We will prove the local existence result and weaker version of uniqueness: any two solutions agree *near* t_0 in I, where "near" may depend on the two solutions. This weaker form is amenable to shrinking I around t_0. At the end we will return to the proof of global uniqueness (without shrinking I) via connectivity considerations.

Fix a norm on V. Let $B = \overline{B}_{2r}(v_0)$ be a compact ball around v_0 that is contained in U with $0 < r < 1$. For the purpose of local existence and local uniqueness, we may also replace I with a compact subinterval that is a neighborhood of t_0 in the given interval I, so we can assume I is compact, say with length c. Let $M > 0$ be an upper bound on $\|\phi(t, v)\|$ for $(t, v) \in I \times B$. Also, let $L > 0$ be an upper bound on the "operator norm" $\|D\phi(t, v)\|$ for $(t, v) \in I \times B$; here, $D\phi(t, v) : \mathbf{R} \times V \to V$ is the total derivative of ϕ at $(t, v) \in I \times U$. Such upper bounds exist because ϕ is C^1 and $I \times B$ is compact.

Our interest in L is that it serves as a "Lipschitz constant" for $\phi(t, \cdot) : B \to V$ for each $t \in I$. That is, for points $v, v' \in B$,

$$\|\phi(t, v) - \phi(t, v')\| \leq L\|v - v'\|.$$

This inequality is due to the Fundamental Theorem of Calculus: letting $g(x) = \phi(t, xv + (1-x)v')$ be the C^1 restriction of $\phi(t, \cdot)$ to the line segment in B joining v and v' (for a fixed t), we have

$$\phi(t, v) - \phi(t, v') = g(1) - g(0) = \int_0^1 g'(y)\mathrm{d}y = \int_0^1 D\phi(t, yv + (1-y)v')(v - v')\mathrm{d}y,$$

so

$$\|\phi(t, v) - \phi(t, v')\| \leq \int_0^1 \|D\phi(t, yv + (1-y)v')(v - v')\|\mathrm{d}y \leq \int_0^1 L\|v - v'\|\mathrm{d}y = L\|v - v'\|.$$

Since the solution u that we seek to construct near t_0 has to be contiuous, the Fundamental Theorem of Calculus allows us to rephrase the ODE with initial condition near t_0 in I

as an "integral equation"

$$u(t) = v_0 + \int_{t_0}^{t} \phi(y, u(y))\mathrm{d}y$$

for a continuous mapping

$$u : I \cap [t_0 - a, t_0 + a] \to B = \overline{B}_{2r}(v_0) \subseteq U$$

for small $a > 0$. How small do we need to take a to get existence? We shall require $a \leq \min(1/2L, r/M)$.

Let $W = C(I \cap [t_0 - a, t_0 + a], V)$ be the space of continuous maps from $I \cap [t_0 - a, t_0 + a]$ to V, endowed with the sup norm. This is a complete metric space. Let $X \subseteq W$ be the subset of such maps f with image in the closed subset $B \subseteq V$ and with $f(t_0) = v_0$, so X is closed in W and hence is also a complete metric space. For any $f \in X$, define $T(f) : I \cap [t_0 - a, t_0 + a] \to V$ by

$$(T(f))(t) = v_0 + \int_{t_0}^{t} \phi(y, f(y))\mathrm{d}y;$$

this makes sense because $f(y) \in B \subseteq U$ for all $y \in I \cap [t_0 - a, t_0 + a]$ (as $f \in X$). By the continuity of ϕ and f, it follows that $T(f)$ is continuous. Note that $(T(f))(t_0) = v_0$ and $\|\phi(y, f(y)) - \phi(y, v_0)\| \leq L\|f(y) - v_0\| \leq 2rL$. Since

$$(T(f))(t) - v_0 = \int_{t_0}^{t} (\phi(t, f(y)) - \phi(y, v_0))\mathrm{d}y + \int_{t_0}^{t} \phi(y, v_0)\mathrm{d}y$$

and $|t - t_0| \leq a$, we therefore get

$$\|(T(f))(t) - v_0\| \leq 2arL + Ma \leq 2r.$$

This shows $(T(f))(t) \in B$ for all $t \in I \cap [t_0 - a, t_0 + a]$, so $T(f) \in X$.

We conclude that the "integral operator" $f \mapsto T(f)$ is a self-map of the complete metric space X. The usefulness of this is that it is a contraction mapping: for $f, g \in X$,

$$\begin{aligned} \|T(f) - T(g)\|_{\text{sup}} &\leq \sup_t \left\| \int_{t_0}^{t} (\phi(y, f(y)) - \phi(y, g(y)))\mathrm{d}y \right\| \\ &\leq a \sup_y \|\phi(y, f(y)) - \phi(y, g(y))\|, \end{aligned}$$

where $t, y \in I \cap [t_0 - a, t_0 + a]$, and this is at most $aM\|f - g\|_{\text{sup}}$ due to the choice of M. Since $aM \leq r < 1$, the contraction property is verified. Thus, there exists a *unique* fixed point $f_0 \in X$ for our integral operator. Such a fixed point gives a solution to the initial-value problem on $I \cap (t_0 - a, t_0 + a)$, and so settles the local existence result.

Since the value of a could have been taken to be arbitrarily small (for a given L, M, r), we also get a local uniqueness result: if $u_1, u_2 : J \rightrightarrows U$ are two solutions, then they coincide near t_0. Indeed, we may take a as above so small that $I \cap [t_0 - a, t_0 + a] \subseteq J$ and both u_1 and u_2 map $I \cap [t_0 - a, t_0 + a]$ into B. (Here we use continuity of u_1 and u_2, as $u_1(t_0) = u_2(t_0) = v_0 \in \mathrm{int}_V(B)$.) For such small a, u_1 and u_2 lie in the complete metric space X as above and they are each fixed points for the same contraction operator. Hence, $u_1 = u_2$ on $I \cap [t_0 - a, t_0 + a]$. That is, u_1 and u_2 agree on J near t_0. This completes the proof of local uniqueness near t_0.

Finally, we must prove "global uniqueness": if $J \subseteq I$ is an open connected set containing t_0 and u_1 and u_2 are solutions on all of J, then we want $u_1 = u_2$ on J. By local uniqueness, they agree on an open around t_0 in J. We now argue for uniqueness to the right of t_0, and the same method will apply to the left. Pick any $t \in J$ with $t \geq t_0$. We want $u_1(t) = u_2(t)$. The case $t = t_0$ is trivial (by the initial condition!), so we may assume $t > t_0$. (In particular, t_0 is not a right endpoint of J.) Let $S \subseteq [t_0, t]$ be the subset of those $\tau \in [t_0, t]$ such that u_1 and u_2 coincide on $[t_0, \tau]$. For example, $t_0 \in S$. Also local uniqueness at t_0 implies that $[t_0, t_0 + \varepsilon] \subseteq S$ for some small $\varepsilon > 0$. It is clear that S is a subinterval of $[t_0, t]$ and that it is closed (as u_1 and u_2 are continuous), so if $\rho = \sup S \in (t_0, t]$ then $S = [t_0, \rho]$. Thus, the problem is to prove $\rho = t$.

We assume $\rho < t$ and we seek a contradiction. Let $v = u_1(\rho) = u_2(\rho)$. Hence, near the point ρ that lies on the *interior* of J in $\mathbf{R}$, we see that u_1 and u_2 are solutions to the same initial-value problem

$$u'(\tau) = \phi(\tau, u(\tau)), \;\; u(\rho) = v$$

on an open interval around ρ contained in J. By local uniqueness applied to *this* new problem (with initial condition at ρ), it follows that u_1 and u_2 coincide near ρ in J. But ρ is not a right endpoint of J, so we get points to the right of ρ lying in S. This contradicts the definition of ρ, so the hypothesis $\rho < t$ is false; that is, $\rho = t$. Hence, $u_1(t) = u_2(t)$. Since $t \geq t_0$ in J was arbitrary, this proves equality of u_1 and u_2 on J to the right of t_0. □

Example E.2.3 shows that when u' shows up non-linearly in the ODE, we can fail to have a solution across the entire interval I even in the absence of unboundedness problems. However, in Example E.2.4 we had difficulties extending the solution across the entire interval due to unboundedness problems. Let us show that in the general case of initial-value problems of the form $u'(t) = \phi(t, u(t))$ with $u(t_0) = v_0$, "unboundedness" is the only obstruction to the local solution propagating across the entire domain. By "unboundedness" we really mean that the local solution approaches the boundary of U, which is to say that it fails to remain in a compact subset of U as we evolve the solution over time.

Corollary E.2.5. *With notation as in Theorem E.2.1, let u be a solution to (E.2.1) on some connected open subset $J \subseteq I$ around t_0. Let $K \subseteq U$ be a compact subset. If $\tau_0 \in I$ is an endpoint of the closure of J in I such that u has image in K at all points near τ_0 in $J - \{\tau_0\}$, then u extends to a solution of (E.2.1) around τ_0. That is, $J_{\max}$ contains a neighborhood of τ_0 in I.*

The condition on $u(t)$ as $t \in J$ tends to τ_0 is trivially necessary if there is to be an extension of the solution around τ_0; the interesting feature is that it is sufficient, and more specifically that we do not need to assume anything stronger such as existence of a limit for $u(t)$ as $t \in J$ approaches τ_0: containment in a compact subset of U is all we are required to assume. In the special case $U = V$, remaining in a compact is the same as boundedness, and so the meaning of this corollary in such cases is that if we have a solution on J and the solution is bounded as we approach an endpoint of J in I, then the solution extends to an open neighborhood of that endpoint in J. Of course, if U is a more general open subset of V then we could run into other problems: perhaps $u(t)$ approaches the boundary of U as $t \to \tau_0 \in \partial_I J$. This possibility is ruled out by the compactness hypothesis in the corollary.

Proof. If $I' \subseteq I$ is a compact connected neighborhood of τ_0, we may replace I with I', so we can assume I is compact. Fix a norm on V. In the proof of local existence, the conditions on a involved upper bounds in terms of parameters r, L, and M. Since K is compact and U is open, there exists $r_0 \in (0, 1)$ such that for *any* $v \in K$, $\overline{B}_{2r_0}(v) \subseteq U$. Let $K' \subseteq V$ be the set of points with distance $\le 2r_0$ from K, so K' is compact and $K' \subseteq U$ (since compactness of K implies that the "distance" to K for any point of V is attained by some point of K).

By compactness of $I \times K'$, there exist $L_0, M_0 > 0$ that are respectively upper bounds on $\|D\phi(t,v)\|$ and $\|\phi(t,v)\|$ for all $(t,v) \in I \times K'$. In particular, for all $v \in B$ we have $\overline{B}_{2r_0}(v) \subseteq K'$ and hence the parameters r_0, L_0, M_0 are suitable for the proof of local existence with an initial condition $u(\tau) = v$ for *any* $\tau \in I$ and *any* $v \in K$. In particular, letting $a = \min(1/2L_0, r_0/M_0)$, for any $\tau \in I$ and $v \in K$ there is a solution to the initial-value problem

$$\widetilde{u}'(x) = \phi(x, \widetilde{u}(x)), \ \ \widetilde{u}(\tau) = v$$

on $I \cap [\tau - a, \tau + a]$, where $a > 0$ is *independent* of $(\tau, v) \in I \times K$.

Returning to the endpoint τ_0 near which we want to extend the solution to the initial-value problem, consider points $\tau \in J$ near τ_0. By hypothesis we have $u(\tau) \in K$ for all such τ. We may find such τ with $|\tau - \tau_0| < a$, so $\tau_0 \in I \cap (\tau - a, \tau + a)$. Thus, if we let $v = u(\tau)$ then the initial-value problem

$$\widetilde{u}'(x) = \phi(x, \widetilde{u}(x)), \ \ \widetilde{u}(\tau) = v$$

has a solution on the open subset $I \cap (\tau - a, \tau + a)$ in I that contains τ_0 (here we are using the "universality" of a for an initial-value condition at any point in I with the initial value equal to any point in K). However, on $J \cap (\tau - a, \tau + a)$ such a solution is given by u! Hence, we get a solution to our differential equation near τ_0 in I such that on J near τ_0 it coincides with u. This solves our extension problem around τ_0 for our original initial-value problem. $\square$

For our needs in differential geometry, it remains to address two rather different phenomena for first-order initial-value problems of the form $u'(t) = \phi(t, u(t))$ with $u(t_0) = v_0$ for C^p mappings $\phi : I \times U \to V$:

- the global existence result in the linear case on all of I (with $U = V$),
- the C^p dependence of u on the initial condition v_0 as well as on auxiliary parameters (when ϕ has C^p dependence on such auxiliary parameters) in the general case.

In the final section of this appendix, we take up the first of these problems.

E.3. **Linear ODE.** Our next goal is to prove a global form of Theorem E.2.1 in the *linear* case:

Theorem E.3.1. *Let $I \subseteq \mathbf{R}$ be a non-trivial interval and let V be a finite-dimensional vector space over $\mathbf{R}$. Let $A : I \to \mathrm{Hom}(V, V)$ and $f : I \to V$ be C^p mappings with $p \ge 0$. Choose $t_0 \in I$ and $v_0 \in V$. The initial-value problem*

$$u'(t) = (A(t))(u(t)) + f(t), \ \ \ u(t_0) = v_0$$

has a unique solution u on I, and $u : I \to V$ is C^{p+1}.

Remark E.3.2. Observe that in this theorem we allow A and f to be merely continuous, not necessarily differentiable. Thus, the corresponding $\phi(t,v) = (A(t))(v) + f(t)$ is merely continuous and not necessarily differentiable. But this ϕ has the special feature that $\phi(t, \cdot) : V \to V$ is affine-linear, and so its possible lack of differentiability is not harmful: the *only* reason we needed ϕ to be at least C^1 in the proof of the existence and uniqueness theorem was to locally have the Lipschitz property

$$\|\phi(t,v) - \phi(t,v')\| \le L\|v - v'\|$$

for some $L > 0$, with t near t_0 and any v, v' near v_0.

For ϕ of the special "affine-linear" form, this Lipschitz condition can be verified without conditions on A and f beyond continuity:

$$\|\phi(t, v) - \phi(t, v')\| = \|(A(t))(v - v')\| \leq \|A(t)\| \cdot \|v - v'\|,$$

so we just need a uniform positive upper bound on the operator norm of $A(t)$ for t near t_0. The existence of such a bound follows from the continuity of A and the continuity of the operator norm on the finite-dimensional vector space $\mathrm{Hom}(V, V)$. In particular, for such special ϕ we may apply the results of the preceding section without differentiability restrictions on the continuous A and f.

One point we should emphasize is that the proof of Theorem E.3.1 will *not* involve reproving the local existence theorem in a manner that exploits linearity to get a bigger domain of existence. Rather, the proof will use the local existence/uniqueness theorem as *input* (via Remark E.3.2) and exploit the linear structure of the differential equation to push out the domain on which the solution exists. In particular, the contraction mapping technique (as used for the local version in the general case without linearity assumptions) does not construct the global solution in the linear case: the integral operator used in the local proof fails to be a contraction mapping when the domain is "too big", and hence we have to use an entirely new idea to prove the global existence theorem in the linear case. (It is conceivable that a better choice of norms could rescue the contraction method, but I do not see such a choice, and anyway it is instructive to see how alternative methods can bypass this difficulty.)

Let us now begin the proof of Theorem E.3.1. The uniqueness and C^{p+1} properties are special cases of Theorem E.2.1. The problem is therefore one of existence on I, and so in view of the uniqueness it suffices to solve the problem on arbitrary *bounded* subintervals of I around t_0. Moreover, in case t_0 is an endpoint of I we may extend A and f to C^p mappings slightly past this endpoint (for $p > 0$ either use the usual cheap definition of the notion of C^p mapping on a half-closed interval or else use the Whitney extension theorem if you are a masochist, and for $p = 0$ extend by a "constant" past the endpoint), so in such cases it suffices to consider the existence problem on a larger interval with t_0 as an interior point. Hence, for the existence problem we may and do assume I is a bounded open interval in $\mathbf{R}$. A linear change of variable on t is harmless (why?), so we may and do now suppose $I = (t_0 - r, t_0 + r)$ for some $r > 0$. By the *local* existence theorem, we can solve the problem on $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $0 < \varepsilon \leq r$.

Let $\rho \in (0, r]$ be the supremum of the set S of $c \in (0, r]$ such that our initial-value problem has a solution on $I_c = (t_0 - c, t_0 + c)$. Of course, the solution on such an I_c is unique, and by uniqueness in general its restriction to any open subinterval centered at t_0 is the unique solution on that subinterval. Hence, $S = (0, \rho)$ or $S = (0, \rho]$. Before we compute ρ, let us show that the latter option must occur. For $0 < c < \rho$ we have a unique solution u_c on I_c for our initial-value problem, and uniqueness ensures that if $c < c' < \rho$ then $u_{c'}|_{I_c} = u_c$. Thus, the u_c's "glue" to a solution u_ρ on the open union I_ρ of the I_c's for $c \in (0, \rho)$. This forces $\rho \in S$, so $S = (0, \rho]$. Our problem is therefore to prove $\rho = r$. Put another way, since we have a solution on I_ρ, to get a contradiction if $\rho < r$ it suffices to prove generally that if there is a solution u on I_c for some $c \in (0, r)$ then there is a solution on $I_{c'}$ for some $c < c' < r$. This is what we shall now prove.

The key to the proof is that the solution does not "blow up" in finite time. More specifically, by Corollary E.2.5 with $U = V$, it suffices to fix a norm on V and prove:

Lemma E.3.3. *The mapping* $\|u\| : I_c \to \mathbf{R}$ *defined by* $t \mapsto \|u(t)\|$ *is bounded.*

To prove the lemma, first note that since $c < r$ we have $\overline{I}_c = [t_0 - c, t_0 + c] \subseteq I$, so by compactness of $\overline{I}_c$ there exist constants $M, m > 0$ such that $\|A(t)\| \leq M$ and $\|f(t)\| \leq m$ for all $t \in \overline{I}_c$; we are using the sup-norm on $\mathrm{Hom}(V, V)$ arising from the choice of norm on V. By the differential equation,

$$\|u'(t)\| \leq M\|u(t)\| + m$$

for all $t \in I_c$. By the Fundamental Theorem of Calculus (for maps $I \to V$), for $t \in I_c$ we have

$$u(t) = u(t_0) + \int_{t_0}^{t} u'(y)\mathrm{d}y = v_0 + \int_{t_0}^{t} u'(y)\mathrm{d}y,$$

so for $t \geq t_0$

$$\|u(t)\| \leq \|v_0\| + \int_{t_0}^{t} (M\|u(y)\| + m)\mathrm{d}y = (\|v_0\| + m|t - t_0|) + M\int_{t_0}^{t} \|u(y)\|\mathrm{d}y$$

and likewise for $t \leq t_0$ with the final integral given by $\int_t^{t_0}$. Let us briefly grant the following lemma:

Lemma E.3.4. *Let* $\alpha, \beta, h : [0, a] \to \mathbf{R}_{\geq 0}$ *be continuous functions* $(with\ a > 0)$ *such that*

$$h(\tau) \leq \alpha(\tau) + \int_0^{\tau} h(y)\beta(y)\mathrm{d}y \tag{E.3.1}$$

for all $\tau \in [0, a]$. *Then* $h(\tau) \leq \alpha(\tau) + \int_0^{\tau} \alpha(y)\beta(y)e^{\int_y^{\tau} \beta}\mathrm{d}y$ *for all* $\tau \in [0, a]$.

Using this Lemma with $h = \|u\|$, $\alpha(\tau) = \|v_0\| + m|\tau - t_0|$, and $\beta(\tau) = M$ for $\tau \in [t_0, t]$ (with fixed $t \in (t_0, t_0 + c)$), by letting $\mu = \|v_0\| + mc$ we get

$$\|u(t)\| \leq \mu + \int_{t_0}^{t} M\mu e^{M(t-y)}\mathrm{d}y \leq \mu + \mu(e^{M(t-t_0)} - 1) \leq \mu e^{Mc}$$

for all $t \in [t_0, t_0 + c)$. The same method gives the same constant bound for $t \in (t_0 - c, t_0]$. Thus, this proves the boundedness of $\|u(t)\|$ as t varies in $(t_0 - c, t_0 + c)$, conditional on Lemma E.3.4 that we must now prove:

Proof. Let $I(\tau) = \int_0^{\tau} h(y)\beta(y)\mathrm{d}y$; this is a C^1 function of $\tau \in [0, a]$ since βh is continuous on $[0, a]$. By direct calculation,

$$I' - \beta I = \beta \cdot (h - I) \leq \beta\alpha$$

by (E.3.1) (and the non-negativity of β). Hence, letting $q(\tau) = I(\tau)e^{-\int_0^{\tau} \beta}$, clearly q is C^1 on $[0, a]$ with $q(0) = 0$ and

$$q'(\tau) = e^{-\int_0^{\tau} \beta}(I'(\tau) - \beta(\tau)I(\tau)) \leq e^{-\int_0^{\tau} \beta}\beta(\tau)\alpha(\tau).$$

Since $q(0) = 0$, so $q(\tau) = \int_0^{\tau} q'$, we have $q(\tau) \leq \int_0^{\tau} \alpha(y)\beta(y)e^{-\int_0^{y} \beta}\mathrm{d}y$, whence multiplying by the number $e^{-\int_0^{\tau} \beta}$ gives

$$I(\tau) \leq \int_0^{\tau} \alpha(y)\beta(y)e^{\int_y^{\tau} \beta}\mathrm{d}y.$$

By (E.3.1) and the definition of I, we are done. □

This completes the proof of the global existence/uniqueness theorem in the linear case. Let us record a famous consequence and give an example to illustrate it.

Corollary E.3.5. *Let $I \subseteq \mathbf{R}$ be a nontrivial interval and let $a_0, \ldots, a_{n-1} : I \to \mathbf{R}$ be smooth functions. Let $D : C^\infty(I) \to C^\infty(I)$ be the $\mathbf{R}$-linear map*

$$u \mapsto u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_1 u' + a_0 u.$$

The equation $Du = h$ has a solution for all $h \in C^\infty(I)$, and $\ker D$ *is n-dimensional. More specifically, for any $t_0 \in I$ the mapping*

$$u \mapsto (u(t_0), u'(t_0), \ldots, u^{(n-1)}(t_0)) \in \mathbf{R}^n \tag{E.3.2}$$

is a bijection from the set of solutions to $Du = h$ onto the "space" $\mathbf{R}^n$ of initial conditions.

Proof. The existence of a solution to $Du = h$ on all of I follows from Theorem E.3.1 applied to the first-order reformulation of our problem with $V = \mathbf{R}^n$. Since $D(u_1) = D(u_2)$ if and only if $u_1 - u_2 \in \ker D$, the proposed description of the set of all solutions is exactly the statement that the vector space $\ker D$ maps isomorphically onto $\mathbf{R}^n$ via the mapping (E.3.2). Certainly this is an $\mathbf{R}$-linear mapping, so the problem is one of bijectivity. But this is precisely the statement that the equation $Du = 0$ admits a unique solution for each specification of the $u^{(j)}(t_0)$'s for $0 \le j \le n-1$, and this follows from applying Theorem E.3.1 to the first-order reformulation of our problem (using $V = \mathbf{R}^n$). □

Example E.3.6. Consider the general second-order equation with *constant* coefficients

$$y'' + Ay' + By = 0$$

on $\mathbf{R}$. Let $\delta = A^2 - 4B$. The nature of the solution space depends on the trichotomy of possibilities $\delta > 0$, $\delta < 0$, and $\delta = 0$. Such simple equations are easily solved by the method of the characteristic polynomial if we admit $\mathbf{C}$-valued functions (in case $\delta < 0$), but rather than discuss that technique (which applies to any constant-coefficient linear ODE) we shall simply exhibit some solutions by inspection and use a dimension-count to ensure we've found all solutions. Corollary E.3.5 ensures that the equation has a 2-dimensional solution space in $C^\infty(\mathbf{R})$, with each solution uniquely determined by $y(0)$ and $y'(0)$ (or even $y(x_0)$ and $y'(x_0)$ for any particular $x_0 \in \mathbf{R}$), so to find all solutions we just have to exhibit 2 linearly independent solutions.

If $\delta = 0$ then the left side of the equation is $y'' + Ay' + (A/2)^2 y = (\partial_x + A/2)^2 y$. In this case inspection or iterating solutions to the trivial equation $(\partial_x + C)y = h$ yields solutions $e^{-Ax/2}$ and $xe^{-Ax/2}$. These are independent because if $c_0 e^{-Ax/2} + c_1 x e^{-Ax/2} = 0$ in $C^\infty(\mathbf{R})$ with $c_0, c_1 \in \mathbf{R}$ then multiplying by $e^{Ax/2}$ gives $c_0 + c_1 x = 0$ in $C^\infty(\mathbf{R})$, an impossibility except if $c_0 = c_1 = 0$. If $\delta \neq 0$, let $k = \sqrt{|\delta|}/2 > 0$. Two solutions are $e^{-Ax/2}y_1$ and $e^{-Ax/2}y_2$ where $y_1(x) = e^{kx}$ and $y_2(x) = e^{-kx}$ if $\delta > 0$, and $y_1(x) = \cos(kx)$ and $y_2(x) = \sin(kx)$ if $\delta < 0$. The linear independence is trivial to verify in each case. Thus, we have found all of the solutions.

We push this example a bit further by imposing initial conditions $y(0) = c_0$ and $y'(0) = c_1$ and studying how the solution depends on A and B viewed as "parameters". What is the unique associated solution $y_{A,B}$? Some simple algebra gives

$$y_{A,B} = \begin{cases} e^{-Ax/2}(c_0 + (c_1 + c_0 A/2)x), & \delta = 0 \\ e^{-Ax/2}\left(c_0 \cos(kx) + (c_1 + c_0 A/2)x \cdot \frac{\sin(kx)}{kx}\right), & \delta < 0 \\ e^{-Ax/2}\left(c_0 \cdot \frac{e^{kx}+e^{-kx}}{2} + (c_1 + c_0 A/2)x \cdot \frac{e^{kx}-e^{-kx}}{2kx}\right), & \delta > 0 \end{cases}$$

with $k = \sqrt{|\delta|}/2 = \sqrt{|A^2 - 4B|}/2$. Observe that $(A, B, x) \mapsto y_{A,B}(x)$ is *continuous* in the triple (A, B, x), the real issue being at triples for which $A^2 - 4B = 0$. (Recall that $\sin(v)/v$ is continuous at $v = 0$.) What is perhaps less evident by inspection of these formulas (due to the trichotomous nature of $y_{A,B}(x)$ as a function of (A, B, x), and the intervention of $\sqrt{|A^2 - 4B|}$) is that $y_{A,B}(x)$ is C^∞ in (A, B, x)! In fact, even as a function of the 5-tuple (A, B, c_0, c_1, x) it is C^∞.

Why is there "good" dependence of solutions on initial conditions and auxiliary parameters as they vary? Applications in differential geometry will require affirmative answers to such questions in the non-linear case. Hence, we need to leave the linear setting and turn to the study of properties of solutions to general first-order initial-value problems as we vary the equation (through parameters or initial conditions). This is taken up in Appendix F.

Appendix F. Integral curves

F.1. **Motivation.** Let M be a smooth manifold, and let $\vec{v}$ be a smooth vector field on M. We choose a point $m_0 \in M$. Imagine a particle placed at m_0, with $\vec{v}$ denoting a sort of "unchanging wind" on M. Does there exist a smooth map $c : I \to M$ with an open interval $I \subseteq \mathbf{R}$ around 0 such that $c(0) = m_0$ and at each time t the velocity vector $c'(t) \in \mathrm{T}_{c(t)}(M)$ is equal to the vector $\vec{v}(c(t))$ in the vector field at the point $c(t)$? In effect, we are asking for the existence of a parametric curve whose speed of trajectory through M arranges it to have velocity at each point exactly equal to the vector from $\vec{v}$ at that point. Note that it is natural to focus on the parameterization c and not just the image $c(I)$ because we are imposing conditions on the velocity vector $c'(t)$ at $c(t)$ and not merely on the "direction" of motion at each point. Since we are specifying both the initial position $c(0) = m_0$ and the velocity at all possible positions, physical intuition suggests that such a c should be unique on I *if it exists*.

We call such a c an *integral curve* in M for $\vec{v}$ through m_0. Note that this really is a mapping to M and is not to be confused with its image $c(I) \subseteq M$. (However, we will see in Remark F.5.2 that knowledge of $c(I) \subseteq M$ and $\vec{v}$ suffices to determine c and I uniquely up to additive translation in time.) The reason for the terminology is that the problem of finding integral curves amounts to solving the equation $c'(t) = \vec{v}(c(t))$ that, in local coordinates, is a vector-valued non-linear first-order ODE with the initial condition $c(0) = m_0$. The process of solving such an ODE is called "integrating" the ODE, so classically it is said that the problem is to "integrate" the vector field to find the curve.

One should consider the language of integral curves as the natural geometric and coordinate-free framework in which to think about first-order ODE's of "classical type" $u'(t) = \phi(t, u(t))$, but the local equations in the case of integral curves are of the special "autonomous" form $u'(t) = \phi(u(t))$; we will see later that such apparently special forms are no less general (by means of a change in how we define ϕ). The hard work is this: prove that in the classical theory of initial-value problems

$$u'(t) = \phi(t, u(t)), \;\; u(t_0) = v_0,$$

the solution has reasonable dependence on v_0 when we vary it, and in case ϕ depends on some auxiliary parameters the solution u has good dependence on variation of such parameters (at least as good as the dependence of ϕ).

Our aim in this appendix is twofold: to develop the necessary technical enhancements in the local theory of first-order ODE's in order to prove the basic existence and uniqueness results for integral curves for smooth vector fields on open sets in vector spaces (in §F.2–§F.4), and to then apply these results (in §F.5) to study flow along vector fields on manifolds. In fact after reading §F.1 the reader is urged to skip ahead to §F.5 to see how such applications work out.

Example F.1.1. Lest it seem "intuitively obvious" that solutions to differential equations should have nice dependence on initial conditions and auxiliary parameters, we now explicitly work out an elementary example to demonstrate why one cannot expect a trivial proof of such results.

Consider the initial-value problem

$$u'(t) = 1 + zu^2, \;\; u(0) = v$$

on $\mathbf{R}$ with $(z, v) \in \mathbf{R} \times \mathbf{R}$; in this family of ODE's, the initial time is fixed at 0 but z and v vary. For each $(v, z) \in \mathbf{R}^2$, the solution and its maximal open subinterval of existence $J_{v,z}$ around 0 in $\mathbf{R}$ may be worked out explicitly by the methods of elementary calculus, and there is a trichotomy of formulas for the solution, depending on whether z is positive, negative, or zero. The general solution $u_{v,z} : J_{v,z} \to \mathbf{R}$ is given as follows. We let $\tan^{-1} : \mathbf{R} \to (-\pi/2, \pi/2)$ be the usual arc-tangent function, and we let $\delta_z = \sqrt{|z|}$. The maximal interval $J_{v,z}$ is given by

$$J_{v,z} = \begin{cases} \{|t| < \min(\pi/2\delta_z, \tan^{-1}(1/\delta_z|v|))\}, & z > 0, \\ \mathbf{R}, & z \leq 0, \end{cases}$$

with the understanding that for $v = 0$ we ignore the $\tan^{-1}$ term, and for $t \in J_{v,z}$ we have

$$u_{v,z}(t) = \begin{cases} t + v, & z = 0, \\ (\delta_z^{-1} \cdot \tan(\delta_z v) + v)/(1 - \delta_z v \tan(\delta_z t)), & z > 0, \\ \delta_z^{-1} \cdot ((1 + \delta_z v)e^{2\delta_z t} - (1 - \delta_z v))/((1 + \delta_z v)e^{2\delta_z t} + (1 - \delta_z v)), & z < 0. \end{cases}$$

It is not too difficult to check (draw a picture) that the union $\mathscr{D}$ of the slices

$$J_{v,z} \times \{(v, z)\} \subseteq \mathbf{R} \times \{(v, z)\} \subseteq \mathbf{R}^3$$

is open in $\mathbf{R}^3$; explicitly, $\mathscr{D}$ is the set of triples (t, v, z) such that $u_{v,z}$ propagates to time t. A bit of examination of the trichotomous formula shows that $u : (t, v, z) \mapsto u_{v,z}(t)$ is a continuous mapping from $\mathscr{D}$ to $\mathbf{R}$. What is not at all obvious by inspection (due to the trichotomy of formulas and the intervention of $\sqrt{|z|}$) is that $u : \mathscr{D} \to \mathbf{R}$ is actually smooth (the difficulties are all concentrated along $z = 0$)! This makes it clear that it will not be a triviality to prove theorems asserting that solutions to certain ODE's have nice dependence on parameters and initial conditions.

The preceding example shows that (aside from very trivial examples) the study of differentiable dependence on parameters and initial values is a nightmare when carried out via explicit formulas. We should also point out another aspect of the situation: the *long-term* behavior of the solution can be very sensitive to initial conditions. For example, in Example F.1.1 this is seen via the trichotomous nature of the formula for $u_{v,z}$. Since the C^p property is manifestly *local*, the drastically different long-term ("global") behavior of $u_{v,0}$ and $u_{v,z}$ as $|t| \to \infty$ for $0 < |z| \ll 1$ in Example F.1.1 is *not* inconsistent with the assertion u is a smooth mapping. Another example of this sort of phenomenon was shown in Example E.3.6.

F.2. **Local continuity results.** Our attack on the problem of C^p dependence of solutions on parameters and initial conditions will use induction on p. We first treat a special case for the C^0-aspect of the problem, working only with t near the initial time t_0. There are two continuity problems: in the non-linear case and the linear case. The theory of existence for solutions to first-order initial-value problems is better in the linear case (a solution always exists "globally", on the entire interval of definition for the ODE, as we proved in Theorem E.3.1). Correspondingly, we will have a global continuity result in the linear case. We begin with the general case, where the conclusions are local. (Global results in the general case will be proved in §F.3.)

Let V and V' be finite-dimensional vector spaces, $U \subseteq V$ and $U' \subseteq V'$ open sets. We view U' as a "space of parameters". Let $\phi : I \times U \times U' \to V$ be a C^p mapping with $p \geq 1$. For each $(t_0, v, z) \in I \times U \times U'$, consider the initial-value problem

$$u'(t) = \phi(t, u(t), z), \quad u(t_0) = v \tag{F.2.1}$$

for $u : I \to U$. We regard z as an auxiliary parameter; since z is a point in an open set U' in vector space V', upon choosing a basis for V' we may say that z encodes the data of finitely many auxiliary *numerical* parameters. The power of the vector-valued approach is to reduce gigantic systems of $\mathbf{R}$-valued ODE's with initial conditions and parameters to a *single* ODE, a *single* initial condition, and a *single* parameter (all vector-valued).

Fix the choice of $t_0 \in I$. The local existence theorem (Theorem E.2.1) ensures that for each $z \in U'$ and $v \in U$ there exists a (unique) solution $u_{v,z}$ to (F.2.1) on an interval around t_0 that may depend on z and v, and that there is a unique maximal connected open subset $J_{v,z} \subseteq I$ around t_0 on which this solution exists. Write $u(t, v, z) = u_{v,z}(t)$ for $t \in J_{v,z}$. Since the partials of ϕ are continuous, and continuous functions on compacts are uniformly continuous, an inspection of the *proof* of the local existence/uniqueness theorem (Theorem E.2.1) shows that for each $v_0 \in U$ and $z_0 \in U'$ there is a connected open subset $I_0 \subseteq I$ around t_0 and small opens $U'_0 \subseteq U'$ around z_0 and $U_0 \subseteq U$ around v_0 such that $u_{v,z}$ exists on I_0 for all $(v, z) \in U_0 \times U'_0$. (The sizes of I_0, U_0, and U'_0 depend on the magnitude of the partials of ϕ at points near (t_0, v_0, z_0), but such magnitudes are bounded by uniform constants provided we do not move too far from (t_0, v_0, z_0).) We are interested in studying the mapping

$$u : I_0 \times U_0 \times U'_0 \to U$$

given by $(t, v, z) \mapsto u_{v,z}(t)$; properties of this map near (t_0, v_0, z_0) reflect dependence of the solution on initial conditions and auxiliary parameters if we do not flow too far in time from t_0.

Theorem F.2.1. *For a small connected open $I_0 \subseteq I$ around t_0 and small opens $U'_0 \subseteq U'$ around z_0 and $U_0 \subseteq U$ around v_0, the unique mapping $u : I_0 \times U_0 \times U'_0 \to U$ that is differentiable in I_0 and satisfies*

$$(\partial_t u)(t, v, z) = \phi(t, u(t, v, z), z), \;\; u(t_0, v, z) = v$$

is a continuous mapping.

This local continuity condition will later be strengthened to a global continuity (and even C^p) property, once we work out how variation of (v, z) influences the position of the maximal connected open subset $J_{v,z} \subseteq I$ around t_0 on which the solution $u_{v,z}$ exists.

Proof. Fix a norm on V. The problem is local near $(t_0, v_0, z_0) \in I \times U \times U'$. In particular, we may assume that the interval I is compact. By inspecting the iteration method (contraction mapping) used to construct local solutions near t_0 for the initial-value problem

$$\widetilde{u}'(t) = \phi(t, \widetilde{u}(t), z), \;\; \widetilde{u}(t_0) = v$$

with $(v, z) \in U \times U'$ it is clear (from the C^1 property of ϕ) that the constants that show up in the construction may be chosen "uniformly" for all (v, z) near (v_0, z_0). That is, we may find a small $a > 0$ so that for all z (resp. v) in a small compact neighborhood $K'_0 \subseteq U'$ (resp. $K_0 \subseteq U$) around z_0 (resp. v_0), the integral operator

$$T_z(f) : t \mapsto f(t_0) + \int_{t_0}^{t} \phi(y, f(y), z)\mathrm{d}y$$

is a self-map of the complete metric space

$$X = \{f \in C(I \cap [t_0 - a, t_0 + a], V) \,|\, f(t_0) \in K_0, \mathrm{image}(f) \subseteq \overline{B}_{2r}(f(t_0))\}$$

(endowed with the sup norm) for a suitable small $r \in (0, 1)$ with $\overline{B}_{2r}(K_0) \subseteq U$.

Note that T_z preserves each closed "slice" $X_v = \{f \in X \,|\, f(t_0) = v\}$ for $v \in K_0$. By taking $a > 0$ sufficiently small and K_0 and K'_0 sufficiently small around v_0 and z_0, T_z is a contraction mapping on X_v with a contraction constant in $(0, 1)$ that is independent of $(v, z) \in K_0 \times K'_0$. Hence, for each $(v, z) \in K_0 \times K'_0$, on $I \cap [t_0 - a, t_0 + a]$ there exists a unique solution $u_{v,z}$ to the initial value problem

$$\widetilde{u}'(t) = \phi(t, \widetilde{u}(t), z), \;\; \widetilde{u}(t_0) = v,$$

and $u_{v,z}$ is the unique fixed point of $T_z : X_v \to X_v$.

Let $I_0 = I \cap (t_0 - a, t_0 + a)$, $U_0 = \mathrm{int}_V(K_0)$, $U'_0 = \mathrm{int}_{V'}(K'_0)$. We claim that $(t, v, z) \mapsto u_{v,z}(t)$ is continuous on $I_0 \times U_0 \times U'_0$. By the construction of fixed points in the proof of the contraction mapping theorem, the contraction constant controls the rate of convergence. Starting with the constant mapping $\underline{v} : t \mapsto v$ in X_v, $u_{v,z}(t) \in \overline{B}_{2r}(v)$ is the limit of the points $(T_z^n(\underline{v}))(t) \in \overline{B}_{2r}(v)$, and so it is enough to prove that $(t, v, z) \mapsto (T_z^n(\underline{v}))(t) \in V$ is continuous on $I_0 \times U_0 \times U'_0$ and that these continuous maps uniformly converge to the mapping $(t, v, z) \mapsto u_{v,z}(t)$. In fact, we shall prove such results on the slightly larger (compact!) domain $(I \cap [t_0 - a, t_0 + a]) \times K_0 \times K'_0$.

A bit more generally, for the continuity results it suffices to show that if $g \in X$ then

$$(t, v, z) \mapsto (T_z(g))(t) := g(t_0) + \int_{t_0}^{t} \phi(y, g(y), z)\mathrm{d}y \in V$$

is continuous on $(I \cap [t_0 - a, t_0 + a]) \times K_0 \times K'_0$. This continuity is immediate from uniform continuity of continuous functions on compact sets (check!). As for the uniformity, we want

$T_z^n(\underline{v})$ to converge uniformly to $u_{v,z}$ in X (using the sup norm) with rate of convergence that is uniform in $(v,z) \in K_0 \times K_0'$. But the rate of convergence is controlled by the contraction constant for T_z on X_v, and we have noted above that this small constant may be taken to be the same for all $(v,z) \in K_0 \times K_0'$. $\square$

There is a stronger result in the linear case, and this will be used in §F.4:

Theorem F.2.2. *With notation as above, suppose $U = V$ and $\phi(t,v,z) = (A(t,z))(v) + f(t,z)$ for continuous maps $A : I \times U' \to \mathrm{Hom}(V,V)$ and $f : I \times U' \to V$. For $(v,z) \in U \times U'$, let $u_{v,z} : I \to V$ be the unique solution to the linear initial-value problem*

$$\widetilde{u}'(t) = \phi(t,\widetilde{u}(t),z) = (A(t,z))(\widetilde{u}(t)) + f(t,z), \ \ \widetilde{u}(t_0) = v.$$

The map $u : (t,v,z) \mapsto u_{v,z}(t)$ is continuous on $I \times U \times U'$.

We make some preliminary comments before proving Theorem F.2.2. Recall from Appendix E that since we are in the linear case the solution $u_{v,z}$ exists across the *entire* interval I (as a C^1 function of t) even though ϕ is now merely continuous in (t,v,z). This is why in the setup in Theorem F.2.2 we really do have u defined on $I \times U \times U'$ (whereas in the general non-linear case the maximal connected open domain of $u_{v,z}$ around t_0 in I may depend on (v,z)). In Theorem F.2.2 we only assume continuity of A and f, not even differentiability; such generality will be critical in the application of this theorem in §F.4. (The method of proof of Theorem F.3.6 gives a direct proof of Theorem F.2.2, so we could have opted to postpone the statement of Theorem F.2.2 until later, deducing it from Theorem F.3.6; however, it seems better to give a quick direct proof here.)

It should also be noted that although Theorem F.2.2 is local in (v,z) near any particular (v_0,z_0), it is *not* local in t because the initial condition is at a fixed time t_0. Thus, the theorem is not a formal consequence of the local result in the general non-linear case in Theorem F.2.1, as that result only gives continuity results for (t,z) near (t_0,z_0), with t_0 the fixed initial time. In the formulation of Theorem F.2.2 we are not free to move the initial time, and so we need to essentially revisit our method of proof that the solution extends to all of I (*beyond* the range of applicability of the contraction method) for linear ODE's.

Proof. Fix a norm on V. Our goal is to prove continuity at each point $(t,v_0,z_0) \in I \times U \times U'$, so we may choose $(v_0,z_0) \in U \times U'$ and focus our attention on $(v,z) \in U \times U'$ near (v_0,z_0). Since I is a rising union of compact interval neighborhoods I_n around t_0, with each $t \in I$ admitting I_n as a neighborhood in I for large n (depending on t), it suffices to treat the I_n's separately. That is, we can assume I is a compact interval. Hence, since $u_{v_0,z_0} : I \to V$ is continuous and I is compact, there is a constant $M > 0$ such that $\|u_{v_0,z_0}(t)\| \leq M$ for all $t \in I$. We wish to prove continuity of u at each point $(t,v_0,z_0) \in I \times U \times U'$. Choose a compact neighborhood $K' \subseteq U'$ around z_0, so the continuous $A : I \times K' \to \mathrm{Hom}(V,V)$ and $f : I \times K' \to V$ satisfy $\|A(t,z)\| \leq N$ (operator norm!) and $\|f(t,z)\| \leq \nu$ for all $(t,z) \in I \times K'$ and suitable $N, \nu > 0$.

By uniform continuity of A and f on the compact $I \times K'$, upon choosing $\varepsilon > 0$ we may find a sufficiently small open U_ε' around z_0 in $\mathrm{int}_{V'}(K')$ so that

$$\|A(t,z) - A(t,z_0)\| < \varepsilon, \ \ \|f(t,z) - f(t,z_0)\| < \varepsilon$$

for all $(t,z) \in I \times U_\varepsilon'$. Let $U_\varepsilon \subseteq U$ be an open around v_0 contained in the open ball of radius ε. Using the differential equations satisfied by $u_{v,z}$ and u_{v_0,z_0} on I, we conclude that for all $(t, v, z) \in I \times U_\varepsilon \times U'_\varepsilon$,

$$\begin{aligned} u'_{v,z}(t) - u'_{v_0,z_0}(t) &= (A(t,z))(u_{v,z}(t) - u_{v_0,z_0}(t)) + \\ &\quad (A(t,z) - A(t,z_0))(u_{v_0,z_0}(t)) + (f(t,z) - f(t,z_0)). \end{aligned}$$

Thus, for all $(t, z) \in I \times U'_\varepsilon$ we have

$$\|u'_{v,z}(t) - u'_{v_0,z_0}(t)\| \le N\|u_{v,z}(t) - u_{v_0,z_0}(t)\| + \varepsilon(M+1). \tag{F.2.2}$$

We now fix $z \in U'_\varepsilon$ and study the behavior of the restriction of $u_{v,z} - u_{v_0,z_0}$ to the closed subinterval $I_t \subseteq I$ with endpoints t and t_0 (and length $|t - t_0|$). By the Fundamental Theorem of Calculus applied to the C^1 mapping $g = u_{v,z} - u_{v_0,z_0} : I \to V$ whose value at t_0 is $v - v_0$ (initial conditions!), for any $t \in I$ we have $g(t) = (v - v_0) + \int_{t_0}^{t} g'$. Thus, the upper bound (F.2.2) on the pointwise norm of the integrand $g' = u'_{v,z} - u'_{v_0,z_0}$ therefore yields

$$\|g(t)\| \le \|v - v_0\| + \int_{I_t} (N\|g(y)\| + \varepsilon(M+1))\mathrm{d}y = \varepsilon(1 + (M+1)|t - t_0|) + N\int_{I_t} \|g(y)\|\mathrm{d}y.$$

Since I is compact, there is an $R > 0$ such that $|t - t_0| \le R$ for all $t \in I$. Hence,

$$\|g(t)\| \le \varepsilon(1 + (M+1)R) + \int_{I_t} \|g(y)\| \cdot N\mathrm{d}y.$$

By Lemma E.3.4, applied to (a translate of) the interval I_t (with h there taken to be the the continuous function $y \mapsto \|g(y)\|$ and α, β respectively taken to be the constant functions $\varepsilon(1 + (M+1)R)$, and N), we get

$$\begin{aligned} \|g(t)\| &\le \varepsilon(1 + (M+1)R)(1 + \int_{I_t} Ne^{N(t-y)}\mathrm{d}y) \\ &= \varepsilon(1 + (M+1)R)e^{N(t-t_0)} \le \varepsilon(M+1)Re^{NR} \end{aligned}$$

for all $t \in I$.

To summarize, for all $(t, v, z) \in I \times U_\varepsilon \times U'_\varepsilon$,

$$\|u(t, v, z) - u(t, v_0, z_0)\| \le \varepsilon Q$$

for a uniform constant $Q = (1 + (M+1)R)e^{NR} > 0$ independent of ε. Thus, for $(t', v, z) \in I \times U_\varepsilon \times U'_\varepsilon$ with t' near t, $\|u(t', v, z) - u(t, v_0, z_0)\|$ is bounded above by

$$\|u(t', v, z) - u(t', v_0, z_0)\| + \|u(t', v_0, z_0) - u(t, v_0, z_0)\| \le \varepsilon Q + \|u_{v_0,z_0}(t') - u_{v_0,z_0}(t)\|. \tag{F.2.3}$$

Since u_{v_0,z_0} is continuous at $t \in I$, it follows from (F.2.3) (by taking ε to be sufficiently small) that $u(t', v, z)$ can be made as close as we please to $u(t, v_0, z_0)$ for (t', v, z) near enough to (t, v_0, z_0). In other words, $u : (t, v, z) \mapsto u(t, v, z) \in V$ on $I \times U \times U'$ is continuous at each point lying in a slice $I \times \{(v_0, z_0)\}$ with $(v_0, z_0) \in U \times U'$ arbitrary. Hence, u is continuous. □

F.3. **Domain of flow, main theorem on C^p-dependence, and reduction steps.** Let $I \subseteq \mathbf{R}$ be a non-trivial interval (i.e., not a point but perhaps half-open/closed, unbounded, etc.), $U \subseteq V$ and $U' \subseteq V'$ open subsets of finite-dimensional vector spaces, and $\phi : I \times U \times U' \to V$ a C^p mapping with $p \ge 1$. For each $(t_0, v_0, z_0) \in I \times U \times U'$, we let $J_{t_0,v_0,z_0} \subseteq I$ be the maximal connected open subset around t_0 on which the initial-value problem

$$\widetilde{u}'(t) = \phi(t, \widetilde{u}(t), z_0), \ \ \widetilde{u}(t_0) = v_0 \tag{F.3.1}$$

has a solution; this solution will be denoted $u_{t_0,v_0,z_0} : J_{t_0,v_0,z_0} \to U$. For example, if the mapping $\phi(t_0,\cdot,z_0) : U \to V$ is the restriction of an affine-linear self-map $x \mapsto (A(t_0,z_0))(x) + f(t_0,z_0)$ of V for each $(t_0,z_0) \in I \times U'$ then $J_{t_0,v_0,z_0} = I$ for all (t_0,v_0,z_0) because linear ODE's on I (with an initial condition) have a (unique) solution on all of I.

We refer to equations of the form (F.3.1) as *time-dependent flow with parameters* in the sense that for each $(t,z) \in I \times U'$ the C^p-vector field $v \mapsto \phi(t,v,z) \in U \subseteq V \simeq \mathrm{T}_v(U)$ depends on both the time t and the auxiliary parameter z. That is, the visual picture for the equation (F.3.1) is that it describes the motion $t \mapsto \widetilde{u}(t)$ of a particle such that the velocity $\widetilde{u}'(t) = \phi(t,\widetilde{u}(t),z_0) \in V \simeq \mathrm{T}_{\widetilde{u}(t)}(U)$ at any time depends on not just the position $\widetilde{u}(t)$ and the fixed value of z_0 but also on the time t (via the "first variable" of ϕ). We wish to now state the most general result on C^p-dependence of such solutions as we vary the auxiliary parameter z, the initial time $t_0 \in I$, and the initial position $v_0 \in U$ at time t_0. In order to give a clean statement, we first need to introduce a new concept:

Definition F.1. The *domain of flow* is

$$\mathscr{D}(\phi) = \{(t,\tau,v,z) \in I \times I \times U \times U' \mid t \in J_{\tau,v,z}\}.$$

In words, for each possible initial position $v_0 \in U$ and initial time $t_0 \in I$ and auxiliary parameter $z_0 \in U'$, we get a maximal connected open subset $J_{t_0,v_0,z_0} \subseteq I$ on which (F.3.1) has a solution, and this is where $\mathscr{D}(\phi)$ meets $I \times \{(t_0,v_0,z_0)\}$. For example, if $\phi(t_0,\cdot,z_0) : U \to V$ is the restriction of an affine-linear self-map of V for all $(t_0,z_0) \in I \times U'$ then $\mathscr{D}(\phi) = I \times I \times U \times U'$.

There is a natural set-theoretic mapping

$$u : \mathscr{D}(\phi) \to V$$

given by $(t,\tau,v,z) \mapsto u_{\tau,v,z}(t)$; this is called the *universal solution* to the given family (F.3.1) of time-dependent parametric ODE's with varying initial positions and initial times. On each "slice" $J_{t_0,v_0,z_0} = \mathscr{D}(\phi) \cap (I \times \{(t_0,v_0,z_0)\})$ this mapping is the unique solution to (F.3.1) on its maximal connected open subset around t_0 in I. Studying this mapping and its differentiability properties is tantamount to the most general study of how time-dependent flow with parameters depends on the initial position, initial time, and auxiliary parameters. Our goal is to prove that $\mathscr{D}(\phi)$ is *open* in $I \times I \times U \times U'$ and that if ϕ is C^p then so is $u : \mathscr{D}(\phi) \to V$. Such a C^p property for u on $\mathscr{D}(\phi)$ is the precise formulation of the idea that solutions to ODE's should depend "nicely" on initial conditions and auxiliary parameter.

In Example F.1.1 we saw that even for rather simple ϕ's the nature of $\mathscr{D}(\phi)$ and the good dependence on parameters and initial conditions can look rather complicated when written out in explicit formulas. Before we address the openness and C^p problems, we verify an elementary topological property of the domain of flow.

Lemma F.3.2. *If U and U' are connected then $\mathscr{D}(\phi) \subseteq I \times I \times U \times U'$ is connected.*

Proof. Pick $(t,t_0,v_0,z_0) \in \mathscr{D}(\phi)$. This lies in the connected subset $J_{t_0,v_0,z_0} \times \{(t_0,v_0,z_0)\}$ in $\mathscr{D}(\phi)$, so moving along this segment brings us to the point (t_0,t_0,v_0,z_0). But $\mathscr{D}(\phi)$ meets the subset of points $(t_0,t_0,v,z) \in I \times I \times U \times U'$ in exactly $\{(t_0,t_0)\} \times U \times U'$ because for *any* initial position $v \in U$ and auxiliary parameter $z \in U'$ the initial-value problem for the parameter z and initial condition $\widetilde{u}(t_0) = v$ does have a solution for $t \in I$ near t_0 (with nearness perhaps depending on (v,z)). Hence, the problem is reduced to connectivity of $U \times U'$, which follows from the assumption that U and U' are connected. $\square$

The main theorem in this appendix is:

Theorem F.3.3. *The domain of flow $\mathscr{D}(\phi) \subseteq I \times I \times U \times U'$ is open and the mapping*

$$u : \mathscr{D}(\phi) \to V$$

given by $(t, \tau, v, z) \mapsto u_{\tau,v,z}(t)$ is C^p.

What does such openness really mean? The point is this: if we begin at some time t_0 with an initial position v_0 and parameter-value z_0, and if the resulting solution exists out to a time t (i.e., $(t, t_0, v_0, z_0) \in \mathscr{D}(\phi)$), then by slightly changing all three of these starting values we can still flow the solution to all times near t (in particular to time t). This fact is *not* obvious, though it is intuitively reasonable. Of course, as $|t| \to \infty$ we expect to have less and less room in which to slightly change t_0, v_0, and z_0 if we wish to retain the property of the solution flowing out to time t.

The proof of Theorem J.3.1 will require two entirely different kinds of reduction steps. For the openness result it will be convenient to reduce the problem to the case when there are no auxiliary parameters ($U' = V' = \{0\}$), the interval I is open, the initial time is always 0, and the flow is not time-dependent (i.e., ϕ has domain U); in other words, the only varying quantity is the initial position. For the C^p aspect of the problem, it will be convenient (for purposes of induction on p) to reduce to the case when I is open, the initial time and position are fixed, and there is both an auxiliary parameter *and* time-dependent flow.

For applications to Lie groups we only require the case of open intervals. For the reader who is interested in allowing I to have endpoints, we now explain how to reduce Theorem J.3.1 for general I to the case when $I \subseteq \mathbf{R}$ is open. (Other readers should skip over the lemma below.) Observe that to prove Theorem J.3.1, we may pick $(v_0, z_0) \in U \times U'$ and study the problem on $I \times I \times U_0 \times U'_0$ for open subsets $U_0 \subseteq U$ and $U'_0 \subseteq U'$ around v_0 and z_0 with $U_0 \subseteq K_0$ and $U'_0 \subseteq K'_0$ for compact subsets $K_0 \subseteq U$ and $K'_0 \subseteq U'$. Hence, to reduce to the case of an open interval I we just need to prove:

Lemma F.3.4. *Let $K_0 \subseteq U$ and $K'_0 \subseteq U'$ be compact neighborhoods of points $v_0 \in U$ and $z_0 \in U'$. Let $U_0 = \mathrm{int}_V(K_0)$ and $U'_0 = \mathrm{int}_{V'}(K'_0)$. There exists an open interval $J \subseteq \mathbf{R}$ containing I and a C^p mapping*

$$\widetilde{\phi} : J \times U_0 \times U'_0 \to V$$

restricting to ϕ on $I \times U \times U'$.

The point is that once we have such a $\widetilde{\phi}$, it is clear that $\mathscr{D}(\widetilde{\phi}) \subseteq J \times J \times U_0 \times U'_0$ satisfies

$$\mathscr{D}(\widetilde{\phi}) \cap (I \times I \times U_0 \times U'_0) = \mathscr{D}(\phi) \cap (I \times I \times U_0 \times U'_0)$$

and the "universal solution" $\mathscr{D}(\widetilde{\phi}) \to V$ agrees with u on the common subset

$$\mathscr{D}(\phi) \cap (I \times I \times U_0 \times U'_0).$$

Hence, proving Theorem J.3.1 for $\widetilde{\phi}$ will imply it for ϕ.

Proof. We may assume I is not open, and it suffices to treat the endpoints separately (if there are two of them). Thus, we fix an endpoint t_0 of $\overline{I}$ and we work locally on $\mathbf{R}$ near t_0. That is, it suffices to make J around t_0. For each point $(v, z) \in U \times U'$, by the Whitney extension theorem (or a cheap definition of the notion of "C^p mapping" on a half-space) there is an open neighborhood $W_{v,z}$ of (v, z) in $U \times U'$ and an open interval $I_{v,z} \subseteq \mathbf{R}$

around t_0 such the mapping $\phi|_{(I\cap I_{v,z})\times W_{v,z}}$ extends to a C^p mapping $\phi_{t,v,z} : I_{v,z} \times W_{v,z} \to V$. Finitely many $W_{v,z}$'s cover the compact $K \times K'$, say W_{v_n,z_n} for $1 \leq n \leq N$. Let J be the intersection of the *finitely many* I_{v_n,z_n}'s for these W_{v_n,z_n}'s that cover $K \times K'$, so J is an open interval around t_0 in $\mathbf{R}$ such that there are C^p mappings $\phi_n : J \times W_{v_n,z_n} \to V$ extending $\phi|_{(I\cap J)\times W_{v_n,z_n}}$ for each n.

Let X be the union of the W_{v_n,z_n}'s in $U \times U'$. Let $\{\alpha_i\}$ be a C^∞ partition of unity subordinate to the collection of opens $J \times W_{v_n,z_n}$ that covers $J \times X$, with α_i compactly supported in $J \times W_{v_{n(i)},z_{n(i)}}$. Thus, $\alpha_i\phi_{n(i)}$ is C^p and compactly supported in the open $J \times W_{v_{n(i)},z_{n(i)}} \subseteq J \times X$. It therefore "extends by zero" to a C^p mapping $\widetilde{\phi}_i : J \times X \to V$. Let $\widetilde{\phi} = \sum_i \widetilde{\phi}_i : J \times X \to V$; this is a locally finite sum since the supports of the α_i's are a locally finite collection. We claim that on $(J \cap I) \times K \times K'$ (and hence on $(J \cap I) \times U_0 \times U_0')$ the map $\widetilde{\phi}$ is equal to ϕ. By construction ϕ_n agrees with ϕ on $(J \cap I) \times W_{v_n,z_n}$ for all n, and hence $\widetilde{\phi}_i$ agrees with $\alpha_i\phi$ on $(J \cap I) \times W_{v_{n(i)},z_{n(i)}}$. Hence, $\widetilde{\phi}_i|_{(J\cap I)\times K\times K'}$ vanishes outside of the support of α_i and on this support it equals $\alpha_i\phi$. Thus, for $(t, v, z) \in (J \cap I) \times K \times K'$ we have $\widetilde{\phi}_i(t, v, z) = \alpha_i(t, v, z)\phi(t, v, z)$ for all i. Adding this up over all i (a finite sum), we get $\widetilde{\phi}(t, v, z) = \phi(t, v, z)$. $\square$

In view of this lemma, we may and do *assume I is open in* $\mathbf{R}$. We now exploit such openness to show how Theorem J.3.1 may be reduced to each of two kinds of special cases.

Example F.3.5. We first reduce the general case to that of time-independent flow without parameters and with a fixed initial time $t = 0$. Define the open subset

$$Y = \{(t, \tau) \in I \times \mathbf{R} \mid t + \tau \in I\} \subseteq \mathbf{R} \times \mathbf{R}$$

and let $W = \mathbf{R}^2 \oplus V \oplus V'$, so $U'' = Y \times U \times U'$ is an open subset of W. Define $\psi : U'' \to W$ to be the C^p mapping

$$(t, \tau, v, z) \mapsto (1, 0, \phi(t + \tau, v, z), 0).$$

Consider the initial-value problem

$$\widetilde{u}'(t) = \psi(\widetilde{u}'(t)), \;\; \widetilde{u}(0) = (t_0, 0, v_0, z_0) \in W \tag{F.3.2}$$

as a W-valued mapping on an unspecified open interval J around the origin in $\mathbf{R}$. A solution to this initial-value problem on J has the form $\widetilde{u} = (u_0, u_1, u_2, u_3)$ where $u_0, u_1 : J \rightrightarrows \mathbf{R}$, $u_2 : J \to U$, and $u_3 : J \to U'$ satisfy

$$(u_0'(t), u_1'(t), u_2'(t), u_3'(t)) = (1, 0, \phi(u_0(t) + u_1(t), u_2(t), u_3(t)), 0)$$

and

$$(u_0(0), u_1(0), u_2(0), u_3(0)) = (t_0, 0, v_0, z_0),$$

so $u_0(t) = t + t_0$, $u_1(t) = 0$, $u_3(t) = z_0$, and

$$u_2'(t) = \phi(t + t_0, u_2(t), z_0), \;\; u_2'(0) = v_0.$$

In other words, $u_2(t - t_0)$ is a solution to (F.3.1).

We define the *domain of flow* $\mathscr{D}(\psi) \subseteq \mathbf{R} \times U'' \subseteq \mathbf{R} \times W$ much like in Definition F.1, except that we now consider initial-value problems

$$\widetilde{u}'(t) = \psi(\widetilde{u}(t)), \;\; \widetilde{u}(0) = (t_0, \tau_0, v_0, z_0) \in U'' \tag{F.3.3}$$

for which the initial time is fixed at 0. That is, $\mathscr{D}(\psi)$ is the set of points $(t_0, w_0) \in \mathbf{R} \times U''$ such that t_0 lies in the maximal open interval $J_{w_0} \subseteq \mathbf{R}$ on which the initial-value problem

$$\widetilde{u}'(t) = \psi(\widetilde{u}(t)), \ \ \widetilde{u}(0) = w_0$$

has a solution.

The above calculations show that the C^∞ isomorphism

$$U'' \simeq I \times I \times U \times U'$$

given by $(t, \tau, v, z) \mapsto (t + \tau, \tau, v, z)$ carries $\mathscr{D}(\psi) \cap (\mathbf{R} \times \mathbf{R} \times \{0\} \times U \times U')$ over to $\mathscr{D}(\phi)$ and carries the restriction of the "universal solution" to (F.3.3) on $\mathscr{D}(\psi)$ over to the universal solution u on $\mathscr{D}(\phi)$. Hence, if we can prove Theorem J.3.1 for ψ then it follows for ϕ. In this way, by studying ψ rather than ϕ we see that to prove Theorem J.3.1 in general it suffices to consider time-independent parameter-free flow with initial time 0. Note also that the study of $\mathscr{D}(\psi)$ uses the time interval $I = \mathbf{R}$ since ψ is "time-independent".

The appeal of the preceding reduction step is that time-independent parameter-free flow with a varying initial position but fixed initial time is *exactly* the setup that is relevant for the local theory of integral curves to smooth vector fields on manifolds (with a varying initial point)! Thus, this apparently "special" case of Theorem J.3.1 is in fact no less general (provided we allow ourselves to consider *all* cases at once). Unfortunately, in this special case it seems difficult to push through the C^p aspects of the argument. Hence, we will take care of the openness and continuity aspects of the problem in this special case (thereby giving such results in the general case), and then we will use an entirely different reduction step to dispose of the C^p property of the universal solution on the domain of flow.

We now restate the situation to which we have reduced ourselves, upon renaming ψ as ϕ. Let $I = \mathbf{R}$, and let U be an open subset in a finite-dimensional vector space V. Let $\phi : U \to V$ be a C^p mapping ($p \geq 1$) and consider the family of initial-value problems

$$u'(t) = \phi(u(t)), \ \ u(0) = v_0 \in U \tag{F.3.4}$$

with varying v_0. We define $J_{v_0} \subseteq \mathbf{R}$ to be the maximal open interval around the origin on which the unique solution u_{v_0} exists, and we define the *domain of flow*

$$\mathscr{D}(\phi) = \{(t, v) \in \mathbf{R} \times U \mid t \in J_v\} \subseteq \mathbf{R} \times U.$$

The openness and continuity aspects of Theorem J.3.1 are a consequence of:

Theorem F.3.6. *In the special situation just described, $\mathscr{D}(\phi)$ is an open subset of $\mathbf{R} \times U$ and u is continuous.*

We hold off on the proof of Theorem F.3.6, because it requires knowing some continuity results of a local nature near the initial time. The continuity input we need was essentially proved in Theorem F.2.1, except for the glitch that Theorem F.2.1 has auxiliary parameters and a fixed initial position whereas Theorem F.3.6 has no auxiliary parameters but a varying initial position! Thus, we will now explain how to reduce the general problems in Theorem J.3.1 and Theorem F.3.6 to another kind of special case.

Example F.3.7. We already know it is sufficient to prove Theorem J.3.1 in the special setup considered in Theorem F.3.6: time-independent parameter-free flows with varying initial position but fixed initial time. Let us now show how the problem of proving Theorem F.3.6 or even its strengthening with continuity replaced by the C^p property (and hence Theorem J.3.1 in general) can be reduced to the case of a family of ODE's with time-dependent flow and auxiliary parameters but a fixed initial time and initial position. The idea is this: we rewrite the (F.3.4) so that the varying initial position becomes an auxiliary parameter! Using notation as in (F.3.4), let $\widetilde{V} = V \oplus V$ and define the open subset

$$\widetilde{U} = \{\widetilde{v} = (v_1, v_2) \in \widetilde{V} \mid v_1 + v_2 \in U, v_2 \in U\}.$$

Also define the "parameter space" $\widetilde{U}' = U$ as an open subset of $\widetilde{V}' = V$. Finally, define the C^p mapping $\widetilde{\phi} : \widetilde{U} \times \widetilde{U}' \to \widetilde{V} = V \oplus V$ by

$$\widetilde{\phi}((v_1, v_2), z) = (\phi(v_1 + z), 0).$$

Consider the parametric family of time-independent flows

$$\widetilde{u}'(t) = \widetilde{\phi}(\widetilde{u}(t), v), \;\; \widetilde{u}(0) = 0 \tag{F.3.5}$$

with varying $v \in \widetilde{U}' = U$ (now serving as an auxiliary parameter!). The unique solution (on a maximal open subinterval of $\mathbf{R}$ around 0) is seen to be $\widetilde{u}_v(t) = (u_v(t), 0)$ with $u_v(t) + v$ the unique solution (on a maximal open subinterval of $\mathbf{R}$ around 0) to

$$u'(t) = \phi(u(t)), \;\; u(0) = v.$$

Thus, the domain of flow $\mathscr{D}(\phi) \subseteq \mathbf{R} \times U$ for this latter family is a "slice" of the domain of flow $\mathscr{D}(\widetilde{\phi}) \subseteq \mathbf{R} \times \widetilde{U} \times \widetilde{U}' = \mathbf{R} \times \widetilde{U} \times U$, namely the subset of points of the form $(t, (0, v), v)$. The universal solution to (F.3.4) on $\mathscr{D}(\phi)$ is easily computed (by simple affine-linear formulas) in terms of the restriction of the universal solution to (F.3.5) on $\mathscr{D}(\widetilde{\phi})$.

It follows that the continuity property for the universal solution on the domain of flow for (F.3.5) with v as an auxiliary parameter implies the same for (F.3.4) with v as an initial position. The same implication works for the openness property of the domain of flow, as well as for the C^p property of the universal solution on the domain of flow. This reduces Theorem J.3.1 (and even Theorem F.3.6) to the special case of time-independent flows on $I = \mathbf{R}$ with an auxiliary parameter and fixed initial conditions. It makes the problem more general to permit the flow ϕ to even be time-dependent (again, keeping $I = \mathbf{R}$) and for the proof of Theorem J.3.1 we will want such generality because it will be forced upon us by the method of proof that is going to be used for the C^p aspects of the problem.

We emphasize that although the preceding example proposes returning to the study of time-dependent flows with parameters (and $I = \mathbf{R}$), in so doing we have gained something on the generality in Theorem J.3.1: the initial time and initial position are *fixed*. As has just been explained, the study of this special situation (with the definition of domain of flow adapted accordingly, namely as a subset of $\mathbf{R} \times U' = I \times U'$ rather than as a subset of $I \times I \times U \times U'$) is sufficient to solve the most general form of the problem as stated in Theorem J.3.1. This new special situation is exactly the framework considered in §F.2.

Remark F.3.8. For applications to manifolds, it is the case of time-independent flow with varying initial position but fixed initial time and no auxiliary parameters that is the relevant one. That is, as we have noted earlier, the setup in Theorem F.3.6 is the one that describes the local situation for the theory of integral curves for smooth vector fields on a smooth manifold. However, such a setup is inadequate for the *proof* of the smooth-dependence properties on the varying initial conditions. This is why it was crucial for us to formulate Theorem J.3.1 in the level of generality that we did: if we had stated it only in the context for which it would be applied on manifolds, then the inductive aspects of the proof would take us out of that context (and thereby create confusion as to what exactly we are aiming to prove), as we shall see in Remark F.4.3.

We conclude this preliminary discussion by proving Theorem F.3.6:

Proof. Obviously each point $(0, v_0) \in \mathbf{R} \times U$ lies in $\mathscr{D}(\phi)$, and we first make the local claim that there is a neighborhood of $(0, v_0)$ in $\mathbf{R} \times U$ contained in $\mathscr{D}(\phi)$ and on which u is continuous. That is, there exists $\varepsilon > 0$ such that for $v \in U$ sufficiently near v_0, the unique solution u_v to

$$u'(t) = \phi(u(t)), \;\; u(0) = v$$

exists on $(-\varepsilon, \varepsilon)$ and the map $(t, v) \mapsto u_v(t) \in V$ is continuous for $|t| < \varepsilon$ and $v \in U$ near v_0. If we ignore the continuity aspect, then the existence of ε follows from the method of proof of the local existence theorem for ODE's; this sort of argument was already used in the build-up to Theorem F.2.1. Hence, $\mathscr{D}(\phi)$ contains an open set around $\{0\} \times U \subseteq \mathbf{R} \times U$, so the problem is to show that we acquire continuity for u on $\mathscr{D}(\phi)$ near $(0, v_0)$. The reduction technique in Example F.3.7 reduces the continuity problem to the case when the initial position is fixed (as is the initial time at 0) but v is an auxiliary parameter. This is *exactly* the problem that was solved in Theorem F.2.1!

We now fix a point $v_0 \in U$ and aim to prove that for *all* $t \in J_{v_0} \subseteq \mathbf{R}$ the point $(t, v_0) \in \mathscr{D}(\phi)$ is an interior point (relative to $\mathbf{R} \times U$) and u is continuous on an open around (t, v_0) in $\mathscr{D}(\phi)$. Let T_{v_0} be the set of $t \in J_{v_0}$ for which such openness and local continuity properties hold at (t', v_0) for $0 \le t' < t$ when $t > 0$ and for $t < t' \le 0$ when $t < 0$. It is a triviality that T_{v_0} is an open connected subset of J_{v_0}, but one has to do work to show it is non-empty! Fortunately, in the preceding paragraph we proved that $0 \in T_{v_0}$. Our goal is to prove $T_{v_0} = J_{v_0}$. Once this is shown, then since $v_0 \in U$ was arbitrary it will follow from the definition of $\mathscr{D}(\phi)$ that this domain of flow is a neighborhood of all of its points relative to the ambient space $\mathbf{R} \times U$ (hence it is open in $\mathbf{R} \times U$) and that $u : \mathscr{D}(\phi) \to V$ is continuous near each point of $\mathscr{D}(\phi)$ and hence is continuous. In other words, we would be done.

To prove that the open subinterval T_{v_0} in the open interval $J_{v_0} \subseteq \mathbf{R}$ around 0 satisfies $T_{v_0} = J_{v_0}$, we try to go as far as possible in both directions. Since $0 \in T_{v_0}$, we may separately treat the cases of moving to the right and moving to the left. We consider going to the right, and leave it to the reader to check that the same method applies in the left direction. If T_{v_0} does not exhaust all positive numbers in J_{v_0} then since T_{v_0} contains 0 it follows that the supremum of T_{v_0} is a finite positive number $t_0 \in J_{v_0}$ and we seek a contradiction by studying flow near (t_0, v_0). More specifically, since $t_0 \in J_{v_0}$ and J_{v_0} is open, the solution u_{v_0} does propagate past t_0. Define $v_1 = u_{v_0}(t_0) = u(t_0, v_0) \in U$.

The local openness and continuity results that we established at the beginning of the proof are applicable to time-independent parameter-free flows with varying initial positions and any fixed initial time in $\mathbf{R}$ (there is nothing sacred about the origin). Hence, there is some positive $\varepsilon > 0$ such that for $v \in B_\varepsilon(v_1) \subseteq U$ and $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ the point (t, v) is in the domain of flow for the family of ODE's (with varying v)

$$\widetilde{u}'(y) = \phi(\widetilde{u}(y)), \;\; \widetilde{u}(t_0) = v,$$

and the universal solution $\widetilde{u} : (t, v) \mapsto \widetilde{u}_v(t)$ is continuous on the subset

$$(t_0 - \varepsilon, t_0 + \varepsilon) \times B_\varepsilon(v_1)$$

in its domain of flow.

By continuity of the differentiable (even C^1) mapping $u_{v_0} : J_{v_0} \to V$, there is a $\delta > 0$ so that $u_{v_0}(t) \in B_{\varepsilon/4}(v_1)$ for $t_0 - \delta < t < t_0$. We may assume $\delta < \varepsilon/4$. Since t_0 is the supremum of the open interval T_{v_0}, we may choose δ sufficiently small so that $(t_0 - \delta, t_0) \subseteq T_{v_0}$. Pick any $t_1 \in (t_0 - \delta, t_0) \subseteq T_{v_0}$, so by definition of T_{v_0} there is an open interval J_1 around t_1 and an open U_1 around v_0 in U such that $J_1 \times U_1 \subseteq \mathscr{D}(\phi)$ and u is continuous on $J_1 \times U_1$. Since $u(t_1, v_0) = u_{v_0}(t_1) \subseteq B_{\varepsilon/4}(v_1)$ (as $t_1 \in (t_0 - \delta, t_0)$) and u is *continuous* (!) at $(t_1, u_0) \in J_1 \times U_1$ (by definition of T_{v_0}), we may shrink U_1 around v_0 and J_1 around t_1 so that

$$J_1 \subseteq (t_0 - \delta, t_0) \subseteq (t_0 - \varepsilon, t_0 + \varepsilon)$$

and $u(J_1 \times U_1) \subseteq B_{\varepsilon/2}(v_1)$. But $B_{\varepsilon/2}(v_1) \subseteq B_\varepsilon(v_1)$, so for all $v \in U_1$ the mapping

$$t \mapsto \widetilde{u}_{u(t_1, v_1)}(t + (t_0 - t_1))$$

extends u_v near t_1 out to time $t_0 + \varepsilon$ as a solution to the original initial-value problem

$$u'(t) = \phi(u(t)), \;\; u(0) = v.$$

We have shown $(0, t_0 + \varepsilon) \subseteq J_v$ for all $v \in U_1$ and that if $(t, v) \in (t_0 - \varepsilon, t_0 + \varepsilon) \times U_1$ then

$$u(t, v) = \widetilde{u}(t + (t_0 - t_1), u(t_1, v)) \tag{F.3.6}$$

(with $|t_0 - t_1| < \delta < \varepsilon/4$). But u is continuous on $J_1 \times U_1$, $\widetilde{u}$ is continuous on

$$(t_0 - \varepsilon, t_0 + \varepsilon) \times B_\varepsilon(v_1),$$

and $u(J_1 \times U_1) \subseteq B_\varepsilon(v_1)$, so by inspection of continuity properties of the ingredients in the right side of (F.3.6) we conclude that

$$(t_0 - \varepsilon/4, t_0 + \varepsilon/4) \times U_1 \subseteq \mathscr{D}(\phi)$$

and that u is continuous on this domain. In particular, T_{v_0} contains $(t_0 - \varepsilon/4, t_0 + \varepsilon/4)$, and this contradicts the definition $t_0 = \sup T_{v_0}$. $\square$

Note that the proof of Theorem F.3.6 would not have worked if we had not simultaneously proved continuity on the domain of flow. This continuity will be recovered in much stronger form below (namely, the C^p property), but the proof of the stronger properties rests on Theorem F.3.6.

F.4. C^p **dependence.** Since Theorem F.3.6 is now proved, in the setup of Theorem J.3.1 the domain of flow $\mathscr{D}(\phi)$ is open and the universal solution $u : \mathscr{D}(\phi) \to V$ on it is continuous. To wrap up Theorem J.3.1, we have to show that this universal solution is C^p. In view of the reduction step in Example F.3.7, it suffices to solve the following analogous problem. Let $U \subseteq V$ and $U' \subseteq V'$ be open subsets of finite-dimensional vector spaces with $0 \in U$, and let $\phi : \mathbf{R} \times U \times U' \to V$ be a C^p mapping. Consider the family of ODE's

$$\widetilde{u}'(t) = \phi(t, \widetilde{u}(t), z), \;\; \widetilde{u}(0) = 0 \tag{F.4.1}$$

for varying $z \in U'$. Let $u_z : J_z \to \mathbf{R}$ be the unique solution on its maximal open interval of definition around the origin in $\mathbf{R}$, and let $\mathscr{D}(\phi) \subseteq \mathbf{R} \times U'$ be the domain of flow: the set of points $(t, z) \in \mathbf{R} \times U'$ such that $t \in J_z$. This is an open subset of $\mathbf{R} \times U'$ since the openness aspect of Theorem J.3.1 has been proved. We define the universal solution

$$u : \mathscr{D}(\phi) \to V$$

by $u(t, z) = u_z(t)$, so this is known to be continuous. Our goal is to prove that u is C^p.

Consider the problem of proving that u is C^p near a particular point (t_0, z_0). (**Warning**. The initial time in (F.4.1) is fixed at 0. Thus, t_0 does not denote an "initial time".) Obviously the parameter space U' only matters near z_0 for the purposes of the C^p property of u near (t_0, z_0). Let J be the compact interval in $\mathbf{R}$ with endpoints 0 and t_0. Since $\mathscr{D}(\phi)$ is *open* in $\mathbf{R} \times U'$ and contains the compact $J \times \{z_0\}$, we may shrink U' around z_0 and find $\varepsilon > 0$ such that $J_\varepsilon \times U' \subseteq \mathscr{D}(\phi)$ with $J_\varepsilon \subseteq \mathbf{R}$ the open interval obtained from J by appending open intervals of length ε at both ends. In other words, we may assume $I \times U' \subseteq \mathscr{D}(\phi)$ for an *open* interval $I \subseteq \mathbf{R}$ containing 0 and t_0.

To get the induction on p off the ground we claim that u is C^1 on $I \times U'$ (and so in particular at the arbitrarily chosen $(t_0, z_0) \in \mathscr{D}(\phi)$). This follows from a stronger result in the C^1 case that will be essential for the induction on p:

Theorem F.4.1. *Let $I \subseteq \mathbf{R}$ be an open interval containing 0, and assume that the initial-value problem*

$$u'(t) = \phi(t, u(t), z), \ \ u(0) = v_0$$

with C^1 mapping $\phi : \mathbf{R} \times U \times U' \to V$ has a solution $u_z : I \to U$ for all $z \in U'$. (That is, the domain of flow $\mathscr{D}(\phi) \subseteq \mathbf{R} \times U'$ contains $I \times U'$). Choose $z_0 \in U'$.

1. *For any connected open neighborhood $I_0 \subseteq I$ around 0 with compact closure in I, there is an open $U'_0 \subseteq U'$ around z_0 and an open interval $I'_0 \subseteq I$ containing $\overline{I}_0$ such that $u : (t, z) \mapsto u_z(t)$ is C^1 on $I'_0 \times U'_0$.*
2. *For any such U'_0 and I'_0, the map $I'_0 \to \mathrm{Hom}(V', V)$ given by the total U'-derivative $t \mapsto (D_2 u)(t, z)$ of the mapping $u(t, \cdot) : U'_0 \to V$ at $z \in U'$ is the solution to the $\mathrm{Hom}(V', V)$-valued linear initial-value problem*

$$Y'(t) = A(t, z) \circ Y(t) + F(t, z), \ \ Y(0) = 0 \tag{F.4.2}$$

with $A(t, z) = (D_2\phi)(t, u_z(t), z) \in \mathrm{Hom}(V, V)$ and $F(t, z) = (D_3\phi)(t, u_z(t), z) \in \mathrm{Hom}(V', V)$ continuous in $(t, z) \in I'_0 \times U'_0$.

The continuity of A and F follows from the C^1 property of ϕ and the continuity of $u_z(t)$ in (t, z) (which is ensured by the partial results we have obtained so far toward Theorem J.3.1, especially Theorem F.3.6). Our proof of Theorem F.4.1 requires a lemma on the growth of "approximate solutions" to an ODE:

Lemma F.4.2. *Let $J \subseteq \mathbf{R}$ be a non-empty open interval and let $\phi : J \times U \to V$ be a C^1 mapping, with U a convex open set in a finite-dimensional vector space V. Fix a norm on V, and assume that for all $(t, v) \in J \times U$ the linear map $(D_2\phi)(t, v) \in \mathrm{Hom}(V, V)$ has operator norm satisfying $\|(D_2\phi)(t, v)\| \leq M$ for some $M > 0$.*

Pick $\varepsilon_1, \varepsilon_2 \geq 0$ and assume that $u_1, u_2 : J \rightrightarrows U$ are respectively ε_1-approximate and ε_2-approximate solutions to $y'(t) = \phi(t, y(t))$ in the sense that

$$\|u_1'(t) - \phi(t, u_1(t))\| \leq \varepsilon_1, \ \ \|u_2'(t) - \phi(t, u_2(t))\| \leq \varepsilon_2$$

for all $t \in J$. For any $t_0 \in J$,

$$\|u_1(t) - u_2(t)\| \leq \|u_1(t_0) - u_2(t_0)\| e^{M|t - t_0|} + (\varepsilon_1 + \varepsilon_2)(e^{M|t - t_0|} - 1)/M. \tag{F.4.3}$$

In the special case $\varepsilon_1 = \varepsilon_2 = 0$ and $u_1(t_0) = u_2(t_0)$, the upper bound is 0 and hence we recover the global uniqueness theorem for a given initial condition. Thus, this lemma is to be understand as an analogue of the general uniqueness theorem when we move the initial condition (allow $u_1(t_0) \neq u_2(t_0)$) and allow approximate solutions to the ODE.

Proof. Using a translation allows us to assume $t_0 = 0$, and by negating if necessary it suffices to treat the case $t \geq t_0 = 0$. Since $u_j(t) = u_j(0) + \int_0^t u_j'(x)\mathrm{d}x$, the ε_j-approximation condition gives

$$\|u_j(t) - u_j(0) - \int_0^t \phi(x, u_j(x))\mathrm{d}x\| = \| \int_0^t (u_j'(x) - \phi(x, u_j(x)))\mathrm{d}x\| \leq \int_0^t \varepsilon_j \mathrm{d}x = \varepsilon_j t.$$

Thus, using the triangle inequality we get

$$\|u_1(t) - u_2(t)\| \leq \|u_1(0) - u_2(0)\| + \int_0^t \|\phi(x, u_1(x)) - \phi(x, u_2(x))\|\mathrm{d}x + (\varepsilon_1 + \varepsilon_2)t.$$

Consider the C^1 restriction $g(z) = \phi(x, zu_1(x) + (1-z)u_2(z))$ of $\phi(x, \cdot)$ on the line segment in V joining the points $u_1(x), u_2(x) \in U$ (a segment lying entirely in U, since U is assumed to be convex). By the Fundamental Theorem of Calculus and the Chain Rule, $\phi(x, u_1(x)) - \phi(x, u_2(x))$ is equal to

$$g(1) - g(0) = \int_0^1 g'(z)\mathrm{d}z = \int_0^1 ((D_2\phi)(x, zu_1(x) + (1-z)u_2(z)))(u_1(x) - u_2(x))\mathrm{d}z.$$

Thus, the assumed bound of M on the operator norm of $(D_2\phi)(t, v)$ for all $(t, v) \in J \times U$ gives

$$\|\phi(x, u_1(x)) - \phi(x, u_2(x))\| \leq M \cdot \int_0^1 \|u_1(x) - u_2(x)\|\mathrm{d}z = M\|u_1(x) - u_2(x)\|$$

for all $x \in J$. Hence, for $h(t) = \|u_1(t) - u_2(t)\|$ we have

$$h(t) \leq h(0) + (\varepsilon_1 + \varepsilon_2)t + \int_0^t Mh(x)\mathrm{d}x$$

for all $t \in J$ satisfying $t \geq 0$. By Lemma E.3.4, we thereby conclude that for all such t there is the bound

$$h(t) \leq h(0) + (\varepsilon_1 + \varepsilon_2)t + \int_0^t (h(0) + (\varepsilon_1 + \varepsilon_2)x)Me^{M(t-x)}\mathrm{d}x,$$

and by direct calculation this upper bound is exactly the one given in (F.4.3). □

Now we prove Theorem F.4.1:

Proof. Fix norms on V and V'. Since A and F are continuous, by Theorem F.2.2 there is a *continuous* mapping $y : I \times U' \to \mathrm{Hom}(V', V)$ such that $y(\cdot, z) : I \to \mathrm{Hom}(V', V)$ is the solution to (F.4.2) for all $z \in U'$. We need to prove (among other things) that if $z \in U'$ is near z_0 and $t \in I$ is near $\overline{I}_0$ then $y(t, z) \in \mathrm{Hom}(V', V)$ serves as a total U'-derivative for $u : I \times U' \to V$ at (t, z). This rests on getting estimates for the norm of $u(t, z+h) - u(t, z) - (y(t, z))(h)$ for $h \in V'$ near 0, at least with (t, z) near $\overline{I}_0 \times \{z_0\}$. Our estimate on this difference will be obtained via an application of Lemma F.4.2. We first require some preliminary considerations to find the right I_0' and U_0' in which t and z should respectively live. The continuity of y on $I \times U'$ will not be used until near the end of the proof.

Since I_0 has compact closure in I, by shrinking I around $\overline{I}_0$, U around v_0, and U' around z_0 we may arrange that the operators norms of $(D_2\phi)(t, v, z) \in \mathrm{Hom}(V, V)$ and $(D_3\phi)(t, v, z) \in \mathrm{Hom}(V', V)$ are bounded above by some positive constants M and N for all $(t, v, z) \in I \times U \times U'$. We may also assume that U and U' are open balls centered at

v_0 and z_0, so each is convex. For any points $(v_1, z_1), (v_2, z_2) \in U \times U'$ and $t \in I$, if we let $h(x) = \phi(t, xv_1 + (1-x)v_2, z_1)$ and $g(x) = \phi(t, v_2, xz_1 + (1-x)z_2)$ then

$$\phi(t, v_1, z_1) - \phi(t, v_2, z_2) = (h(1) - h(0)) + (g(1) - g(0)) = \int_0^1 h'(x)\mathrm{d}x + \int_0^1 g'(x)\mathrm{d}x$$

with

$$\begin{aligned} h'(x) &= ((D_2\phi)(t, xv_1 + (1-x)v_2, z))(v_1 - v_2), \\ g'(x) &= ((D_2\phi)(t, v_2, xz_1 + (1-x)z_2))(z_1 - z_2) \end{aligned}$$

by the Chain Rule. Hence, the operator-norm bounds give

$$\|\phi(t, v_1, z_1) - \phi(t, v_2, z_2)\| \leq M\|v_1 - v_2\| + N\|z_1 - z_2\|.$$

Setting $v_1 = v_2 = u_{z_1}(t)$ and using the equation $u'_{z_1}(t) = f(t, u_{z_1}(t), z_1)$ we get

$$\|u'_{z_1}(t) - f(t, u_{z_1}(t), z_2)\| \leq N\|z_1 - z_2\|$$

for all $t \in I$.

For $c = N\|z_1 - z_2\|$ we have shown that $u_{z_1} : I \to U$ is a c-approximate solution to the ODE $f'(t) = \phi(t, f(t), z_2)$ on I and its value at $t_0 = 0$ coincides with that of the 0-approximate (i.e., exact) solution u_{z_2} to the same ODE on I. Shrink I around $\overline{I}_0$ so that it has finite length, say bounded above by R. Hence, by Lemma F.4.2 (U is convex!), for all $t \in I$ we have

$$\|u_{z_1}(t) - u_{z_2}(t)\| \leq c \cdot (e^{M|t|} - 1)/M \leq Q\|z_1 - z_2\|$$

with $Q = N(e^{MR} - 1)/M$.

Choose $\varepsilon > 0$. By working near the compact set $\overline{I}_0 \times \{(v_0, z_0)\}$, for h sufficiently near 0 the difference $u(t, z+h) - u(t, z)$ is as uniformly small as we please for all (t, z) near $\overline{I}_0 \times \{z_0\}$ because u is *continuous* on $I \times U \times U'$ (and hence uniformly continuous around compacts). Hence, by taking h sufficiently small (depending on ε!) we may form a first-order Taylor approximation to

$$\phi(t, u(t, z+h), z+h) = \phi(t, u(t, z) + (u(t, z+h) - u(t, z)), z+h)$$

with error bounded in norm by $\varepsilon\|h\|$ for h near enough to 0 such that $u(t, z+h) - u(t, z)$ is uniformly small for (t, z) near $\overline{I}_0 \times \{z_0\}$. That is, for a suitable open ball $U'_0 \subseteq U'$ around z_0 and an open interval $I'_0 \subseteq I$ around $\overline{I}_0$ we have that for h sufficiently near 0 there is an estimate

$$\begin{aligned} \|\phi(t, u(t, z+h), z+h) - \phi(t, u(t, z), z) - \\ (A(t, z))(u(t, z+h) - u(t, z)) - (F(t, z))(h)\| &\leq \varepsilon\|h\| \end{aligned}$$

for all $(t, v) \in I'_0 \times U'_0$. Here, we are of course using the *definitions* of A and F in terms of partials of ϕ. In view of the ODE's satisfied by u_z and u_{z+h}, we therefore get

$$\|u'_{z+h}(t) - u'_z(t) - (A(t, z))(u_{z+h}(t) - u_z(t)) - (F(t, z))(h)\| \leq \varepsilon\|h\| \tag{F.4.4}$$

for all $(t, v) \in I'_0 \times U'_0$ and h sufficiently near 0 (independent of (t, z)).

For $h \in V'$ near 0 and $(t, z) \in I'_0 \times U'_0$, let

$$\delta(t, z, h) = u(t, z+h) - u(t, z) - (y(t, z))(h)$$

where $y_z = y(\cdot, z)$ is the solution to (F.4.2) on I. Using the ODE satisfied by y_z we get

$$\begin{aligned}(\partial_t\delta)(t,z,h) &= u'_{z+h}(t) - u'_z(t) - (y'_z(t))(h)\\ &= u'_{z+h}(t) - u'_z(t) - (A(t,z))((y_z(t))(h)) - (F(t,z))(h).\end{aligned}$$

Hence, (F.4.4) says

$$\|(\partial_t\delta)(t,z,h) - (A(t,z))(\delta(t,z,h))\| \leq \varepsilon\|h\|$$

for all $(t,z) \in I'_0 \times U'_0$ and h sufficiently near 0 (where "sufficiently near" is independent of (t,z)). This says that for $z \in U'_0$ and h sufficiently near 0, $\delta(\cdot, z, h)$ is an $\varepsilon\|h\|$-approximate solution to the V-valued ODE

$$X'(t) = (A(t,z))(X(t))$$

on I'_0 with initial value $\delta(0,z,h) = u_{z+w}(0) - u_z(0) - (y_z(0))(h) = v_0 - v_0 - 0 = 0$ at $t = 0$. The exact solution with this initial value is $X = 0$, and so Lemma F.4.2 gives

$$\|\delta(t,z,h)\| \leq q\varepsilon\|h\|$$

for all $(t,z) \in I'_0 \times U'_0$ and sufficiently small h, with $q = (e^{MR} - 1)/M$ for an upper bound R on the length of I. The "sufficient smallness" of h depends on ε, but neither q nor $I'_0 \times U'_0$ have dependence on ε. Thus, we have proved that for $(t,z) \in I'_0 \times U'_0$

$$u(t, z+h) - u(t,z) - (y(t,z))(h) = \delta(t,z,h) = o(\|h\|)$$

in V as $h \to 0$ in V'. Hence, $(D_2u)(t,z)$ exists for all $(t,z) \in I'_0 \times U'_0$ and it is equal to $y(t,z) \in \mathrm{Hom}(V', V)$. But y depends *continuously* on (t,z), so $D_2u : I'_0 \times U'_0 \to \mathrm{Hom}(V', V)$ is continuous. Meanwhile, the ODE for u_z gives

$$(D_1u)(t,z) = u'_z(t) = \phi(t, u(t,z), z)$$

in $\mathrm{Hom}(\mathbf{R}, V) = V$, so by continuity of ϕ and of u in (t,z) it follows that $D_1u : I'_0 \times U'_0 \to V$ exists and is continuous.

We have shown that at each point of $I'_0 \times U'_0$ the mapping $u : I'_0 \times U'_0 \to V$ admits partials in the I'_0 and U'_0 directions with D_1u and D_2u both continuous on $I'_0 \times U'_0$. Thus, u is C^1. The preceding argument also yields that $(D_2u)(\cdot, z)$ is the solution to (F.4.2) on I'_0 for all $z \in U'_0$. $\square$

It has now been proved that, in the setup of Theorem J.3.1, on the open domain of flow $\mathscr{D}(\phi)$ the universal solution u is always C^1. We shall use induction on p and the description of D_2u in Theorem F.4.1 to prove that u is C^p when ϕ is C^p, with $1 \leq p \leq \infty$.

Remark F.4.3. The inductive hypothesis will be applied to the ODE's (F.4.2) that are *time-dependent* and depend on parameters even if the initial ODE for the u_z's is time-independent. It is exactly for this aspect of induction that we have to permit time-dependent flow: without incorporating time-dependent flow into the inductive hypothesis, the argument would run into problems when we try to apply the inductive hypothesis to (F.4.2). (Strictly speaking, we could have kept time-dependence out of the inductive hypothesis by making repeated use of the reduction steps of the sort that preceded Theorem F.3.6; however, it seems simplest to cut down on the use of such reduction steps when they're not needed.)

Since the domain of flow $\mathscr{D}(\phi)$ for a C^p mapping ϕ with $1 \leq p \leq \infty$ is "independent of p" (in the sense that it remains the domain of flow even if ϕ is viewed as being C^r with $1 \leq r < p$, as we must do in inductive arguments), it suffices to treat the case of *finite* $p \geq 1$. Thus, we now fix $p > 1$ and assume that the problem has been solved in the C^{p-1} case in general. As we have already seen, it suffices to treat the C^p case in the same setup considered in Theorem F.4.1, which is to say time-dependent flow with an auxiliary parameter but fixed initial conditions. Moreover, since the domain $\mathscr{D}(\phi)$ is an *open* set in $\mathbf{R} \times U'$, the C^p problem near any particular point $(t_0, z_0) \in \mathscr{D}(\phi)$ is local around the compact product $I_{t_0} \times \{z_0\}$ in $\mathbf{R} \times U'$ where I_{t_0} is the compact interval in $\mathbf{R}$ with endpoints 0 and t_0. In particular, it suffices to prove:

Corollary F.4.4. *Keep notation as in Theorem F.4.1, and assume ϕ is C^p with $1 \leq p < \infty$. For sufficiently small open $U'_0 \subseteq U_0$ around z_0 and an open subinterval $I'_0 \subseteq I$ around $\overline{I}_0$, $(t,z) \mapsto u_z(t)$ is C^p as a mapping from $I'_0 \times U'_0$ to V.*

As we will see in the proof, each time we use induction on p we will have to *shrink* U'_0 and I'_0 further. Hence, the method of proof does not directly give a result for $p = \infty$ across a neighborhood of $\overline{I}_0 \times \{z_0\}$ in $I \times U'$ because a shrinking family of opens (in $I \times U'$) around $\overline{I}_0 \times \{z_0\}$ need not have its intersection contain an open (in $I \times U'$) around $\overline{I}_0 \times \{z_0\}$. The reason we get a result in the C^∞ case is because we did the hard work to prove that the global domain of flow $\mathscr{D}(\phi)$ has good topological structure (i.e., it is an open set in $\mathbf{R} \times U'$); in the discussion preceding the corollary we saw how this openness enabled us to reduce the C^∞ case to the C^p case for *finite* $p \geq 1$. If we had not introduced the concept of domain of flow that is "independent of p" and proved its openness *a priori*, then we would run into a brick wall in the C^∞ case (the case we need in differential geometry!).

Proof. We proceed by induction, the case $p = 1$ being Theorem F.4.1. Thus, we may and do assume $p > 1$. We emphasize (for purposes of the inductive step later) that our induction is really to be understood to be simultaneously applied to *all* time-dependent flows with an auxiliary parameter and a fixed initial condition.

By the inductive hypothesis, we can find open U'_0 around z_0 in U' and an open interval $I'_0 \subseteq I$ around $\overline{I}_0$ so that $u : (t,z) \mapsto u(t,z)$ is C^{p-1} on $I'_0 \times U'_0$. Since u is C^{p-1} with $p - 1 \geq 1$, to prove that it is C^p on $I''_0 \times U''_0$ for some open $U''_0 \subseteq U'_0$ around z_0 and some open subinterval $I''_0 \subseteq I'_0$ around $\overline{I}_0$ it is equivalent to check that (as a V-valued mapping) for suitable such I''_0 and U''_0 the partials of u along the directions of I_0 and U' (via a basis of V', say) are all C^{p-1} at each point $(t,z) \in I''_0 \times U''_0$. By construction, $(D_1 u)(t,z) \in \mathrm{Hom}(\mathbf{R}, V) \simeq V$ is $\phi(t, u(t,z), z)$, and this has C^{p-1}-dependence on (t,z) because ϕ is C^p on $I \times U \times U'$ and $u : I'_0 \times U'_0 \to U$ is C^{p-1}.

To show that $(D_2 u)(t,z) \in \mathrm{Hom}(V', V)$ has C^{p-1}-dependence on $(t,z) \in I''_0 \times U''_0$ for suitable I''_0 and U''_0, first recall from Theorem F.4.1 (viewing ϕ as a C^1 mapping) that on $I'_0 \times U'_0$ the map $(t,z) \mapsto (D_2 u)(t,z)$ is the solution to the $\mathrm{Hom}(V', V)$-valued initial-value problem

$$Y'(t) = A(t,z) \circ Y(t) + F(t,z), \ \ Y(0) = 0 \tag{F.4.5}$$

with

$$A(t,z) = (D_2\phi)(t, u_z(t), z) \in \mathrm{Hom}(V,V), \ \ F(t,z) = (D_3\phi)(t, u_z(t), z) \in \mathrm{Hom}(V', V)$$

depending continuously on $(t,z) \in I'_0 \times U'_0$. Since $u_z(t)$ has C^{p-1}-dependence on $(t,z) \in I'_0 \times U'_0$ and ϕ is C^p, both A and F have C^{p-1}-dependence on $(t,z) \in I_0 \times U'_0$. But $p - 1 \geq 1$ and the compact $\overline{I}_0$ is contained in I'_0, so we may invoke the inductive hypothesis on $I'_0 \times U'_0$

for the *time-dependent* flow (F.4.5) with a *varying parameter* but a *fixed* initial condition. More precisely, we have a "universal solution" D_2u to this latter family of ODE's across $I_0' \times U_0'$ and so by induction there exists an open $U_0'' \subseteq U_0'$ around z_0 and an open subinterval $I_0'' \subseteq I_0'$ around $\overline{I}_0$ such that the restriction to $I_0'' \times U_0''$ of the family of solutions $(D_2u)(\cdot, z)$ to (F.4.5) for $z \in U_0''$ has C^{p-1}-dependence on $(t, z) \in I_0'' \times U_0''$.

We have proved that for the C^1 map $u : I_0'' \times U_0'' \to U$ the maps $D_1u : I_0'' \times U_0'' \to V$ and $D_2u : I_0'' \times U_0'' \to \mathrm{Hom}(V', V)$ are C^{p-1}. Hence, u is C^p. □

F.5. **Smooth flow on manifolds.** Up to now we have proved some rather general results on the structure of solutions to ODE's (in both Appendix E and in §F.2–§F.4 above). We now intend to use these results to study integral curves for smooth vector fields on smooth manifolds. The diligent reader will see that (with some modifications to statements of results) in what follows we can relax smoothness to C^p with $2 \le p < \infty$, but such cases with finite p lead to extra complications (due to the fact that vector fields cannot be better than class C^{p-1}). Thus, we shall now restrict our development to the smooth case – all of the real ideas are seen here anyway, and it is by far the most important case in geometric applications. Let M be a smooth manifold, and let $\vec{v}$ be a smooth vector field on M. The first main theorem is the existence and uniqueness of a maximal integral curve to $\vec{v}$ through a specified point at time 0.

The following theorem is a manifold analogue of the existence and uniqueness theorem on maximal intervals around the initial time in the "classical" theory of ODE's (see §E.2). The extra novelty is that in the manifold setting we cannot expect the integral curve to lie in a single coordinate chart and so to prove the existence/uniqueness theorem (for the maximal integral curve) on manifolds we need to artfully reduce the problem to one in a single chart where we can exploit the established theory in open subsets of vector spaces.

Theorem F.5.1. *Let $m_0 \in M$ be a point. There exists a unique maximal integral curve for $\vec{v}$ through m_0. That is, there exists an open interval $J_{m_0} \subseteq \mathbf{R}$ around 0 and a smooth mapping $c_{m_0} : J_{m_0} \to M$ satisfying*

$$c_{m_0}'(t) = \vec{v}(c_{m_0}(t)), \;\; c_{m_0}(0) = m_0$$

such that if $I \subseteq \mathbf{R}$ is any open interval around 0 and $c : I \to M$ is an integral curve for $\vec{v}$ with $c(0) = m_0$ then $I \subseteq J_{m_0}$ and $c_{m_0}|_I = c$.

Moreover, the map $c_{m_0} : J_{m_0} \to M$ is an immersion except if $\vec{v}(m_0) = 0$, in which case it is the constant map $c_{m_0}(t) = m_0$ for all $t \in J_{m_0} = \mathbf{R}$.

Beware that in general J_{m_0} may not equal $\mathbf{R}$. (It could be bounded, or perhaps bounded on one side.) For the special case $M = \mathbf{R}^n$, this just reflects the fact (as in Example F.1.1) that solutions to non-linear initial-value problems $u'(t) = \phi(u(t))$ can fail to propagate for all time. Also, Example F.5.4 below shows that c_{m_0} may fail to be injective. The immersion condition says that the image $c_{m_0}(J_{m_0})$ does not have "corners". In Example F.5.7 we show that $c_{m_0}(J_{m_0})$ cannot "cross itself". However, c_{m_0} can fail to be an embedding: this occurs for certain integral curves on a doughnut (for which the image is a densely-wrapped line).

Proof. We first construct a maximal integral curve, and then address the immersion aspect. Upon choosing local C^∞ coordinates around m_0, the problem of the existence of an integral curve for $\vec{v}$ through m_0 on a small open time interval around 0 is "the same" as the problem of solving (for small $|t|$) an ODE of the form

$$u'(t) = \phi(u(t)), \;\; u(0) = v_0$$

for $\phi : U \to V$ a C^∞ mapping on an open set U in a finite-dimensional vector space V (with $v_0 \in U$). Hence, the classical local existence/uniqueness theorem for ODE's (Theorem E.2.1) ensures that for some $\varepsilon > 0$ there is an integral curve $c : (-\varepsilon, \varepsilon) \to M$ to $\vec{v}$ through m_0 (at time $t = 0$) and that any two such integral curves to $\vec{v}$ through m_0 (at time $t = 0$) coincide for t near 0.

For the existence of the maximal integral curve that recovers all others, all we have to show is that if $I_1, I_2 \subseteq \mathbf{R}$ are open intervals around 0 and $c_j : I_j \to M$ are integral curves for $\vec{v}$ through m_0 at time 0 then $c_1|_{I_1 \cap I_2} = c_2|_{I_1 \cap I_2}$. (Indeed, once this is proved then we can "glue" c_1 and c_2 to get an integral curve on the open interval $I_1 \cup I_2$, and more generally we can "glue" *all* such integral curves; on the union of their open interval domains we obviously get the unique maximal integral curve of the desired sort.) By replacing c_1 and c_2 with their restrictions to $I_1 \cap I_2$, we may rephrase the problem as a uniqueness problem: $I \subseteq \mathbf{R}$ is an open interval around 0 and $c_1, c_2 : I \rightrightarrows M$ are both solutions to the same "initial-value problem"

$$c'(t) = \vec{v}(c(t)), \;\; c(0) = m_0$$

with values in the manifold M. We wish to prove $c_1 = c_2$ on I. As we saw at the beginning of the present proof, by working in a local coordinate system near m_0 we may use the classical local uniqueness theorem to infer that $c_1(t) = c_2(t)$ for $|t|$ near 0.

To get equality on all of I we will treat the case $t > 0$ (the case $t < 0$ goes similarly). If $c_1(t) \neq c_2(t)$ for some $t > 0$ then the set $S \subseteq I$ of such t has an infimum $t_0 \in I$. Since c_1 and c_2 agree near the origin, necessarily $t_0 > 0$. Thus, c_1 and c_2 coincide on $[0, t_0)$, whence they agree on $[0, t_0]$. Let $x_0 \in M$ be the common point $c_1(t_0) = c_2(t_0)$. We can view c_1 and c_2 as integral curves for $\vec{v}$ through x_0 at time t_0. The local uniqueness for integral curves through a specified point at a specified time (the time $t = 0$ is obviously not sacred; $t = t_0$ works the same) implies that c_1 and c_2 must coincide for t near t_0. Hence, we get an ε-interval around t_0 in I on which c_1 and c_2 agree, whence they agree on $[0, t_0 + \varepsilon)$. Thus, t_0 cannot be the infimum of S after all. This contradiction completes the construction of maximal integral curves.

If $\vec{v}(m_0) = 0$ then the constant map $c(t) = m_0$ for all $t \in \mathbf{R}$ satisfies the conditions that uniquely characterize an integral curve for $\vec{v}$ through m_0. Thus, it remains to prove that if $\vec{v}(m_0) \neq 0$ then c is an immersion. By definition, $c'(t_0) = \mathrm{d}c(t_0)(\partial_t|_{t_0})$ for any $t_0 \in I$, with $\partial_t|_{t_0} \in \mathrm{T}_{t_0}(I)$ a basis vector. Thus, by the immersion theorem, c is an immersion around t_0 if and only if the tangent map $\mathrm{d}c(t_0) : \mathrm{T}_{t_0}(I) \to \mathrm{T}_{c(t_0)}(M)$ is injective, which is to say that the velocity $c'(t_0)$ is nonzero. In other words, we want to prove that if $\vec{v}(m_0) \neq 0$ then $c'(t_0) \neq 0$ for all $t_0 \in I$.

Assuming $c'(t_0) = 0$ for some $t_0 \in I$, in local coordinates near $c(t_0)$ the "integral curve" condition expresses c near t_0 as a solution to an initial-value problem of the form

$$u'(t) = \phi(u(t)), \;\; u(t_0) = v_0$$

with $c'(t_0) = 0$. Since $c'(t_0) = \vec{v}(c(t_0))$, in the initial-value problem we get the extra property $\phi(v_0) = \phi(u(t_0)) = 0$. The constant mapping $\xi : t \mapsto v_0$ for all $t \in \mathbf{R}$ therefore satisfies the initial-value problem (as $\phi(\xi(t)) = \phi(v_0) = 0$ and $\xi'(t) = 0$ for all t). Hence, by uniqueness it follows that c is constant for t near t_0, and so in particular c has vanishing velocity vectors for t near t_0. Since t_0 was an arbitrary point at which c has velocity zero, this shows that the subset $Z \subseteq I$ of t such that $c'(t) = 0$ is an *open* subset of I. However, by the local nature of closedness we may work on open parts of I carried by c into coordinate domains to see that Z is also a closed subset of I, and so since (by hypothesis) Z is non-empty we conclude from connectivity of I that $Z = I$. In particular, $0 \in Z$. This contradicts the assumption that $c'(0) = \vec{v}(m_0)$ is nonzero. Hence, c has to be an immersion when $\vec{v}(m_0) \neq 0$. $\square$

Remark F.5.2. From a geometric point of view, it is unnatural to specify a "base point" on integral curves. Dropping reference to a specified "base point" at time 0, we can redefine the concept of integral curve for $\vec{v}$: a smooth map $c : I \to M$ on a non-empty open interval $I \subseteq \mathbf{R}$ (possibly not containing 0) such that $c'(t) = \vec{v}(c(t))$ for all $t \in I$. We have simply omitted the requirement that a particular number (such as 0) lies in I and that c has a specific image at that time. It makes sense to speak of maximal integral curves $c : I \to M$ for $\vec{v}$, namely integral curves that cannot be extended as such on a strictly larger open interval in $\mathbf{R}$. It is obvious (via Theorem F.5.1) that any integral curve in this new sense uniquely extends to a maximal integral curve, and the only novelty is that the analogue of the uniqueness aspect of Theorem F.5.1 requires a mild reformulation: if two maximal integral curves $c_1 : I_1 \to M$ and $c_2 : I_2 \to M$ for $\vec{v}$ have images that meet at a point, then there exists a unique $t_0 \in \mathbf{R}$ (usually nonzero) such that two conditions hold: $t_0 + I_1 = I_2$ (this determines t_0 if $I_1, I_2 \neq \mathbf{R}$) and $c_2(t_0 + t) = c_1(t)$ for all $t \in I_1$. This verification is left as a simple exercise via Theorem F.5.1. (Hint: If $c_1(t_1) = c_2(t_2)$ for some $t_j \in I_j$, consider $t \mapsto c_1(t + t_1)$ and $t \mapsto c_2(t + t_2)$ on the open intervals $-t_1 + I_1$ and $-t_2 + I_2$ around 0.) In this new sense of integral curve, with no fixed base point, we consider the "interval of definition" in $\mathbf{R}$ to be well-defined up to additive translation. (That is, we tend to "identify" two integral curves that are related through additive translation in time.) Note that it is absolutely essential throughout the discussion that we are specifying velocities (via $\vec{v}$), as otherwise we cannot expect the subset $c(I) \subseteq M$ to determine its "time parameterization" uniquely up to additive translation in time.

Example F.5.3. Consider the "inward" unit radial vector field

$$\vec{v} = -\partial_r = -\frac{x}{\sqrt{x^2 + y^2}}\partial_x - \frac{y}{\sqrt{x^2 + y^2}}\partial_y$$

on $M = \mathbf{R}^2 - \{(0, 0)\}$. The integral curves are straight-line trajectories toward the origin at unit speed. Explicitly, an integral curve $c(t) = (c_1(t), c_2(t))$ satisfies an initial condition $c(0) = (x_0, y_0) \in M$ and an evolution equation

$$c'(t) = -\partial_r|_{c(t)} = -\frac{c_1(t)}{\sqrt{c_1(t)^2 + c_2(t)^2}}\partial_x|_{c(t)} - \frac{c_2(t)}{\sqrt{c_1(t)^2 + c_2(t)^2}}\partial_y|_{c(t)},$$

so since (by the Chain Rule) for any t_0 we must have

$$c'(t_0) := \mathrm{d}c(t_0)(\partial_t|_{t_0}) = c_1'(t_0)\partial_x|_{c(t_0)} + c_2'(t_0)\partial_y|_{c(t_0)}$$

the differential equation says Hom

$$c_1' = -\frac{c_1}{\sqrt{c_1^2 + c_2^2}},\ \ c_2' = -\frac{c_2}{\sqrt{c_1^2 + c_2^2}},\ \ (c_1(0), c_2(0)) = (x_0, y_0).$$

These differential equations become a lot more transparent in terms of local polar coordinates (r, θ), with θ ranging through less than a full "rotation": $r'(t) = -1$ and $\theta'(t) = 0$. (Strictly speaking, whenever one computes an integral curve in local coordinates one must never forget the possibility that the integral curve might "escape the coordinate domain" in finite time, and so if the flow ceases at some time with the path approaching the boundary then the flow may well propagate within the manifold beyond the time for which it persists in the chosen coordinate chart. In the present case we get "lucky": the flow stops for global reasons unrelated to the chosen coordinate domain in which we compute.) It follows that in the coordinate domain the path must have $r(t) = r_0 - t$ with $r_0 = \sqrt{x_0^2 + y_0^2}$ and θ is constant (on the coordinate domain under consideration). In other words, the path is a half-line with motion towards the origin with a linear parameterization in time. Explicitly, if we let

$$(u_0, u_1) = (x_0/\sqrt{x_0^2 + y_0^2}, y_0/\sqrt{x_0^2 + y_0^2})$$

be the "unit vector" pointing in the same direction as $(x_0, y_0) \neq (0, 0)$ (i.e., the unique scaling of (x_0, y_0) by a positive number to have length 1) then in the chosen sector for polar coordinates we have

$$c_{(x_0,y_0)}(t) = (x_0 - u_0 t, y_0 - u_1 t) = (1 - t/(x_0^2 + y_0^2)^{1/2})x_0, (1 - t/(x_0^2 + y_0^2)^{1/2})y_0)$$

on the interval $J_{(x_0,y_0)} = (-\infty, r(x_0, y_0)) = (-\infty, \sqrt{x_0^2 + y_0^2})$ is the maximal integral curve through (x_0, y_0) at time $t = 0$. The failure of the solution to persist in the manifold to time $\sqrt{x_0^2 + y_0^2}$ is obviously not due to working in a coordinate sector, but rather because the flow viewed in $\mathbf{R}^2$ (containing M as an open subset) is approaching the point $(0, 0)$ not in the manifold (and so there cannot even be a continuous extension of the flow to time $\sqrt{x_0^2 + y_0^2}$ in the manifold M). Hence, the global integral curve really does cease to exist in M at this time. Of course, from the viewpoint of a person whose universe is M (and so cannot "see" the point $(0, 0) \in \mathbf{R}^2$), as they flow along this integral curve they will have a hard time understanding why spaceships moving along this curve encounter difficulties at this time.

As predicted by Theorem F.5.1 and Remark F.5.2, since the vector field $\vec{v}$ is everywhere non-vanishing there are no constant integral curves and all of them are immersions, with any two having images that are either disjoint or equal in M, and for those that are equal we see that varying the position at time $t = 0$ only has the effect of changing the parameterization mapping by an additive translation in time.

If we consider a vector field $\vec{v} = h(r)\partial_r$ for a non-vanishing smooth function h on $(0, \infty)$, then we naturally expect the integral curves to again be given by these rays, except that the direction of motion will depend on the constant sign of h (positive or negative) and the speed along the ray will depend on h. Indeed, by the same method as above it is clear that the integral curve $t \mapsto c_{(x_0,y_0)}(t)$ passing through (x_0, y_0) at time $t = 0$ is $c_{(x_0,y_0)}(t) = (x_0 + u_0 H(t), y_0 + u_1 H(t))$ where (u_0, u_1) is the unit-vector obtained through positive scaling of (x_0, y_0) and $H(t) = \int_{r_0}^{t+r_0} h$ for $r_0 = r(x_0, y_0) = \sqrt{x_0^2 + y_0^2}$.

Example F.5.4. Let $M = \mathbf{R}^2$ and consider the "circular" (non-unit!) vector field
$$\vec{v} = \partial_\theta = -y\partial_x + x\partial_y$$
that is smooth on the entire plane (including the origin). The integral curve through the origin is the constant map to the origin, and the integral curve through any $(x_0, y_0) \neq (0,0)$ at time 0 is the circular path
$$c_{(x_0,y_0)}(t) = (r_0 \cos(t+\theta_0), r_0 \sin(t+\theta_0)) = (x_0 \cos t - y_0 \sin t, x_0 \sin t + y_0 \cos t)$$
for $t \in \mathbf{R}$ with constant speed of motion $r_0 = r(x_0, y_0) = \sqrt{x_0^2 + y_0^2}$ (θ_0 is the "angle" parameter, only well-defined up to adding an integral multiple of 2π).

In Example F.5.3 the obstruction to maximal integral curves being defined for all time is related to the hole at the origin. Quite pleasantly, on compact manifolds such a difficulty never occurs:

Theorem F.5.5. *If M is compact, then maximal integral curves have interval of definition* $\mathbf{R}$.

Proof. We may assume M has constant dimension n. For each $m \in M$ we may choose a local coordinate chart $(\{x_1, \dots, x_n\}, U_m)$ with parameterization by an open set $B_m \subseteq \mathbf{R}^n$ (i.e., B_m is the open image of U_m under the coordinate system). In such coordinates, the ODE for an integral curve takes the form $u'(t) = \phi(u(t))$ for a smooth mapping $\phi : B_m \to \mathbf{R}^n$. Give $\mathbf{R}^n$ its standard norm. By shrinking the coordinate domain U_m around m, we can arrange that $\phi(B_m)$ is bounded, say contained in a ball of radius R_m around the origin in $\mathbf{R}^n$, and that the total derivative for ϕ at each point of B_m has operator-norm bounded by some constant $L_m > 0$. Finally, choose $r_m \in (0,1)$ and an open U'_m around m in U_m so that B_m contains the set of points in $\mathbf{R}^n$ with distance at most $2r_m$ from the image of U'_m in B_m. Let $a_m = \min(1/2L_m, r_m/R_m) > 0$. By the *proof* of the local existence theorem for ODE's (Theorem E.2.1), it follows that the equation $c'(t) = \vec{v}(c(t))$ with an initial condition $c(t_0) \in U'_m$ (for $c : I \to M$ on an unspecified open interval I around $0 \in \mathbf{R}$) can always be solved on $(t_0 - a_m, t_0 + a_m)$ for c with values in U_m. That is, if an integral curve has a point in U'_m at a time t_0 then it persists in U_m for a_m units of time in *both* directions.

The opens $\{U'_m\}_{m \in M}$ cover M, so by compactness of M there is a finite subcover $U'_{m_1}, \dots, U'_{m_N}$. Let $a = \min(a_{m_1}, \dots, a_{m_N}) > 0$. Let $c : I \to M$ be a *maximal* integral curve for $\vec{v}$. For any $t \in I$ we have $c(t) \in U'_{m_i}$ for some i, and so by the preceding argument (and maximality of c!)
$$(t - a_{m_i}, t + a_{m_i}) \subseteq I$$
with c having image inside of U_{m_i} on this interval. Hence, $(t - a, t + a) \subseteq I$. Since a is a positive constant independent of $t \in I$, this shows that for *all* $t \in I$ the interval $(t - a, t + a)$ is contained in I. Obviously (argue with supremums and infimums, or use the Archimedean property of $\mathbf{R}$) the only non-empty open interval (or even subset!) in $\mathbf{R}$ with such a property is $\mathbf{R}$. $\square$

Definition F.6. A smooth vector field $\vec{v}$ on a smooth manifold M is *complete* if all of its maximal integral curves are defined on $\mathbf{R}$.

Theorem F.5.5 says that on a compact C^∞ manifold all smooth vector fields are complete. Example F.5.3 shows that some non-compact C^∞ manifolds can have non-complete smooth vector fields. In Riemannian geometry, the notion of completeness for (certain) vector fields is closely related to the notion of completeness in the sense of metric spaces (hence the terminology!).

Example F.5.7. We now work out the interesting geometry when a maximal integral curve $c : I \to M$ is not injective. That is, $c(t_1) = c(t_2)$ for some distinct t_1 and t_2 in I. As one might guess, the picture will be that of a circle wound around infinitely many times (forward and backwards in time). Let us now prove that this is exactly what must happen.

Denote the point $c(t_1) = c(t_2)$ by x, so on $(t_2 - t_1) + I$ the map $\widetilde{c} : t \mapsto c(t + t_1 - t_2)$ is readily checked to be an integral curve whose value at t_2 is $c(t_1) = c(t_2)$. We can likewise run the process in reverse to recover c on I from $\widetilde{c}$, so the integral curve $\widetilde{c}$ is also maximal. The maximal integral curves c and $\widetilde{c}$ agree at t_2, so they must coincide: same interval domain and same map. In particular, $(t_2 - t_1) + I = I$ in $\mathbf{R}$ and $c = \widetilde{c}$ on this open interval. Since $t_2 - t_1 \neq 0$, the invariance of I under additive translation by $t_2 - t_1$ forces $I = \mathbf{R}$. We conclude that c is defined on $\mathbf{R}$ and (since $c = \widetilde{c}$) the map c is periodic with respect to additive time translation by $\tau = t_2 - t_1$. To keep matters interesting we assume c is not a constant map, and so (by Theorem F.5.1) c is an immersion. In particular, for any $t \in I$ we have that c is injective on a neighborhood of t in I. Hence, there must be a *minimal* period $\tau > 0$ for the map c. (Indeed, if $t \in \mathbf{R}$ is a nonzero period for c then $c(t) = c(0)$ and hence t cannot get too close to zero. Since a limit of periods is a period, the infimum of the set of periods is both positive and a period, hence the least positive period.)

Any integral multiple of τ is clearly a period for c (by induction). Conversely, if τ' is any other period for c, it has to be an integral multiple of τ. Indeed, pick $n \in \mathbf{Z}$ so that $0 \leq \tau' - n\tau < \tau$ (visualize!), so we want $\tau' - n\tau$ to vanish. Any $\mathbf{Z}$-linear combination of periods for c is a period for c (why?), so $\tau' - n\tau$ is a non-negative period less than the least positive period. Hence, it vanishes. In view of the preceding proof that if $c(t_1) = c(t_2)$ with $t_1 \neq t_2$ then the difference $t_2 - t_1$ is a nonzero period, it follows that $c(t_1) = c(t_2)$ if and only if t_1 and t_2 have the same image in $\mathbf{R}/\mathbf{Z}\tau$. Since $c : \mathbf{R} \to M$ is a smooth map that is invariant under the additive translation by τ, it factors uniquely through the projection $\mathbf{R} \to \mathbf{R}/\mathbf{Z}\tau$ via a *smooth* mapping

$$\overline{c} : \mathbf{R}/\mathbf{Z}\tau \to M$$

that we have just seen is injective. The injective map $\overline{c}$ is an immersion because c is an immersion and $\mathbf{R} \to \mathbf{R}/\mathbf{Z}\tau$ is a local C^∞ isomorphism, and since $\mathbf{R}/\mathbf{Z}\tau$ is compact (it's a circle!) any injective continuous map from $\mathbf{R}/\mathbf{Z}\tau$ to a Hausdorff space is automatically a homeomorphism onto its (compact) image. In other words, $\overline{c}$ is an embedded smooth submanifold.

To summarize, we have proved that any maximal integral curve that "meets itself" in the sense that the trajectory eventually returns to the same point twice (i.e., $c(t_1) = c(t_2)$ for some $t_1 \neq t_2$) must have a very simple form: it is a smoothly embedded circle parameterized by modified notion of angle (as $\tau > 0$ might not equal 2π). This does *not* say that the velocity vectors along the curve look "constant" if M is given as a submanifold of some $\mathbf{R}^n$, but rather than the time parameter induces a C^∞-embedding $\mathbf{R}/\mathbf{Z}\tau \hookrightarrow M$ for the minimal positive period τ.

The reader will observe that we have not yet used any input from ODE beyond the existence/uniqueness results from Appendix E. That is, none of the work in §F.2–§F.4 has played any role in the present considerations. This shall now change: the geometry becomes very interesting when we allow m_0 to *vary*. This is the global analogue of varying the initial condition. Before we give the global results for manifolds, we make a definition:

Definition F.8. The *domain of flow* is the subset $\mathscr{D}(\vec{v}) \subseteq \mathbf{R} \times M$ consisting of pairs (t, m) such that the integral curve to $\vec{v}$ through m (at time 0) flows out to time t. That is, $(t, m) \in \mathscr{D}(\vec{v})$ if and only if the maximal integral curve $c_m : I_m \to M$ for $\vec{v}$ with $c_m(0) = m$ has t contained in its open interval of definition I_m. For each nonzero $t \in \mathbf{R}$, $\mathscr{D}(\vec{v})_t \subseteq M$ denotes the set of $m \in M$ such that $t \in I_m$.

Clearly $\mathbf{R} \times \{m\} \subseteq \mathbf{R} \times M$ meets $\mathscr{D}(\vec{v})$ in the domain of definition $I_m \subseteq \mathbf{R}$ for the maximal integral curve of $\vec{v}$ through m (at time 0). In particular, if M is connected then $\mathscr{D}(\vec{v})$ is connected (as in Lemma F.3.2).

Example F.5.9. Let us work out $\mathscr{D}(\vec{v})$ and $\mathscr{D}(\vec{v})_t$ for Example F.5.3. Let $M = \mathbf{R}^2 - \{(0,0)\}$. In this case, $\mathscr{D}(\vec{v}) \subseteq \mathbf{R} \times M$ is the subset of pairs $(t, (x, y))$ with $t < \sqrt{x^2 + y^2}$. This is obviously an open subset. For each $t \in \mathbf{R}$, $\mathscr{D}(\vec{v})_t \subseteq M$ is the subset of points $(x, y) \in M$ such that $\sqrt{x^2 + y^2} > t$; hence, it is equal to M precisely for $t \leq 0$ and it is a proper open subset of M otherwise (exhausting M as $t \to 0^+$).

Example F.5.10. For $t > 0$ we have $(-\varepsilon, t + \varepsilon) \times \{m\} \subseteq \mathscr{D}(\vec{v})$ for some $\varepsilon > 0$ if and only if $m \in \mathscr{D}(\vec{v})_t$, and similarly for $t < 0$ using $(t - \varepsilon, \varepsilon)$. Hence, if $t > 0$ then $\mathscr{D}(\vec{v})_t$ is the image under $\mathbf{R} \times M \to M$ of the union of the overlaps $\mathscr{D}(\vec{v}) \cap ((-\varepsilon, t + \varepsilon) \times M)$ over all $\varepsilon > 0$, and similarly for $t < 0$ using intervals $(t - \varepsilon, \varepsilon)$ with $\varepsilon > 0$.

The subsets $\mathscr{D}(\vec{v})_t \subseteq M$ grow as $t \to 0^+$, and the union of these loci is all of M: this just says that for each $m \in M$ there exists $\varepsilon_m > 0$ such that the maximal integral curve $c_m : J_m \to M$ has domain J_m that contains $(-\varepsilon_m, \varepsilon_m)$ (so $m \in \mathscr{D}(\vec{v})_t$ for $0 < |t| < \varepsilon_m$). Obviously $\mathscr{D}(\vec{v})_0 = M$.

Example F.5.11. If M is compact, then by Theorem F.5.5 the domain of flow $\mathscr{D}(\vec{v})$ is equal to $\mathbf{R} \times M$. Such equality is the notion of "completeness" for a smooth vector field on a smooth manifold, as in Definition F.6.

In the case of opens in vector spaces, the above notion of domain of flow recovers the notion of domain of flow (as in §F.3) for time-independent parameter-free vector fields with fixed initial time (at 0) but varying initial position (a point on M at time 0). One naturally expects an analogue of Theorem J.3.1; the proof is a mixture of methods and results from §F.3 (on opens in vector spaces):

Theorem F.5.12. *The domain of flow $\mathscr{D}(\vec{v})$ is open in $\mathbf{R} \times M$, and the locus $\mathscr{D}(\vec{v})_t \subseteq M$ is open for all $t \in \mathbf{R}$. Moreover, if we give $\mathscr{D}(\vec{v})$ its natural structure of open C^∞ submanifold of $\mathbf{R} \times M$ then the set-theoretic mapping*

$$X_{\vec{v}} : \mathscr{D}(\vec{v}) \to M$$

defined by $(t, m) \mapsto c_m(t)$ is a smooth mapping (here; $c_m : I_m \to M$ is the maximal integral curve for $\vec{v}$ through m at time 0).

The mapping $X_{\vec{v}}$ is "vector flow along integral curves of $\vec{v}$"; it is the manifold analogue of the universal solution to a family of ODE's over the domain of flow in the classical case in §F.3. The openness in the theorem has a very natural intepretation: the ability to flow the solution to a given time is unaffected by small perturbations in the initial position. The smoothness of the mapping $X_{\vec{v}}$ is a manifold analogue of the C^∞-dependence on initial conditions in the classical case (as in Theorem J.3.1, restricted to $\mathbf{R} \times \{0\} \times U \times \{0\}$ with $U' = V' = \{0\}$).

Proof. Since the map $\mathbf{R} \times M \to M$ is open and in Example F.5.10 we have described $\mathscr{D}(\vec{v})_t$ as the image of a union of overlaps of $\mathscr{D}(\vec{v})$ with open subsets of $\mathbf{R} \times M$, the openness result for $\mathscr{D}(\vec{v})_t$ will follow from that for $\mathscr{D}(\vec{v})$ is open. Our problem is therefore to show that each $(t_0, m_0) \in \mathscr{D}(\vec{v})$ is an interior point (with respect to $\mathbf{R} \times M$) and that the set-theoretic mapping $(t, m) \mapsto c_m(t)$ is smooth near (t_0, m_0).

We first handle the situation for $t_0 = 0$. Pick a point $(0, m_0) \in \mathscr{D}(\vec{v})$, and choose a coordinate chart (φ, U) around m_0. In such coordinates the condition to be an integral curve with position at a point $m \in U$ at time 0 becomes a family of initial-value problems $u'(t) = \phi(u(t))$ with initial condition $u(0) = v_0 \in \varphi(U)$ for varying v_0 and $u : I \to \varphi(U)$ a C^∞ map on an unspecified open interval in $\mathbf{R}$ around 0. By using Theorem J.3.1 (with $V' = \{0\}$) and the restriction to the slice $\mathbf{R} \times \{0\} \times \varphi(U) \times \{0\}$) it follows that for some $\varepsilon > 0$ and some $U_0 \subseteq U$ around m_0 the integral curve through any $m \in U_0$ at time 0 is defined on $(-\varepsilon, \varepsilon)$ and moreover the flow mapping $(t, m) \mapsto c_m(t)$ is smooth on $(-\varepsilon, \varepsilon) \times U_0$. Hence, the problem near points of the form $(0, m_0)$ is settled.

We now explain how to handle points $(t_0, m_0) \in \mathscr{D}(\vec{v})$ with $t_0 > 0$; the case $t_0 < 0$ will go in exactly the same way. Let $T_{m_0} \subseteq I_{m_0}$ be the subset of positive $\tau \in I_{m_0}$ such that $[0, \tau] \times \{m_0\}$ is interior to $\mathscr{D}(\vec{v})$ in $\mathbf{R} \times M$ and $(t, m) \mapsto c_m(t)$ is smooth at around (t', m_0) for all $t' \in [0, \tau)$. For example, the argument of the preceding paragraph shows $(0, \varepsilon) \subseteq T_{m_0}$ for some $\varepsilon > 0$ (depending on m_0). We want T_{m_0} to exhaust the set of positive elements of I_{m_0}, so we assume to the contrary and let τ_0 be the infimum of the set of positive numbers in $I_{m_0} - T_{m_0}$. Hence, $\tau_0 > 0$. Since $\tau_0 \in I_{m_0}$, the maximal integral curve c_{m_0} does propagate past τ_0. In particular, we get a well-defined point $m_1 = c_{m_0}(\tau_0) \in M$. Around m_1 we may choose a C^∞ coordinate chart (φ, U) with domain given by some open $U \subseteq M$ around m_1.

The integral curves for $\vec{v}|_U$ are described by time-independent parameter-free flow with a varying initial condition in the C^∞ coordinates on U. Thus, the argument from the final two paragraphs in the proof of Theorem F.3.6 may now be carried over essentially *verbatim*. The only modification is that at all steps where the earlier argument said "continuous" (which was simultaneously being proved) we may use the word "smooth" (since Theorem J.3.1 is now available to us on the open subset $\varphi(U)$ in a vector space). One also has to choose opens in $\varphi(U)$ so as to not wander outside of $\varphi(U)$ during the construction (as leaving this open set loses touch with the manifold M). We leave it to the reader to check that indeed the method of proof carries over with no substantive changes. $\square$

Definition F.13. On the open subset $\mathscr{D}(\vec{v})_t \subseteq M$, the *flow to time t* is the mapping

$$X_{\vec{v},t} : \mathscr{D}(\vec{v})_t \to M$$

defined by $m \mapsto c_m(t)$.

In words, the points of $\mathscr{D}(\vec{v})_t$ are exactly those $m \in M$ such that the maximal integral curve for $\vec{v}$ through m at time 0 does propagate to time t (possibly $t < 0$), and $X_{\vec{v},t}(m)$ is the point $c_m(t)$ that this curve reaches after flowing for t units of time along $\vec{v}$ starting at m. For example, obviously $X_{\vec{v},0}$ is the identity map (since $c_m(0) = m$ for all $m \in M$). The perspective of integral curves is to focus on variation in time, but the perspective the mapping $X_{\vec{v},t}$ is to fix the time at t and to focus on variation in initial positions (at least for those initial positions for which the associated maximal integral curve persists to time t).

Corollary F.5.14. *The mapping $X_{\vec{v},t}$ is a C^∞ isomorphism onto $\mathscr{D}(\vec{v})_{-t}$ with inverse given by $X_{\vec{v},-t}$.*

The meaning of the smoothness in this corollary is that the position the flow reaches at time t (if it lasts that long!) has smooth dependence on the initial position.

Proof. Set-theoretically, let us first check that the image of $X_{\vec{v},t}$ is $\mathscr{D}(\vec{v})_{-t}$ and that $X_{\vec{v},-t}$ is an inverse. For any $m \in \mathscr{D}(\vec{v})_t$ the point $X_{\vec{v},t}(m) = c_m(t) \in M$ sits on the maximal integral curve $c_m : I_m \to M$, and so (see Remark F.5.2) the additive translate

$$c_m(t + (\cdot)) : (-t + I_m) \to M$$

is the maximal integral curve for $\vec{v}$ through $m' = c_m(t)$ at time 0. Thus, $I_{m'} = -t + I_m$ and $c_{m'}(t') = c_m(t + t')$. Clearly $-t \in -t + I_m = I_{m'}$, so $m' \in \mathscr{D}(\vec{v})_{-t}$. Flowing by $-t$ units of time along this integral curve brings us from $m' = c_m(t)$ to $c_{m'}(-t) = c_m(t + (-t)) = c_m(0) = m$ as it should. Thus, $m = X_{\vec{v},-t}(m') = X_{\vec{v},-t}(c_t(m))$, as desired. We can run through the same argument with $-t$ in the role of t, and so in this way we see that $X_{\vec{v},t}$ and $X_{\vec{v},-t}$ are indeed inverse bijections between the open subsets $\mathscr{D}(\vec{v})_t$ and $\mathscr{D}(\vec{v})_{-t}$ in M. Hence, once we prove that $X_{\vec{v},t}$ and $X_{\vec{v},-t}$ are smooth maps (say when considered with target as M) then they are C^∞ isomorphisms between these open domains in M.

It remains to prove that $X_{\vec{v},t} : \mathscr{D}(\vec{v})_t \to M$ is smooth. Pick a point $m \in \mathscr{D}(\vec{v})_t$, so $(t, m) \in \mathscr{D}(\vec{v})$. By openness of $\mathscr{D}(\vec{v})$ in $\mathbf{R} \times M$ (Theorem F.5.12), there exists $\varepsilon > 0$ and an open $U \subseteq M$ around m such that

$$(t - \varepsilon, t + \varepsilon) \times U \subseteq \mathscr{D}(\vec{v})$$

inside of $\mathbf{R} \times M$. Thus, $U \subseteq \mathscr{D}(\vec{v})_t$. On U, the mapping X_t is the composite of the C^∞ inclusion

$$U \to (t - \varepsilon, t + \varepsilon) \times U$$

given by $u \mapsto (t, u)$ and the restriction to this open target of the vector flow mapping $X_{\vec{v}} : \mathscr{D}(\vec{v}) \to M$ that has been proved to be C^∞ in Theorem F.5.12. $\square$

Example F.5.15. Suppose $\vec{v}$ is complete, so $\mathscr{D}(\vec{v}) = \mathbf{R} \times M$; i.e., $\mathscr{D}(\vec{v})_t = M$ for all $t \in \mathbf{R}$. (By Theorem F.5.5, this is the case when M is compact.) For all $t \in \mathbf{R}$ we get a C^∞ automorphism $X_{\vec{v},t} : M \to M$ that flows each $m \in M$ to the point $c_m(t) \in M$ that is t units in time further away on the maximal integral curve of $\vec{v}$ through m. Explicitly, the vector flow mapping has the form $X_{\vec{v}} : \mathbf{R} \times M \to M$ and restricting it to the "slice" $\{t\} \times M$ in the source (or rather, composing X with the smooth inclusion $M \to \mathbf{R} \times M$ given by $m \mapsto (t, m)$) gives $X_{\vec{v},t}$.

This family of automorphisms $\{X_{\vec{v},t}\}_{t \in \mathbf{R}}$ is the 1-*parameter group generated by* $\vec{v}$. It is called a group because under composition it interacts well with the additive group structure on $\mathbf{R}$. More specifically, we have noted that $X_{\vec{v},0}$ is the identity and that $X_{\vec{v},t}$ is inverse to $X_{\vec{v},-t}$. We claim that $X_{\vec{v},t'} \circ X_{\vec{v},t} = X_{\vec{v},t'+t}$ for all $t, t' \in \mathbf{R}$ (so $X_{\vec{v},t'} \circ X_{\vec{v},t} = X_{\vec{v},t} \circ X_{\vec{v},t'}$ for all $t, t' \in \mathbf{R}$). In view of Remark F.5.2, this says that if $c : \mathbf{R} \to M$ is (up to additive translation) the unique maximal integral curve for $\vec{v}$ with image containing $m \in M$, say $m = c(t_0)$, then $c(t' + (t + t_0)) = c((t' + t) + t_0)$; but this is (even physically) obvious!

Remark F.5.16. In the case of non-complete $\vec{v}$ one can partially recover the group-like aspects of the $X_{\vec{v},t}$'s as in Example F.5.15, except that one has to pay careful attention to domains of definition.

F.6. **Applications to proper families.** Here is a marvelous application of the mappings $X_{\vec{v},t}$. Suppose that $f : M' \to M$ is a surjective submersion between smooth manifolds. Since each fiber $f^{-1}(m)$ is a smooth closed submanifold of M' (submersion theorem!) and these fibers cover M' without overlaps as $m \in M$ varies, we visualize the map f as a "smoothly varying family of manifolds" $\{f^{-1}(m)\}_{m\in M}$ indexed by the points of M. Such maps show up quite a lot in practice. In general, if a surjective C^∞ submersion $f : M' \to M$ is *proper* then we consider f to be a "smoothly varying family of compact manifolds": not only is each fiber compact, but the properness of the total mapping f gives an extra "relative compactness" yielding very pleasant consequences (as we shall see soon).

For any C^∞ submersion $f : M' \to M$, the submersion theorem says that if we work locally on both M *and* M' then (up to C^∞ isomorphism) the mapping f looks like projection to a factor space. However, this description requires us to work locally on the source and hence it loses touch with the *global* structure of the smooth fibers $f^{-1}(m)$ for $m \in M$. It is an important fact that for *proper* surjective submersions $f : M' \to M$, the local (on M' and M) description of f as projection to the factor of a product can be achieved by shrinking *only* on M:

Theorem F.6.1 (Ehresmann Fibration Theorem)**.** *If $f : M' \to M$ is a proper surjective submersion, then M is covered by opens U_i such that for each i there is a C^∞ isomorphism $f^{-1}(U_i) \simeq U_i \times X_i$ for a compact manifold X_i with this isomorphism carrying the C^∞ map f on $f^{-1}(U_i)$ over to the standard projection $U_i \times X_i \to U_i$.*

We make some general remarks before proving the theorem. One consequence is that $f^{-1}(m_i)$ is C^∞-isomorphic to X_i for all $m_i \in U_i$, and hence the C^∞-isomorphism class of a fiber $f^{-1}(m)$ is "locally constant" in M. By using path-connectivity of connected components, it follows that if $f : M' \to M$ is a proper surjective C^∞ submersion to a connected smooth base M then *all* fibers $f^{-1}(m)$ are C^∞-isomorphic to each other!

The property of f as given in the conclusion of the theorem is usually summarized by saying that f is a "C^∞ fiber bundle with compact fibers". Such a fibration result is an incredibly powerful topological tool, especially in the study of families of compact manifolds, and the technique of its proof (vector flow) is a basic ingredient in getting Morse theory off the ground.

Proof. We may work locally over M, so without loss of generality M is the open unit ball in $\mathbf{R}^n$ with coordinates $x_1, \dots, x_n$, and that we work around the origin m in this ball. We first need to construct smooth vector fields $\vec{v}_1, \dots, \vec{v}_n$ on M' such that $\mathrm{d}f(m') : \mathrm{T}_{m'}(M) \to \mathrm{T}_{f(m')}(M)$ sends $\vec{v}_i(m')$ to $\partial_{x_i}|_{f(m')}$ for all $m' \in M'$. Using the submersion property of f, such vector fields can be constructed.

The open set $\mathscr{D}(\vec{v}_i) \subseteq \mathbf{R} \times M'$ contains $\{0\} \times M'$, and so it contains the subset $\{0\} \times f^{-1}(m)$ that is *compact* (since f is proper). Hence, it contains $(-\varepsilon_i, \varepsilon_i) \times U_i'$ for some open set $U_i' \subseteq M'$ around $f^{-1}(m)$. But since $f : M' \to M$ is *proper*, an open set around a fiber $f^{-1}(m)$ must contain an open of the form $f^{-1}(U_i)$ for an open $U_i \subseteq M$ around m. Thus, we conclude that $\mathscr{D}(\vec{v}_i)$ contains $(-\varepsilon_i, \varepsilon_i) \times f^{-1}(U_i)$ for some $\varepsilon_i > 0$ and some open U_i around m. Let $\varepsilon = \min_i \varepsilon_i > 0$ and $U = \cap_i U_i$, so for all i the domain of flow $\mathscr{D}(\vec{v}_i)$ contains

$(-\varepsilon, \varepsilon) \times f^{-1}(U)$, with $\varepsilon > 0$ and $U \subseteq M$ an open around m. Hence, there is a flow mapping

$$X_{\vec{v}_i} : (-\varepsilon, \varepsilon) \times \pi^{-1}(U) \to M'$$

for $1 \leq i \leq n$.

Fix $1 \leq i \leq n$, and consider the composite mapping $h_i : M' \to M \xrightarrow{p_i} (-1, 1)$ where p_i is projection to the ith coordinate on the open unit ball $M \subseteq \mathbf{R}^n$. Since $\mathrm{d}f(m')(\vec{v}_i(m')) = \partial_{x_i}|_{f(m')}$ for all $m' \in M'$, integral curves for $\vec{v}_i$ in M' map to integral curves for ∂_{x_i} in M, and these are straight lines in the open unit ball M. For any $m' \in f^{-1}(U) \cap h_i^{-1}(t_0)$ with $|t_0| < \varepsilon$, for $|t| < \varepsilon - |t_0|$ the integral curve for $\vec{v}_i$ passing through m' therefore flows out to time t with $X_{\vec{v}_i, t}(m') \in h_i^{-1}(t)$. Provided that we begin at m' sufficiently close to the compact $f^{-1}(m)$, this endpoint $X_{\vec{v}_i, t}(m')$ will be in the open set $\pi^{-1}(U)$ around $f^{-1}(m)$.

Arguing in this way and again using properness of f (to know that opens around $f^{-1}(m)$ contain f-preimages of opens around m), we can find $\varepsilon_0 \in (0, \varepsilon)$ and an open $U_0 \subseteq U$ around m such that flow along $\vec{v}_1$ over time $(-\varepsilon_0, \varepsilon_0)$ with initial point in $\pi^{-1}(U_0)$ ends at a point in $\pi^{-1}(U)$. We repeat this procedure for $\vec{v}_2$ with U_0 in the role of U, and so on, to eventually arrive (after n iterations of this argument) at a very small $\eta \in (0, \varepsilon)$ and open sets $U_n \subseteq U_{n-1} \subseteq \cdots \subseteq U_0$ such that $X_{\vec{v}_i, t}(\pi^{-1}(U_i)) \subseteq \pi_i^{-1}(U_{i-1})$ for $|t| < \eta$. Hence, we arrive at an "iterated flow" mapping

$$(-\eta, \eta)^n \times \pi^{-1}(U_n) \to \pi^{-1}(U)$$

defined by

$$(t_1, \ldots, t_n, m') \mapsto (X_{\vec{v}_1, t_1} \circ X_{\vec{v}_2, t} \circ \cdots \circ X_{\vec{v}_n, t_n})(m')$$

that is certainly C^∞. We restrict this by replacing $\pi^{-1}(U_n)$ with the closed smooth submanifold $f^{-1}(m) \subseteq \pi^{-1}(U_n)$ to arrive at a smooth mapping

$$(-\eta, \eta)^n \times f^{-1}(m) \to \pi^{-1}(U)$$

given by the same iterated flow formula.

Geometrically, the map we have just constructed is a flow away from the fiber $f^{-1}(m)$ by flowing for time t_i in the ith coordinate direction over the base, done in the order "first x_1-direction, then x_2-direction, and so on." The image is contained in $\cap h_i^{-1}(-\eta, \eta) \subseteq f^{-1}((-\varepsilon, \varepsilon)^n)$. More specifically, by recalling that flow along $\vec{v}_i$ on M' lies over straight-line flow in the ith coordinate direction in the ball M (with the same time parameter!), we have built a smooth mapping

$$(-\eta, \eta)^n \times f^{-1}(m) \to f^{-1}((\varepsilon, \varepsilon)^n)$$

that lies over the inclusion

$$(-\eta, \eta)^n \hookrightarrow (-\varepsilon, \varepsilon)^n.$$

Hence, this map has image contained in $f^{-1}((-\eta, \eta)^n)$, and so for the open set $U = (-\eta, \eta)^n$ around $m \in M$ we have a smooth map

$$\psi : U \times f^{-1}(m) \to f^{-1}(U)$$

compatible with the projections from each side onto U (using $f : f^{-1}(U) \to U$).

We want to prove that after shrinking U around m the map ψ becomes a smooth isomorphism. By the *definition* of ψ (and of the iterated flow!), the restriction of ψ to the fiber over the origin m is the *identity map* on $f^{-1}(m)$. Since the $\vec{v}_i$'s were constructed to "lift" the ∂_{x_i}'s, it follows that for any $m' \in f^{-1}(m)$ the tangent mapping

$$\mathrm{d}\psi((0,\ldots,0),m') : \mathbf{R}^n \times \mathrm{T}_{m'}(f^{-1}(m)) \to \mathrm{T}_{m'}(M)$$

is an isomorphism: it carries $\mathrm{T}_{m'}(f^{-1}(m)) = \ker(\mathrm{d}f(m'))$ to itself by the identity, and carries the standard basis of $\mathbf{R}^n$ to the vectors $\vec{v}_i(m')$ in $\mathrm{T}_{m'}(M')$ whose images in

$$\mathrm{T}_{m'}(M')/\mathrm{T}_{m'}(f^{-1}(m)) \simeq \mathrm{T}_m(M)$$

are the $\partial_{x_i}|_m$'s that are a basis of this quotient! Thus, by the inverse function theorem we conclude that ψ is a local C^∞ isomorphism near all points over $m \in M$. Since the projection $U \times f^{-1}(m) \to U$ and the map $f : f^{-1}(U) \to U$ are proper C^∞ submersions, we may conclude the result by Theorem F.6.2 below. $\square$

The following interesting general theorem was used in the preceding proof:

Theorem F.6.2. *Let Z be a smooth manifold and let $\pi' : X' \to Z$ and $\pi : X \to Z$ be proper C^∞ submersions of smooth manifolds. Let*

$$h : X' \to X$$

be a mapping "over Z" (in the sense that $\pi \circ h = \pi'$). If $z_0 \in Z$ is a point such that h restricts to a C^∞ isomorphism ${\pi'}^{-1}(z_0) \simeq \pi^{-1}(z_0)$ and h is a local C^∞ isomorphism around points of ${\pi'}^{-1}(z_0)$, then there exists an open subset $U \subseteq Z$ around z_0 such that the mapping ${\pi'}^{-1}(U) \to \pi^{-1}(U)$ induced by the Z-map h is a C^∞-isomorphism.

The principle of this result is that for maps between proper objects over a base space, whatever happens on the fibers over a single point of the base space also happens over an open around the point. This is not literally a true statement in such generality, but in many contexts it can be given a precise meaning.

Proof. There is an open set in X' around ${\pi'}^{-1}(z_0)$ on which h is a local smooth isomorphism. Such an open set contains the π'-preimage of an open around z_0, due to properness of π'. Hence, by replacing Z with this open subset around z_0 and X and X' with the preimages of this open, we may assume that $h : X' \to X$ is a local C^∞-isomorphism. Since X and X' are proper over Z and all spaces under consideration are Hausdorff, it is not difficult to check that h must also be proper! Hence, h is a proper local isomorphism. The fibers of h are compact (by properness) and discrete (by the local isomorphism condition), whence they are *finite*.

Consider the function $s : X \to \mathbf{Z}$ that sends x to the size of $h^{-1}(x)$. This function is equal to 1 on $\pi^{-1}(z_0)$ by the hypothesis on h. I claim that it is a locally constant function. Grant this for a moment, so $s^{-1}(1)$ is an open set in X around $\pi^{-1}(z_0)$. By properness of π we can find an open set $U \subseteq Z$ around z_0 such that $\pi^{-1}(U) \subseteq s^{-1}(1)$. Replacing Z, X, and X' with U, $\pi^{-1}(U)$, and ${\pi'}^{-1}(U) = h^{-1}(\pi^{-1}(U))$ we get to the case when $h : X' \to X$ is a local C^∞-isomorphism whose fibers all have size 1. Such an h is bijective, and hence a C^∞ isomorphism (as desired).

How are we to show that s is locally constant? Rather generally, if $h : X' \to X$ is any proper local C^∞ isomorphism between smooth manifolds (so h has finite fibers, by the same argument used above), then we claim that the size of the fibers of h is locally constant on X. We can assume that X is connected, and in this case we claim that all fibers have the same size. Suppose $x_1, x_2 \in X$ are two points over which the fibers of h have different sizes. Make a continuous path $\sigma : [0,1] \to X$ with $\sigma(0) = x_1$ and $\sigma(1) = x_2$. For each $t \in [0,1]$ we can count the size of $h^{-1}(\sigma(t))$, and this has distinct values at $t = 0, 1$. Thus, the subset of $t \in [0,1]$ such that $\#h^{-1}(\sigma(t)) \neq \#h^{-1}(\sigma(0))$ is non-empty. We let t_0 be its infimum, and $x_0 = \sigma(t_0)$. Hence, there exist points t arbitrarily close to t_0 such that $\#h^{-1}(\sigma(t)) = \#h^{-1}(\sigma(0))$ and there exist other points t arbitrarily close to t_0 such that $\#h^{-1}(\sigma(t)) \neq \#h^{-1}(\sigma(0))$. (Depending on whether or not $h^{-1}(\sigma(t_0))$ has the same size as $h^{-1}(\sigma(0))$, we can always take $t = t_0$ for one of these two cases.)

It follows that the size of $h^{-1}(x_0)$ is distinct from that of $h^{-1}(\xi_n)$ for a sequence $\xi_n \to x_0$ in X. Since h is a local C^∞ isomorphism, if there are exactly r points in $h^{-1}(x_0)$ (perhaps $r = 0$) and we enumerate them as $x'_1, \ldots, x'_r$ then by the Hausdorff and local C^∞-isomorphism conditions we may choose pairwise disjoint small opens U'_i around x'_i mapping isomorphically onto a common open U around x_0. Thus, for all $x \in U$ there are at least r points in $h^{-1}(x)$. It follows that for large n (so $\xi_n \in U$) the fiber $h^{-1}(\xi_n)$ has size at least r and hence has size strictly larger than r. In particular, $h^{-1}(\xi_n)$ contains a point ξ'_n *not* equal to any of the r points where $h^{-1}(\xi_n)$ meets $\coprod U'_i$ ($h : U'_i \to U$ is bijective for all i). In particular, for each n we have that $\xi'_n \notin U'_i$ for all i.

Let K be a compact neighborhood of x_0 in X, so $K' = h^{-1}(K)$ is a compact subset of X' by properness. Taking large n so that $\xi_n \in K$, the sequence ξ'_n lies in the compact K'. Passing to a subsequence, we may suppose $\{\xi'_n\}$ has a limit $\xi' \in K'$. But $h(\xi'_n) = \xi_n \to x_0$, so $h(\xi') = x_0$. In other words, $\xi' = x'_i$ for some i. This implies that $\xi'_n \in U'_i$ for large n, a contradiction! □

Appendix G. More features and applications of the exponential map

This appendix addresses smoothness of the exponential map and two applications: continuous homomorphisms between Lie groups are C^∞ and a neat proof of the Fundamental Theorem of Algebra. We also discuss another context in which an exponential-like expression naturally arises: Lie's local formula for vector flow in the real-analytic case.

G.1. **Smoothness.** Let G be a Lie group, and $\exp_G : \mathfrak{g} \to G$ the exponential map. By definition, $\exp_G(v) = \alpha_v(1)$ is the flow 1 unit of time along the integral curve through e for the global left-invariant vector field $\widetilde{v}$ extending v at e. As we noted in class, it isn't immediately evident from general smoothness properties of solutions to ODE's (in terms of dependence on initial conditions and/or auxiliary parameters) that $\exp_G$ is C^∞. Of course, it is entirely unsurprising that such smoothness should hold. We now justify this smoothness, elaborating a bit on the proof given as [BtD, Ch. I, Prop. 3.1].

Consider the manifold $M = G \times \mathfrak{g}$, and the set-theoretic global vector field X on M given by

$$X(g, v) = (\widetilde{v}(g), 0) = (d\ell_g(e)(v), 0) \in T_{(g,v)}(M).$$

Note that X is C^∞ because $g \mapsto d\ell_g(e)$ is the adjoint representation $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ that we know is C^∞. We are going to directly construct integral curves to X and see by inspection that these are defined for all time. Hence, the associated flow is defined for all time. The general smoothness property for such flow on its domain of definition (see Theorem F.5.12, whose proof involved some real work) will then yield what we want.

For each $m = (g, v) \in M$, consider the mapping $c_m : \mathbf{R} \to M$ defined by $c_m(t) = (g \cdot \alpha_v(t), v)$. This is a C^∞ mapping since the 1-parameter subgroup $\alpha_v : \mathbf{R} \to G$ is C^∞. Let's calculate the velocity vector at each time for this parametric curve in M: by the Chain Rule we have

$$c'_m(t) = ((d\ell_g(\alpha_v(t)))(\alpha'_v(t)), 0) = ((d\ell_g(\alpha_v(t)))(\widetilde{v}(\alpha_v(t))), 0)$$

where the final equality uses the identity $\alpha'_v(t) = \widetilde{v}(\alpha_v(t))$ which expresses that α_v is an integral curve to $\widetilde{v}$ (as holds by design of α_v). But $\widetilde{v}$ is a left-invariant vector field on G by design, so

$$(d\ell_g(h))(\widetilde{v}(h)) = \widetilde{v}(gh)$$

for all $h \in G$. Setting h to be $\alpha_v(t)$, we obtain

$$(d\ell_g(\alpha_v(t)))(\widetilde{v}(\alpha_v(t))) = \widetilde{v}(g \cdot \alpha_v(t)),$$

so

$$c'_m(t) = (\widetilde{v}(g \cdot \alpha_v(t)), 0) = X(g \cdot \alpha_v(t), v) = X(c_m(t)).$$

We have shown that c_m is an integral curve to the global smooth vector field X on M with value $c_m(0) = (g, v) = m$ at $t = 0$. In other words, c_m is an integral curve to X through m at time 0. But by design the smooth parametric curve c_m in M is defined on the entire real line, so we conclude that the open domain in $\mathbf{R} \times M$ for the flow associated to X is the entirety of $\mathbf{R} \times M$. In other words, the global flow is a mapping

$$\Phi : \mathbf{R} \times M \to M$$

and (as we noted already) this flow is known to always be C^∞ on its domain. That is, Φ is a C^∞ map. By design, $\Phi(t, m)$ is the flow to time t of the integral curve to X that is at m at time 0; i.e., $\Phi(t, m) = c_m(t)$.

We conclude that the restriction of Φ to the closed C^∞-submanifold $\mathbf{R} \times \{1\} \times \mathfrak{g}$ is also C^∞. This map

$$\mathbf{R} \times \mathfrak{g} \to M = G \times \mathfrak{g}$$

is $(t, v) \mapsto c_{(1,v)}(t) = (\alpha_v(t), v)$. Composing with projection to G, we conclude that the map $\mathbf{R} \times \mathfrak{g} \to G$ defined by $(t, v) \mapsto \alpha_v(t)$ is C^∞. Now restricting this to the slice $t = 1$ gives a further C^∞ map $\mathfrak{g} \to G$, and this is exactly the map $v \mapsto \alpha_v(1) = \exp_G(v)$ that we wanted to show is C^∞.

G.2. **Continuous homomorphisms.** Let $f : G' \to G$ be a continuous homomorphism between Lie groups. We aim to prove that f is necessarily C^∞. Consider the graph map

$$\Gamma_f : G' \to G' \times G$$

defined by $g' \mapsto (g', f(g'))$. This is a continuous injection with inverse given by the continuous restriction of pr_1, so it is a homeomorphism onto its image. Moreover, this image is closed, since it is the preimage of the diagonal in $G \times G$ under the continuous map $G' \times G \to G \times G$ defined by $(g', g) \mapsto (f(g'), g)$. The homomorphism property of f ensures that Γ_f is a subgroup inclusion, so Γ_f identifies G' topologically with a closed subgroup of $G' \times G$.

As we mentioned in class (and is proved in [BtD, Ch. I, Thm. 3.11], using the exponential map in an essential manner), a closed subgroup of a Lie group is necessarily a closed smooth submanifold. Thus, the graph has a natural smooth manifold structure, and we shall write $H \subseteq G' \times G$ to denote this manifold structure on $\Gamma_f(G')$.

Observe that topologically, f is the composition of the *inverse* of the homeomorphism $\mathrm{pr}_1 : H \to G$ with the other projection $\mathrm{pr}_2 : H \to G$. Since the inclusion of H into $G' \times G$ is C^∞ and $\mathrm{pr}_2 : G' \times G \to G$ is C^∞, it follows that $\mathrm{pr}_2 : H \to G$ is C^∞. Hence, to deduce that f is C^∞ it suffices to show that the C^∞ homeomorphism $\mathrm{pr}_1 : H \to G$ is a diffeomorphism (so its inverse is C^∞). But this latter map is a bijective Lie group homomorphism, so by an application of Sard's theorem and *homogeneity* developed in Exercise 5(iii) of HW3 it is necessarily a diffeomorphism! (Beware that in general a C^∞ homeomorphism between smooth manifolds need not be a diffeomorphism, as illustrated by $x \mapsto x^3$ on $\mathbf{R}$.)

G.3. **Fundamental Theorem of Algebra.** Now we present a beautiful proof of the Fundamental Theorem of Algebra that was discovered by Witt (and rediscovered by Benedict Gross when he was a graduate student). Let F be a finite extension of $\mathbf{C}$ with degree $d \geq 1$. Viewing F as an $\mathbf{R}$-vector space, $F \simeq \mathbf{R}^{2d}$ with $2d \geq 2$. Our aim is to prove $d = 1$.

Topologically, $F - \{0\} = \mathbf{R}^{2d} - \{0\}$ is connected. For any finite-dimensional associative $\mathbf{R}$-algebra (equipped with its natural manifold structure as an $\mathbf{R}$-vector space), the open submanifold of units is a Lie group. So $F^\times$ is a *connected* commutative Lie group.

In class we have seen via the exponential map that every connected commutative Lie group is a product of copies of S^1 and $\mathbf{R}$. Thus, $F^\times \simeq (S^1)^r \times \mathbf{R}^{2d-r}$ as Lie groups, for some $0 \leq r \leq 2d$. Necessarily $r > 0$, as the vector space factor $\mathbf{R}^{2d-r}$ is torsion-free whereas $F^\times$ has nontrivial torsion (any of the nontrivial roots of unity in $\mathbf{C}^\times$). By contracting the factor $\mathbf{R}^{2d-r}$ to a point, we see that $F^\times$ retracts onto $(S^1)^r$, and so by the homotopy-invariance and direct product functoriality of the fundamental group it follows that $\pi_1(F^\times) = \pi_1((S^1)^r)) = \pi_1(S^1)^r = \mathbf{Z}^r \neq 0$. But we noted that $F^\times = \mathbf{R}^{2d} - \{0\}$, and for $n > 1$ the group $\pi_1(\mathbf{R}^n - \{0\})$ is nontrivial only for $n = 2$, so $2d = 2$ and hence $d = 1$.

G.4. **Lie's exponential flow formula.** We know that if X is a C^∞ vector field on a C^∞ manifold M then the associated domain of flow $\Omega \subset \mathbf{R} \times M$ is an open subset containing $\{0\} \times M$ such that the flow $\Phi : \Omega \to M$ (also written as $\Phi_t(m) := \Phi(t, m)$) is C^∞ and satisfies $\Phi_0 = \mathrm{id}_M$ and

$$\Phi_{t'} \circ \Phi_t = \Phi_{t'+t}$$

locally on M for t, t' near 0 (depending on a small region in M on which we work). By considerations with several complex variables (to handle convergence issues for power series via differentiability conditions), one can refine this to see that if M and X are real-analytic then so is Φ.

The procedure can be run essentially in reverse if we focus on $t \approx 0$ (and hence don't fret about the maximality property of Ω): if M is a C^∞-manifold and $\Omega \subset \mathbf{R} \times M$ is an open subset containing $\{0\} \times M$ on which a C^∞-map $\Phi : \Omega \to M$ is given that satisfies $\Phi_0 = \mathrm{id}_M$ and $\Phi_{t'} \circ \Phi_t = \Phi_{t'+t}$ locally on M for t, t' near 0 (nearness to 0 depending on a small region in M on which we work) then near $\{0\} \times M$ we claim that Φ *is* the flow for an associated C^∞ vector field! Indeed, by smoothness of Φ the set-theoretic vector field

$$m \mapsto X(m) = \Phi_0'(m) \in \mathrm{T}_m(M)$$

is easily seen to be C^∞, and we claim that for each $m \in M$ the parametric curve $t \mapsto \Phi_t(m)$ (defined for t near 0 since Ω is open containing $\{0\} \times M$, and passing through $\Phi_0(m) = m$ at $t = 0$) is an integral curve to X on the open subset $\Omega \cap (\mathbf{R} \times \{m\}) \subset \mathbf{R}$ that contains 0 (but might be disconnected). To see this, we consider the identity

$$\Phi(t', \Phi(t, m)) = \Phi(t + t', m)$$

for t, t' near 0. Following Lie, we fix t to make each side a C^∞ parametric curve in t' and compute the velocity vector at $t' = 0$. Using the Chain Rule and the hypothesis $\Phi(0, \cdot) = \mathrm{id}_M$, this becomes $\Phi_0'(\Phi_t(m)) = \Phi_t'(m)$ for t near 0. By definition of X this says

$$X(\Phi_t(m)) = \Phi_t'(m),$$

or in other words $t \mapsto \Phi_t(m)$ is an integral curve to X for t near 0, as desired.

Thus, as long as we're willing to work in a small region $U \subset M$ and stick to time near 0, two viewpoints are interchangeable: the vector field X on U and the "action" Φ on U by a small interval in $\mathbf{R}$ around 0 (the perspective of "local Lie group" that was all Lie ever considered). In the real-analytic case (which is all Lie ever considered) the flow is also real-analytic as we noted above, and we claim that the flow Φ_t can then be described in terms of X in rather concrete terms. This rests on Lie's precursor to the Baker-Campbell-Hausdorff formula, a kind of "exponential map" at the level of operators on functions to reconstruct the flow for small time from the associated vector field:

Proposition G.4.1 (Lie). *For any point $m \in M$ and real-analytic f defined near m, we have*

$$f \circ \Phi_t = (e^{tX})(f) := \sum_{j=0}^{\infty} \frac{t^j}{j!} X^j(f)$$

on a small open neighborhood of m for all t near 0.

This "exponential operator" notation e^{tX} is just suggestive. The proof below ensures the convergence since we work real-analytically throughout.

Proof. The left side is real-analytic on $I \times U$ for some open interval $I \subset \mathbf{R}$ around 0 and some open subset $U \subset M$ around m. Thus, it admits a convergent power series expansion $\sum_{j \geq 0} c_j(t^j/j!)$ for t near 0 and *real-analytic functions c_j* defined on a common small open neighborhood of m in M. By analytic continuation, it suffices to show that for each j we have $c_j = X^j(f)$ near m.

Considering $f \circ \Phi_t$ as an $\mathbf{R}$-valued real-analytic function on some domain $I \times U$, if we differentiate it with respect to t then by the Chain Rule we get the real-analytic function on U given by

$$((df)(\Phi_t(u)))(\Phi_t'(u)) = ((df)(\Phi_t(u)))(X(\Phi_t(u))),$$

where the equality uses that the parametric curve $t \mapsto \Phi_t(u)$ is an integral curve to X (by design of X). The right side is exactly $(Xf)(\Phi_t(u))$, so we conclude that

$$\partial_t(f \circ \Phi_t) = (Xf) \circ \Phi_t$$

on $I \times U$. Now iterating this inductively,

$$\partial_t^j(f \circ \Phi_t) = (X^j f) \circ \Phi_t.$$

Setting t to be 0 on the right collapses this to be $X^j f$ since $\Phi_0 = \mathrm{id}_M$, so the coefficient functions in $C^\omega(U)$ for the series expansion of $f \circ \Phi_t$ in t are exactly the $X^j f$'s as desired. $\square$

If $\{x_1, \dots, x_n\}$ are local real-analytic coordinates on M on a small open neighborhood U of a point $m \in M$, we conclude that for t near 0 the flow Φ_t near m is given in these coordinates by

$$\Phi_t(u) = (y_1(t,u), \dots, y_n(t,u))$$

where

$$y_i(t,u) = e^{tX}(x_i) := \sum_{j=0}^{\infty} (X^j(x_i))(u)\frac{t^j}{j!}.$$

This is Lie's explicit description of the flow along X for small time in a small region of M when M and X are real-analytic.

Appendix H. Local and global Frobenius theorems

In this appendix, we explain how to relate Lie subalgebras to connected Lie subgroups by using a higher-dimensional version of the theory of integral curves to vector fields. This rests on some serious work in differential geometry, for the core parts of which (beyond our earlier work on integral curves to vector fields and the associated global flow) we refer to a well-written book for a good self-contained account.

H.1. **Subbundles.**

For ease of discussion we want to use the language of "vector bundles", but we don't need much about this beyond a few basic things of "algebraic" nature in the specific setting of smooth vector fields on a smooth manifold. Thus, we now introduce some notions with an eye towards what is relevant later.

In this appendix, the *tangent bundle* TM of a smooth manifold is shorthand for the assignment to each open set $U \subseteq M$ of the $C^\infty(U)$-module $\mathrm{Vec}_M(U)$ of smooth vector fields on U; this may also be denoted as $(TM)(U)$. In differential geometry there is geometric object given the same name, so for the sake of completeness let's briefly describe how that goes (which we will never use in what follows). The geometers build a smooth manifold TM equipped with a submersion $q : TM \to M$ for which several properties are satisfied. Firstly, there is an identification $q^{-1}(m) = \mathrm{T}_m(M)$ for all $m \in M$. Second, for every open set $U \subseteq M$ the set $\Gamma(U, TM)$ of *smooth* cross-sections $s : U \to TM$ to q (i.e., $s(u) \in q^{-1}(u) = \mathrm{T}_u(M)$ for all $u \in U$) is identified with $\mathrm{Vec}_M(U)$ *compatibly with* restriction to open subsets $U' \subseteq U$. Visually, such an s can be viewed as a map filling in a commutative diagram:

$$\begin{tikzcd} & TM \arrow[d, "q"] \\ U \arrow[ur, "s"] \arrow[r, "j"'] & M \end{tikzcd}$$

where $j : U \hookrightarrow M$ is the natural inclusion. Finally, there is a compatibility between these two kinds of identifications (pointwise and over open subsets of M): for any smooth cross-section $s \in \Gamma(U, TM)$ and its associated smooth vector field $X \in \mathrm{Vec}_M(U)$ the identification $q^{-1}(u) = \mathrm{T}_u(M)$ carries $s(u)$ over to the tangent vector $X(u) \in \mathrm{T}_u(M)$ for all $u \in U$. The interplay of pointwise notions and "over varying open sets" notions is very important.

Definition H.1. A *subbundle* E of TM is a choice of subspace $E(m) \subseteq \mathrm{T}_m(M)$ for each $m \in M$ that "varies smoothly in m" in the following precise sense: M is covered by open sets U_α over which there are *smooth* vector fields $X_1^{(\alpha)}, \dots, X_{r_\alpha}^{(\alpha)} \in \mathrm{Vec}_M(U_\alpha)$ such that the vectors $X_i^{(\alpha)}(m) \in \mathrm{T}_m(M)$ are a basis of $E(m)$ for all $m \in U_\alpha$. Such $\{U_\alpha\}$ is called a *trivializing cover* of M for E, and $\{X_i^\alpha\}_i$ is called a *trivializing frame* for E over U_α.

Informally, the subspaces $E(m)$ for $m \in U_\alpha$ admit a "common" basis $\{X_i^\alpha(m)\}_i$, and the "smooth variation" of the subspaces $E(m) \subset \mathrm{T}_m(M)$ for varying $m \in U_\alpha$ is encoded in the fact that the X_i^α are *smooth*.

Example H.1.2. If $\{x_1, \dots, x_n\}$ is a C^∞ coordinate system on an open subset V of M then the vector fields ∂_{x_i} constitute a trivializing frame for TM over V.

To think about the notion of trivializing frame in a more algebraic manner, modeled on the notion of a basis of a module, it is convenient to introduce some notation: for any open set $U \subseteq M$ we define $E(U)$ to be the set of *smooth* vector fields $X \in \mathrm{Vec}_M(U)$ satisfying $X(m) \in E(m)$ for all $m \in U$. (For example, if $E = TM$ then $E(U)$ is just $\mathrm{Vec}_M(U)$ by another name.) Clearly $E(U)$ is a $C^\infty(U)$-module. The key point is this:

Lemma H.1.3. *Let $\{U_\alpha\}$ be a trivializing cover of M for E, and $\{X_i^\alpha\}$ a trivializing frame for E over U_α. For any open subset $U \subseteq U_\alpha$, a set-theoretic vector field X on U (i.e., an assignment $u \mapsto X(u) \in \mathrm{T}_u(M)$ for all $u \in U$) lies in the subset $E(U) \subseteq \mathrm{Vec}_M(U)$ precisely when $X = \sum a_i X_i^{(\alpha)}|_U$ for smooth functions a_i. In other words, $E(U)$ is a free $C^\infty(U)$-module with basis $\{X_i^\alpha|_U\}$.*

We refer to $\{X_i^{(\alpha)}\}$ $(1 \le i \le r_\alpha)$ as a "trivializing frame" for E over U_α: the restrictions X_i^α over *any* open $U \subset U_\alpha$ are a basis for $E(U)$ as a module over $C^\infty(U)$, not just for $U = U_\alpha$!

Proof. Since $X_i^\alpha(m) \in E(m)$ for all $m \in U$, clearly any $C^\infty(U)$-linear combination of the $X_i^\alpha|_U$'s lies in $E(U)$. The real content is to prove the converse: if X is a *smooth* vector field on U and $X(m) \in E(m)$ for all $m \in U$ then X is a $C^\infty(U)$-linear combination of the vector fields $X_i^\alpha|_U$. It is harmless to rename U as M, so we can rephrase the problem as follows. Consider smooth vector fields $X_1, \dots, X_r$ on M that are *pointwise* linearly independent, and a set-theoretic vector field X that is pointwise a linear combination of the X_i's (much as $X(m)$ above is in the span $E(m)$ of the linearly independent set $\{X_i^\alpha(m)\}_i$ for each $m \in U \subseteq U_\alpha$). For the unique pointwise expansions

$$X(m) = \sum a_i(m) X_i(m),$$

we claim that the functions $a_i : M \to \mathbf{R}$ are *smooth* when X is smooth.

Our problem is local on M, so it suffices to work on individual open coordinate domains that cover M. In other words, we can assume that M is an open subset of some $\mathbf{R}^n$, say with coordinates $x_1, \dots, x_n$. The idea, roughly speaking, is to reduce to the case when (after rearranging coordinates) $X_i = \partial_{x_i}$ for $1 \le i \le r$, in which case the smoothness of the coefficient functions a_i follows from the *definition* of smoothness for the vector field X (which may be checked using any *single* coordinate system). This isn't quite what we will do, but it gives the right idea.

Consider the expansions

$$X_i = \sum c_{ij} \partial_{x_j}$$

where the coefficient functions $c_{ij} : M \to \mathbf{R}$ are smooth by the smoothness hypothesis on each X_i. For each $m \in M$, the $r \times n$ matrix $(c_{ij}(m))$ has rows that are linearly independent:

this expresses that the r vectors $X_i(m) \in \mathrm{T}_m(M)$ are linearly independent for all m. By the equality of "row rank" and "column rank", there are r linearly independent columns. In other words, there is an $r \times r$ submatrix that is invertible. The specific choices of r columns that give an invertible $r \times r$ submatrix of $(c_{ij}(m))$ may vary as we change m, but whatever such choice of columns gives an invertible submatrix at a specific $m_0 \in M$ also works at nearby m since the associated $r \times r$ determinant is nonzero at m_0 and hence is nonzero nearby (since each c_{ij} is continuous). Thus, we can *cover* M by open subsets on each of which a fixed set of r columns gives a pointwise invertible $r \times r$ submatrix (with the specific r columns depending on the open subset).

Since our smoothness assertion for the a_i's is of local nature on M, by passing to such individual open subsets we may now assume that there are r specific columns for which the associated $r \times r$ submatrix is *everywhere* invertible. By rearranging the coordinates, we may assume this is the leftmost $r \times r$ submatrix. Now consider the expansion

$$X(m) = \sum_i a_i(m) X_i(m) = \sum_i \sum_j a_i(m) c_{ij}(m) \partial_{x_j} = \sum_j (\sum_i a_i(m) c_{ij}(m)) \partial_{x_j}.$$

The smoothness hypothesis on X means that the coefficient functions $\sum_i a_i c_{ij}$ are smooth on M for each j. Taking $1 \le j \le r$ and letting A be the commutative ring of all $\mathbf{R}$-valued functions on M (no smoothness conditions!), we see that the matrix $C := (c_{ij})_{1 \le i,j \le r}$ over A carries $(a_1, \ldots, a_r) \in A^r$ into $C^\infty(M)^r$. But C is a matrix with entries in the subring $C^\infty(M)$ and its determinant is in $C^\infty(M)^\times$, so by Cramer's Formula its inverse C^{-1} exists over $C^\infty(M)$. Hence, any $(a_1, \ldots, a_r) \in C^{-1}(C^\infty(M)^r) = C^\infty(M)^r$ inside A^r. $\square$

H.2. **Integrable subbundles.** For smooth vector fields on M we have the notion of an integral curve, and through any point $m \in M$ there exists a unique "maximal" integral curve $c : I \to M$ to X at m with $I \subseteq \mathbf{R}$ an open interval around 0 (i.e., $c(0) = m$ and $c'(t) = X(c(t))$ for all $t \in I$). There is a higher-dimensional generalization of this notion for subbundles E of TM. Consider a *submanifold* N of M, by which we mean an injective immersion $j : N \to M$. This might not be a topological embedding (consider a densely wrapped line on a torus), but one knows from basic manifold theory (essentially, the Immersion Theorem) that it satisfies a good mapping property: if M' is a smooth manifold and a C^∞-map $M' \to M$ from a smooth manifold M' factors through j *continuously* (i.e., M' lands inside $j(N)$ and the associated map $M' \to N$ is continuous, as is automatic if j is topological embedding) then the resulting map $M' \to N$ is actually C^∞. Hence, a submanifold works as nicely as a subset from the viewpoint of C^∞ maps *provided* we keep track of continuity aspects of maps. This continuity condition cannot be dropped:

Example H.2.1. Let $M = \mathbf{R}^2$ and let $N = (-2\pi, 2\pi)$ with $i : N \hookrightarrow M$ an injective immersion whose image is a "figure 8", where $i((-2\pi, 0))$ and $i((0, 2\pi))$ are the two "open halves" of the figure 8 complementary to the crossing point $m_0 = i(0)$, so for small $c > 0$ the restriction $i|_{(-c,c)}$ traces out *one* of the two short intervals in the figure 8 passing through m_0. For an explicit example, we can define

$$i(t) = \begin{cases} (-1 + \cos(t), \sin(t)) \text{ if } t \le 0, \\ (1 - \cos(t), \sin(t)) \text{ if } t \ge 0. \end{cases}$$

Note that i is *not* a topological embedding (i.e., not a homeomorphism onto its image with the subspace topology) since it carries points very near to $\pm 2\pi$ arbitrarily close to $i(0)$.

Let $M' = (-c, c)$ for small $c > 0$ and let $f : M' \to M$ be an injective immersion that is a homeomorphism onto the *other* short interval in the figure 8 passing through m_0. Then $f(M') \subset i(N)$ but the resulting set-theoretic factorization $M' \to N$ of f through i isn't even continuous (let alone C^∞), due to the same reasoning given for why i isn't a topological embedding.

Definition H.2. A connected submanifold $j : N \to M$ is an *integral manifold* to a subbundle E of TM if $E(j(n)) = \mathrm{d}j(n)(\mathrm{T}_n(N))$ inside $\mathrm{T}_{j(n)}(M)$ for all $n \in N$; informally, $E(j(n)) = \mathrm{T}_n(N)$ inside $\mathrm{T}_n(M)$ for all $n \in N$. We say such an N is a *maximal* integral manifold to E if it is not contained in a strictly larger one.

The basic question we wish to address is this: when does a given E admit an integral submanifold through any point of M, and when do such integral submanifolds lie in uniquely determined maximal ones? In the special case that the subspaces $E(m) \subseteq \mathrm{T}_m(M)$ are lines, this generalizes a weakened form of the theory of integral curves to smooth vector fields (and their associated "maximal" integral curves): we're considering a situation similar to that of integral curves to *nowhere-vanishing* smooth vector fields, and in fact we're *ignoring* the specific vectors in the vector field and focusing only on the lines spanned by each vector. In particular, we are throwing away the information of the parameterization of the curve and focusing on its actual image inside M.

In contrast with the case of vector fields, for which integral curves *always* exist (at least for short time) through any point, when the vector spaces $E(m)$ have dimension larger than 1 it turns out to be a highly nontrivial condition for integral manifolds to exist through any point. The local and global Frobenius theorems will identify a sufficient condition for such existence, including the refinement of maximal integral manifolds to E through any point. This sufficient condition (which is also necessary, by an elementary argument that we omit) involves the bracket operation $[X, Y] = X \circ Y - Y \circ X$ on $(TM)(U) = \mathrm{Vec}_M(U)$ that is $\mathbf{R}$-bilinear (but *not* $C^\infty(U)$-bilinear!):

Definition H.3. A subbundle E of the tangent bundle TM is *integrable* when for all open sets $U \subseteq M$ and smooth vector fields $X, Y \in E(U)$ over U lying in the subbundle, we have $[X, Y] \in E(U)$.

This notion is only of interest when $E(m)$ has dimension larger than 1 for $m \in M$: if the $E(m)$'s are lines then locally any element of $E(U)$ is a $C^\infty(U)$-multiple of a single non-vanishing smooth vector field and hence the integrability condition is automatic: $[fX, gX] = (fX(g) - gX(f))X$ for smooth functions f and g on a common open set in M.

Remark H.2.4. The local and global Frobenius theorems will make precise the sense in which integrability of E is sufficient for E to be "integrated" in the sense of admitting an integral manifold through any point of m, and in fact a unique maximal one.

A particularly interesting example of an integrable subbundle is the following. Let $M = G$ be a Lie group and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$. In this case, we have a *global* trivialization of TG via the construction of left-invariant vector fields. More specifically, to any $v \in \mathfrak{g}$ we have the associated left-invariant smooth vector field $\widetilde{v}$ on G, and for a basis $\{v_i\}$ of $\mathfrak{g}$ the resulting collection $\{\widetilde{v}_i\}$ is a trivializing frame for TG over the entirety of G. In particular, for *any* open U in M we have an isomorphism

$$C^\infty(U) \otimes_{\mathbf{R}} \mathfrak{g} = \mathrm{Vec}_G(U) =: (TG)(U).$$

We define $\mathfrak{h}(m) \subseteq \mathrm{T}_m(M)$ to be $\mathrm{d}\ell_g(e)(\mathfrak{h})$ and define

$$\widetilde{\mathfrak{h}}(U) := C^\infty(U) \otimes_{\mathbf{R}} \mathfrak{h} \subseteq C^\infty(U) \otimes_{\mathbf{R}} \mathfrak{g} = \mathrm{Vec}_G(U).$$

Explicitly, $\widetilde{\mathfrak{h}}(U)$ is the set of $C^\infty(U)$-linear combinations of the U-restrictions of the left-invariant vector fields arising from $\mathfrak{h}$.

To justify this notation, let us check that a smooth vector field $X \in \mathrm{Vec}_M(U)$ lies in $\widetilde{\mathfrak{h}}(U)$ if and only if $X(u) \in \widetilde{\mathfrak{h}}(u)$ for all $u \in U$. Choose a basis $\{v_1, \ldots, v_n\}$ of $\mathfrak{g}$ extending a basis $\{v_1, \ldots, v_r\}$ of $\mathfrak{h}$. Clearly $\widetilde{\mathfrak{h}}(U)$ is a free $C^\infty(U)$-module on the basis $\widetilde{v}_1|_U, \ldots, \widetilde{v}_r|_U$, whereas $\mathrm{Vec}_G(U)$ is a free $C^\infty(U)$-module on the basis of all $\widetilde{v}_i|_U$, so the condition of membership in $\widetilde{\mathfrak{h}}(U)$ for a general smooth vector field X on U is that the unique $C^\infty(U)$-linear expansion $X = \sum a_i \widetilde{v}_i$ has $a_i = 0$ in $C^\infty(U)$ for all $i > r$. But such vanishing is equivalent to $a_i(u) = 0$ for all $i > r$ and $u \in U$, which in turn says precisely that $X(u) \in \widetilde{\mathfrak{h}}(u)$ inside $\mathrm{T}_u(M)$ for all $u \in U$.

So far we have not used that $\mathfrak{h}$ is a Lie subalgebra, just that it is a linear subspace of $\mathfrak{g}$. The Lie subalgebra property underlies:

Proposition H.2.5. *The subbundle $\widetilde{\mathfrak{h}}$ of TG is integrable.*

Proof. First observe that by construction if $v \in \mathfrak{h} \subseteq \mathfrak{g}$ then the associated left-invariant vector field $\widetilde{v}$ on G lies in $\widetilde{\mathfrak{h}}(G) \subseteq \mathrm{Vec}_G(G)$ since this can be checked pointwise on G. If $v_1, \ldots, v_n$ is a basis of $\mathfrak{h}$ then $\widetilde{v}_1, \ldots, \widetilde{v}_n$ is a global trivializing frame for $\widetilde{\mathfrak{h}}$.

In general, to prove integrability of a subbundle of the tangent bundle it suffices to prove that the bracket operation applied to members of a trivializing frame over the constituents of an open covering of the base space yields output that is a section of the subbundle. (Why?) In our case there is the global trivializing frame $\widetilde{v}_1, \ldots, \widetilde{v}_n$ of $\widetilde{\mathfrak{h}}$, so to prove integrability of $\widetilde{\mathfrak{h}} \subseteq TG$ we just have to prove that $[\widetilde{v}_i, \widetilde{v}_j] \in \widetilde{\mathfrak{h}}(G)$ inside of $(TG)(G) = \mathrm{Vec}_G(G)$. But $\widetilde{v}_i$ and $\widetilde{v}_j$ are left-invariant vector fields on G, and by the very definition of the Lie algebra structure on $\mathfrak{g} = \mathrm{T}_e(G)$ in terms of the commutator operation on global left-invariant vector fields we have $[\widetilde{v}_i, \widetilde{v}_j] = [v_i, v_j]^\sim$.

That is, the bracket of $\widetilde{v}_i$ and $\widetilde{v}_j$ is equal to the left-invariant vector field associated to the tangent vector $[v_i, v_j] \in \mathfrak{g}$. But $v_i, v_j \in \mathfrak{h}$ and by hypothesis $\mathfrak{h}$ *is a Lie subalgebra of* $\mathfrak{g}$. Hence, $[v_i, v_j] \in \mathfrak{h}$, so by construction when this is propagated to a left-invariant vector field on G the resulting global vector field lies in $\widetilde{\mathfrak{h}}(G)$ inside $\mathrm{Vec}_G(G)$ (as may be checked pointwise on G). $\square$

We shall see later in this appendix that a maximal integral submanifold H in G through e to the integrable subbundle $\widetilde{\mathfrak{h}}$ is a connected *Lie subgroup* of G (by which we mean an injective immersion of Lie groups $i : H \to G$ that respects the group structures), and that its associated Lie subalgebra $\mathrm{Lie}(H) \subseteq \mathfrak{g}$ is the initial choice of Lie subalgebra $\mathfrak{h}$.

In what follows we shall state the local and global Frobenius theorems, discuss some aspects of the proofs, and work out the application to the existence and uniqueness of a *connected* Lie subgroup H of a Lie group G such that $\mathrm{Lie}(H) \subseteq \mathrm{Lie}(G)$ coincides with a given Lie subalgebra $\mathfrak{h} \subseteq \mathrm{Lie}(G)$.

H.3. **Statement of main results.** If E is a subbundle of TM then the existence of local trivializing frames implies that $\dim E(m)$ is locally constant in m, and so is constant on each connected component of M. In practice one often focuses on E of constant rank (i.e.,

$\dim E(m)$ is the same for all $m \in M$), though passage to that case is also achieved by passing to separate connected components of M.

Here is the local Frobenius theorem.

Theorem H.3.1 (Frobenius)**.** *Let E be an integrable subbundle of TM with* $\dim E(m) = r$ *for all $m \in M$ with some $r > 0$. There exists a covering of M by C^∞ charts (U, φ) with $\varphi = \{x_1, \ldots, x_n\}$ a C^∞ coordinate system satisfying $\varphi(U) = \prod(a_i, b_i) \subseteq \mathbf{R}^n$ such that the embedded r-dimensional slice submanifolds $\{x_i = c_i\}_{i>r}$ for $(c_{r+1}, \ldots, c_n) \in \prod_{i>r}(a_i, b_i)$ are integral manifolds for E. Moreover, all (connected!) integral manifolds for E in U lie in a unique such slice set-theoretically, and hence lie in these slices as C^∞ submanifolds of U.*

Geometrically, the local coordinates in the theorem have the property that E is the subbundle spanned by the vector fields $\partial_{x_1}, \ldots, \partial_{x_r}$. The *proof* of this local theorem proceeds by induction on the rank r of E, and to get the induction started in the case $r = 1$ it is necessary to prove a local theorem concerning a non-vanishing vector field (chosen to locally trivialize the line subbundle E in TM):

Theorem H.3.2. *For any non-vanishing smooth vector field on an open subset of a smooth manifold, there are local coordinate systems in which the vector field is ∂_{x_1}.*

We give a complete proof of this theorem in §H.5, using the technique of vector flow from the theory of integral curves. This base case for the inductive proof of the local Frobenius theorem uses the entire force of the theory of ODE's, especially smooth dependence of solutions on varying initial conditions. Given such a local coordinate system as in Theorem H.3.2, it is clear from the existence and uniqueness of integral curves for vector fields that the x_1-coordinate lines in a coordinate box (all other coordinates held fixed) *do* satisfy the requirements of the local Frobenius integrability theorem in the case of rank 1. That is, Theorem H.3.2 does settle the rank 1 case of the local Frobenius theorem. The general geometric inductive proof of the local Frobenius theorem, building on the special case for rank 1, is given in section 1.60 in Warner's book "Foundations of differentiable manifolds and Lie groups".

We now turn to the statement of the global Frobenius theorem (see sections 1.62 and 1.64 in Warner's book). We state it in a slightly stronger form than in Warner's book (but his proof yields the stronger form, as we will explain), and it is certainly also stronger than the version in many other references (which is why we prefer to reference Warner's book for the proof):

Theorem H.3.3 (Frobenius)**.** *Let E be an integrable subbundle of TM.*

(1) For all $m \in M$, there is a (unique) maximal integral submanifold $i : N \hookrightarrow M$ through m_0.
(2) For any C^∞ mapping $M' \to M$ landing in $i(N)$ set-theoretically, the unique factorization $M' \to N$ is continuous and hence smooth.
(3) Any connected *submanifold $i' : N' \hookrightarrow M$ satisfying $\mathrm{T}_{n'}(N') \subseteq E(i'(n'))$ for all $n' \in N'$ lies in a maximal integral submanifold for E.*

Note that in (3), we allow for the possibility that N' might be "low-dimensional" with respect to the rank of E, and so it is a definite strengthening of the property of maximal integral submanifolds for E in M (which are only required to be maximal with respect to other integral submanifolds for E in M, not with respect to connected submanifolds whose tangent spaces are pointwise just contained in – rather than actually equal to – the corresponding fiber of E).

To appreciate how special the automatic continuity is in (2) even if i is *not* a topological embedding (as often occurs even in Lie-theoretic settings as in Theorem H.4.3), see Example H.2.1. Also, in (2) we do not require that the map from M' to M be injective. The deduction of smoothness from continuity in (2) follows from the fact that the only obstruction to smoothness for a C^∞ map factoring set-theoretically through an injective immersion is topological (i.e., once the first step of the factorization is known to be continuous, the immersion theorem can be used *locally on the source* to infer its smoothness).

In Warner's book, the above global theorem is proved except that he omits (3). However, his proof of the "maximal integral submanifold" property in (1) does not use the "maximal dimension" condition on the connected submanifold source, and so it actually proves (3). We will use (3) at one step below.

H.4. **Applications to Lie subgroups and homomorphisms.** Before we turn to the task of proving Theorem H.3.2, let us explain how to use the global Frobenius theorem to prove a striking result on the existence of connected Lie subgroups realizing a given Lie subalgebra as its Lie algebra. First, a definition:

Definition H.1. A *Lie subgroup* of a Lie group G is a subgroup H equipped with a C^∞ submanifold structure that makes it a Lie group.

In other words, a Lie subgroup "is" (up to unique isomorphism) an injective immersion $i : H \to G$ of Lie groups with i a group homomorphism. The example of the real line densely wrapped around the torus by the mapping $i : \mathbf{R} \to S^1 \times S^1$ defined by $t \mapsto ((\cos t, \sin t), (\cos(bt), \sin(bt)))$ for b not a rational multiple of π is a Lie subgroup that is *not* an embedded submanifold.

As we have seen in class, if $f : H \to G$ is a map of Lie groups (i.e., smooth map of manifolds that is also a group homomorphism) then $(\mathrm{d}f)(e_H) : \mathrm{T}_{e_H}(H) \to \mathrm{T}_{e_G}(G)$ respects the brackets on both sides (i.e., it is a "Lie algebra" map). Hence, in the immersion case we get $\mathrm{Lie}(H)$ as a Lie subalgebra of $\mathrm{Lie}(G)$.

Remark H.4.2. The passage between Lie subalgebras and Lie subgroups pervades many arguments in the theory of (non-compact) Lie groups. In particular, as we will make precise in the next theorem, any connected Lie subgroup $i : H \to G$ is uniquely determined (with its topology and C^∞ structure!) by the associated Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Beware that in general it is very hard to tell in terms of $\mathfrak{h}$ (in an abstract situation) whether or not H is an embedded submanifold (i.e., has the subspace topology), in which case it turns out to be necessarily a *closed* submanifold. However, there are some convenient criteria on a Lie subalgebra $\mathfrak{h}$ in $\mathrm{Lie}(G)$ that are sufficient to ensure closedness. For example, if G is closed in some $\mathrm{GL}_n(\mathbf{R})$ and the subspace in $\mathfrak{h}$ spanned by all "brackets" $[x, y]$ with $x, y \in \mathfrak{h}$ is equal to $\mathfrak{h}$ then closedness is automatic (this implication is not at all obvious). It may seem that this criterion for closedness is a peculiar one, but it is actually a rather natural one from the perspective of the general structure theory of semisimple Lie algebras. Moreover, in practice it is a very mild condition.

Theorem H.4.3. *Let G be a Lie group, with Lie algebra $\mathfrak{g}$. For every Lie subalgebra $\mathfrak{h}$ there exists a unique connected Lie subgroup H in G with Lie algebra $\mathfrak{h}$ inside of $\mathfrak{g}$. Moreover, if H and H' are connected Lie subgroups then* $\mathrm{Lie}(H) \subseteq \mathrm{Lie}(H')$ *if and only if $H \subseteq H'$ as subsets of G, in which case the inclusion is C^∞.*

Before we explain the proof of this theorem (using the Frobenius theorems), we make some comments. The connectivity is crucial in the theorem. For example, the closed subgroup $\mathrm{O}_n(\mathbf{R})$ of orthogonal matrices in $\mathrm{GL}_n(\mathbf{R})$ for the standard inner product is a Lie subgroup (even a closed submanifold), but it is disconnected with identity component given by the index-2 open subgroup $\mathrm{SO}_n(\mathbf{R})$ of orthogonal matrices with determinant 1. Both $\mathrm{O}_n(\mathbf{R})$ and $\mathrm{SO}_n(\mathbf{R})$ agree near the identity inside of $\mathrm{GL}_n(\mathbf{R})$, so they give the same Lie subalgebra of $\mathfrak{gl}_n(\mathbf{R})$ (consisting of the skew-symmetric matrices in $\mathfrak{gl}_n(\mathbf{R}) = \mathrm{Mat}_{n\times n}(\mathbf{R})$). Beware that there are non-injective immersions of Lie groups, such as $\mathrm{SL}_2(\mathbf{R}) \to \mathrm{SL}_2(\mathbf{R})/\langle -1\rangle$ that induce isomorphisms of Lie algebras. Hence, the passage between the isomorphism problem for connected Lie groups and for Lie algebras is a little subtle and we will not get into it here. The moral of the story is that a good understanding of the structure of $\mathrm{Lie}(G)$ as a Lie algebra does encode a lot of information about the Lie group G. In this way, the structure theory of finite-dimensional Lie algebras over $\mathbf{R}$ (which is a purely algebraic theory that makes sense over any field, though is best behaved in characteristic 0) plays a fundamental role in the theory of Lie groups.

Proof. Let $i : H \to G$ be an arbitrary connected Lie subgroup. Since the inclusion $i : H \to G$ is a group homomorphism and hence is compatible with left translations by elements of H, it follows that for $h \in H$ the mapping $(\mathrm{d}\ell_{i(h)})(e_G)$ carries $\mathrm{T}_{e_H}(H) \subseteq \mathrm{T}_{e_G}(G)$ (inclusion via $(\mathrm{d}i)(e_H)$) over to the subspace $\mathrm{T}_h(H) \subseteq \mathrm{T}_{i(h)}(G)$ (inclusion via $(\mathrm{d}i)(h)$). In other words (since $\dim(\mathrm{Lie}(H)) = \dim H$), the connected submanifold H is an integral manifold for the integrable subbundle of $TG \simeq G \times \mathfrak{g}$ given by $G \times \mathrm{Lie}(H)$. It therefore follows that the integral manifold H must factor smoothly through the maximal integral submanifold through e_G for the subbundle $G \times \mathrm{Lie}(H)$ in TG. In particular, once we know that H agrees with this maximal integral submanifold we will get the assertion that one Lie subgroup factors smoothly through another if and only if there is a corresponding inclusion of their Lie algebras inside of $\mathfrak{g}$ (as an inclusion of such Lie subalgebras forces a corresponding inclusion of subbundles of TG, and hence a smooth inclusion of maximal integral submanifolds through e_G by the *third part* of the global Frobenius theorem). This gives the uniqueness (including the manifold structure!) for a connected Lie subgroup of G with a specified Lie algebra inside of $\mathfrak{g}$.

Our problem is now reduced to: given a Lie subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ we seek to prove that the maximal integral submanifold H for the integrable subbundle $\widetilde{\mathfrak{h}}$ of TG is the unique connected Lie subgroup of G with $\mathfrak{h}$ as Lie algebra. First, we prove that this maximal integral submanifold H is in fact a Lie subgroup. That is, we must prove that H is algebraically a subgroup of G and then that the induced group law and inversion mappings are smooth for the manifold structure on H (and on $H \times H$). The stability of H under the group law and inversion will use the maximality, and the uniqueness will use a trick for connected groups.

Pick $h \in H$. We want $hH \subseteq H$. In other words, if $i : H \to G$ is the inclusion for H as a submanifold of G, we want the composite injective immersive mapping $\ell_{i(h)} \circ i : H \to G$ to factor through $i : H \to G$ set-theoretically (but we'll even get such a factorization smoothly). To make the picture a little clearer, instead of considering the maps $\ell_{i(h)}$ that are C^∞ automorphisms of the manifold G, let us consider a general smooth automorphism φ of a general manifold M and a general integrable subbundle $E \subseteq TM$. The mapping $\mathrm{d}\varphi$ is an automorphism of TM over φ, so $(\mathrm{d}\varphi)(E)$ is a subbundle of TM, and if N is an integral manifold in M for E then the submanifold $\varphi(N)$ is clearly an integral manifold for $(\mathrm{d}\varphi)(E)$ in M. If N is a maximal integral manifold for E then the integral manifold $\varphi(N)$ must be *maximal* for the subbundle $(\mathrm{d}\varphi)(E)$. Indeed, if it is not maximal then (by the global Frobenius theorem!) $\varphi(N) \to M$ factors smoothly through a strictly larger integral submanifold $N' \to M$ for the subbundle $(\mathrm{d}\varphi)(E)$, and so applying φ^{-1} then gives $\varphi^{-1}(N')$ as an integral submanifold for E in M that strictly contains N, contradicting the assumed maximality of N. (Here we have used that $\mathrm{d}\varphi$ and $\mathrm{d}\varphi^{-1}$ are inverse maps on TM with matrices having C^∞ entries over local coordinate domains, as follows from the Chain Rule.)

Now in our special situation, the integrable subbundle $E := \widetilde{\mathfrak{h}}$ of TG satisfies $(\mathrm{d}\ell_g)(E) = E$ for all $g \in G$ in the sense that $(\mathrm{d}\ell_g)(g')$ carries $E(g')$ to $E(gg')$ inside of $\mathrm{T}_{gg'}(G)$. This holds because the subspaces $E(g) \subset \mathrm{T}_g(G)$ were *constructed* using the left translation maps on tangent spaces. Hence, the preceding generalities imply that ℓ_g carries maximal integral manifolds for $E = \widetilde{\mathfrak{h}}$ to maximal integral manifolds for E. In particular, for the maximal integral manifold $i : H \to G$, we conclude that $\ell_{i(h)} \circ i : H \to G$ is also a maximal integral manifold for E. But the image of the latter contains the point $he = h \in i(H)$, so these two integral submanifold touch! Hence, by uniqueness of maximal integral manifolds through any single point (such as that point of common touch) they must coincide as submanifolds, which is to say that left multiplication by h on G carries H smoothly isomorphically back to itself (as a smooth submanifold of G).

This not only proves that H is algebraically a subgroup, but also that for all $h \in H$ the left multiplication mapping on G restricts to a bijective self-map (even smooth automorphism) of H. Since the identity e lies in H, it follows that $hh' = e$ for some $h' \in H$, which is to say that the unique inverse $h^{-1} \in G$ lies in H. That is, H is stable under inversion, and so it is algebraically a subgroup of G. If we let $\mathrm{inv} : G \simeq G$ be the smooth inversion mapping, then this says that the composite of inv with the smooth inclusion of H into G lands in the subset $H \subseteq G$ set-theoretically. Hence, by (2) in the global Frobenius theorem (applied to the maximal integral manifold H for E in G) we conclude that $\mathrm{inv}|_H : H \to G$ factors *smoothly* through the inclusion of H into G, which is to say that inversion on the subgroup H of G is a *smooth* self-map of the manifold H.

To conclude that H is a Lie subgroup, it remains to check smoothness for the group law. That is, we want the composite smooth mapping

$$H \times H \stackrel{i \times i}{\to} G \times G \to G$$

(the second step being the smooth group law of G) to factor smoothly through the inclusion i of H into G. But it does factor through this inclusion set-theoretically because H is a subgroup of G, and so again by (2) in the Frobenius theorem we get the desired smooth factorization. Hence, H is a Lie subgroup of G.

Finally, we have to prove the uniqueness aspect: if H' is a connected Lie subgroup of H with Lie algebra equal to the Lie algebra $\mathfrak{h}$ of H inside of $\mathfrak{g}$, then we want $H' = H$ as Lie subgroups of G. The discussion at the beginning of the proof shows that H' must at least smoothly factor through the maximal integral submanifold through the identity for the integrable subbundle $E = \widetilde{\mathfrak{h}} \subseteq TG$, which is to say that H' factors smoothly through H. Hence, we have a smooth injective immersion $H' \hookrightarrow H$ (as submanifolds of G) and we just need this to be an isomorphism. Any Lie group has the same dimension at all points (due to left translation automorphisms that identify all tangent spaces with the one at the identity), so H' and H have the same dimension at all points (as their tangent spaces at the identity coincide inside of $\mathfrak{g}$). Thus, the injective tangent mappings for the immersion $H' \to H$ are isomorphisms for dimension reasons, so the injective map $H' \to H$ is a local C^∞ isomorphism by the inverse function theorem! As such, it has *open* image onto which it is bijective, so H' is an open submanifold of H and thus is an open Lie subgroup of H.

Now comes the magical trick (which is actually a powerful method for proving global properties of a connected group): a *connected* topological group (such as H) has no proper open subgroups. This will certainly force the open immersion $H' \to H$ to be surjective and thus $H' = H$ as Lie subgroups of G. Rather more generally, an open subgroup of a topological group is always *closed* (giving what we need in the connected case). To see closedness, it is equivalent to prove openness of the complement, and by group theory we know that the complement of a subgroup of a group is a disjoint union of left cosets. Since any coset for an open subgroup is open (as it is an image of the open subgroup under a left-translation map that is necessarily a homeomorphism), any union of such cosets is open. □

In [BtD, Ch. I, Thm. 3.11], the exponential map in the theory of Lie groups (that has no logical dependence on the "Lie subgroups to Lie subalgebras" correspondence discussed in this appendix) is used to prove that closed subgroups of Lie groups are smooth submanifolds (with the subspace topology). Here is a further application of the exponential map:

Proposition H.4.4. *Let $f : G \to G'$ be a Lie group homomorphism. For any closed Lie subgroup $H' \subseteq G'$, the preimage $f^{-1}(H')$ is a closed Lie subgroup of G, with Lie algebra equal to $\mathrm{Lie}(f)^{-1}(\mathfrak{h}') \subseteq \mathfrak{g}$. In particular, taking $H' = \{e'\}$, $\ker f$ is a closed Lie subgroup of G with Lie algebra $\ker(\mathrm{Lie}(f))$.*

If G_1 and G_2 are closed Lie subgroups of G then the closed Lie subgroup $G_1 \cap G_2$ has Lie algebra $\mathfrak{g}_1 \cap \mathfrak{g}_2$ inside $\mathfrak{g}$.

Beware that in the second part of this proposition, G_1 and G_2 do *not* necessarily have transverse intersection inside G at their common points. A counterexample is $G = \mathbf{R}^4$ with G_1 and G_2 planes through the origin that share a common line. Consequently, it is remarkable that $G_1 \cap G_2$ is always a submanifold (with the subspace topology).

Proof. The closed Lie subgroup $f^{-1}(H')$ in G is a submanifold (with the subspace topology), but it is not obvious that its tangent space exhausts $\mathrm{Lie}(f)^{-1}(\mathfrak{h}')$ (rather than being a proper subspace). So we forget the manifold structure on $f^{-1}(H')$ and aim to construct a connected Lie subgroup that *does* have the desired Lie algebra, and eventually inferring that it recovers $f^{-1}(H')^0$.

Since $\mathrm{Lie}(f)^{-1}(\mathfrak{h}')$ is a Lie subalgebra of G, it has the form $\mathrm{Lie}(H)$ for a unique connected Lie subgroup H of G (which we do *not* yet know to have the subspace topology from G). We want to show that the Lie group homomorphism $f|_H : H \to G'$ smoothly factors through the connected Lie subgroup H'^0. By the mapping property for H'^0 as the maximal integral submanifold at e' to the subbundle $\widetilde{\mathfrak{h}}'$ of $T(G')$ (see the second part of the global Frobenius theorem), it suffices to show that $f|_H$ factors through H' (and hence H'^0) set-theoretically.

It is enough to show that an open neighborhood U of e in H is carried by f into H', as H is generated by U algebraically (since H is *connected*, so it has no open subgroup aside from itself). But consider the commutative diagram of exponential maps

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "\mathrm{Lie}(f)"] \arrow[d, "\exp_H"'] & \mathfrak{g}' \arrow[d, "\exp_{G'}"] \\
H \arrow[r, "f|_H"'] & G'
\end{tikzcd}$$

with $\mathfrak{h} := \mathrm{Lie}(H) = \mathrm{Lie}(f)^{-1}(\mathfrak{h}')$. The right vertical map carries an open neighborhood of 0 in $\mathfrak{h}'$ diffeomorphically onto an open neighborhood of e' in the connected Lie subgroup H'^0, since $\exp_{G'}|_{H'} = \exp_{H'}$ (functoriality of the exponential map). But $\mathrm{Lie}(f)$ carries $\mathfrak{h}$ into $\mathfrak{h}'$, so since $\exp_H$ carries an open neighborhood of 0 in $\mathfrak{h}$ diffeomorphically onto an open neighborhood of e in H we see from the commutativity of the diagram that $f|_H$ must carry an open neighborhood of e in H into H'^0 as required. This completes the proof that $f|_H$ factors through H'^0, so $H \subseteq f^{-1}(H'^0) \subseteq f^{-1}(H')$. Note that we do *not* yet know if H has the subspace topology from G!

Recall that $f^{-1}(H')$ is a closed submanifold of G (with its subspace topology). The injective immersion $H \hookrightarrow f^{-1}(H')^0$ between connected Lie groups is an isomorphism on Lie algebras since we have the reverse inclusion

$$\mathrm{Lie}(f^{-1}(H')^0) = \mathrm{Lie}(f^{-1}(H')) \subseteq \mathrm{Lie}(f)^{-1}(H') =: \mathfrak{h}$$

(the middle inclusion step due to the fact that $f : G \to G'$ carries $f^{-1}(H')$ into H'), so $H = f^{-1}(H')^0$. It follows that $\mathrm{Lie}(f^{-1}(H')) = \mathfrak{h} = \mathrm{Lie}(f)^{-1}(H')$, as desired.

Now we turn to the behavior of intersections for closed Lie subgroups G_1 and G_2 of G. By the preceding generalities applied to $G_1 \hookrightarrow G$ in the role of f and to the closed Lie subgroup G_2 of G, the preimage $G_1 \cap G_2$ is a closed Lie subgroup of G whose Lie algebra is the preimage of $\mathfrak{g}_2$ under the inclusion $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$. This says exactly that $\mathfrak{g}_1 \cap \mathfrak{g}_2 = \mathrm{Lie}(G_1 \cap G_2)$. $\square$

Corollary H.4.5. *Let $\{G_\alpha\}$ be closed Lie subgroups of a Lie group G. The intersection $H = \bigcap G_\alpha$ is a closed Lie subgroup of G (with the subspace topology) and its Lie algebra is $\bigcap \mathfrak{g}_\alpha$.*

Proof. By finite-dimensionality of $\mathfrak{g}$, we can find a finite collection of indices $\alpha_1, \dots, \alpha_n$ so that

$$\bigcap \mathfrak{g}_\alpha = \mathfrak{g}_{\alpha_1} \cap \cdots \cap \mathfrak{g}_{\alpha_n}.$$

The previous proposition ensures that *finite* intersections among closed Lie subgroups are closed Lie subgroups with the expected Lie algebra, so $G' := G_{\alpha_1} \cap \cdots \cap G_{\alpha_n}$ is a closed Lie subgroup of G with $\mathfrak{g}' = \cap_\alpha \mathfrak{g}_\alpha$. Hence, for any α outside of the α_i's, the intersection $G' \cap G_\alpha$ is a closed Lie subgroup whose Lie algebra is the same as that of G', so its identity component coincides with that of G'. In other words, $G' \cap G_\alpha$ is a union of cosets of G'^0 inside G'. Thus, intersecting over all such α, we conclude that H is a union of G'^0-cosets inside G'. Any such union is an open and closed submanifold of G', so it is a closed submanifold of G (with the subspace topology) and its Lie algebra is $\mathrm{Lie}(G'^0) = \mathrm{Lie}(G') = \cap \mathfrak{g}_\alpha$. $\square$

Corollary H.4.6. *Let H be a closed Lie subgroup of G. The centralizer $Z_G(H)$ is a closed Lie subgroup of G, and* $\mathrm{Lie}(Z_G(H)) = \mathfrak{g}^{\mathrm{Ad}_G(H)}$.

If H is connected then the normalizer $N_G(H)$ is a closed Lie subgroup of G and

$$\mathrm{Lie}(N_G(H)) = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) := \{v \in \mathfrak{g} \mid [v, \mathfrak{h}] \subseteq \mathfrak{h}\}, \ \mathrm{Lie}(Z_G(H)) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) := \{v \in \mathfrak{g} \mid [v, \mathfrak{h}] = 0\}.$$

Before proving this corollary, we make some remarks. When defining $N_G(H)$, the equality $gHg^{-1} = H$ (ensures stability of $N_G(H)$ under inversion!) is equivalent to the inclusion $gHg^{-1} \subseteq H$ (more convenient to use) for dimension reasons due to *connectedness* of H, and so also holds if $\pi_0(H)$ is finite; it is used without comment. The expressions $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ are the *Lie algebra normalizer* and *Lie algebra centralizer*. By the Jacobi identity, each is a Lie subalgebra of $\mathfrak{g}$.

Also, the equality $\mathrm{Lie}(Z_G(H)) = \mathfrak{g}^{\mathrm{Ad}_G(H)}$ can be proved very quickly using the exponential map: the inclusion "$\subseteq$" is obvious by functoriality of the tangent space at the identity since H-conjugation on G has no effect on $Z_G(H)$, and the reserve inclusion holds because any $v \in \mathfrak{g}^{\mathrm{Ad}_G(H)}$ is the velocity at $t = 0$ to the 1-parameter subgroup $\alpha_v : t \mapsto \exp_G(tv)$ that is visibly valued in the closed submanifold $Z_G(H)$ (by functoriality of α_v in (G, v)). We give a rather different proof of this equality below because we prefer arguments that minimize the role of the exponential map (that we have limited to its role in the proof of Proposition H.4.4) because such arguments are more robust for adapting to other contexts such as the algebro-geometric study of linear algebraic groups over general fields (where one has a result like Proposition H.4.4 in characteristic 0 for purely algebraic reasons unrelated to an "exponential map").

Finally, the equality $\mathrm{Lie}(Z_G(H)) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ generally *fails* when H is not assumed to be connected (e.g., consider discrete H). Likewise, if $\pi_0(H)$ is infinite then the condition "$gHg^{-1} \subseteq H$" on g is generally not equivalent to "$gHg^{-1} = H$" (equivalently, this inclusion condition is not stable under inversion on g); an example is $G \subset \mathrm{GL}_2(\mathbf{R})$ the closed subgroup of upper-triangular matrices, H the subgroup of unipotent matrices $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ with $n \in \mathbf{Z}$, and diagonal $g = \mathrm{diag}(m, 1)$ for $m \in \mathbf{Z} - \{0, 1, -1\}$.

Proof. The closedness is clear since G is Hausdorff and H is closed in G. Clearly $Z_G(H) = \bigcap_{h \in H} Z_G(h)$, so by Corollary H.4.5 we have

$$\mathrm{Lie}(Z_G(H)) = \bigcap_{h \in H} \mathrm{Lie}(Z_G(h)).$$

Since $\mathfrak{g}^{\mathrm{Ad}_G(H)} = \bigcap_{h \in H} \mathfrak{g}^{\mathrm{Ad}_G(h)=1}$, to prove the equality $\mathrm{Lie}(Z_G(H)) = \mathfrak{g}^{\mathrm{Ad}_G(H)}$ it suffices to show that $\mathrm{Lie}(Z_G(g)) = \mathfrak{g}^{\mathrm{Ad}_G(g)=1}$ for any $g \in G$. The centralizer $Z_G(g)$ is the preimage of the diagonal $\Delta : G \hookrightarrow G \times G$ under the Lie group homomorphism $f : G \to G \times G$ given by $x \mapsto (gxg^{-1}, x)$. Thus, by Proposition H.4.4, $\mathrm{Lie}(Z_G(g))$ is the preimage of the diagonal $\mathrm{Lie}(\Delta) : \mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$ under $\mathrm{Lie}(f) = (\mathrm{Ad}_G(g), \mathrm{id})$. This preimage is exactly $\mathfrak{g}^{\mathrm{Ad}_G(g)=1}$.

Now we assume H is connected for the rest of the argument. The conjugation action of $N_G(H)$ on G carries H into itself, so the restriction of Ad_G to $N_G(H)$ carries $\mathfrak{h}$ into itself. That is,

$$\mathrm{Ad}_G : N_G(H) \to \mathrm{GL}(\mathfrak{g})$$

lands inside the closed subgroup $\mathrm{Stab}(\mathfrak{h})$ of linear automorphisms that carry $\mathfrak{h}$ into itself. Thus, the map $\mathrm{ad}_{\mathfrak{g}} = \mathrm{d}(\mathrm{Ad}_G)(e)$ from $\mathfrak{g}$ into $\mathrm{End}(\mathfrak{g})$ carries $\mathrm{Lie}(N_G(H))$ into $\mathrm{Lie}(\mathrm{Stab}(\mathfrak{h}))$.

A computation with block upper triangular matrices inside $\mathrm{End}(\mathfrak{g})$ using a basis of $\mathfrak{g}$ extending a basis of $\mathfrak{h}$ shows that the Lie algebra of $\mathrm{Stab}(\mathfrak{h})$ is equal to the vector space of linear endomorphisms of $\mathfrak{g}$ that carry $\mathfrak{h}$ into itself. This establishes the containment "$\subseteq$"

for $N_G(H)$. Arguining similarly with $Z_G(H)$-conjugation on G that restricts to the identity on H, and replacing $\mathrm{Stab}(\mathfrak{h})$ with the closed subgroup $\mathrm{Fix}(\mathfrak{h})$ of elements of $\mathrm{GL}(\mathfrak{g})$ that fix $\mathfrak{h}$ pointwise, we deduce $\mathrm{Lie}(Z_G(H)) \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$.

So far we have not used that H is connected! We explain how to establish the desired equality for $N_G(H)$ by using the connectedness of H, and the same method works (check!) to show that $\mathrm{Lie}(Z_G(H)) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$. Consider the connected Lie subgroup N' of G whose Lie algebra is the Lie subalgebra $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{g}$. Beware that we do not yet know that N' has the subspace topology (equivalently, is closed in G). Since $N_G(H)^0$ and N' are connected Lie subgroups of G and

$$\mathrm{Lie}(N_G(H)^0) = \mathrm{Lie}(N_G(H)) \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathrm{Lie}(N'),$$

we have $N_G(H)^0 \subseteq N'$. Hence, to prove the desired equality of Lie algebras we just have to show that $N' \subseteq N_G(H)$ as subsets of G. That is, for $n' \in N'$ we want that $c_{n'} : G \simeq G$ carries H into itself. By the connectedness of H, $c_{n'}(H) \subseteq H$ if and only if there is such a containment on Lie algebras, which is to say $\mathrm{Ad}_G(n')(\mathfrak{h}) = \mathfrak{h}$ inside $\mathfrak{g}$. In other words, we want that $\mathrm{Ad}_{G'}$ carries N' into $\mathrm{Stab}(\mathfrak{h})$. But since N' is connected by design, it is equivalent to show that the map $\mathrm{ad}_{\mathfrak{g}} = \mathrm{d}(\mathrm{Ad}_G)(e)$ carries $\mathrm{Lie}(N')$ into the vector space $\mathrm{Lie}(\mathrm{Stab}(\mathfrak{h}))$ of linear endomorphisms of $\mathfrak{g}$ that carry $\mathfrak{h}$ into itself. But N' was created precisely so that $\mathrm{Lie}(N') = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$. $\square$

We conclude our tour through the dictionary between Lie groups and Lie algebras by posing a natural question: if G and G' are *connected* Lie groups, does any map of the associated Lie algebras $T : \mathfrak{g}' \to \mathfrak{g}$ ($\mathbf{R}$-linear respecting the brackets) necessarily arise from a map $f : G' \to G$ of Lie groups (smooth map of manifolds and group homomorphism)? If such an f exists then it is unique, but there is a topological obstruction to existence. To analyze this, we use the method of graphs.

To see the uniqueness, first note that $f : G' \to G$ gives rise to a smooth graph mapping $\Gamma_f : G' \to G' \times G$ (via $g' \mapsto (g', f(g'))$) that is a homeomorphism onto its image (using pr_1 as an inverse), and its image is closed (the preimage of the diagonal in $G \times G$ under the map $f \times \mathrm{id} : G' \times G \to G \times G$). Thus, Γ_f defines a connected *closed* Lie subgroup of $G' \times G$ when the latter is made into a group with product operations, and so it is a closed submanifold (as for any injective immersion between manifolds that is a homeomorphism onto a closed image).

Via the method of left-translations, the natural identification $\mathrm{T}_{(e',e)}(G' \times G) \simeq \mathrm{T}_{e'}(G') \oplus \mathrm{T}_e(G)$ carries the Lie bracket on the left over to the direct sum of the Lie brackets on the right. That is, $\mathrm{Lie}(G' \times G) \simeq \mathrm{Lie}(G') \oplus \mathrm{Lie}(G)$. In this way, the mapping

$$\mathrm{Lie}(\Gamma_f) : \mathrm{Lie}(G') \to \mathrm{Lie}(G' \times G) \simeq \mathrm{Lie}(G') \oplus \mathrm{Lie}(G)$$

is identified with the linear-algebra "graph" of the map $\mathrm{Lie}(f) = (\mathrm{d}f)(e') : \mathrm{Lie}(G') \to \mathrm{Lie}(G)$ that is assumed to be T. Hence, Γ_f corresponds to a connected Lie subgroup of $G' \times G$ whose associated Lie subalgebra in $\mathfrak{g}' \oplus \mathfrak{g}$ is the graph of the linear map T. By the uniqueness aspect of the passage from connected Lie subgroups to Lie subalgebras, it follows that the mapping $\Gamma_f : G' \to G' \times G$ is uniquely determined (if f is to exist), and so composing it with the projection $G' \times G \to G$ recovers f. This verifies the uniqueness of f.

How about existence? To this end, we try to reverse the above procedure: we use the injective graph mapping $\Gamma_T : \mathfrak{g}' \to \mathfrak{g}' \oplus \mathfrak{g}$ that is a mapping of Lie algebras precisely because T is a map of Lie algebras (and because the direct sum is given the "componentwise" bracket). By the general existence/uniqueness theorem, there is a unique connected Lie subgroup $H \subseteq G' \times G$ whose associated Lie subalgebra is the image of Γ_T. In particular, the first projection $H \to G'$ induces an isomorphism on Lie algebras, and if this mapping of connected Lie groups were an isomorphism then we could compose its inverse with the other projection $H \to G$ to get the desired mapping. (Conversely, it is clear that if the existence problem is to have an affirmative answer, then the first projection $H \to G'$ *must* be an isomorphism.) Hence, the problem is reduced to this: can a mapping $\pi : H \to G'$ between *connected* Lie groups induce an isomorphism on Lie algebras without being an isomorphism?

Such a mapping must be a local isomorphism near the identities (by the inverse function theorem), and so the image subgroup is open (as it contains an open in G' around the identity, and hence around all of its points via left translation in the image subgroup). But we have seen above that *connected* topological groups have no proper open subgroups, so the mapping π must be surjective. Also, $\ker(\pi)$ is a closed subgroup that meets a neighborhood of the identity in H in exactly the identity point (as π is a local isomorphism near identity elements), so the identity is an open point in $\ker(\pi)$. It follows by translations that the closed topological normal subgroup $\ker(\pi)$ must have the *discrete* topology. But if Γ is a discrete closed normal subgroup of a connected Lie group H then we can make the quotient H/Γ as a C^∞ manifold and in fact a Lie group. The induced C^∞ map $H/\Gamma \to G$ is a bijective Lie group map that is an isomorphism on Lie algebras and so (via translations!) is an isomorphism between tangent spaces at all points, whence by the inverse function theorem it is a C^∞ isomorphism (whence is an isomorphism of Lie groups).

In the general construction of quotients N'/N for Lie groups modulo closed subgroups, there is a covering of N'/N by open sets U_α over which the preimage in N' can be identified N-equivariantly with $U_\alpha \times N$ in such a way that the quotient map back to U_α is pr_1. Thus, the identification $H/\Gamma \simeq G$ and the discreteness of Γ imply that G admits an open covering $\{U_\alpha\}$ whose preimage in H is a disjoint union of copies of U_α (indexed by Γ). In topology, a *covering space* of a topological space X is a surjective continuous map $q : E \to X$ so that for some open covering $\{U_\alpha\}$ of X each preimage $q^{-1}(U_\alpha)$ is a disjoint union of copies of U_α. Thus, $H \to G$ is a *connected covering space*.

To summarize, we have found the precise topological obstruction to our problem: if G admits nontrivial connected covering spaces then there may be problems in promoting Lie algebra homomorphisms $\mathfrak{g} \to \mathfrak{g}'$ to Lie group homomorphisms $G \to G'$.

For a connected manifold, the existence of a nontrivial connected covering space is equivalent to the nontriviality of the fundamental group (this will be immediate from HW5 Exercise 3 and HW9 Exercise 3). Thus, for any connected Lie group G the map $\mathrm{Hom}(G, G') \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}')$ is bijective when $\pi_1(G) = 1$. (This holds for $\mathrm{SL}_n(\mathbf{C})$, $\mathrm{Sp}_{2n}(\mathbf{C})$, $\mathrm{SU}(n)$, and $\mathrm{Sp}(n)$, but fails for $\mathrm{SL}_n(\mathbf{R})$ and $\mathrm{SO}(n)$ for $n \geq 2$.)

H.5. **Vector fields and local coordinates.** We now turn to the task of proving Theorem H.3.2. First we consider a more general situation. Let M be a smooth manifold and let $\vec{v}_1, \dots, \vec{v}_n$ be pointwise linearly independent smooth vector fields on an open subset $U \subseteq M$ ($n \geq 1$). One simple example of such vector fields is $\partial_{x_1}, \dots, \partial_{x_n}$ on a coordinate domain for local smooth coordinates $\{x_1, \dots, x_N\}$ on an open set U in M. Can all examples be described in this way (locally) for suitable smooth coordinates?

Choose a point $m_0 \in U$. It is very natural (e.g., to simplify local calculations) to ask if there exists a local C^∞ coordinate system $\{x_1, \dots, x_N\}$ on an open subset $U_0 \subseteq U$ around

m_0 such that $\vec{v}_i|_{U_0} = \partial_{x_i}$ in $\mathrm{Vec}_M(U_0) = (TM)(U_0)$ for $1 \leq i \leq n$. The crux of the matter is to have such an identity across an entire open neighborhood of m_0. If we only work in the tangent space *at* the point m_0, which is to say we inquire about the identity $\vec{v}_i(m_0) = \partial_{x_i}|_{m_0}$ in $\mathrm{T}_{m_0}(U_0) = \mathrm{T}_{m_0}(M)$ for $1 \leq i \leq n$, then the answer is trivial (and not particularly useful): we choose local C^∞ coordinates $\{y_1, \dots, y_N\}$ near m_0 and write $\vec{v}_j(m_0) = \sum c_{ij}\partial_{y_i}|_{m_0}$, so the $N \times n$ matrix (c_{ij}) has independent columns. We extend this to an invertible $N \times N$ matrix, and then make a *constant* linear change of coordinates on the y_j's via the inverse matrix to get to the case $c_{ij} = \delta_{ij}$ for $i \leq n$ and $c_{ij} = 0$ for $i > n$. Of course, such new coordinates are only adapted to the situation at m_0. If we try to do the same construction by considering the matrix of functions (h_{ij}) with $\vec{v}_j = \sum h_{ij}\partial_{y_i}$ near m_0, the change of coordinates will now typically have to be *non-constant*, thereby leading to a big mess due to the appearance of differentiation in the transformation formulas for ∂_{t_i}'s with respect to change of local coordinates (having "non-constant" coefficients).

There is a very good reason why the problem over an open set (as opposed to at a single point) is complicated: usually no such coordinates exist! Indeed, if $n \geq 2$ then the question generally has a negative answer because there is an obstruction that is often non-trivial: since the commutator vector field $[\partial_{x_i}, \partial_{x_j}]$ vanishes for any i, j, if such coordinates are to exist around m_0 then the commutator vector fields $[\vec{v}_i, \vec{v}_j]$ must vanish near m_0. (Note that the concept of commutator of vector fields is meaningless when working on a single tangent space; it only has meaning when working with vector fields over open sets. This is "why" we had no difficulties when working at a single point m_0.)

For $n \geq 2$, the necessary condition of vanishing of commutators for pointwise independent vector fields usually fails. For example, on an open set $U \subseteq \mathbf{R}^3$ consider a pair of smooth vector fields

$$\vec{v} = \partial_x + f\partial_z, \quad \vec{w} = \partial_y + g\partial_z$$

for smooth functions f and g on U. These are visibly pointwise independent vector fields but

$$[\vec{v}, \vec{w}] = ((\partial_x g + f\partial_z g) - (\partial_y f + g\partial_z f))\partial_z,$$

so a necessary condition to have $\vec{v} = \partial_{x_1}$ and $\vec{w} = \partial_{x_2}$ for local C^∞ coordinates $\{x_1, x_2, x_3\}$ near $m_0 \in U$ is

$$\partial_x g + f\partial_z g = \partial_y f + g\partial_z f$$

near m_0. There is a special case in which the vanishing condition on the commutators $[\vec{v}_i, \vec{v}_j]$ for all i, j is vacuous: $n = 1$. Indeed, since $[\vec{v}, \vec{v}] = 0$ for any smooth vector field, in the case $n = 1$ we see no obvious reason why our question cannot always have an affirmative answer. The technique of vector flow along integral curves will prove such a result.

In the case $n = 1$, pointwise-independence for the singleton $\{\vec{v}_1\}$ amounts to pointwise non-vanishing. Hence, we may restate the goal: if $\vec{v}$ is a smooth vector field on an open set $U \subseteq M$ and $\vec{v}(m_0) \neq 0$ for some $m_0 \in U$ (so $\vec{v}(m) \neq 0$ for m near m_0, by continuity of $\vec{v} : U \to TM$), then there exists a local C^∞ coordinate system $\{x_1, \dots, x_N\}$ near m_0 in U such that $\vec{v} = \partial_{x_1}$ near m_0.

Example H.5.1. Consider the circular vector field $\vec{v} = -y\partial_x + x\partial_y$ on $M = \mathbf{R}^2$ with constant speed $r \geq 0$ on the circle of radius r centered at the origin. This vector field vanishes at the origin, but for $m_0 \neq (0, 0)$ we have $\vec{v}(m_0) \neq 0$. Let $U_0 = \mathbf{R}^2 - L$ for a closed half-line L emanating from the origin and not passing through m_0. For a suitable

θ_0, trigonometry provides a C^∞ parameterization $(0,\infty)\times(\theta_0,\theta_0+2\pi)\simeq U_0$ given by $(r,\theta)\mapsto(r\cos\theta, r\sin\theta)$, and $\partial_\theta=\vec{v}|_{U_0}$. Thus, in this special case we get lucky: we already "know" the right coordinate system to solve the problem. But what if we didn't already know trigonometry? How would we have been able to figure out the answer in this simple special case?

Example H.5.2. In order to appreciate the non-trivial nature of the general assertion we are trying to prove, let us try to prove it in general "by hand" (i.e., using just basic definitions, and no substantial theoretical input such as the theory of vector flow along integral curves). We shrink U around m_0 so that there exist local C^∞ coordinates $\{y_1,\dots,y_N\}$ on U. Hence, $\vec{v}=\sum h_j\partial_{y_j}$, and since $\vec{v}(m_0)=\sum h_j(m_0)\partial_{y_j}|_{m_0}$ is nonzero, we have $h_j(m_0)\neq 0$ for some j. By relabelling, we may assume $h_1(m_0)\neq 0$. By shrinking U around m_0, we may assume h_1 is non-vanishing on U (so $\vec{v}$ is non-vanishing on U). We wish to find a C^∞ coordinate system $\{x_1,\dots,x_N\}$ near m_0 inside of U such that $\vec{v}=\partial_{x_1}$ near m_0.

What conditions are imposed on the x_i's in terms of the y_j's? For any smooth coordinate system $\{x_i\}$ near m_0, $\partial_{y_j}=\sum(\partial_{y_j}x_i)\partial_{x_i}$ near m_0, so near m_0 we have

$$\vec{v}=\sum_j h_j\sum_i(\partial_{y_j}x_i)\partial_{x_i}=\sum_i(\sum_j h_j\partial_{y_j}(x_i))\partial_{x_i}.$$

Thus, the necessary and sufficient conditions are two-fold: $x_1,\dots,x_N$ are smooth functions near m_0 such that $\det((\partial_{y_j}x_i)(m_0))\neq 0$ (this ensures that the x_i's are local smooth coordinates near m_0, by the inverse function theorem) and

$$\sum_j h_j\partial_{y_j}(x_i)=\delta_{ij}$$

for $1\le i\le N$. This is a system of linear first-order PDE's in the N unknown functions $x_i=x_i(y_1,\dots,y_N)$ near m_0. We have already seen that the theory of first-order linear ODE's is quite substantial, and here were are faced with a PDE problem. Hence, our task now looks to be considerably less straightforward than it may have seemed to be at the outset.

The apparent complications are an illusion: it is because we have written out the explicit PDE's in local coordinates that things look complicated. As will be seen in the proof below, when we restate our problem in *geometric* language the idea for how to solve the problem essentially drops into our lap without any pain at all.

The fundamental theorem is this (a restatement of Theorem H.3.2):

Theorem H.5.3. *Let M be a smooth manifold and $\vec{v}$ a smooth vector field on an open set $U\subseteq M$. Let $m_0\in U$ be a point such that $\vec{v}(m_0)\neq 0$. There exists a local C^∞ coordinate system $\{x_1,\dots,x_N\}$ on an open set $U_0\subseteq U$ containing m_0 such that $\vec{v}|_{U_0}=\partial_{x_1}$.*

Proof. What is the geometric meaning of what we are trying to do? We are trying to find local coordinates $\{x_i\}$ an open open U_0 in U around m_0 so that the integral curves for $\vec{v}|_{U_0}$ are exactly flow along the x_1-direction at unit speed. That is, in this coordinate system for any point ξ near m_0 the integral curve for $\vec{v}$ through ξ is coordinatized as $c_\xi(t)=(t+x_1(\xi),x_2(\xi),\dots,x_N(\xi))$ for t near 0. This suggests that we try to find a local coordinate system around m_0 such that the first coordinate is "time of vector flow". The study of flow along integral curves in manifolds shows that for a sufficiently small open $U_0\subseteq U$ around m_0 there exists $\varepsilon>0$ such that for all $\xi\in U_0$ the maximal interval of definition for the integral curve c_ξ contains $(-\varepsilon, \varepsilon)$. More specifically, the vector-flow mapping

$$\Phi : \Omega \to M$$

defined by $(t, \xi) \mapsto c_\xi(t)$ (using t varying through the maximal open interval of definition I_ξ around 0 for each ξ) has *open* domain of definition $\Omega \subset \mathbf{R} \times M$ and is a smooth mapping on Ω. Thus, for small $\varepsilon > 0$ and small $U_0 \subseteq U$ around m_0, we have that $(-\varepsilon, \varepsilon) \times U_0$ is contained in Ω (as $\{0\} \times M \subseteq \Omega$). The mapping Φ, restricted to $(-\varepsilon, \varepsilon) \times U_0$, will be the key to creating a coordinate system on M near m_0 such that the time-of-flow parameter t is the first coordinate.

Here is the construction. We first choose an arbitrary smooth coordinate system $\phi : W \to \mathbf{R}^N$ on an open around m_0 that "solves the problem at m_0". That is, if $\{y_1, \ldots, y_N\}$ are the component functions of ϕ, then $\partial_{y_1}|_{m_0} = \vec{v}(m_0)$. This is the trivial pointwise version of the problem that we considered at the beginning of this appendix (and it has an affirmative answer precisely because the singleton $\{\vec{v}(m_0)\}$ in $\mathrm{T}_{m_0}(M)$ is an independent set; i.e., $\vec{v}(m_0) \neq 0$). Making a constant translation (for ease of notation), we may assume $y_j(m_0) = 0$ for all j. In general this coordinate system will fail to "work" at any other points, and we use vector flow to fix it. Consider points on the slice $W \cap \{y_1 = 0\}$ in M near m_0. In terms of y-coordinates, these are points $(0, a_2, \ldots, a_N)$ with small $|a_j|$'s. By openness of the domain of flow $\Omega \subseteq \mathbf{R} \times M$, there exists $\varepsilon > 0$ such that, after perhaps shrinking W around m_0, $(-\varepsilon, \varepsilon) \times W \subseteq \Omega$.

By the definition of the y_i's in terms of ϕ, $\phi(W \cap \{y_1 = 0\})$ is an open subset in $\{0\} \times \mathbf{R}^{N-1} = \mathbf{R}^{N-1}$, and ϕ restricts to a C^∞ isomorphism from the smooth hypersurface $W \cap \{y_1 = 0\}$ onto $\phi(W \cap \{y_1 = 0\})$. Consider the vector-flow mapping

$$\Psi : (-\varepsilon, \varepsilon) \times \phi(W \cap \{y_1 = 0\}) \to M$$

defined by

$$(t, a_2, \ldots, a_N) \mapsto \Phi(t, \phi^{-1}(0, a_2, \ldots, a_N)) = c_{\phi^{-1}(0, a_2, \ldots, a_N)}(t).$$

By the theory of vector flow, this is a *smooth* mapping. (This is the family of solutions to a first-order initial-value problem with varying initial parameters $a_2, \ldots, a_N$ near 0. Thus, the smoothness of the map is an instance of smooth dependence on varying initial conditions for solutions to first-order ODE's.) Geometrically, we are trying to parameterize M near m_0 by starting on the hypersurface $H = \{y_1 = 0\}$ in W (with coordinates given by the restrictions $y_2', \ldots, y_N'$ of $y_2, \ldots, y_N$ to H) and flowing away from H along the vector field $\vec{v}$; the time t of flow provides the first parameter in our attempted parameterization of M near m_0.

Note that $\Psi(0, 0, \ldots, 0) = c_{m_0}(0) = m_0$. Is Ψ a parameterization of M near m_0? That is, is Ψ a local C^∞ isomorphism near the origin? If so, then its local inverse near m_0 provides a C^∞ coordinate system $\{x_1, \ldots, x_N\}$ with $x_1 = t$ measuring time of flow along integral curves for $\vec{v}$ with their *canonical* parameterization (as integral curves). Thus, it is "physically obvious" that in such a coordinate system we will have $\vec{v} = \partial_{x_1}$ (but we will also derive this by direct calculation below). To check the local isomorphism property for Ψ near the origin, we use the inverse function theorem: we have to check $\mathrm{d}\Psi(0, \ldots, 0)$ is invertible. In terms of the local C^∞ coordinates $\{t, y_2', \ldots, y_N'\}$ near the origin on the source of Ψ and $\{y_1, \ldots, y_N\}$ near $m_0 = \Psi(0, \ldots, 0)$ on the target of Ψ, the $N \times N$ Jacobian matrix for $\mathrm{d}\Psi(0, \ldots, 0)$ has lower $(N-1) \times (N-1)$ block given by the identity matrix (i.e., $(\partial_{y'_j} y_i)(0,\dots,0) = \delta_{ij}$) because $\partial_{y'_j} y_i = \delta_{ij}$ at points on $W \cap \{y_1 = 0\}$ (check! It is *not* true at most other points of W).

What is the left column of the Jacobian matrix at $(0,\dots,0)$? Rather generally, if ξ is the point with y-coordinates $(t_0, a_2, \dots, a_N)$ then the t-partials $(\partial_t y_i)(t_0, a_2, \dots, a_N)$ are the coefficients of the velocity vector $c'_\xi(t_0)$ to the integral curve c_ξ of $\vec{v}$ at time t_0, and such a velocity vector is equal to $\vec{v}(c_\xi(t_0))$ by the *definition* of the concept of integral curve. Hence, setting $t_0 = 0$, $c'_\xi(0) = \vec{v}(c_\xi(0)) = \vec{v}(\xi)$, so taking $\xi = m_0 = \Psi(0,\dots,0)$ gives that $(\partial_t y_i)(0,\dots,0)$ is the coefficient of $\partial_{y_i}|_{m_0}$ in $\vec{v}(m_0)$. Aha, but recall that we *chose* $\{y_1,\dots,y_N\}$ at the outset so that $\vec{v}(m_0) = \partial_{y_1}|_{m_0}$. Hence, the left column of the Jacobian matrix at the origin has $(1,1)$ entry 1 and all other entries equal to 0. Since the lower right $(N-1)\times(N-1)$ block of the Jacobian matrix is the identity, this finishes the verification of invertibility of $\mathrm{d}\Psi(0,\dots,0)$, so Ψ gives a local C^∞ isomorphism between opens around $(0,\dots,0)$ and m_0.

Let $\{x_1,\dots,x_N\}$ be the C^∞ coordinate system near m_0 on M given by the local inverse to Ψ. We claim that $\vec{v} = \partial_{x_1}$ near m_0. By definition of the x-coordinate system, $(a_1,\dots,a_n)$ is the tuple of x-coordinates of the point $\Phi(a_1, \phi^{-1}(0, a_2, \dots, a_n)) \in M$. Thus, ∂_{x_1} is the field of velocity vectors along the parameteric curves $\Phi(t, \phi^{-1}(0, a_2, \dots, a_n)) = c_{\phi^{-1}(0,a_2,\dots,a_n)}(t)$ that are the integral curves for the smooth vector field $\vec{v}$ with initial positions (time 0) at points

$$\phi^{-1}(0, a_2, \dots, a_n) \in W \cap \{y_1 = 0\}$$

near m_0. Thus, the velocity vectors along these parametric curves are exactly the vectors from the smooth vector field $\vec{v}$! This shows that the smooth vector fields ∂_{x_1} and $\vec{v}$ coincide near m_0. $\square$

Appendix I. Elementary properties of characters

I.1. **Basic definitions and facts.** Let G be a Lie group, and consider a nonzero left-invariant top-degree differential form ω. For any $g \in G$, $(r_g)^*(\omega)$ is also left-invariant (as r_g commutes with left translations!), so $(r_g)^*(\omega) = c(g)\omega$ for some constant $c(g) \in \mathbf{R}^\times$ (computed by comparing at the identity point). Clearly $c(g)$ is unaffected by replacing ω with an $\mathbf{R}^\times$-multiple, and since such multiples exhaust all choices for ω it follows that $c(g)$ depends only on g and not ω. It is also easy to check that $g \mapsto c(g)$ is a homomorphism.

In class, $|c|$ was called the *modulus character* of G. (In some references c is called the modulus character, or the "algebraic" modulus character, due to certain considerations with matrix groups over general fields, lying beyond the level of this course.) On HW4 you show that $|c|$ is continuous, and the same method of proof will certainly show that c itself is continuous (hence C^∞). As was noted in class, such continuity forces $|c|$ to be trivial when G is compact. Also, if G is *connected* then c has constant sign and hence $c = |c|$ (as $c(e) = 1$), so in general if G is compact then ω is right-invariant under G^0 but in general the potential sign problems for the effect of right-translation on ω by points of G in other connected components really can occur:

Example I.1.1. Let $G = \mathrm{O}(n) = \mathrm{SO}(n) \rtimes \langle \iota \rangle$ for the diagonal $\iota := \mathrm{diag}(-1, 1, 1, \dots, 1)$. The adjoint action of ι on the subspace $\mathfrak{so}(n) \subseteq \mathfrak{gl}_n(\mathbf{R})$ of skew-symmetric matrices is the restriction of the adjoint action of $\mathrm{GL}_n(\mathbf{R})$ on $\mathfrak{gl}_n(\mathbf{R}) = \mathrm{Mat}_n(\mathbf{R})$, which is the ordinary conjugation-action of $\mathrm{GL}_n(\mathbf{R})$ on $\mathrm{Mat}_n(\mathbf{R})$ (why?). Thus, we see that $\mathrm{Ad}_G(\iota)$ acts on

$\mathfrak{so}(n)$ by negating the first row and first column (keep in mind that the diagonal entries vanish, due to skew-symmetry). Since $\mathfrak{so}(n)$ projects isomorphically onto the vector space of strictly upper-triangular matrices, we conclude that $\mathrm{Ad}_G(\iota)$ has -1-eigenspace of dimension $n-1$, so its determinant is $(-1)^{n-1}$. But this determinant is the scaling effect of $\mathrm{Ad}_G(\iota)$ on the top exterior power of $\mathfrak{so}(n)$ and hence likewise on the top exterior power of its dual.

In the language of differential forms, it follows that $r_\iota^*(\omega) = (-1)^{n-1}(\omega)$ for any left-invariant top-degree differential form on G due to two facts: $\omega = \ell_\iota^*(\omega)$ and the composition of $r_\iota = r_{\iota^{-1}}$ with ℓ_ι is ι-conjugation (whose effect on the tangent space at the identity is $\mathrm{Ad}_G(\iota)$). Thus, if n is even it follows that left-invariant top-degree differential forms are *not* right-invariant.

For the remainder of this appendix, assume G is compact (but not necessarily connected, since we wish to include finite groups as a special case of our discussion). Thus, for any left-invariant ω, the measure $|\omega|$ is bi-invarant (as $|c| = 1$) even though ω might not be right-invariant under the action of some connected components away from G^0. However, this bi-invariant measure is not uniquely determined since if we scale ω by some $a > 0$ then the measure scales by a. But there is a canonical choice! Indeed, the integral $\int_G |\omega|$ converges to some positive number since G is compact, so by scaling ω it can be uniquely determined up to a sign by the condition $\int_G |\omega| = 1$. This defines a *canonical* bi-invariant measure on G, called the "volume 1 measure". We denote this measure with the suggestive notation $\mathrm{d}g$ (though it is *not* a differential form, and has no dependence on orientations).

Example I.1.2. If $G = S^1$ inside $\mathbf{R}^2$ then $\mathrm{d}g = |\mathrm{d}\theta|/2\pi$. If G is a finite group then $\mathrm{d}g$ is the measure that assigns each element of the group the mass $1/|G|$, so $\int_G f(g)\,\mathrm{d}g = (1/|G|)\sum_{g\in G} f(g)$. Thus, integration against $\mathrm{d}g$ in the finite case is precisely the averaging process that pervades the representation theory of finite groups (in characteristic 0).

For an irreducible finite-dimensional $\mathbf{C}$-linear representation $\rho : G \to \mathrm{GL}(V)$, we define the *character* χ_V (or χ_ρ) to be the function $g \mapsto \mathrm{Tr}(\rho(g))$. This is a smooth $\mathbf{C}$-valued function on G since ρ is C^∞ (so its matrix entries relative to a $\mathbf{C}$-basis of V are smooth $\mathbf{C}$-valued functions). Obviously $\chi_{V\oplus W} = \chi_V + \chi_W$, and in class some other related identities were explained:

$$\chi_{V^*} = \overline{\chi}_V,\ \ \chi_{V\otimes W} = \chi_V \cdot \chi_W,\ \ \chi_{\mathrm{Hom}(V,W)} = \overline{\chi}_V \cdot \chi_W$$

where $\mathrm{Hom}(V, W)$ denotes the space of $\mathbf{C}$-linear maps $V \to W$ and is equipped with a *left* G-action via $g.T = \rho_W(g) \circ T \circ \rho_V(g)^{-1}$. (This ensures the crucial fact that the subspace $\mathrm{Hom}(V, W)^G$ is precise the subspace $\mathrm{Hom}_G(V, W)$ of G-equivariant homomorphisms.)

To save notation, we shall now write $g.v$ rather than $\rho(g)(v)$. The following lemma will be used a lot.

Lemma I.1.3. *Let $L : W' \to W$ be a $\mathbf{C}$-linear map between finite-dimensional $\mathbf{C}$-vector spaces and $f : G \to W'$ a continuous function. Then $L(\int_G f(g)\,\mathrm{d}g) = \int_G (L \circ f)(g)\,\mathrm{d}g$.*

In this statement $\int_G f(g)\,\mathrm{d}g$ is a "vector-valued" integral.

Proof. By computing relative to $\mathbf{C}$-bases of W and W', we reduce to the case $W = W' = \mathbf{C}$ and $L(z) = cz$ for some $c \in \mathbf{C}$. This case is obvious. $\square$

I.2. **Applications.** Consider the linear operator $T : V \to V$ defined by the vector-valued integral $v \mapsto \int_G g.v \, \mathrm{d}g$. In the case of finite groups, this is the usual averaging projector onto V^G. Let's see that it has the same property in general. Since $\int_G \mathrm{d}g = 1$, it is clear (e.g., by computing relative to a $\mathbf{C}$-basis of V) that $T(v) = v$ if v lies in the subspace V^G of G-invariant vectors. In fact, T lands inside V^G: by Lemma I.1.3, we may compute

$$g'.T(v) = \int_G g'.(g.v) \, \mathrm{d}g = \int_G (g'g).v \, \mathrm{d}g = \int_G g.v \, \mathrm{d}g = T(v),$$

where the second to last equality uses the invariance of the measure under left translation by g'^{-1}.

Since T is a linear projector on V^G, its trace as an endomorphism of V is $\dim(V^G)$. In terms of integration of continuous fuctions $G \to \mathrm{End}(V)$, we see that $T = \int_G \rho(g) \, \mathrm{d}g$. Thus, applying Lemma I.1.3 to the linear map $\mathrm{Tr} : \mathrm{End}(V) \to \mathbf{C}$, we conclude that

$$\dim(V^G) = \mathrm{Tr}(T) = \int_G \mathrm{Tr}(\rho(g)) \, \mathrm{d}g = \int_G \chi_V(g) \, \mathrm{d}g.$$

Applying this identity to the G-representation space $\mathrm{Hom}(V, W)$ mentioned earlier (with V and W any two finite-dimensional G-representations), we obtain:

Proposition I.2.1.

$$\dim_{\mathbf{C}} \mathrm{Hom}_G(V, W) = \int_G \overline{\chi}_V(g) \chi_W(g) \, \mathrm{d}g.$$

Rather generally, for any continuous functions $\psi, \phi : G \rightrightarrows \mathbf{C}$ we define

$$\langle \psi, \phi \rangle = \int_G \overline{\psi}(g) \phi(g) \, \mathrm{d}g,$$

so $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W)$. Thus, by Schur's Lemma for irreducible (continuous) finite-dimensional $\mathbf{C}$-linear representations of G (same proof as for finite groups), we conclude that if V and W are *irreducible* then $\langle \chi_V, \chi_W \rangle$ is equal to 1 if $V \simeq W$ and it vanishes otherwise, exactly as in the special case of finite groups. These are the *orthogonality relations* among the characters.

By using integration in place of averaging, any quotient map $V \twoheadrightarrow W$ between finite-dimensional continuous representations of G admits a G-equivariant section, so a general finite-dimensional continuous representation V of G over $\mathbf{C}$ decomposes up to isomorphism as a direct sum $\oplus V_j^{\oplus n_j}$ where the V_j's are the pairwise non-isomorphic irreducible subrepresentations of G inside V and n_j is the multiplicity with which it occurs (explicitly, $n_j = \dim \mathrm{Hom}_G(V_j, V)$ due to Schur's Lemma, as for finite groups). Thus, in view of the orthogonality relations,

$$\langle \chi_V, \chi_V \rangle = \sum n_j^2.$$

This yields:

Corollary I.2.2. *A representation (ρ, V) of G is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$; in the reducible case this pairing is larger than* 1*. In particular, χ_V determines whether or not V is an irreducible representation of G.*

In fact, we can push this computation a bit further: if W is an irreducible representation of G and $W \not\simeq V_j$ for any j (i.e., W does not occur inside V) then $\langle \chi_V, \chi_W \rangle = 0$, so an irreducible representation W of G occurs inside V if and only if $\langle \chi_V, \chi_W \rangle \neq 0$, in which case this pairing is equal to the multiplicity of W inside V. Thus, writing $n_{V,W} := \langle \chi_V, \chi_W \rangle$, we have

$$V \simeq \bigoplus_{n_{V,W} \neq 0} W^{\oplus n_{V,W}}.$$

This reconstructs V from data depending solely on χ_V (and general information associated to every irreducible representation of G). In particular, it proves:

Corollary I.2.3. *Every representation V of G is determined up to isomorphism by χ_V.*

Appendix J. Representations of $\mathfrak{sl}_2$

J.1. **Introduction.** In this appendix we work out the finite-dimensional k-linear representation theory of $\mathfrak{sl}_2(k)$ for any field k of characteristic 0. (There are also infinite-dimensional irreducible k-linear representations, but here we focus on the finite-dimensional case.) This is an introduction to ideas that are relevant in the general classification of finite-dimensional representations of "semisimple" Lie algebras over fields of characteristic 0, and is a *crucial* technical tool for our later work on the structure of general connected compact Lie groups (especially to explain the ubiquitous role of $\mathrm{SU}(2)$ in the general structure theory).

When k is fixed during a discussion, we write $\mathfrak{sl}_2$ to denote the Lie algebra $\mathfrak{sl}_2(k)$ of traceless 2×2 matrices over k (equipped with its usual Lie algebra structure via the commutator inside the associative k-algebra $\mathrm{Mat}_2(k)$). Recall the standard k-basis $\{X^-, H, X^+\}$ of $\mathfrak{sl}_2$ given by

$$X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

satisfying the commutation relations

$$[H, X^\pm] = \pm 2X^\pm, \;\; [X^+, X^-] = H.$$

For an $\mathfrak{sl}_2$-module V over k (i.e., k-vector space V equipped with a map of Lie algebras $\mathfrak{sl}_2 \to \mathrm{End}_k(V)$), we say it is *irreducible* if $V \neq 0$ and there is no nonzero proper $\mathfrak{sl}_2$-submodule. We say that V is *absolutely irreducible* (over k) if for any extension field K/k the scalar extension V_K is irreducible as a K-linear representation of the Lie algebra $\mathfrak{sl}_2(K) = K \otimes_k \mathfrak{sl}_2$ over K.

If V is a finite-dimensional representation of a Lie algebra $\mathfrak{g}$ over k then we define the *dual* representation to be the dual space V^* equipped with the $\mathfrak{g}$-module structure $X.\ell = \ell \circ (-X.v) = -\ell(X.v)$ for $\ell \in V^*$. (The reason for the minus sign is to ensure that the action of $[X, Y]$ on V^* satisfies $[X, Y](\ell) = X.(Y.\ell) - Y.(X.\ell)$ rather than $[X, Y](\ell) = Y.(X.\ell) - X.(Y.\ell)$. The intervention of negation here is similar to the fact that dual represenations of groups G involve evaluation against the action through inversion, to ensure the dual of a left G-action is a left G-action rather than a right G-action.) The natural k-linear isomorphism $V \simeq V^{**}$ is easily checked to be $\mathfrak{g}$-linear. It is also straightforward to check (do it!) that if $\rho : G \to \mathrm{GL}(V)$ is a smooth representation of a Lie group G on a finite-dimensional vector space over $\mathbf{R}$ or $\mathbf{C}$ then $\mathrm{Lie}(\rho^*) = \mathrm{Lie}(\rho)^*$ as representations of $\mathrm{Lie}(G)$ (where V^* is initially made into a G-representation in the usual way). The same holds if G is a complex Lie group (acting linearly on finite-dimensional $\mathbf{C}$-vector space via a holomorphic ρ, using the $\mathbf{C}$-linear identification of $\mathrm{Lie}(\mathrm{GL}(V)) = \mathrm{End}_{\mathbf{C}}(V)$ for $\mathrm{GL}(V)$ as a complex Lie group).

In the same spirit, the *tensor product* of two $\mathfrak{g}$-modules V and V' has underlying k-vector space $V \otimes_k V'$ and $\mathfrak{g}$-action given by

$$X.(v \otimes v') = (X.v) \otimes v' + v \otimes (X.v')$$

for $X \in \mathfrak{g}$ and $v \in V$, $v' \in V'$. It is easy to check (do it!) that this tensor product construction is compatible via the Lie functor with tensor products of finite-dimensional representations of Lie groups in the same sense as formulated above for dual representations.

J.2. **Primitive vectors.** To get started, we prove a basic fact:

Lemma J.2.1. *Let V be a nonzero finite-dimensional $\mathfrak{sl}_2$-module over k. The operators $X^{\pm}$ on V are nilpotent and the H-action on V carries each* $\ker(X^{\pm})$ *into itself.*

Proof. Since $[H, X^{\pm}] = \pm 2X^{\pm}$, the Lie algebra representation conditions give the identity

$$H.(X^{\pm}.v) = [H, X^{\pm}].v + X^{\pm}.(H.v) = \pm 2X^{\pm}.v + X^{\pm}.(H.v)$$

for any $v \in V$. Thus, if $X^{\pm}v = 0$ then $X^{\pm}.(H.v) = 0$, so $H.v \in \ker X^{\pm}$. This gives the H-stability of each $\ker X^{\pm}$.

We now show that $X^{\pm}$ is nilpotent. More generally, if E and H are linear endomorphisms of a finite-dimensional vector space V in characteristic 0 and the usual commutator $[E, H]$ of endomorphisms is equal to cE for some nonzero c then we claim that E must act nilpotently on V. By replacing H with $(1/c)H$ if necessary, we may assume $[E, H] = E$. Since $EH = HE + E = (H+1)E$, we have

$$E^2 H = E(H+1)E = (EH+E)E = ((H+1)E+E)E = (H+2)E^2,$$

and in general $E^n H = (H+n)E^n$ for integers $n \geq 0$. Taking the trace of both sides, the invariance of trace under swapping the order of multiplication of two endomorphisms yields

$$\mathrm{Tr}(E^n H) = \mathrm{Tr}((H+n).E^n) = \mathrm{Tr}(E^n.(H+n)) = \mathrm{Tr}(E^n H) + n\mathrm{Tr}(E^n),$$

so $n\mathrm{Tr}(E^n) = 0$ for all $n > 0$. Since we're in characteristic 0, we have $\mathrm{Tr}(E^n) = 0$ for all $n > 0$.

There are now two ways to proceed. First, we can use "Newton's identities", which reconstruct the symmetric functions of a collection of d elements $\lambda_1, \ldots, \lambda_d$ (with multiplicity) in a field of characteristic 0 (or any commutative $\mathbf{Q}$-algebra whatsoever) from their first d power sums $\sum_j \lambda_j^n$ $(1 \leq n \leq d)$. The formula involves division by positive integers, so the characteristic 0 hypothesis is essential (and the assertion is clearly false without that condition). In particular, if the first d power sums vanish then all λ_j vanish. Applying this to the eigenvalues of E over $\overline{k}$, it follows that the eigenvalues all vanish, so E is nilpotent. However, the only proofs of Newton's identities that I've seen are a bit unpleasant (perhaps one of you can enlighten me as to a slick proof?), so we'll instead use a coarser identity that is easy to prove.

In characteristic 0 there is a general identity (used very creatively by Weil in his original paper on the Weil conjectures) that reconstructs the "reciprocal root" variant of characteristic polynomial of any endomorphism E from the traces of its powers: since the polynomial

$\det(1-tE)$ in $k[t]$ has constant term 1 and $\log(1-tE) = -\sum_{n\geq 1}(tE)^n/n \in t\mathrm{Mat}_d(k[\![t]\!])$, it makes sense to compute the trace $\mathrm{Tr}(\log(1-tE)) \in tk[\![t]\!]$ and we claim that

$$\begin{aligned}\det(1-tE) = \exp(\log(\det(1-tA)) &= \exp(\mathrm{Tr}(\log(1-tA))) \\ &= \exp(\mathrm{Tr}(-\sum_{n\geq 1} t^n E^n/n)) \\ &= \exp(-\sum_{n\geq 1} t^n \mathrm{Tr}(E^n)/n)\end{aligned}$$

as formal power series in $k[\![t]\!]$. (Note that it makes sense to plug an element of $tk[\![t]\!]$ into any formal power series in one variable!) Once this identity proved for general E, it follows that if $\mathrm{Tr}(E^n)$ vanishes for all $n > 0$ then $\det(1-tE) = \exp(0) = 1$. But the coefficients of the positive powers of t in $\det(1-tE)$ are the lower-order coefficients of the characteristic polynomial of E, whence this characteristic polynomial is t^d, so E is indeed nilpotent, as desired.

In the above string of equalities, the only step which requires explanation is the equality

$$\log(\det(1-tE)) = \mathrm{Tr}(\log(1-tE))$$

in $k[\![t]\!]$. To verify this identity we may extend scalars so that k is algebraically closed, and then make a change of basis on k^d so that E is upper-triangular, say with entries $\lambda_1, \dots, \lambda_d \in k$ down the diagonal. Thus, $\det(1-tE) = \prod(1-\lambda_j t)$ and $\log(1-tE) = -\sum_{n\geq 1} t^n E^n/n$ is upper-triangular, so its trace only depends on the diagonal, whose entries are $\log(1-\lambda_j t) \in tk[\![t]\!]$. Summarizing, our problem reduces to the formal power series identity that log applied to a finite product of elements in $1 + tk[\![t]\!]$ is equal to the sum of the logarithms of the terms in the product. By continuity considerations in complete local noetherian rings (think about it!), the rigorous justification of this identity reduces to the equality $\log(\prod(1+x_j)) = \sum \log(1+x_j)$ in $\mathbf{Q}[\![x_1, \dots, x_d]\!]$, which is easily verified by computing coefficients of multivariable Taylor expansions. □

In view of the preceding lemma, if V is any nonzero finite-dimensional $\mathfrak{sl}_2$-module over k we may find nonzero elements $v_0 \in \ker X^+$, and moreover if k is algebraically closed then we can find such v_0 that are eigenvectors for the restriction of H to an endomorphism of $\ker X^+$.

Definition J.2. A *primitive vector* in V is an H-eigenvector in $\ker X^+$.

We shall see that in the finite-dimensional case, the H-eigenvalue on a primitive vector is necessarily a non-negative integer. (For infinite-dimensional irreducible $\mathfrak{sl}_2$-modules one can make examples in which there are primitive vectors but their H-eigenvalue is not an integer.) We call the H-eigenvalue on a primitive eigenvector (or on any H-eigenvector at all) its *H-weight*.

J.3. **Structure of $\mathfrak{sl}_2$-modules.** Here is the main result, from which everything else will follow.

Theorem J.3.1. *Let $V \neq 0$ be a finite-dimensional $\mathfrak{sl}_2$-module over a field k with* $\mathrm{char}(k) = 0$.

(1) *The H-weight of any primitive vector is a non-negative integer.*

(2) *Let $v_0 \in V$ be a primitive vector, its weight an integer $m \geq 0$. The $\mathfrak{sl}_2$-submodule $V' := \mathfrak{sl}_2.v_0$ of V generated by v_0 is absolutely irreducible over k and has dimension $m+1$. Moreover, if we define*

$$v_j = \frac{1}{j!}(X^-)^j(v_0)$$

for $0 \leq j \leq m$ (and define $v_{-1} = 0$, $v_{m+1} = 0$) then

$$H.v_j = (m-2j).v_j, \quad X^+.v_j = (m-j+1)v_{j-1}, \quad X^-.v_j = (j+1)v_{j+1}$$

for $0 \leq j \leq m$. In particular, the H-action on V' is diagonalizable with eigenspaces of dimension 1 having as eigenvalues the $m+1$ integers $\{m, m-2, \dots, -m+2, -m\}$.

(3) *$X^+|_{V'}$ has kernel equal to kv_0, and this line exhibits the unique highest H-weight.*

In particular, if V is irreducible then $V = V'$ is absolutely irreducible and is determined up to isomorphism by its dimension $m+1$, and all H-eigenvalues on V' are integers, with the unique highest weight m having a 1-dimenional eigenspace.

Before proving the theorem, we make some remarks.

Remark J.3.2. In class we saw the visualization of the effect of $X^{\pm}$ on the H-eigenlines, as "raising/lowering" operators with respect to the H-eigenvalues (hence the notation X^+ and X^-, the asymmetry between which is due to our convention to work with $H = [X^+, X^-]$ rather than $-H = [X^-, X^+]$ at the outset).

The conceptual reason that we divide by factorials in the definition of the v_j's is to ensure that the formulas relating $X^{\pm}.v_j$ to $v_{j\mp 1}$ involve *integer* coefficients with the evident monotonicity behavior as we vary j. In view of the fact that we'll later *construct* such an irreducible $(m+1)$-dimensional representation as the mth symmetric power of the *dual* of the standard 2-dimensional representation of $\mathfrak{sl}_2$, what is really going on with the factorial division is that the formation of symmetric powers of finite-dimensional vector spaces does *not* naturally commute with the formation of dual spaces (in contrast with tensor powers and exterior powers): in positive characteristic it fails badly, and in general the symmetric power of a finite-dimensional dual vector space is identified with the dual of a "symmetric divided power" space (and divided powers are identified with symmetric powers in characteristic 0 via suitable factorial divisions); read the Wikipedia page on *divided power structure*.

Remark J.3.3. The final part of Theorem J.3.1, characterizing an irreducible $\mathfrak{sl}_2$-module up to isomorphism by its highest weight, has a generalization to all "semisimple" finite-dimensional Lie algebras over algebraically closed fields of characteristic 0, called the *Theorem of the Highest Weight*. The precise statement of this result requires refined knowledge of the structure theory of semisimple Lie algebras (such as results and definitions concerning Cartan subalgebras), so we do not address it here.

In addition to the consequence that any irreducible $\mathfrak{sl}_2$-module of finite dimension over k is absolutely irreducible and determined up to isomorphism by its dimension (so it is isomorphic to its dual representation!), there is the separate problem of showing that all possible dimensions really occur. That is, one has to make an actual construction of an irreducible $\mathfrak{sl}_2$-module of every positive dimension. Likewise, the precise statement of the Theorem of the Highest Weight for general semisimple finite-dimensional Lie algebras includes an existence aspect, and is very much tied up with a good knowledge of the theory of root systems (a theory that plays an essential role in our later work on the structure theory of connected compact Lie groups).

We will address the existence issue below for $\mathfrak{sl}_2$, and also show that the entire finite-dimensional representation theory of $\mathfrak{sl}_2$ is completely reducible – i.e., every such representation is a direct sum of irreducibles – a fact that we can see over $k = \mathbf{C}$ via using the analytic technique of Weyl's unitarian trick to pass to an analogue for the connected compact Lie group $\mathrm{SU}(2)$. The proof of such complete reducibility over a general field of characteristic 0 (especially not algebraically closed) requires a purely algebraic argument. In particular, this will give a purely algebraic proof of the semisimplicity of the finite-dimensional $\mathbf{C}$-linear representation theory of $\mathfrak{sl}_2(\mathbf{C})$ without requiring the unitarian trick (even though historically the unitarian trick was a milestone in the initial development of the understanding of the representation theory of finite-dimensional semisimple Lie algebras over $\mathbf{C}$).

Now we finally prove Theorem J.3.1.

Proof. In view of the precise statement of the theorem, it is sufficient to prove the result after a ground field extension (check: for this it is essential that we are claiming that certain eigenvalues are integers, not random elements of k). Thus, now we may and do assume k is algebraically closed, so we can make eigenvalues. Consider any $\lambda \in k$ and $v \in V$ satisfying $H.v = \lambda v$. (We are mainly interested in the case $v \neq 0$, but let's not assume that just yet). The condition of V being an $\mathfrak{sl}_2$-module yields the computation

$$H.(X^{\pm}.v) = [H, X^{\pm}].v + X^{\pm}.H.v = \pm 2X^{\pm}.v + \lambda X^{\pm}.v = (\lambda \pm 2)X^{\pm}.v$$

that we saw earlier. In particular, if v is a λ-eigenvector for H then $X^{\pm}.v$ is an eigenvector for H with eigenvalue $\lambda \pm 2j$ provided that $X^{\pm}.v \neq 0$. In particular, the elements in the sequence $\{(X^{\pm})^j.v\}_{j \geq 0}$ that are nonzero are mutually linearly independent since they are H-eigenvectors with pairwise distinct eigenvalues $\lambda \pm 2j$. (Here we use that we're in characteristic 0!)

Let v_0 be a primitive vector, which exists by Lemma J.2.1 since k is algebraically closed. Thus, $H.v_0 = \lambda v_0$ for some $\lambda \in k$. Define $v_j = (1/j!)(X^-)^j.v_0$ for all $j \geq 0$, and define $v_{-1} = 0$. Since X^- is nilpotent on V, we have $v_j = 0$ for sufficiently large $j > 0$, so the set of j such that $v_j \neq 0$ is a sequence of consecutive integers $\{0, 1, \ldots, m\}$ for some $m \geq 0$. Clearly from the definition we have

$$H.v_j = (\lambda - 2j)v_j, \;\; X^-.v_j = (j+1)v_{j+1}$$

for $j \geq 0$. We claim that $X^+.v_j = (\lambda - j + 1)v_{j-1}$ for all $j \geq 0$. This is clear for $j = 0$, and in general we proceed by induction on j. Assuming $j > 0$ and that the result is known for $j - 1 \geq 0$, we have

$$X^+.v_j = (1/j)X^+.X^-.v_{j-1} = (1/j)([X^+, X^-] + X^-.X^+).v_{j-1} = (\lambda - j + 1)v_{j-1},$$

where the final equality is a computation using the inductive hypothesis that we leave to the reader to check.

We know that $v_0, \ldots, v_m$ are linearly independent. Since

$$(\lambda - m)v_m = X^+.v_{m+1} = 0,$$

necessarily $\lambda = m$. This proves that the primitive vector v_0 has H-weight equal to the non-negative integer m, and from our formulas for the effect of H and $X^{\pm}$ on each v_j

($0 \le j \le m$), clearly the $(m+1)$-dimensional k-linear span V' of $v_0, \dots, v_m$ coincides with the $\mathfrak{sl}_2$-submodule of V generated by v_0. The formulas show that $X^+|_{V'}$ has kernel equal to the line spanned by v_0.

It remains to show that V' is irreducible as an $\mathfrak{sl}_2$-module. (By extending to an even larger algebraically closed extension if necessary and applying the same conclusion over that field, the absolute irreducibility would follow.) Consider a nonzero $\mathfrak{sl}_2$-submodule W of V'. Since the H-action on V is diagonalizable with 1-dimensional eigenspaces, the H-stable W must be a span of some of these eigenlines. But the explicit formulas for the effect of $X^\pm$ as "raising/lowering" operators on the lines kv_j makes it clear that a single such line generates the entirety of V' as an $\mathfrak{sl}_2$-module. Hence, V' is irreducible. $\square$

J.4. **Complete reduciblity and existence theorem.** We finish by discussing two refinements: the proof that every finite-dimensional $\mathfrak{sl}_2$-module is a direct sum of irreducibles, and the existence of irreducible representations of each positive dimension. As a consequence, for $k = \mathbf{C}$ we'll recover the connection between irreducible finite-dimensional $\mathrm{SO}(3)$-representations over $\mathbf{C}$ and spherical harmonics. First we prove the existence result.

Proposition J.4.1. *Let V_1 be the standard 2-dimensional representation of $\mathfrak{sl}_2 \subset \mathrm{End}_k(k^2)$. For $m \ge 0$, the symmetric power $V_m = \mathrm{Sym}^m(V_1^*)$ of dimension $m+1$ is irreducible as an $\mathfrak{sl}_2$-module for every $m \ge 1$.*

Proof. Obviously $V_0 = k$ is the 1-dimensional trivial representation, so we may focus on cases with $m \ge 1$. In an evident manner, V_m is the space of homogenous polynomials of degree m in two variables z_1, z_2. By inspection, $v_0 := z_1^m$ is a primitive vector, and the associated v_j's are given by

$$v_j = \binom{m+1}{j} z_1^{m-j} z_2^j$$

for $0 \le j \le m$. These span the entire $(m+1)$-dimensional space V_m, so V_m is irreducible by Theorem J.3.1. $\square$

In the general theory of semisimple Lie algebras, there is a construction called the *Killing form* (named after Wilhelm Killing even though it was introduced by Cartan, much as Cartan matrices were introduced by Killing...). This underlies the conceptual technique by which complete reducibility of representations is proved. In our situation we will use our explicit knowledge of the list of irreducibles to prove the complete reducibility; such a technique is certainly ill-advised in a broader setting (beyond $\mathfrak{sl}_2$):

Theorem J.4.2. *Every nonzero finite-dimensional $\mathfrak{sl}_2$-module over k is a direct sum of irreducibles.*

Proof. We proceed by induction on the dimension, the case of dimension 1 being clear. Consider a general V, so if it is irreducible then there is nothing to do. Hence, we may assume it contains a nonzero proper $\mathfrak{sl}_2$-submodule, so there is a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

of $\mathfrak{sl}_2$-modules, with V' and V'' nonzero of strictly smaller dimension than V. Hence, V' and V'' are each a direct sum of irreducibles. We just want to split this exact sequence of $\mathfrak{sl}_2$-modules.

As we noted in class, for any Lie algebra $\mathfrak{g}$ over k, the category of $\mathfrak{g}$-modules over k is the same as the category of left $U(\mathfrak{g})$-modules where $U(\mathfrak{g})$ is the associative universal enveloping algebra over k. This is the quotient of the tensor algebra $\oplus_{n\geq 0}\mathfrak{g}^{\otimes n}$ modulo the 2-sided ideal generated by relations $X\otimes Y - Y\otimes X = [X,Y]$ for $X, Y \in \mathfrak{g}$. (The Poincaré–Birkhoff–Witt theorem describes a basis of $U(\mathfrak{g})$, but we don't need that.) Short exact sequences of $\mathfrak{g}$-modules are the same as those of left $U(\mathfrak{g})$-modules. Letting $R = U(\mathfrak{sl}_2)$, the short exact sequence of interest is an element of $\mathrm{Ext}^1_R(V'', V')$. We want this Ext-group to vanish.

In an evident manner, since V'' is finite-dimensional over k, we see that for any left R-module W (even infinite-dimensional), $\mathrm{Hom}_R(V'', W) \simeq \mathrm{Hom}_R(k, V''^* \otimes_k W)$. By a universal δ-functor argument (or more hands-on arguments that we leave to the interested reader),

$$\mathrm{Ext}^\bullet_R(V'', \cdot) = \mathrm{Ext}^\bullet_R(k, V''^* \otimes_k (\cdot)).$$

(Recall our discussion at the outset of this appendix concerning duals and tensor products of representations of Lie algebras.) Thus, to prove the vanishing of the left side in degree 1 when evaluated on a finite-dimensional argument, it suffices to prove the vanishing of the right side in such cases. In other words, we are reduced to proving $\mathrm{Ext}^1_R(k, W) = 0$ for all finite-dimensional left $\mathfrak{sl}_2$-modules W. By using short exact sequences in W, we filter down to the case when W is irreducible. Thus, we're reduced to proving the splitting in the special case when $V'' = k$ is the trivial representation and V' is irreducible.

Dualizing is harmless, so we want to split short exact sequences

$$0 \to k \to E \to V_m \to 0$$

for $m \geq 0$. The key trick, inspired by knowledge later in the theory (the structure of the center of the universal enveloping algebra of a semisimple Lie algebra), is to consider the element

$$C := H^2 + 2(X^+X^- + X^-X^+) = H^2 + 2H + 4X^-X^+ \in R = U(\mathfrak{sl}_2).$$

The advanced knowledge that inspires the focus on C is that $C/8$ is a distinguished element in the *center* of $U(\mathfrak{sl}_2)$ (with an analogue in the center of $U(\mathfrak{g})$ for any finite-dimensional semisimple Lie algebra $\mathfrak{g}$ over k), called the *Casimir element*. By centrality it must act as a constant on every absolutely irreducible finite-dimensional representation, due to Schur's Lemma. For $\mathfrak{sl}_2$ one can verify by direct computation (do it!) that C acts as $m(m+2)$ on V_m (be careful about computations with the "boundary" vectors v_0 and v_m in the $(m+1)$-dimensional V_m), so it "picks out" isotypic parts in a direct sum of irreducibles. The centrality of C in $U(\mathfrak{sl}_2)$ will be used in the *proof* of complete reducibility; there is a conceptual proof using the notion of Killing form, but for $\mathfrak{sl}_2$ it's just as easy to give a direct check:

Lemma J.4.3. *The element C in $R = U(\mathfrak{sl}_2)$ is central.*

Proof. This amounts to showing that the commutators $CH - HC$ and $CX^\pm - X^\pm C$ in R vanish. By direct computation with commutators in R and the commutator relations in $\mathfrak{sl}_2$, the second expression for C yields

$$[C, H] = 4([X^-, H]X^+ + X^-[X^+, H]) = 4(2X^-X^+ - 2X^-X^+) = 0.$$

By symmetry in the inital expression for C (and its invariance under swaping the roles of X^+ and X^-, which amounts to replacing H with $-H$), to prove that $[C, X^\pm] = 0$ in R it suffices to treat the case of X^+.

Again using the second expression for C, since $[H, X^+] = 2X^+$ we have

$$[C, X^+] = [H^2, X^+] + 4X^+ + 4[X^- X^+, X^+].$$

But $[H^2, X^+] = 4X^+(H+1)$ because

$$\begin{aligned} H^2 X^+ - X^+ H^2 = H([H, X^+] + X^+ H) - X^+ H^2 &= H(2X^+ + X^+ H) - X^+ H^2 \\ &= HX^+(2+H) - X^+ H^2 \end{aligned}$$

and substituting the identity

$$HX^+ = [H, X^+] + X^+ H = 2X^+ + X^+ H = X^+(2+H)$$

yields

$$H^2 X^+ - X^+ H^2 = X^+(2+H)^2 - X^+ H^2 = X^+(4+4H).$$

Thus, $[C, X^+] = 4X^+(H+1) + 4X^+ + 4(X^-(X^+)^2 - X^+ X^- X^+)$. Since

$$X^- X^+ = [X^-, X^+] + X^+ X^- = -H + X^+ X^-,$$

inside R we have

$$X^-(X^+)^2 - X^+ X^- X^+ = -HX^+ + X^+ X^- X^+ - X^+ X^- X^+ = -HX^+,$$

and hence

$$CX^+ - X^+ C = 4X^+(H+1) + 4X^+ - 4HX^+ = 8X^+ - 4[H, X^+] = 0.$$

□

The upshot of our study of C is that the given short exact sequence is C-equivariant with C acting on V_m via multiplication by $m(m+2)$, and the C-action on E is $\mathfrak{sl}_2$-equivariant since C is *central* in R. Thus, if $m > 0$ (so $m(m+2) \neq 0$ in k) then the $m(m+2)$-eigenspace for C on E is an $\mathfrak{sl}_2$-subrepresentation which does not contain the trivial line k and so must map isomorphically onto the irreducible V_m, thereby splitting the exact sequence as $\mathfrak{sl}_2$-modules. If instead $m = 0$ then the representation of $\mathfrak{sl}_2$ on E corresponds to a Lie algebra homomorphism

$$\mathfrak{sl}_2 \to \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}$$

which we want to vanish (so as to get the "triviality" of the $\mathfrak{sl}_2$-action on E, hence the desired splitting). In other words, it suffices to show that $\mathfrak{sl}_2$ does not admit the trivial 1-dimensional Lie algebra as a quotient. Since $[H, X^\pm] = \pm 2X^\pm$, any abelian Lie algebra quotient of $\mathfrak{sl}_2$ must kill $X^\pm$, so it also kills $[X^+, X^-] = H$ and the abelian quotient vanishes. □

Remark J.4.4. In HW4, explicit models are given for the finite-dimensional irreducible $\mathbf{C}$-linear representations of SO(3) via harmonic polynomials in 3 variables (with coefficients in $\mathbf{C}$). These are the representations V_m of SU(2) of dimension $m = 2\ell + 1$. Under the identification of $\mathfrak{so}(3)_{\mathbf{C}}$ with $\mathfrak{sl}_2(\mathbf{C})$, the set of H-eigenvalues is

$$\{-2\ell, -2\ell + 2, \ldots, 0, \ldots, 2\ell - 2, 2\ell\},$$

and an explicit nonzero element with H-eigenvalue 0 is computed in [BtD, pp. 118–121]. In terms of spherical coordinates (r, θ, φ), such an eigenvector is the ℓth Legendre polynomial evaluated at $\cos\varphi$. (The usual meaning of θ and φ is swapped in [BtD], so it writes $\cos\theta$ for what is usually denoted as $\cos\varphi$.)

When switching between representations of $\mathrm{SO}(3)$ and $\mathfrak{sl}_2(\mathbf{C})$ via the unitarian trick, Lie-theoretic invariance under H translates into invariance under the 1-parameter subgroup of $\mathrm{SO}(3)$ given by rotation of arbitrary angles around the z-axis (an obvious property for polynomials in $\cos\varphi$!) since the velocity at e for this 1-parameter subgroup turns out to be a $\mathbf{C}^{\times}$-multiple of H via the isomorphism $\mathfrak{sl}_2(\mathbf{C}) \simeq \mathfrak{so}(3)_{\mathbf{C}}$.

APPENDIX K. WEYL GROUPS AND CHARACTER LATTICES

K.1. **Introduction.** Let G be a connected compact Lie group, and S a torus in G (not necessarily maximal). In class we saw that the centralizer $Z_G(S)$ of S in G is connected *conditional* on Weyl's general conjugacy theorem for maximal tori in compact connected Lie groups. The normalizer $N_G(S)$ is a closed subgroup of G that contains S. (Read the discussion just below the statement of Corollary H.4.6 for subtleties concerning normalizer subgroups of closed subgroups that have infinite component group.)

For any compact Lie group H, the component group $\pi_0(H) = H/H^0$ is finite since it is compact and discrete (as H^0 is open in H, due to the local connectedness of manifolds). Thus, if H is a compact Lie group then a connected closed subgroup H' exhausts the identity component H^0 precisely when H/H' is finite. (Indeed, in such cases H^0/H' with its quotient topology is a subspace of the Hausdorff quotient H/H' that is finite and hence *discrete*. Thus, H' is open in H, yet also visibly closed, so its finitely many cosets contained in the compact connected H^0 constitute a pairwise disjoint open cover, contradicting connectedness of H^0 unless the cover is a singleton; i.e., $H' = H^0$.)

Note that since $Z_G(S)$ is normal in $N_G(S)$ (why?), its identity component $Z_G(S)^0$ (which we don't know is equal to $Z_G(S)$ until the Conjugacy Theorem is proved) is also normal in $N_G(S)$. In particular, $N_G(S)/Z_G(S)^0$ makes sense as a compact Lie group. In this appendix, we will show that $W(G,S) := N_G(S)/Z_G(S)$ is finite, which is equivalent to the finiteness of the compact Lie group $N_G(S)/Z_G(S)^0$ since $\pi_0(Z_G(S))$ is finite. This finiteness holds if and only if $Z_G(S)^0 = N_G(S)^0$ (why?). The group $W(G,S)$ is the *Weyl group* of G relative to S, and its primary interest is when S is a maximal torus in G (in which case $Z_G(S) = S$ *conditional* on the Conjugacy Theorem, as we saw in class).

We will first look at some instructive examples when S is maximal (by far the most important case for the study of connected compact Lie groups, for which G-conjugation permutes all choices of such a torus, *conditional* on the Conjugacy Theorem). Then we will prove two results: the finiteness of $W(G,S)$ and that $T = Z_G(T)^0$ when T is maximal (so $N_G(T)/T$ is also finite for maximal T, without needing to know that $Z_G(T) = T$). These two proofs will have no logical dependence on the Conjugacy Theorem, an important point because the proof of the Conjugacy Theorem *uses* the finiteness of $N_G(T)/T$ for maximal T.

K.2. **Examples.** Let $G = \mathrm{U}(n) \subset \mathrm{GL}_n(\mathbf{C})$ with $n > 1$ and let $T = (S^1)^n$ be the diagonal maximal torus in G. There are some evident elements in $N_G(T)$ outside T, namely the standard permutation matrices that constitute an S_n inside G. We first show:

Lemma K.2.1. *The centralizer $Z_G(T)$ is equal to T and the subgroup S_n of G maps isomorphically onto $W(G,T) = N_G(T)/T$.*

Proof. We shall work "externally" by appealing to the vector space $\mathbf{C}^n$ on which $G = \mathrm{U}(n)$ naturally acts. The diagonal torus $T = (S^1)^n$ acts linearly on $\mathbf{C}^n$ in this way, with the standard basis lines $\mathbf{C}e_j$ supporting the *pairwise distinct* characters $\chi_j : t \mapsto t_j \in \mathbf{C}^\times$. This gives an intrinsic characterization of these lines via the completely reducible finite-dimensional $\mathbf{C}$-linear representation theory of T. Thus, conjugation on T by any $g \in N_G(T)$ must *permute* these lines! Likewise, an element of $N_G(T)$ that centralizes T must act on $\mathbf{C}^n$ in a manner that *preserves* each of these lines, and so is diagonal in $\mathrm{GL}_n(\mathbf{C})$. In other words, $Z_G(T)$ is the diagonal subgroup of the compact subgroup $U(n) \subset \mathrm{GL}_n(\mathbf{C})$. But by compactness we therefore have $Z_G(T) = T$, since entry-by-entry we can apply the evident fact that inside $\mathbf{C}^\times = S^1 \times \mathbf{R}^\times_{>0}$ any compact subgroup lies inside S^1 (as $\mathbf{R}^\times_{>0}$ has no nontrivial compact subgroups).

Returning to an element $g \in N_G(T)$, whatever its permutation effect on the standard basis lines may be, there is visibly a permutation matrix $s \in S_n \subset G$ that achieves the same effect, so $s^{-1}g$ preserves each of the standard basis lines, which is to say $s^{-1}g$ is diagonal as an element in the ambient $\mathrm{GL}_n(\mathbf{C})$ and so it lies in $Z_G(T) = T$. Hence, $g \in sT$, so every class in $W(G,T)$ is represented by an element of S_n. It follows that $S_n \to W(G,T)$ is surjective.

For injectivity, we just have to note that a nontrivial permutation matrix must move some of the standard basis lines in $\mathbf{C}^n$, so its conjugation effect on $T = (S^1)^n$ is to permute the S^1-factors accordingly. That is a nontrivial automorphism of T since $n > 1$, so $S_n \to W(G,T)$ is indeed injective. $\square$

The preceding lemma has an additional wrinkle in its analogue for $G' = \mathrm{SU}(n)$ in place of $G = \mathrm{U}(n)$, as follows. In G', consider the diagonal torus T' of dimension $n-1$. This is the kernel of the map $(S^1)^n \to S^1$ defined by $(z_1,\dots,z_n) \mapsto \prod z_j$, and the diagonally embedded S^1 is the maximal central torus Z in $G = \mathrm{U}(n)$. It is clear by inspection that $Z \cdot T' = T$, $Z \cdot G' = G$, and $T' = T \cap G'$, so $Z_{G'}(T') = Z_{G'}(T) = G' \cap Z_G(T) = G' \cap T = T'$. Thus, T' is a maximal torus in G', $N_G(T) = Z \cdot N_{G'}(T')$, and the inclusion $Z \subset T$ implies $Z \cap N_{G'}(T') = Z \cap G' = Z[n] \subset T'$. Hence, the Weyl group $W(G',T')$ is naturally identified with $W(G,T) = S_n$.

Under this identification of Weyl groups, the effect of S_n on the codimension-1 subtorus $T' \subset T = (S^1)^n$ via the identification of S_n with $N_{G'}(T')/T'$ is the restriction of the usual S_n-action on $(S^1)^n$ by permutation of the factors. This is reminiscent of the representation of S_n on the hyperplane $\{\sum x_j = 0\}$ in $\mathbf{R}^n$, and literally comes from this hyperplane representation if we view T as $(\mathbf{R}/\mathbf{Z})^n$ (and thereby write points of T as n-tuples $(e^{2\pi i x_j})$).

A key point is that, in contrast with $\mathrm{U}(n)$, in general the Weyl group quotient of $N_{G'}(T')$ does *not* isomorphically lifts into $N_{G'}(T')$ as a subgroup (equivalently, $N_{G'}(T')$ is not a semi-direct product of T' against a finite group). In particular, the Weyl group is generally only a quotient of the normalizer of a maximal torus and cannot be found as a subgroup of the normalizer. The failure of such lifting already occurs for $n = 2$:

Example K.2.2. Let's show that for $G' = \mathrm{SU}(2)$ and T' the diagonal maximal torus, there is no order-2 element in the nontrivial T'-coset of $N_{G'}(T')$. A representative for this non-identity coset is the "standard Weyl representative"

$$\nu = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

that satisfies $\nu^2 = -1$. For any $t' \in T'$, the representative $\nu t'$ for the nontrivial class in the order-2 group $W(G', T')$ satisfies

$$(\nu t')(\nu t') = (\nu t' \nu^{-1})(\nu^2 t') = {t'}^{-1}(-1)t' = -1$$

regardless of the choice of t'. Hence, no order-2 representative exists.

For $n \geq 3$ the situation is more subtle, since (for $G' = \mathrm{SU}(n)$ and its diagonal maximal torus T') the order-2 elements of $W(G', T')$ do always lift (non-uniquely) to order-2 elements of $N_{G'}(T')$. We will build such lifts and use them to show that the Weyl group doesn't lift isomorphically to a subgroup of $N_{G'}(T')$. Inside $W(G', T') = S_n$ acting in its usual manner on the codimension-1 subtorus $T' \subset T = (S^1)^n$, consider the element $\sigma = (12)$. This action is the identity on the codimension-1 subtorus

$$T'' := \{(z_2, z_2, z_3, \ldots, z_{n-1}, (z_2^2 z_3 \cdots z_{n-1})^{-1}) \in (S^1)^n\} \subset T'$$

of T' (understood to consist of points (z_2, z_2, z_2^{-2}) when $n = 3$), so *any* $g' \in N_{G'}(T')$ that lifts $\sigma \in W(G', T')$ must conjugate T' by means of σ and thus must *centralize* T''. Thus, the only place we need to look for possible order-2 lifts of σ is inside $Z_{G'}(T'')$. But thinking in terms of eigenlines once again, we see the action of T'' on $\mathbf{C}^n$ decomposes as a plane $\mathbf{C}e_1 \oplus \mathbf{C}e_2$ with action through z_2-scaling and the additional standard basis lines $\mathbf{C}e_j$ on which T'' acts with pairwise *distinct* non-trivial characters.

Hence, the centralizer of T'' inside the ambient group $\mathrm{GL}_n(\mathbf{C})$ consists of elements that (i) preserve the standard basis lines $\mathbf{C}e_j$ for $j > 2$ and (ii) preserve the plane $\mathbf{C}e_1 \oplus \mathbf{C}e_2$. This is $\mathrm{GL}_2(\mathbf{C}) \times (S^1)^{n-2}$. Intersecting with $\mathrm{SU}(n)$, we conclude that

$$Z_{G'}(T'') = (\mathrm{SU}(2) \cdot S_0) \times S$$

where this $\mathrm{SU}(2)$ is built inside $\mathrm{GL}(\mathbf{C}e_1 \oplus \mathbf{C}e_2)$, S is the codimension-1 subtorus of $(S^1)^{n-2}$ (in coordinates $z_3, \ldots, z_n$) whose coordinates have product equal to 1, and

$$S_0 = \{(z, z, 1, \ldots, 1, z^{-2})\}$$

(so $\mathrm{SU}(2) \cap S_0 = \{\pm 1\}$).

Clearly

$$N_{G'}(T') \cap Z_{G'}(T'') = (N_{\mathrm{SU}(2)}(T_0) \cdot S_0) \times S$$

where $T_0 \subset \mathrm{SU}(2)$ is the diagonal torus. Thus, an order-2 element of $N_{G'}(T')$ whose effect on T' is induced by $\sigma = (12)$ has the form (ν, s) for some 2-torsion $\nu \in N_{\mathrm{SU}(2)}(T_0) \cdot S_0$ acting on T_0 through inversion and some 2-torsion $s \in S$. But ν cannot be trivial, and s centralizes T', so it is equivalent to search for order-2 lifts of σ subject to the hypothesis $s = 1$. This amounts to the existence of an order-2 element $\nu \in N_{\mathrm{SU}(2)}(T_0) \cdot S_0$ that acts on T_0 through inversion. Writing $\nu = \nu_0 \cdot s_0$ for $s_0 \in S_0$ and $\nu_0 \in N_{\mathrm{SU}(2)}(T_0)$, necessarily ν_0 acts on T_0 through inversion (so $\nu_0^2 \in Z_{\mathrm{SU}(2)}(T_0) = T_0$) and $\nu_0^2 = s_0^{-2} \in S_0 \cap T_0 = \langle \tau \rangle$ for $\tau = (-1, -1, 1, \ldots, 1)$.

A direct calculation in $\mathrm{SU}(2)$ shows that every element of $N_{\mathrm{SU}(2)}(T_0) - T_0$ has square equal to the unique element $-1 \in T_0 \simeq S^1$ with order 2, so $s_0^{-2} = \nu_0^2 = \tau$. This says $s_0 = (i, i, 1, \ldots, 1, -1)$ for some $i = \pm\sqrt{-1}$, so if $n = 3$ (so $S = 1$) then the order-2 lifts of

$\sigma = (12)$ are exactly

$$\begin{pmatrix} 0 & it & 0 \\ -i/t & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

for some $i = \pm\sqrt{-1}$ and $t \in S^1$. In particular, these all have trace equal to -1 when $n = 3$.

With a general $n \geq 3$, assume we have a subgroup of $N_{G'}(T')$ isomorphically lifting the Weyl group. That identifies the lift with a subgroup $S_n \subset N_{G'}(T') \subset G' \subset \mathrm{GL}_n(\mathbf{C})$ acting on the standard basis lines through the standard permutation because this subgroup has conjugation effect on $\mathrm{X}(T = T' \cdot Z_G) = \mathbf{Z}^n$ through the standard permutation. (We do *not* claim the effect of this action permutes the set of standard basis vectors; it merely permutes the set of lines they span.) In particular, n-cycles in S_n act by permuting the standard basis lines and not fixing any of them, so such n-cycles act with trace equal to 0. For $n = 3$, inspection of the characters of the 3-dimensional representation of S_3 over $\mathbf{C}$ shows that the only one with that trace property is the standard representation. But in the standard representation of S_3 the order-2 elements act with trace 1, and we saw above that if $n = 3$ then an order-2 lift in $N_{G'}(T')$ of $\sigma = (12)$ (if one exists) must have trace equal to -1. This contradiction settles the case $n = 3$.

Finally, we can carry out an induction with $n \geq 4$ (using that the cases of $n-1, n-2 \geq 2$ are settled). Consider the subgroup $S_{n-1} \subset S_n$ stabilizing n. This acts on $\mathbf{C}^n$ preserving the standard basis line $\mathbf{C}e_n$, so its action on this line is one of the two 1-dimensional representations of S_{n-1} (trivial or sign). If trivial then we have

$$S_{n-1} \subset \mathrm{SU}(n-1) \subset \mathrm{GL}(\mathbf{C}e_1 \oplus \cdots \oplus \mathbf{C}e_{n-1})$$

normalizing the diagonal torus, contradicting the inductive hypothesis. Thus, this $S_{n-1} \subset S_n$ acts through the sign character on $\mathbf{C}e_n$.

Now consider the subgroup $S_{n-2} \subset S_{n-1}$ stabilizing 1, so its action on $\mathbf{C}^n$ preserves $\mathbf{C}e_1$. The action of this subgroup on $\mathbf{C}e_n$ is through the sign character (as the sign character on S_{n-1} restricts to that of S_{n-2}). If its action on $\mathbf{C}e_1$ is also through the sign character then by determinant considerations this S_{n-2} is contained in

$$\mathrm{SU}(n-2) \subset \mathrm{GL}(\mathbf{C}e_2 \oplus \cdots \oplus \mathbf{C}e_{n-1}),$$

contradicting the inductive hypothesis! Hence, its action on $\mathbf{C}e_1$ is trivial. But $(1n)$-conjugation inside S_n normalizes this subgroup S_{n-2} and its effect on $\mathbf{C}^n$ swaps the lines $\mathbf{C}e_1$ and $\mathbf{C}e_n$ (perhaps not carrying e_1 to e_n!). Consequently, this conjugation action on S_{n-2} must swap the characters by which S_{n-2} acts on $\mathbf{C}e_1$ and $\mathbf{C}e_n$. But that is impossible since the action on $\mathbf{C}e_1$ is trivial and the action on $\mathbf{C}e_n$ is the sign character. Hence, no lift exists.

K.3. **Character and cocharacter lattices.** We want to show that $W(G,S) := N_G(S)/Z_G(S)$ is finite for any torus S in G, and discuss some applications of this finite group (especially for maximal S). The group $W(G,S)$ with its quotient topology is compact, and we will construct a realization of this topological group inside a discrete group, so we can appeal to the fact that a discrete compact space is finite.

To analyze finiteness in a systematic way, it is useful to introduce a general concept that will pervade the later part of the course.

Definition K.1. The *character lattice* $\mathrm{X}(T)$ of a torus $T \simeq (S^1)^n$ is the abelian group of (continuous) characters $\mathrm{Hom}(T, S^1) = \mathrm{Hom}(S^1, S^1)^{\oplus n} = \mathbf{Z}^n$ (where $(a_1, \ldots, a_n) \in \mathbf{Z}^n$ corresponds to the character $(t_1, \ldots, t_n) \mapsto \prod t_i^{a_i}$).

Underlying the description of $\mathrm{X}((S^1)^n)$ as $\mathbf{Z}^n$ is the fact that the power characters $z \mapsto z^n$ ($n \in \mathbf{Z}$), are the endomorphisms of S^1 as a compact Lie group (as we saw in class earlier, by using $S^1 = \mathbf{R}/\mathbf{Z}$). In an evident manner, $T \rightsquigarrow \mathrm{X}(T)$ is a *contravariant* functor into the category of finite free $\mathbf{Z}$-modules, with $\mathrm{X}(T)$ having rank equal to $\dim T$.

Often the evaluation of a character $\chi \in \mathrm{X}(T)$ on an element $t \in T$ is denoted in "exponential" form as t^χ, and so correspondingly the group law on the character lattice (i.e., pointwise multiplication of S^1-valued functions) is denoted *additively*: we write $\chi + \chi'$ to denote the character $t \mapsto \chi(t)\chi'(t)$, 0 denotes the trivial character $t \mapsto 1$, and $-\chi$ denotes the reciprocal character $t \mapsto 1/\chi(t)$. This convention allows us to write things like

$$t^{\chi+\chi'} = t^\chi \cdot t^{\chi'}, \;\; t^0 = 1, \;\; t^{-\chi} = 1/t^\chi.$$

The reason for preferring this exponential notation is due to the following observation. As for any connected commutative Lie group (compact or not), the exponential map of a torus is a canonical surjective homomorphism $\exp_T : \mathfrak{t} \to T$ with discrete kernel $\Lambda \subset \mathfrak{t}$ inducing a canonical isomorphism $\mathfrak{t}/\Lambda \simeq T$. The compactness of T implies that Λ has maximal rank inside the $\mathbf{R}$-vector space $\mathfrak{t}$, which is to say that that natural map $\Lambda_{\mathbf{R}} := \mathbf{R} \otimes_{\mathbf{Z}} \Lambda \to \mathfrak{t}$ is an isomorphism. Thus,

$$\mathrm{X}(T) = \mathrm{Hom}(\mathfrak{t}/\Lambda, S^1) = \mathrm{Hom}(\Lambda_{\mathbf{R}}/\Lambda, S^1) \simeq \Lambda^* \otimes_{\mathbf{Z}} \mathrm{Hom}(\mathbf{R}/\mathbf{Z}, S^1)$$

where Λ^* denotes the $\mathbf{Z}$-linear dual of Λ and the final isomorphism (right to left) carries $\ell \otimes f$ to the map $\mathbf{R} \otimes_{\mathbf{Z}} \Lambda \to S^1$ given by $c \otimes \lambda \bmod \Lambda \mapsto f(c) \cdot \ell(\lambda)$. But $\mathbf{R}/\mathbf{Z} = S^1$ via $x \mapsto e^{2\pi i x}$, so plugging in the identification of the ring $\mathrm{Hom}(S^1, S^1)$ with $\mathbf{Z}$ via power maps brings us to the identification

$$\Lambda^* \simeq \mathrm{Hom}(\mathfrak{t}/\Lambda, S^1) = \mathrm{X}(T)$$

given by

$$\ell \mapsto (c \otimes \lambda \bmod \Lambda \mapsto e^{2\pi i c \ell(\lambda)} = e^{2\pi i \ell_{\mathbf{R}}(c\lambda)}).$$

Put another way, by identifying the scalar extension $(\Lambda^*)_{\mathbf{R}}$ with the $\mathbf{R}$-linear dual of $\Lambda_{\mathbf{R}} = \mathfrak{t}$ we can rewrite our identification $\Lambda^* \simeq \mathrm{X}(T)$ as $\ell \mapsto (v \bmod \Lambda \mapsto e^{2\pi i \ell_{\mathbf{R}}(v)})$ ($v \in \mathfrak{t}$). In this manner, we "see" the character lattice $\mathrm{X}(T)$ occurring inside analytic exponentials, in accordance with which the S^1-valued pointwise group law on $\mathrm{X}(T)$ literally is addition inside exponents.

Remark K.3.2. A related functor of T which comes to mind is the covariant *cocharacter lattice* $\mathrm{X}_*(T) = \mathrm{Hom}(S^1, T)$ (finite free of rank $\dim T$ also). This is the $\mathbf{Z}$-linear dual of the character lattice via the evaluation pairing

$$\langle \cdot, \cdot \rangle : \mathrm{X}(T) \times \mathrm{X}_*(T) \to \mathrm{End}(S^1) = \mathbf{Z}$$

defined by $\chi \circ \lambda : z \mapsto \chi(\lambda(z)) = z^{\langle \chi, \lambda \rangle}$ (check this really is a perfect pairing, and note that the order of appearance of the inputs into the pairing $\langle \cdot, \cdot \rangle$ – character on the left and cocharacter on the right – matches the order in which we write them in the composition notation $\chi \circ \lambda$ as functions).

We conclude from this perfect pairing that the covariant functor $\mathrm{X}_*(T)$ in T is naturally identified with $\mathrm{X}(T)^* = \Lambda^{**} = \Lambda$, and so its associated $\mathbf{R}$-vector space $\mathrm{X}_*(T)_{\mathbf{R}}$ is identified with $\Lambda_{\mathbf{R}} = \mathfrak{t}$. Thus, we can expression the exponential parameterization of T in the canonical form

$$\exp_T : \mathfrak{t}/\mathrm{X}_*(T) \simeq T.$$

The preference for $\mathrm{X}(T)$ over $\mathrm{X}_*(T)$ will become apparent later in the course. To emphasize the contravariance of $\mathrm{X}(T)$ in T, one sometimes denotes the character lattice as $\mathrm{X}^*(T)$, akin to the convention in topology that the contravariant cohomology functors are denoted with degree superscripts whereas the covariant homology functors are denoted with degree subscripts. We will generally not do this.

Here is an elaborate triviality:

Lemma K.3.3. *The functor $T \rightsquigarrow \mathrm{X}(T)$ from tori to finite free $\mathbf{Z}$-modules is an equivalence of categories. That is, for any tori T and T' and any map $f : \mathrm{X}(T') \to \mathrm{X}(T)$ of $\mathbf{Z}$-modules there exists exactly one homomorphism $\phi : T \to T'$ such that $\mathrm{X}(\phi) = f$, and every finite free $\mathbf{Z}$-module has the form $\mathrm{X}(T)$ for a torus T.*

In particular, passing to invertible endomorphisms, for an r-dimensional torus T the natural anti-homomorphism

$$\mathrm{Aut}(T) \to \mathrm{GL}(\mathrm{X}(T)) \simeq \mathrm{GL}_r(\mathbf{Z})$$

is an anti-isomorphism.

Proof. It is clear that $\mathbf{Z}^n \simeq \mathrm{X}((S^1)^n)$, so the main issue is bijectivity on Hom-sets. Decomposing T and T' into direct products of S^1's and using the natural identification $\mathrm{X}(T_1 \times T_2) \simeq \mathrm{X}(T_1) \times \mathrm{X}(T_2)$, we may immediately reduce to the case in which $T, T' = S^1$. In this case $\mathrm{X}(S^1) = \mathbf{Z}$ via the power characters and the endomorphism ring of $\mathbf{Z}$ as a $\mathbf{Z}$-module is compatibly identified with $\mathbf{Z}$. □

K.4. **Finiteness.** Let's return to tori in a connected compact Lie group G. For any such S in G, there is a natural action of the abstract group $W(G,S) = N_G(S)/Z_G(S)$ on S via $\overline{n}.s = nsn^{-1}$, and by definition of $Z_G(S)$ this is a *faithful* action in the sense that the associated action homomorphism $W(G,S) \to \mathrm{Aut}(S)$ is injective.

The target group $\mathrm{Aut}(S)$ looks a bit abstract, but by bringing in the language of character lattices it becomes more concrete as follows. Since the "character lattice" functor on tori is fully faithful and contravariant, it follows that the associated action of $W(G,S)$ on $\mathrm{X}(S)$ via $w.\chi : s \mapsto \chi(w^{-1}.s)$ is also faithful: the homomorphism $W(G,S) \to \mathrm{GL}(\mathrm{X}(S)) \simeq \mathrm{GL}_r(\mathbf{Z})$ is injective (with $r = \dim S = \mathrm{rank}_{\mathbf{Z}}(\mathrm{X}(S))$).

Now we are finally in position to prove:

Proposition K.4.1. *The group $W(G,S)$ is finite.*

As we noted at the outset, this finiteness is equivalent to the equality $Z_G(S)^0 = N_G(S)^0$. (We cannot say $Z_G(S) = N_G(S)^0$ until we know that $Z_G(S)$ is connected, which rests on the Conjugacy Theorem.)

Proof. Since $W(G,S)$ with its quotient topology is compact, it suffices to show that the natural injective map $W(G,S) \to \mathrm{GL}(\mathrm{X}(S)) = \mathrm{GL}_r(\mathbf{Z})$ is continuous when the target is given the discrete topology (as then the image of this *injective* map would be compact and discrete, hence finite). Of course, it is equivalent (and more concrete) to say that $N_G(S) \to \mathrm{GL}(\mathrm{X}(S))$ is continuous.

By using duality to pass from automorphisms of $\mathrm{X}(S)$ to automorphisms of $\mathrm{X}_*(S)$, it is equivalent to check that the injective homomorphism $N_G(S) \to \mathrm{GL}(\mathrm{X}_*(S))$ arising from the action of $N_G(S)$ on $\mathrm{X}_*(S)$ defined by $(n.\lambda)(z) = n \cdot \lambda(z) \cdot n^{-1}$ is continuous when the target is viewed discretely. Equivalently, this says that the action map

$$N_G(S) \times \mathrm{X}_*(S) \to \mathrm{X}_*(S)$$

is continuous when we give $\mathrm{X}_*(S)$ the discrete topology and the left side the product topology.

But the discrete topology on $\mathrm{X}_*(S)$ is the subspace topology from naturally sitting inside the finite-dimensional $\mathbf{R}$-vector space $\mathrm{X}_*(S)_{\mathbf{R}}$, so it is equivalent to check that the natural action map

$$f_S : N_G(S) \times \mathrm{X}_*(S)_{\mathbf{R}} \to \mathrm{X}_*(S)_{\mathbf{R}}$$

is continuous relative to the natural manifold topologies on the source and target.

Recall that for any torus T whatsoever, the "realification" $\mathrm{X}_*(T)_{\mathbf{R}}$ of the cocharacter lattice is naturally identified by means of $\exp_T$ with the Lie algebra $\mathrm{Lie}(T)$. Identifying $\mathrm{X}_*(S)_{\mathbf{R}}$ with $\mathrm{Lie}(S)$ in this way, we can express f_S as a natural map

$$N_G(S) \times \mathrm{Lie}(S) \to \mathrm{Lie}(S).$$

What could this natural map be? By using the *functoriality* of the exponential map of Lie groups with respect to the inclusion of S into G, and the definition of Ad_G in terms the effect of conjugation on the tangent space at the identity point of G, one checks (do it!) that this final map carries (n, v) to $\mathrm{Ad}_G(n)(v)$, where we note that $\mathrm{Ad}_G(n)$ acting on $\mathfrak{g}$ preserves the subspace $\mathrm{Lie}(S)$ since n-conjugation on G preserves S. (The displayed commutative diagram just above (1.2) in Chapter IV gives a nice visualization for the automorphism $\varphi = \mathrm{Ad}_G(n)$ of the torus S.)

So to summarize, our map of interest is induced by the restriction to $N_G(S) \times \mathrm{Lie}(S)$ of the action map

$$G \times \mathfrak{g} \to \mathfrak{g}$$

given by $(g, X) \mapsto \mathrm{Ad}_G(g)(X)$. The evident continuity of this latter map does the job. $\square$

Remark K.4.2. What is "really going on" in the above proof is that since the automorphism group of a torus is a "discrete" object, a continuous action on S by a *connected* Lie group H would be a "connected family of automorphisms" of S (parameterized by H) containing the identity automorphism and so would have to be the constant family: a trivial action on S by every $h \in H$. That alone forces $N_G(S)^0$, whatever it may be, to act trivially on S. In other words, this forces $N_G(S)^0 \subset Z_G(S)$, so $N_G(S)^0 = Z_G(S)^0$.

The reason that this short argument is not a rigorous proof as stated is that we have not made a precise logical connection between the idea of "discreteness" for the automorphism group of S and the connectedness of the topological space $N_G(S)^0$; e.g., we have not discussed the task of putting a natural topology on the set of automorphisms of a Lie group. (The proof in IV, 1.5 that $N_G(T)^0 = T$ is incomplete because of exactly this issue, so we prove it below using Proposition K.4.1, whose proof made this discrete/connected dichotomy rigorous in some situations.) There is a systematic and useful way to justify this intuition by means of the idea of "Lie group of automorphisms of a connected Lie group", but it is more instructive for later purposes to carry out the argument as above using the crutch of the isomorphism $\mathrm{X}_*(S)_{\mathbf{R}} \simeq \mathrm{Lie}(S)$ rather than to digress into the topic (actually not very hard) of equipping the automorphism group of a connected Lie group with a structure of (possibly very disconnected) Lie group in a useful way.

Proposition K.4.3. *Let G be a compact connected Lie group, T a maximal torus. The identity component $N_G(T)^0$ is equal to T (so $N_G(T)/T$ is finite).*

Remark K.4.4. The proof of the Conjugacy Theorem works with the finite group $N_G(T)/T$, and it is essential in that proof that we know $N_G(T)/T$ is *finite* (but only after the proof of the Conjugacy Theorem is over and connectedness of torus centralizers thereby becomes available can we conclude that $N_G(T)/T = N_G(T)/Z_G(T) =: W(G,T)$).

Proof. To prove that $N_G(T)^0 = T$ for any choice of maximal torus T in a compact connected Lie group G, we first note that $N_G(T)^0 = Z_G(T)^0$ by Proposition K.4.1. To show that $Z_G(T)^0 = T$, if we rename $Z_G(T)^0$ with G (as we may certainly do!) then we find ourselves in the case that T is *central* in G, and in such cases we wish to prove that $G = T$.

It suffices to show that the inclusion of Lie algebras $\mathfrak{t} \hookrightarrow \mathfrak{g}$ is an equality. Choose $X \in \mathfrak{g}$ and consider the associated 1-parameter subgroup $\alpha_X : \mathbf{R} \to G$. Clearly $\alpha_X(\mathbf{R})$ is a connected commutative subgroup of G, so $\alpha_X(\mathbf{R}) \cdot T$ is also a connected commutative subgroup of G (as T is central in G). Its closure H is therefore a connected commutative compact Lie group, but we know that the only such groups are tori. Thus, H is a torus in G, but by design it contains T which is a maximal torus of G. Thus, $H = T$, so $\alpha_X(\mathbf{R}) \subseteq T$. Hence, $\alpha_X : \mathbf{R} \to G$ factors through T, so $X = \alpha_X'(0) \in \mathfrak{t}$. This proves that $\mathfrak{g} = \mathfrak{t}$ (so $G = T$). $\square$

K.5. **Subtori via character lattices.** We finish this appendix with a discussion of how to use the character lattice to keep track of subtori and quotient tori. This is very convenient in practice, and clarifies the sense in which the functor $T \rightsquigarrow \mathrm{X}(T)$ is similar to duality for finite-dimensional vector spaces over a field. (This is all subsumed by the general Pontryagin duality for locally compact Hausdorff topological abelian groups, but our treatment of tori is self-contained without such extra generalities.) As an application, we shall deduce by indirect means that any torus contains a dense cyclic subgroup.

Let T be a torus. For any closed (not necessarily connected) subgroup $H \subset T$, the quotient T/H is a compact connected commutative Lie group, so it is a torus. The pullback map $\mathrm{X}(T/H) \to \mathrm{X}(T)$ (composing characters with the quotient map $T \twoheadrightarrow T/H$) is an injection between finitely generated $\mathbf{Z}$-modules. For example, if $T = S^1$ and $H = \mu_n$ is the cyclic subgroup of nth roots of unity ($n > 0$) then the map $f : T \to S^1$ defined by $f(t) = t^n$ identifies T/H with S^1 and the subgroup $\mathrm{X}(T/H) = \mathrm{X}(S^1) = \mathbf{Z}$ of $\mathrm{X}(T) = \mathrm{X}(S^1) = \mathbf{Z}$ via composition of characters with f is the injective map $\mathbf{Z} \to \mathbf{Z}$ defined by multiplication by n. (Check this!)

Proposition K.5.1. *For any subtorus $T' \subset T$, the induced map $\mathrm{X}(T) \to \mathrm{X}(T')$ is surjective with kernel $\mathrm{X}(T/T')$. In particular, $\mathrm{X}(T/T')$ is a saturated subgroup of $\mathrm{X}(T)$ in the sense that the quotient $\mathrm{X}(T)/\mathrm{X}(T/T') = \mathrm{X}(T')$ is torsion-free.*

Proof. The elements $\chi \in \mathrm{X}(T)$ killed by restriction to T' are precisely those $\chi : T \to S^1$ that factor through T/T', which is to say $\chi \in \mathrm{X}(T/T')$ inside $\mathrm{X}(T)$. This identifies the kernel, so it remains to prove surjectivity. That is, we have to show that every character $\chi' : T' \to S^1$

extends to a character $T \to S^1$ (with all characters understood to be homomorphisms of Lie groups, or equivalently continuous). We may assume $T' \neq 1$.

Choose a product decomposition $T' = (S^1)^m$, so $\mathrm{X}(T')$ is generated as an abelian group by the characters $\chi'_i : (z_1, \ldots, z_n) \mapsto z_i$. (Check this!) It suffices to show that each χ'_i extends to a character of T. Let $T'' = (S^1)^{m-1}$ be the product of the S^1-factors of T' apart from the ith factor. In an evident manner, χ'_i is identified with a character of $T'/T'' = S^1$ and it suffices to extend this to a character of T/T'' (as such an extension composed with $T \twoheadrightarrow T/T''$ is an extension of χ'_i). Thus, it suffices to treat the case $T' = S^1$ and to extend the identity map $S^1 \to S^1$ to a character of T.

Choosing a product decomposition $T = (S^1)^n$, the given abstract inclusion $S^1 = T' \hookrightarrow T = (S^1)^n$ is a map $\iota : z \mapsto (z^{e_1}, \ldots, z^{e_n})$ for some integers $e_1, \ldots, e_n$, and the triviality of $\ker \iota$ implies that $\gcd(e_i) = 1$. Hence, there exist integers $r_1, \ldots, r_n$ so that $\sum r_j e_j = 1$. Hence, $\chi : T \to S^1$ defined by

$$\chi(z_1, \ldots, z_n) = \prod z_j^{r_j}$$

satisfies $\chi(\iota(z)) = z^{\sum r_j e_j} = z$, so χ does the job. $\square$

Corollary K.5.2. *A map $f : T' \to T$ between tori is a subtorus inclusion (i.e., isomorphism of f onto a closed subgroup of T) if and only if $\mathrm{X}(f)$ is surjective.*

Proof. We have already seen that the restriction map to a subtorus induces a surjection between character groups. It remains to show that if $\mathrm{X}(f)$ is surjective then f is a subtorus inclusion. For any homomorphism $f : G' \to G$ between compact Lie groups, the image is closed, even compact, and the surjective map $G' \to f(G')$ between Lie groups is a Lie group isomorphism of $G'/(\ker f)$ onto $f(G')$. Thus, if $\ker f = 1$ then f is an isomorphism onto a closed Lie subgroup of G. Hence, in our setting with tori, the problem is to show $\ker f = 1$.

By the surjectivity hypothesis, every $\chi' \in \mathrm{X}(T')$ has the form $\chi \circ f$ for some $\chi \in \mathrm{X}(T)$, so $\chi'(\ker f) = 1$. Hence, for any $t' \in \ker f$ we have $\chi'(t') = 1$ for all $\chi' \in \mathrm{X}(T')$. But by choosing a product decomposition $T' \simeq (S^1)^m$ it is clear that for any nontrivial $t' \in T'$ there is some $\chi' \in \mathrm{X}(T')$ such that $\chi'(t') \neq 1$ (e.g., take χ' to be the projection from $T' = (S^1)^m$ onto an S^1-factor for which t' has a nontrivial component). Thus, we conclude that any $t' \in \ker f$ is equal to 1, which is to say that $\ker f = 1$. $\square$

Corollary K.5.3. *The map $T' \mapsto \mathrm{X}(T/T')$ is a bijection from the set of subtori of T onto the set of saturated subgroups of $\mathrm{X}(T)$. Moreover, it is inclusion-reversing in both directions in the sense that $T' \subseteq T''$ inside T if and only if $\mathrm{X}(T/T'') \subseteq \mathrm{X}(T/T')$ inside $\mathrm{X}(T)$.*

Proof. Let $j' : T' \hookrightarrow T$ and $j'' : T'' \hookrightarrow T$ be subtori, so the associated maps $\mathrm{X}(j')$ and $\mathrm{X}(j'')$ on character lattices are surjections from $\mathrm{X}(T)$. Clearly if $T' \subseteq T''$ inside T then we get a quotient map $T/T' \twoheadrightarrow T/T''$ as torus quotients of T, so passing to character lattices gives that $\mathrm{X}(T/T'') \subseteq \mathrm{X}(T/T')$ inside $\mathrm{X}(T)$.

Let's now show the reverse implication: assuming $\mathrm{X}(T/T'') \subseteq \mathrm{X}(T/T')$ inside $\mathrm{X}(T)$ we claim that $T' \subseteq T''$ inside T. Passing to the associated quotients of $\mathrm{X}(T)$, we get a map $f : \mathrm{X}(T'') \to \mathrm{X}(T')$ respecting the surjections $\mathrm{X}(j'')$ and $\mathrm{X}(j')$ onto each side from $\mathrm{X}(T)$. By the categorical equivalence in Lemma K.3.3, the map f has the form $\mathrm{X}(\phi)$ for a map of tori $\phi : T' \to T''$, and $j'' \circ \phi = j'$ since we can check the equality at the level of character groups (by how ϕ was constructed). This says exactly that $T' \subseteq T''$ inside T, as desired.

As a special case, we conclude that $\mathrm{X}(T/T'') = \mathrm{X}(T/T')$ inside $\mathrm{X}(T)$ if and only if $T' = T''$ inside T. Thus, the saturated subgroup $\mathrm{X}(T/T')$ inside $\mathrm{X}(T)$ determines the subtorus T' inside T.

It remains to show that if $\Lambda \subseteq \mathrm{X}(T)$ is a saturated subgroup then $\Lambda = \mathrm{X}(T/T')$ for a subtorus $T' \subseteq T$. By the definition of saturatedness, $\mathrm{X}(T)/\Lambda$ is a finite free $\mathbf{Z}$-module, so it has the form $\mathrm{X}(T')$ for an abstract torus T'. By the categorical equivalence in Lemma K.3.3, the quotient map $q : \mathrm{X}(T) \twoheadrightarrow \mathrm{X}(T)/\Lambda = \mathrm{X}(T')$ has the form $\mathrm{X}(\phi)$ for a map of tori $\phi : T' \to T$. By Corollary K.5.2, ϕ is a subtorus inclusion since $\mathrm{X}(\phi) = q$ is surjective, so T' is identified with a subtorus of T identifying the q with the restriction map on characters. Hence, the kernel Λ of q is identified with $\mathrm{X}(T/T')$ inside $\mathrm{X}(T)$ as desired. □

Corollary K.5.4. *For a map $f : T \to S$ between tori, the kernel is connected (equivalently, is a torus) if and only if $\mathrm{X}(f)$ has torsion-free cokernel.*

Proof. We may replace S with the torus $f(T)$, so f is surjective and hence $S = T/(\ker f)$. Thus, $\mathrm{X}(f)$ is injective. If $T' := \ker f$ is a torus then

$$\mathrm{coker}(\mathrm{X}(f)) = \mathrm{X}(T)/\mathrm{X}(S) = \mathrm{X}(T)/\mathrm{X}(T/T') = \mathrm{X}(T')$$

is torsion-free. It remains to prove the converse.

Assume $\mathrm{coker}(\mathrm{X}(f))$ is torsion-free, so the image of $\mathrm{X}(S)$ in $\mathrm{X}(T)$ is saturated. Thus, this image is $\mathrm{X}(T/T')$ for a subtorus $T' \subset T$. The resulting composite map

$$\mathrm{X}(f) : \mathrm{X}(S) \twoheadrightarrow \mathrm{X}(T/T') \hookrightarrow \mathrm{X}(T)$$

arises from a diagram of torus maps factoring f as a composition

$$T \twoheadrightarrow T/T' \hookrightarrow S$$

of the natural quotient map and a subtorus inclusion. It follows that $\ker f = T'$ is a torus. □

Since $\mathrm{X}(T)$ has only *countably many* subgroups (let alone saturated ones), Corollary K.5.3 implies that T contains only *countably many* proper subtori $\{T_1, T_2, \dots\}$. As an application, we get the existence of (lots of) $t \in T$ for which the cyclic subgroup $\langle t \rangle$ generated by t is dense, as follows. Pick any $t \in T$ and let Z be the closure of $\langle t \rangle$ (which we want to equal T for suitable t), so Z^0 is a subtorus of T. This has finite index in Z, so $t^n \in Z^0$ for some $n \geq 1$. We need to rule out the possibility that $Z^0 \neq T$; i.e., that Z^0 is a proper subtorus. There are countably many proper subtori of T, say denoted $T_1, T_2, \dots$, and we just need to avoid the possibility $t \in [n]^{-1}(T_m)$ for some $n \geq 1$ and some m. Put another way, we just have to show that T is not the union of the preimages $[n]^{-1}(T_m)$ for all $n \geq 1$ and all m. This is immediate via the Baire Category Theorem, but here is a more elementary approach.

If some t^n (with $n > 0$) lies in a proper subtorus $T' \subset T$ then t^n is killed by the quotient map $T \to T/T'$ whose target T/T' is a *non-trivial* torus (being commutative, connected, and compact). Writing T/T' as a product of copies of S^1 and projecting onto one of those factors, we get a quotient map $q : T \twoheadrightarrow S^1 = \mathbf{R}/\mathbf{Z}$ killing t^n. If we identify T with $\mathbf{R}^d/\mathbf{Z}^d$, t is represented by $(t_1, \dots, t_d) \in \mathbf{R}^d$ and $q : \mathbf{R}^d/\mathbf{Z}^d \twoheadrightarrow \mathbf{R}/\mathbf{Z}$ is induced by $(x_1, \dots, x_d) \mapsto \sum a_j x_j$ for some $a_1, \dots, a_d \in \mathbf{Z}$ not all zero. Thus, we just have to pick $(t_1, \dots, t_d) \in \mathbf{R}^d$ avoiding the possibility that for some $n > 0$ and some $a_1, \dots, a_d \in \mathbf{Z}$ not all 0, $\sum a_j(nt_j) \in \mathbf{Z}$. Replacing a_j with na_j, it suffices to find $t \in \mathbf{R}^n$ so that $\sum b_j t_j \notin \mathbf{Z}$ for all $b_1, \dots, b_d \in \mathbf{Z}$ not all zero.

Suppose for some $b_1, \ldots, b_d \in \mathbf{Z}$ not all zero that $\sum b_j t_j = c \in \mathbf{Z}$. Then $\sum b_j t_j + (-c) \cdot 1 = 0$, so $\{t_1, \ldots, t_d, 1\}$ is $\mathbf{Q}$-linearly dependent. Hence, picking a $\mathbf{Q}$-linearly independent subset of $\mathbf{R}$ of size $\dim(T) + 1$ (and scaling throughout so 1 occurs in the set) provides such a t. In this way we see that there are very many such t.

APPENDIX L. THE WEYL JACOBIAN FORMULA

L.1. **Introduction.** Let G be a connected compact Lie group, and T a maximal torus in G. Let $q : (G/T) \times T \to G$ be the map $(\overline{g}, t) \mapsto gtg^{-1}$. Fix choices of nonzero left-invariant top-degree differential forms $\mathrm{d}g$ on G and $\mathrm{d}t$ on T, and let $\mathrm{d}\overline{g}$ be the associated left-invariant top-degree differential form on G/T, determined by the property that

$$\mathrm{d}\overline{g}(e) \otimes \mathrm{d}t(\overline{e}) = \mathrm{d}g(e)$$

via the canonical isomorphism $\det(\mathrm{Tan}^*_{\overline{e}}(G/T)) \otimes \det(\mathrm{Tan}^*_e(T)) \simeq \det(\mathrm{Tan}^*_e(G))$. These differential forms are used to define orientations on G, T, and G/T, hence also on $(G/T) \times T$ via the product orientation.

Remark L.1.1. Recall that for oriented smooth manifolds X and X' with respective dimensions d and d', the *product orientation* on $X \times X'$ declares the positive ordered bases of

$$\mathrm{T}_{(x,x')}(X \times X') = \mathrm{T}_x(X) \oplus \mathrm{T}_{x'}(X')$$

to be the ones in the common orientation class of the ordered bases $\{v_1, \ldots, v_d, v'_1, \ldots, v'_{d'}\}$, where $\{v_i\}$ is an oriented basis of $\mathrm{T}_x(X)$ and $\{v'_{j'}\}$ is an oriented basis of $\mathrm{T}_{x'}(X')$. If we swap the order of the factors (i.e., we make an ordered basis for $\mathrm{T}_{(x,x')}(X \times X')$ by putting the $v'_{j'}$'s ahead of the v_i's) then the orientation on $X \times X'$ changes by a sign of $(-1)^{dd'}$. Consequently, as long as one of d or d' is *even*, there is no confusion about the orientation on $X \times X'$. Fortunately, it will turn out that G/T has even dimension!

Since $\mathrm{d}t \wedge \mathrm{d}\overline{g}$ is a nowhere-vanishing top-degree C^∞ differential form on $(G/T) \times T$, there is a unique C^∞ function F on $(G/T) \times T$ satisfying

$$q^*(\mathrm{d}g) = F \cdot \mathrm{d}t \wedge \mathrm{d}\overline{g}.$$

(In class we used $\mathrm{d}\overline{g} \wedge \mathrm{d}t$, as in [BtD]; this discrepancy will not matter since we'll eventually show that $\dim(G/T)$ is even.) Let's describe the meaning of $F(\overline{g}_0, t_0) \in \mathbf{R}$ as a Jacobian determinant. For any point $(\overline{g}_0, t_0) \in (G/T) \times T$, we may and do choose an oriented ordered basis of $\mathrm{Tan}_{t_0}(T)$ whose ordered dual basis has wedge product equal to $\mathrm{d}t(t_0)$. We also may and do choose an oriented ordered basis of $\mathrm{Tan}_{\overline{g}_0}(G/T)$ whose ordered dual basis has wedge product equal to $\mathrm{d}\overline{g}(\overline{g}_0)$. Use these to define an ordered basis of $\mathrm{Tan}_{(\overline{g}_0, t_0)}((G/T) \times T)$ by following the convention for product orientation putting T ahead of G/T. Finally, choose an oriented ordered basis of $\mathrm{Tan}_{q(\overline{g}_0, t_0)}(G)$ whose associated ordered dual basis has wedge product equal to $\mathrm{d}g(q(\overline{g}_0, t_0))$.

Consider the matrix of the linear map

$$\mathrm{d}q(\overline{g}_0, t_0) : \mathrm{Tan}_{(\overline{g}_0, t_0)}((G/T) \times T) \to \mathrm{Tan}_{q(\overline{g}_0, t_0)}(G)$$

relative to the specified ordered bases on the source and target. The determinant of this matrix is exactly $F(\overline{g}_0, t_0)$. (Check this!) In particular, $F(\overline{g}_0, t_0) \neq 0$ precisely when q is a local C^∞ isomorphism near $(\overline{g}_0, t_0)$, and $F(\overline{g}_0, t_0) > 0$ precisely when q is an orientation-preserving local C^∞ isomorphism near $(\overline{g}_0, t_0)$. Provided that $\dim(G/T)$ is even, it won't matter which way we impose the orientation on $(G/T) \times T$ (i.e., which factor we put "first").

In view of the preceding discussion, it is reasonable to introduce a suggestive notation for F: we shall write $\det(\mathrm{d}q(\overline{g}, t))$ rather than $F(\overline{g}, t)$. Of course, this depends on more than just the linear map $\mathrm{d}q(\overline{g}, t)$: it also depends on the initial choices of invariant differential forms $\mathrm{d}g$ on G and $\mathrm{d}t$ on T, as well as the convention to orient $(G/T) \times T$ by putting T ahead of G/T (the latter convention becoming irrelevant once we establish that $\dim(G/T)$ is even). Such dependence is suppressed in the notation, but should not be forgotten.

The purpose of this appendix is to establish an explicit formula for the Jacobian determinant $\det(\mathrm{d}q)$ associated to the map q. It is expressed in terms of the map

$$\mathrm{Ad}_{G/T} : T \to \mathrm{GL}(\mathrm{Tan}_{\overline{e}}(G/T)),$$

so let's recall the definition of $\mathrm{Ad}_{G/T}$ more generally (given dually in Exercise 2 of HW5).

If H is a closed subgroup of a Lie group G then $\mathrm{Ad}_{G/H} : H \to \mathrm{GL}(\mathrm{T}_{\overline{e}}(G/H))$ is the C^∞ homomorphism that carries $h \in H$ to the linear automorphism $\mathrm{d}(\ell_h)(e)$ of $\mathrm{T}_{\overline{e}}(G/H) = \mathfrak{g}/\mathfrak{h}$ arising from the left translation on G/H by h (which fixes the coset $\overline{e} = \{H\}$ of h). It is equivalent to consider the effect on $\mathfrak{g}/\mathfrak{h}$ of the conjugation map $c_h : x \mapsto hxh^{-1}$ since right-translation on G by h^{-1} has no effect upon passage to G/H. Put in other terms, $\mathrm{Ad}_{G/H}(h)$ is the effect on $\mathfrak{g}/\mathfrak{h}$ of the automorphism $\mathrm{d}c_h(e) = \mathrm{Ad}_G(h)$ of $\mathfrak{g}$ that preserves $\mathfrak{h}$ (since $\mathrm{Ad}_G(h)|_{\mathfrak{h}} = \mathrm{Ad}_H(h)$ due to functoriality in the Lie group for the adjoint representation).

L.2. **Main result.** We shall prove the following formula for the "Weyl Jacobian" $\det(\mathrm{d}q)$:

Theorem L.2.1. *For any* $(\overline{g}, t) \in (G/T) \times T$,

$$\det(\mathrm{d}q(\overline{g}, t)) = \det(\mathrm{Ad}_{G/T}(t^{-1}) - 1).$$

In class we will show that $\dim(G/T)$ is even, so there is no possible sign ambiguity in the orientation on $(G/T) \times T$ that underlies the definition of $\det(\mathrm{d}q)$. Our computation in the proof of the Theorem will use the orientation which puts T ahead of G/T; this is why we used $\mathrm{d}t \wedge \mathrm{d}\overline{g}$ rather than $\mathrm{d}\overline{g} \wedge \mathrm{d}t$ when initially defining $\det(\mathrm{d}q)$. Only after the evenness of $\dim(G/T)$ is proved in class will the Theorem hold without sign ambiguity. (The proof of such evenness will be insensitive to such sign problems.)

Proof. We first use left translation to pass to the case $\overline{g} = \overline{e}$, as follows. For $g_0 \in G$, we have a commutative diagram

$$\begin{tikzcd} (G/T) \times T \arrow[r, "q"] \arrow[d, "\lambda"'] & G \arrow[d, "c_{g_0}"] \\ (G/T) \times T \arrow[r, "q"'] & G \end{tikzcd}$$

where $\lambda(\overline{g}, t) = (g_0.\overline{g}, t)$. The left-translation by g_0 on G/T pulls $\mathrm{d}\overline{g}$ back to itself due to left invariance of this differential form on G/T. The conjugation c_{g_0} pulls $\mathrm{d}g$ back to itself because $\mathrm{d}g$ is *bi-invariant* (i.e., also right-invariant) due to the triviality of the algebraic modulus character Δ_G (since G is compact and *connected*). It therefore follows via the definition of $\det(\mathrm{d}q)$ that

$$\det(\mathrm{d}q(g_0.\overline{g}, t)) = \det(\mathrm{d}q(\overline{g}, t)).$$

Hence, by choose g_0 to represent the inverse of a representative of $\overline{g}$, we may and do restrict attention to the case $\overline{g} = \overline{e}$.

Our aim is to show that for any $t \in T$, $\det(\mathrm{d}q(\overline{e}, t_0)) = \det(\mathrm{Ad}_{G/T}(t_0^{-1}) - 1)$ for all $t_0 \in T$. Consider the composite map

$$f : (G/T) \times T \xrightarrow{1 \times \ell_{t_0}} (G/T) \times T \xrightarrow{q} G \xrightarrow{\ell_{t_0^{-1}}} G.$$

This carries $(\overline{g}, t)$ to $(t_0^{-1} g t_0)(t g^{-1})$ (which visibly depends only on gT rather than on g, as it must); this map clearly carries $(\overline{e}, e)$ to e. The first and third steps pull the chosen differential forms on $(G/T) \times T$ and G (namely, $\mathrm{d}t \wedge \mathrm{d}\overline{g}$ and $\mathrm{d}g$) back to themselves due to the arranged left-invariance properties. Consequently, $\det(\mathrm{d}q(\overline{e}, t_0)) = \det(\mathrm{d}f(\overline{e}, e))$, where the "determinant" of

$$\mathrm{d}f(\overline{e}, e) : (\mathfrak{g}/\mathfrak{t}) \oplus \mathfrak{t} \to \mathfrak{g}$$

is defined using oriented ordered bases of $\mathfrak{t}$, $\mathfrak{g}/\mathfrak{t}$, and $\mathfrak{g}$ whose wedge products are respectively dual to $\mathrm{d}t(e)$, $\mathrm{d}\overline{g}(\overline{e})$, and $\mathrm{d}g(e)$ (and the direct sum is oriented by putting $\mathfrak{t}$ ahead of $\mathfrak{g}/\mathfrak{t}$).

The *definition* of $\mathrm{d}\overline{g}(\overline{e})$ uses several pieces of data: $\mathrm{d}g(e)$, $\mathrm{d}t(e)$, and the natural isomorphism $\det(V') \otimes \det(V'') \simeq \det(V)$ associated to a short exact sequence of finite-dimensional vector spaces $0 \to V' \to V \to V'' \to 0$ (applied to $0 \to \mathfrak{t} \to \mathfrak{g} \to \mathrm{Tan}_{\overline{e}}(G/T) \to 0$). Thus, we can *choose* the oriented ordered basis of $\mathrm{Tan}_{\overline{e}}(G/T) = \mathfrak{g}/\mathfrak{t}$ adapted to $\mathrm{d}\overline{g}(\overline{e})$ to be induced by an oriented ordered basis of $\mathfrak{g}$ adapted to $\mathrm{d}g(e)$ that has as its inital part an ordered oriented basis of $\mathfrak{t}$ adapted to $\mathrm{d}t(e)$. To summarize, the matrix for $\mathrm{d}f(\overline{e}, e) : (\mathfrak{g}/\mathfrak{t}) \oplus \mathfrak{t} \to \mathfrak{g}$ rests on: an ordered basis of $\mathfrak{t}$, an extension of this to an ordered basis of $\mathfrak{g}$ by appending additional vectors at the end of the ordered list, and the resulting quotient ordered basis of $\mathfrak{g}/\mathfrak{t}$.

The restriction $\mathrm{d}f(\overline{e}, e)|_{\mathfrak{t}}$ to the direct summand $\mathfrak{t}$ is the differential of $f(\overline{e}, \cdot) : T \to G$ that sends $t \in T$ to $(t_0^{-1} e t_0) t e^{-1} = t \in G$. In other words, this restriction is the natural inclusion of $\mathfrak{t}$ into $\mathfrak{g}$.

Composing the restriction $\mathrm{d}f(\overline{e}, e)|_{\mathfrak{g}/\mathfrak{t}}$ to the direct summand $\mathfrak{g}/\mathfrak{t}$ with the quotient map $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{t}$ is the endomorphism of $\mathfrak{g}/\mathfrak{t}$ that is the differential at $\overline{e}$ of the map $k : G/T \to G/T$ defined by

$$\overline{g} \mapsto (t_0^{-1} g t_0) e g^{-1} \bmod T = c_{t_0^{-1}}(g) g^{-1} \bmod T = m(c_{t_0^{-1}}(g), g^{-1}) \bmod T.$$

Since the group law $m : G \times G \to G$ has differential at (e, e) equal to addition in $\mathfrak{g}$, and inversion $G \to G$ has differential at the identity equal to negation on $\mathfrak{g}$, clearly

$$\mathrm{d}k(\overline{e}) = (\mathrm{Ad}_G(t_0^{-1}) \bmod \mathfrak{t}) - 1 = \mathrm{Ad}_{G/T}(t_0^{-1}) - 1.$$

Our ordered basis of $\mathfrak{g}$ begins with an ordered basis of $\mathfrak{t}$ and the remaining part lifts our chosen ordered basis of $\mathfrak{g}/\mathfrak{t}$, so the matrix used for $\mathrm{d}f(\overline{e}, e)$ has the upper triangular form

$$\begin{pmatrix} 1 & * \\ 0 & M(t_0) \end{pmatrix}$$

where the lower-right square $M(t_0)$ is the matrix of the endomorphism $\mathrm{Ad}_{G/T}(t_0^{-1}) - 1$ of $\mathfrak{g}/\mathfrak{t}$ relative to the *same* ordered basis on its source and target. Consequently, the determinant of this matrix for $\mathrm{d}f(\overline{e}, e)$ is equal to $\det M(t_0) = \det(\mathrm{Ad}_{G/T}(t_0^{-1}) - 1)$ (the intrinsic determinant using *any* fixed choice of ordered basis for the common source and target $\mathfrak{g}/\mathfrak{t}$). □

APPENDIX M. CLASS FUNCTIONS AND WEYL GROUPS

M.1. **Main result.** As an application of the Conjugacy Theorem, we can describe the continuous class functions $f : G \to \mathbf{C}$ on a connected compact Lie group G in terms of a choice of maximal torus $T \subset G$. This will be an important "Step 0" in our later formulation of the Weyl character formula. If we consider G acting on itself through conjugation, the quotient $\mathrm{Conj}(G)$ by that action is the space of conjugacy classes. We give it the quotient topology from G, so then the $\mathbf{C}$-algebra of continuous $\mathbf{C}$-valued class functions on G is the same as the $\mathbf{C}$-algebra $C^0(\mathrm{Conj}(G))$ of continuous $\mathbf{C}$-valued class functions on $\mathrm{Conj}(G)$.

Let $W = N_G(T)/T$ be the (finite) Weyl group, so W naturally acts on T. The W-action on T is induced by the conjugation action of $N_G(T)$ on G, so we get an induced continuous map of quotient spaces $T/W \to \mathrm{Conj}(G)$.

Proposition M.1.1. *The natural continuous map $T/W \to \mathrm{Conj}(G)$ is bijective.*

M.2. **Proof of Proposition M.1.1 and applications.** By the Conjugacy Theorem, every element of G belongs to a maximal torus, and such tori are G-conjugate to T, so surjectivity is clear. For injectivity, consider $t, t' \in T$ that are conjugate in G. We want to show that they below to the same W-orbit in T.

Pick $g \in G$ so that $t' = gtg^{-1}$. The two tori T, gTg^{-1} then contain t', so by connectedness and commutativity of tori we have $T, gTg^{-1} \subset Z_G(t')^0$. But these are *maximal* tori in $Z_G(t')^0$ since they're even maximal in G, and $Z_G(t')^0$ is a connected compact Lie group. Hence, by the Conjugacy Theorem applied to this group we can find $z \in Z_G(t')^0$ such that $z(gTg^{-1})z^{-1} = T$. That is, zg conjugates T onto itself, or in other words $zg \in N_G(T)$. Moreover,

$$(zg)t(zg)^{-1} = z(gtg^{-1})z^{-1} = zt'z^{-1} = t',$$

the final equality because $z \in Z_G(t')$. Thus, the class of zg in $W = N_G(T)/T$ carries t to t', as desired. This completes the proof of Propositon M.1.1.

To fully exploit the preceding result, we need the continuous bijection $T/W \to \mathrm{Conj}(G)$ to be a homeomorphism. Both source and target are compact spaces, so to get the homeomorphism property we just need to check that each is Hausdorff. The Hausdorff property for these is a special case of:

Lemma M.2.1. *Let X be a locally compact Hausdorff topological space equipped with a continuous action by a compact topological group H. The quotient space X/H with the quotient topology is Hausdorff.*

Proof. This is an exercise in definitions and point-set topology. □

Combining this lemma with the proposition, it follows that the $\mathbf{C}$-algebra $C^0(\mathrm{Conj}(G))$ of continuous $\mathbf{C}$-valued class functions on G is naturally identified with $C^0(T/W)$, and it is elementary (check!) to identify $C^0(T/W)$ with the the $\mathbf{C}$-algebra $C^0(T)^W$ of W-invariant continuous $\mathbf{C}$-valued functions on T. Unraveling the definitions, the composite identification

$$C^0(\mathrm{Conj}(G)) \simeq C^0(T)^W$$

of the **C**-algebras of continuous **C**-valued class functions on G and W-invariant continuous **C**-valued functions on T is given by $f \mapsto f|_T$.

APPENDIX N. WEYL GROUP COMPUTATIONS

N.1. **Introduction.** In §K.2, for $n \geq 2$ it is shown that for $G = \mathrm{U}(n) \subset \mathrm{GL}_n(\mathbf{C})$ and $T = (S^1)^n$ the diagonal maximal torus (denoted $\Delta(n)$ in [BtD]), we have $N_G(T) = T \rtimes S_n$ using the symmetric group S_n in its guise as $n \times n$ permutation matrices. (In [BtD] this symmetric group is denoted as $S(n)$.)

The case of $\mathrm{SU}(n)$ and its diagonal maximal torus $T' = T \cap \mathrm{SU}(n)$ (denoted as $S\Delta(n)$ in [BtD]) was also worked out there, and its Weyl group is also S_n. This case is more subtle than in the case of $\mathrm{U}(n)$ since we showed that the Weyl group of $\mathrm{SU}(n)$ does *not* lift isomorphically to a subgroup of the corresponding torus normalizer inside $\mathrm{SU}(n)$.

Remark N.1.1. Consider the inclusion $T' \hookrightarrow T$ between respective diagonal maximal tori of $\mathrm{SU}(n)$ and $\mathrm{U}(n)$. Since $T = T' \cdot Z$ for the *central* diagonally embedded circle $Z = S^1$ in $\mathrm{U}(n)$, we have $N_{\mathrm{SU}(n)}(T') \subset N_{\mathrm{U}(n)}(T)$ and thus get an injection $W(\mathrm{SU}(n), T') \hookrightarrow W(\mathrm{U}(n), T)$ that is an equality for size reasons. During our later study of root systems we will explain this equality of Weyl groups for $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ in a broader setting.

For each of the additional classical compact groups $\mathrm{SO}(n)$ ($n \geq 3$) and $\mathrm{Sp}(n)$ ($n \geq 1$), we found an explicit self-centralizing and hence maximal torus in HW3 Exercise 4; the maximal torus found in this way for $\mathrm{SO}(2m)$ is also a maximal torus in $\mathrm{SO}(2m+1)$. The aim of this appendix is to work out the Weyl group in these additional cases.

This material is explained in [BtD, Ch. V, 3.3–3.8]. Our presentation is different in some minor aspects, but the underlying technique is the same: just as the method of determination of the Weyl groups for $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ in Appendix K rested on a consideration of eigenspace decompositions relative to the action of the maximal torus on a "standard" **C**-linear representation of the ambient compact connected Lie group, we shall do the same for the special orthogonal and symplectic cases with appropriate "standard" representations over **C**.

N.2. **Odd special orthogonal groups.** Let's begin with $G = \mathrm{SO}(2m+1) \subset \mathrm{GL}_{2m+1}(\mathbf{R})$ with $m \geq 1$. In this case, a maximal torus $T = (S^1)^m$ was found in HW3 Exercise 4: it consists of a string of 2×2 rotation matrices $r_{2\pi\theta_1}, \ldots, r_{2\pi\theta_n}$ laid out along the diagonal of a $(2m+1) \times (2m+1)$ matrix, with the lower-right entry equal to 1 and $\theta_j \in \mathbf{R}/\mathbf{Z}$. In other words, a typical $t \in T$ can be written as

$$t = \begin{pmatrix} r_{2\pi\theta_1} & 0 & \ldots & 0 & 0 \\ 0 & r_{2\pi\theta_2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & \ldots & r_{2\pi\theta_m} & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

(This torus is denoted as $T(m)$ in [BtD].)

View $\mathbf{C}^n$ as the complexification of the standard representation of $\mathrm{SO}(n)$, so the decomposition of the rotation matrix r_θ into its diagonal form over **C** implies that the action of T on $\mathbf{C}^{2m+1}$ has as its eigencharacters

$$\{\chi_1, \chi_{-1}, \ldots, \chi_m, \chi_{-m}, 1\}$$

where $\chi_{\pm j}(t) = e^{\pm 2\pi i \theta_j}$. More specifically, any $t \in T$ acts on each plane $P_j = \mathbf{R}e_{2j-1} \oplus \mathbf{R}e_{2j}$ via a rotation r_{θ_j}, so t acts on $(P_j)_{\mathbf{C}}$ with eigenvalues $\chi_{\pm j}(t)$ (with multiplicity).

These T-eigencharacters are pairwise distinct with 1-dimensional eigenspaces in $\mathbf{C}^{2m+1}$, so any $n \in N_G(T)$ must have action on $\mathbf{C}^{2m+1}$ that *permutes* these eigenlines in accordance with its permutation effect on the eigencharacters in $\mathrm{X}(T)$. In particular, n preserves the eigenspace $(\mathbf{C}^{2m+1})^T$ for the trivial chracters, and this eigenspace is the basis line $\mathbf{C}e_{2m+1}$.

Since the action of G on $\mathbf{C}^{2m+1}$ is defined over $\mathbf{R}$, if n acting on T (hence on $\mathrm{X}(T)$) carries χ_k to $\chi_{k'}$ then by compatibility with the componentwise complex conjugation on $\mathbf{C}^{2m+1}$ we see that n acting on T (hence on $\mathrm{X}(T)$) carries the complex conjugate $\overline{\chi}_k = \chi_{-k}$ to $\overline{\chi}_{k'} = \chi_{-k'}$. Keeping track of the χ_k-eigenline via the index $k \in \{\pm 1, \dots, \pm m\}$, the effect of $W(G,T)$ on the set of eigenlines defines a homomorphism f from $W(G,T)$ into the group $\Sigma(m)$ of permutations σ of $\{\pm 1, \cdots \pm m\}$ that permute the numbers $\pm j$ in pairs; equivalently, $\sigma(-k) = -\sigma(k)$ for all k. (In [BtD], $\Sigma(m)$ is denoted as $G(m)$.)

The permutation within each of the m pairs of indices $\{j, -j\}$ constitutes a $\mathbf{Z}/2\mathbf{Z}$, and the permutation induced by σ on the set of m such pairs is an element of S_m, so we see that $\Sigma(m) = (\mathbf{Z}/2\mathbf{Z})^m \rtimes S_m$ with the standard semi-direct product structure.

Proposition N.2.1. *The map*
$$f : W(G,T) \to \Sigma(m) = (\mathbf{Z}/2\mathbf{Z})^m \rtimes S_m$$
is an isomorphism.

For injectivity, note that any $g \in N_G(T) \subset \mathrm{GL}_{2m+1}(\mathbf{C})$ representing a class in the kernel has effect on $\mathbf{C}^{2m+1}$ preserving every (1-dimensional) eigenspace of T and so must be diagonal over $\mathbf{C}$ (not just diagonalizable) with entries in S^1 by compactness. Membership in $G = \mathrm{SO}(2m+1) \subset \mathrm{GL}_{2m+1}(\mathbf{R})$ forces the diagonal entries of g to be ± 1. Such g with $\det(g) = 1$ visibly belongs to $\mathrm{SO}(2m+1) = G$ and hence lies in $Z_G(T) = T$, so the injectivity of $W(G,T) \to \Sigma(m)$ is establshed.

To prove surjectivity, first note that a permutation among the m planes P_j is obtained from a $2m \times 2m$ matrix that is an $m \times m$ "permutation matrix" in copies of the 2×2 identity matrix. This $2m \times 2m$ matrix has determinant 1 since each transposition $(ij) \in S_m$ acts by swapping the planes P_i and P_j via a direct sum of *two* copies of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus, by expanding this to a determinant-1 action on $\mathbf{R}^{2m+1}$ via action by the trivial action on $\mathbf{R}e_{2m+1}$ gives an element of $G = \mathrm{SO}(2m+1)$ that lies in $N_G(T)$ and represents any desired element of $S_m \subset \Sigma(m)$. Likewise, since the eigenlines for $\chi_{\pm j}$ in $(P_j)_{\mathbf{C}}$ are the lines $\mathbf{C}(e_{2j-1} + ie_{2j})$ and $\mathbf{C}(e_{2j-1} - ie_{2j}) = \mathbf{C}(e_{2j} + ie_{2j-1})$ that are swapped upon swapping e_{2j-1} and e_{2j} *without* a sign intervention, we get an element of $N_G(T)$ representing any $(\epsilon_1, \dots, \epsilon_m) \in (\mathbf{Z}/2\mathbf{Z})^m \subset \Sigma(m)$ by using the action of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^{\epsilon_j} \in \mathrm{O}(2)$ on the plane P_j for each j and using the action by $(-1)^{\sum \epsilon_j}$ on $\mathbf{R}e_{2m+1}$ to ensure an overall sign of 1. This completes our determination of the Weyl group of $\mathrm{SO}(n)$ for odd n.

N.3. **Even special orthogonal groups.** Now suppose $G = \mathrm{SO}(2m)$. We have a similar description of a maximal torus $T = (S^1)^m$: it is an array of m rotation matrices $r_{\theta_j} \in \mathrm{SO}(2)$ (without any singleton entry of 1 in the lower-right position). The exact same reasoning as in the case $n = 2m+1$ defines a homomorphism
$$f : W(G,T) \to \Sigma(m)$$

that is injective due to the exact same argument as in the odd special orthogonal case.

The proof of surjectivity in the case $n = 2m + 1$ does not quite carry over (and in fact f will not be surjective, as is clear when $m = 1$ since $\mathrm{SO}(2)$ is commutative), since we no longer have the option to act by a sign on $\mathbf{R}e_{2m+1}$ in order to arrange for an overall determinant to be equal to 1 (rather than -1).

Inside $\Sigma(m) = (\mathbf{Z}/2\mathbf{Z})^m \rtimes S_m$ we have the index-2 subgroup $A(m)$ that is the kernel of the homomorphism $\delta_m : \Sigma(m) \to \{\pm 1\}$ defined by

$$((\epsilon_1, \dots, \epsilon_m), \sigma) \mapsto (-1)^{\sum \epsilon_j}.$$

(In [BtD] this group is denoted as $SG(m)$.) Explicitly, $A(m) = H_m \rtimes S_m$ where $H_m \subset (\mathbf{Z}/2\mathbf{Z})^m$ is the hyperplane defined by $\sum \epsilon_j = 0$.

Note that T is a maximal torus in $\mathrm{SO}(2m+1)$, and $N_{\mathrm{SO}(2m)}(T) \subset N_{\mathrm{SO}(2m+1)}(T)$ via the natural inclusion $\mathrm{GL}_{2m}(\mathbf{R}) \hookrightarrow \mathrm{GL}_{2m+1}(\mathbf{R})$ using the decomposition $\mathbf{R}^{2m+1} = H \oplus \mathbf{R}e_{2m+1}$ for the hyperplane H spanned by $e_1, \dots, e_{2m}$. Hence, we get an injection

$$W(\mathrm{SO}(2m), T) \hookrightarrow W(\mathrm{SO}(2m+1), T).$$

Projection to the lower-right matrix entry defines a character $N_{\mathrm{SO}(2m+1)}(T) \twoheadrightarrow \{\pm 1\}$ that encodes the sign of the action of this normalizer on the line $\mathbf{R}e_{2m+1}$ of T-invariants. This character kills T and visibly has as its kernel exactly $N_{\mathrm{SO}(2m)}(T)$.

Upon passing to the quotient by T, we have built a character

$$\Sigma(m) = W(\mathrm{SO}(2m+1), T) \twoheadrightarrow \{\pm 1\}$$

whose kernel is $W(\mathrm{SO}(2m), T)$. This character on $\Sigma(m)$ is checked to coincide with δ_m by using the explicit representatives in $N_{\mathrm{SO}(2m+1)}(T)$ built in our treatment of the odd special orthogonal case. Thus, we have proved:

Proposition N.3.1. *The injection $f : W(\mathrm{SO}(2m), T) \to \Sigma(m)$ is an isomorphism onto* $\ker \delta_m = A(m)$.

N.4. **Symplectic groups.** Finally, we treat the case $G = \mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{GL}_n(\mathbf{H})$. Recall that G consists of precisely the matrices

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in \mathrm{U}(2n),$$

and (from Exercise 4 in HW3) a maximal torus $T = (S^1)^n$ of G is given by the set of elements

$$\mathrm{diag}(z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_n)$$

for $z_j \in S^1$. (This torus is denoted as T^n in [BtD].) Note that the "standard" representation of G on $\mathbf{C}^{2n}$ has $T = (S^1)^n$ acting with $2n$ distinct eigencharacters: the component projections $\chi_j : T \to S^1$ and their reciprocals $1/\chi_j = \overline{\chi}_j$. Denoting $1/\chi_j$ as χ_{-j}, the action of $N_G(T)$ on T via conjugation induces a permutation of this set of eigencharacters $\chi_{\pm 1}, \dots, \chi_{\pm n}$.

Keeping track of these eigencharacters via their indices, we get a homomorphism from $W(G, T) = N_G(T)/T$ into the permutation group of $\{\pm 1, \dots, \pm n\}$. Recall that this permutation group contains a distinguished subgroup $\Sigma(n)$ consisting of the permutations σ satisfying $\sigma(-k) = -\sigma(k)$ for all k. We claim that $W(G, T)$ lands inside $\Sigma(n)$. This says exactly that if the action on $\mathrm{X}(T)$ by $w \in W(G,T)$ carries χ_k to $\chi_{k'}$ (with $-n \leq k, k' \leq n$) then it carries χ_{-k} to $\chi_{-k'}$. But by definition we have $\chi_{-k} = 1/\chi_k$, so this is clear.

Proposition N.4.1. *The map $W(G,T) \to \Sigma(n)$ is an isomorphism.*

This equality with the same Weyl group as for $\mathrm{SO}(2n+1)$ is not a coincidence, but its conceptual explanation rests on a duality construction in the theory of root systems that we shall see later.

Proof. Suppose $w \in W(G,T)$ is in the kernel. Then for a representative $g \in G$ of w, the g-action on $\mathbf{C}^{2n}$ preserves the χ_k-eigenline for all $-n \leq k \leq n$, so g is diagonal in $\mathrm{GL}_{2n}(\mathbf{C})$. Thus, $g \in Z_G(T) = T$, so $w = 1$. Using the inclusion $\mathrm{U}(n) \subset \mathrm{Sp}(n)$ via

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}$$

that carries the diagonal maximal torus T_n of $\mathrm{U}(n)$ isomorphically onto our chosen maximal torus T of $\mathrm{Sp}(n)$, we get an injection

$$S_n = W(\mathrm{U}(n), T_n) \hookrightarrow W(G,T)$$

that coincides (check!) with the natural inclusion of S_n into $\Sigma(n) = (\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n$.

It remains to show that each of the standard direct factors $\mathbf{Z}/2\mathbf{Z}$ of $(\mathbf{Z}/2\mathbf{Z})^n$ lies in the image of $W(G,T)$ inside $\Sigma(n)$. This is a problem inside each

$$\mathrm{SU}(2) = \mathrm{Sp}(1) \subset \mathrm{GL}(\mathbf{C}e_j \oplus \mathbf{C}e_{j+n})$$

for $1 \leq j \leq n$, using its 1-dimensional diagonal maximal torus that is one of the standard direct factors S^1 of $T = (S^1)^n$. But we already know $W(\mathrm{SU}(2), S^1) = \mathbf{Z}/2\mathbf{Z}$, with nontrivial class represented by the unit quaternion $j \in \mathrm{SU}(2) \subset \mathbf{H}^\times$ whose conjugation action normalizes the unit circle $S^1 \subset \mathbf{C}^\times$ via inversion, so we are done. $\square$

Appendix O. Fundamental groups of Lie groups

O.1. **Injectivity result.** Let G be a connected Lie group, and Γ a discrete normal (hence closed and central) subgroup. Let $G' = G/\Gamma$. Equip each of these connected manifolds with its identity point as the base point for computing its fundamental group. In HW5 you constructed a natural surjection $\pi_1(G') \twoheadrightarrow \Gamma$ and showed it is an isomorphism when $\pi_1(G) = 1$. The aim of this appendix is to prove in general that the natural map $\pi_1(G) \to \pi_1(G')$ is injective and the resulting diagram of groups

$$1 \to \pi_1(G) \to \pi_1(G') \to \Gamma \to 1$$

is short exact; i.e., $\pi_1(G)$ maps isomorphically onto $\ker(\pi_1(G') \twoheadrightarrow \Gamma)$. Along the way, we'll construct the surjective homomorphism $\pi_1(G') \twoheadrightarrow \Gamma$ whose kernel is identified with $\pi_1(G)$. These matters can be nicely handled by using the formalism of universal covering spaces (especially that the universal cover $\widetilde{H}$ of a connected Lie group H is uniquely equipped with a Lie group structure compatible with one on H upon choosing a point $\widetilde{e} \in \widetilde{H}$ over the identity $e \in H$ to serve as the identity of the group law on $\widetilde{H}$). In this appendix we give a more hands-on approach that avoids invoking universal covering spaces.

Proposition O.1.1. *The natural map $\pi_1(G) \to \pi_1(G')$ is injective.*

Proof. Suppose $\sigma : (S^1, 1) \to (G, e)$ is a loop whose composition with the quotient map $q : G \to G'$ is homotopic to the constant loop based at e'. View σ as a continuous map $[0,1] \to G$ carrying 0 and 1 to e, and let

$$F : [0,1] \times [0,1] \to G'$$

be such a homotopy, so $F(\cdot, 0) = \sigma$, $F(x, 1) = e'$, and $F(0,t) = F(1,t) = e'$ for all $t \in [0,1]$. Letting S be the 3/4-square $\partial_{\mathbf{R}^2}([0,1]^2) - (0,1) \times \{1\}$ obtained by removing the right edge, $F|_S : S \to G'$ lifts to the continuous map $\widetilde{F}_S : S \to G$ defined by $\widetilde{F}_S(x, 0) = \sigma(x)$, $\widetilde{F}_S(0,t) = \widetilde{F}_S(1,t) = e$.

By the homotopy lifting lemma in HW5 Exercise 3(iii), $\widetilde{F}_S$ extends to a lift $\widetilde{F} : [0,1]^2 \to G$ of F; i.e., $q \circ \widetilde{F} = F$. In particular, $\widetilde{F}$ gives a homotopy between $\widetilde{F}(\cdot, 0) = \sigma$ and the path $\tau := \widetilde{F}(\cdot, 1)$ in G which lifts the constant path $F(\cdot, 1) = \{e'\}$. Hence, $\tau : [0,1] \to G$ is a path joining $\tau(0) = \widetilde{F}(0,1) = e$ to $\tau(1) = \widetilde{F}(1,1) = e$ in G and supported entirely inside the fiber $q^{-1}(e') = \Gamma$. But Γ is *discrete*, so the path τ valued in Γ must be constant. Since $\tau(0) = e$, it follows that τ is the constant path $\tau(x) = e$, so F is a homotopy between σ and the constant path in G based at e. In other words, the homotopy class of the initial σ is trivial, and this is precisely the desired injectivity. $\square$

O.2. **Surjectivity result.** Next, we construct a surjective homomorphism $\pi_1(G') \to \Gamma$. For any continuous map $\sigma : (S^1, 1) \to (G', e')$, by using compactness and connectedness of $[0,1]$ the method of proof of the homotopy lifting lemma gives that σ admits a lifting $\widetilde{\sigma} : [0,1] \to G$ with $\widetilde{\sigma}(0) = e$. In fact, since $G \to G'$ is a covering space (as Γ is discrete), the connectedness of $[0,1]$ implies that such a lift $\widetilde{\sigma}$ is *unique*. The terminal point $\widetilde{\sigma}(1)$ is an element of $q^{-1}(e') = \Gamma$. If $\sigma' \sim \sigma$ is a homotopic loop then the homotopy lifting argument in the previous paragraph adapts to show that a homotopy $F : [0,1]^2 \to G$ from σ to σ' lifts to a continuous map $\widetilde{F} : [0,1]^2 \to G'$ satisfying $\widetilde{F}(\cdot, 0) = \widetilde{\sigma}$, $\widetilde{F}(0,t) = \widetilde{\sigma}(0) = e$, and $\widetilde{F}(1,t) = \widetilde{\sigma}(1)$. Consequently, $\widetilde{F}(\cdot, 1) : [0,1] \to G$ is a continuous lift of $F(\cdot, 1) = \sigma'$ that begins at $\widetilde{F}(0,1) = e$. By the *uniqueness* of the lift $\widetilde{\sigma}'$ of σ' beginning at e, it follows that $\widetilde{F}(\cdot, 1) = \widetilde{\sigma}'$. In particular, $\widetilde{\sigma}'(1) = \widetilde{F}(1,1) = \widetilde{\sigma}(1)$ and $\widetilde{\sigma}'$ is homotopic to $\widetilde{\sigma}$ (as paths in G with initial point e and the same terminal point). Hence, $\widetilde{\sigma}(1)$ only depends on the homotopy class $[\sigma]$ of σ, so we get a well-defined map of sets

$$f : \pi_1(G') \to \Gamma$$

via $[\sigma] \mapsto \widetilde{\sigma}(1)$.

The map f is surjective. Indeed, choose $g_0 \in \Gamma = q^{-1}(e')$ and a path $\tau : [0,1] \to G$ linking e to g_0 (as we may do since G is path-connected). Define $\sigma := q \circ \tau : [0,1] \to G'$, visibly a *loop* based at e'. We have $\widetilde{\sigma} = \tau$ due to the uniqueness of the lift $\widetilde{\sigma}$ of σ beginning at $\tau(0) = e$. Consequently, $f([\sigma]) = \widetilde{\sigma}(1) = \tau(1) = g_0$, so f is surjective. From the definition of f it is clear that $f(\pi_1(G)) = \{e\}$ (since if σ is the image of a loop in G based on e then this latter *loop* must be $\widetilde{\sigma}$ and hence its terminal point $\widetilde{\sigma}(1)$ is equal to e). Conversely, if $\widetilde{\sigma}(1) = e$ then $\widetilde{\sigma}$ is a loop $(S^1, 1) \to (G, e)$ whose projection into $G' = G/\Gamma$ is σ, so $f^{-1}(e) = \pi_1(G)$. Thus, if f is a group homomorphism then it is surjective with kernel $\pi_1(G)$, so we would be done.

It remains to show that f is a homomorphism. For loops $\sigma_1, \sigma_2 : (S^1, 1) \rightrightarrows (G', e)$, we want to show that

$$\widetilde{\sigma}_1(1)\widetilde{\sigma}_2(1) = \widetilde{\sigma_1 * \sigma_2}(1)$$

in Γ, where the left side uses the group law in G and $\sigma_1 * \sigma_2 : S^1 \to G'$ is the loop made via concatenation (and time reparameterization). In other words, we wish to show that the unique lift of $\sigma_1 * \sigma_2$ to a path $[0,1] \to G$ beginning at e has as its terminal point exactly the product $\widetilde{\sigma}_1(1)\widetilde{\sigma}_2(1)$ computed in the group law of G.

Consider the two paths $\widetilde{\sigma}_2 : [0,1] \to G$ and $\widetilde{\sigma}_1(\cdot)\widetilde{\sigma}_2(1) : [0,1] \to G$. The first of these lifts σ_2 with initial point e and terminal point $\widetilde{\sigma}_2(1)$, and the second is the right-translation by $\widetilde{\sigma}_2(1) \in q^{-1}(e')$ of the unique path lifting σ_1 with initial point e and terminal point $\widetilde{\sigma}_1(1)$. Hence, $\widetilde{\sigma}_1(\cdot)\widetilde{\sigma}_2(1)$ is the unique lift of σ_1 with initial point $\widetilde{\sigma}_2(1)$ that is the terminal point of $\widetilde{\sigma}_2$, and its terminal point is $\widetilde{\sigma}_1(1)\widetilde{\sigma}_2(1)$. Thus, the concatenation path

$$(\widetilde{\sigma}_1(\cdot)\widetilde{\sigma}_2(1)) * \widetilde{\sigma}_2$$

is the unique lift of $\sigma_1 * \sigma_2$ with initial point e, so it is $\widetilde{\sigma_1 * \sigma_2}$ (!), and its terminal point is $\widetilde{\sigma}_1(1)\widetilde{\sigma}_2(1)$ as desired.

Appendix P. Clifford algebras and spin groups

Clifford algebras were discovered by Clifford in the late 19th century as part of his search for generalizations of quaternions. He considered an algebra generated by $V = \mathbf{R}^n$ subject to the relation $v^2 = -\|v\|^2$ for all $v \in V$. (For $n = 2$ this gives the quaternions via $i = e_1$, $j = e_2$, and $k = e_1e_2$.) They were rediscovered by Dirac. In this appendix we explain some general features of Clifford algebras beyond the setting of $\mathbf{R}^n$, including its role in the definition of spin groups. This may be regarded as a supplement to the discussion in [BtD, Ch. I, 6.1–6.19], putting those constructions into a broader context. Our discussion is generally self-contained, but we punt to [BtD] for some arguments.

P.1. **Quadratic spaces and associated orthogonal groups.** Let V be a finite-dimensional nonzero vector space over a field k and let $q : V \to k$ be a *quadratic form*; i.e., $q(cv) = c^2q(v)$ for all $v \in V$ and $c \in k$, and the symmetric function $B_q : V \times V \to k$ defined by $B_q(v,w) := q(v+w) - q(v) - q(w)$ is k-bilinear. This is less abstract than it may seem to be: if $\{e_i\}$ is a k-basis of V then

$$q(\sum x_ie_i) = \sum q(e_i)x_i^2 + \sum_{i<j} B_q(e_i,e_j)x_ix_j.$$

Thus, a quadratic form is just a degree-2 homogeneous polynomial function in linear coordinates. Conversely, any such function on k^n is a quadratic form. You should intrinsically define the *scalar extension* quadratic form $q_{k'} : V_{k'} \to k'$ for any field extension k'/k.

As long as $\mathrm{char}(k) \neq 2$ (the cases of most interest to use will be $\mathbf{R}$ and $\mathbf{C}$) we can reconstruct q from B_q since the equality $B_q(v,v) = q(2v) - 2q(v) = 2q(v)$ yields that $q(v) = B_q(v,v)/2$. It is easy to check that if $\mathrm{char}(k) \neq 2$ and $B : V \times V \to k$ is *any* symmetric k-bilinear form then $q_B : v \mapsto B(v,v)/2$ is a quadratic form whose associated symmetric bilinear form is B. In other words, if $\mathrm{char}(k) \neq 2$ then we have a natural isomorphism of k-vector spaces

$$\mathrm{Quad}(V) \simeq \mathrm{SymBil}(V)$$

given by $q \mapsto B_q$ and $B \mapsto q_B$ between the k-vector spaces of quadratic forms on V and symmetric bilinear forms on V.

By successive "completion of the square" (check!), if $\mathrm{char}(k) \neq 2$ then any quadratic form q on V admits an B_q*-orthogonal basis*: a basis $\{e_i\}$ such that $B_q(e_i,e_j) = 0$ for all $i \neq j$, which is to say that

$$q(\sum x_i e_i) = \sum q(e_i) x_i^2;$$

in other words, q can be put in "diagonal form". (The spectral theorem gives a much deeper result over $k = \mathbf{R}$, namely that there exists a B_q-orthogonal basis that is also orthogonal for a choice of positive-definite inner product. That has nothing to do with the purely algebraic diagonalization of quadratic forms, for which there is no geometric meaning akin to that in the Spectral Theorem when $k = \mathbf{R}$.)

In what follows we shall *always assume* $\mathrm{char}(k) \neq 2$ for ease of discussion (we only need the cases $k = \mathbf{R}, \mathbf{C}$). If one is attentive to certain technical details and brings in ideas from algebraic geometry then it is possible to appropriately formulate the definitions, results, and proofs so that essentially everything works in characteristic 2 (and most arguments wind up being characteristic-free). This is genuinely useful in number theory for the integral theory of quadratic forms, as it is important to have p-adic results for *all* primes p, including $p = 2$, for which the integral theory requires input over the residue field $\mathbf{F}_p$.

Definition P.1. A quadratic space (V, q) is *non-degenerate* if the symmetric bilinear form $B_q : V \times V \to k$ is non-degenerate; i.e., the linear map $V \to V^*$ defined by $v \mapsto B_q(v, \cdot) = B_q(\cdot, v)$ is an isomorphism.

Example P.1.2. When working with a B_q-orthogonal basis, non-degeneracy is just the condition that the diagonal form $q = \sum c_i x_i^2$ for q has all coefficients $c_i \neq 0$. Thus, the notion of "non-degenerate quadratic form" on an n-dimensional vector space is just a coordinate-free way to think about diagonal quadratic forms $q = \sum c_i x_i^2$ in n variables with all $c_i \neq 0$.

Given a non-degenerate (V, q), the *orthogonal group* and *special orthogonal group* are respectively defined to be the groups

$$\mathrm{O}(q) = \{L \in \mathrm{GL}(V) \mid q \circ L = q\} = \{L \in \mathrm{GL}(V) \mid B_q \circ (L \times L) = B_q\}$$

and

$$\mathrm{SO}(q) = \mathrm{O}(q) \cap \mathrm{SL}(V).$$

In down-to-earth terms, if we choose a basis $\{e_i\}$ of V and let $[B_q]$ be the *symmetric* matrix $(B_q(e_i, e_j))$ then identifying $\mathrm{GL}(V)$ with $\mathrm{GL}_n(k)$ via $\{e_i\}$ identifies $\mathrm{O}(q)$ with the set of $L \in \mathrm{GL}_n(k)$ such that

$$L^\top [B_q] L = [B_q]$$

and $\mathrm{SO}(q)$ entails the additional condition $\det(L) = 1$. (The groups $\mathrm{O}(q)$ and $\mathrm{SO}(q)$ can certainly be defined without requiring non-degeneracy, but they have rather poor properties in the degenerate case.)

Applying the determinant to both sides of the matrix equation describing $\mathrm{O}(q)$ gives that $\det(L)^2 \det([B_q]) = \det([B_q])$, and $\det([B_q]) \neq 0$ by the non-degeneracy hypothesis, so $\det(L)^2 = 1$. In other words, $\det(L) = \pm 1$. Thus, $\mathrm{SO}(q)$ has index in $\mathrm{O}(q)$ at most 2.

Define $n := \dim(V)$. If n is odd then $-1 \in \mathrm{O}(q)$ and $\det(-1) = (-1)^n = -1$ in k, so $\mathrm{O}(q) = \{\pm 1\} \times \mathrm{SO}(q)$ in such cases. If n is even then $-1 \in \mathrm{SO}(q)$ and so to show that $\mathrm{O}(q) \neq \mathrm{SO}(q)$ in general we choose a basis diagonalizing q and use $\mathrm{diag}(-1, 1, 1, \dots, 1)$. This is the B_q-orthogonal reflection in e_1, and suitable reflections always provide elements of $\mathrm{O}(q) - \mathrm{SO}(q)$:

Proposition P.1.3. *For any $v_0 \in V$ such that $q(v_0) \neq 0$, the subspace $L^\perp := \ker B_q(v_0, \cdot)$ that is B_q-orthogonal to the line $L = kv_0$ is a hyperplane in V not containing v_0 and the endomorphism*

$$r_{v_0} : v \mapsto v - \frac{B_q(v, v_0)}{q(v_0)} v_0$$

of V lies in $\mathrm{O}(q)$ and fixes $L^\perp$ pointwise but acts as negation on L. In particular, $r_{v_0} \in \mathrm{O}(q)$ and $\det r_{v_0} = -1$.

The formula for r_{v_0} is an algebraic analogue of the formula for orthogonal reflection through a hyperplane $H = \mathbf{R}v_0^\perp$ in $\mathbf{R}^n$:

$$x \mapsto x - 2\left(x \cdot \frac{v_0}{\|v_0\|}\right)\frac{v_0}{\|v_0\|} = x - \frac{2(x \cdot v_0)v_0}{v_0 \cdot v_0},$$

noting that $B_q(x, v_0) = \|x + v_0\|^2 - \|x\|^2 - \|v_0\|^2 = 2(x \cdot v_0)$.

Proof. Since $B_q(v_0, v_0) = 2q(v_0) \neq 0$, certainly $v_0 \notin L^\perp$. Thus, the linear functional $B_q(v_0, \cdot)$ on V is not identically zero, so its kernel $L^\perp$ is a hyperplane in V (not containing v_0). From the definition it is clear that r_{v_0} fixes $L^\perp$ pointwise, and

$$r_{v_0}(v_0) = v_0 - \frac{B_q(v_0, v_0)}{q(v_0)} v_0 = v_0 - 2v_0 = -v_0.$$

Finally, to check that $r_{v_0} \in \mathrm{O}(q)$ we want to show that $q(r_{v_0}(v)) = q(v)$ for all $v \in V$. Let's write $v = v' + cv_0$ for $v' \in L^\perp$ and $c \in k$. We have $r_{v_0}(v) = v' - cv_0$, so

$$q(r_{v_0}(v)) = q(v' - cv_0) = q(v') + c^2 q(v_0) - 2cB_q(v', v_0) = q(v') + c^2 q(v_0)$$

since $B_q(v', v_0) = 0$ (as $v' \in L^\perp$). This is clearly equal to $q(v' + cv_0) = q(v)$ by a similar calculation. $\square$

We conclude that $\mathrm{SO}(q)$ has index 2 in $\mathrm{O}(q)$. Our main aim is to construct an auxiliary "matrix group" $\mathrm{Spin}(q)$ equipped with a surjective homomorphism $\mathrm{Spin}(q) \to \mathrm{SO}(q)$ whose kernel is central with order 2, and to show that in a specific case over $k = \mathbf{R}$ this group is *connected*. (It is also "Zariski-connected" when constructed in a more general algebro-geometric framework, but that lies beyond the level of this course.) Actually, in the end we will only succeed when the nonzero values of $-q$ are squares in k, such as when q is negative-definite over $k = \mathbf{R}$ (e.g., $V = \mathbf{R}^n$, $q = -\sum x_i^2$, with $\mathrm{SO}(q) = \mathrm{SO}(-q) = \mathrm{SO}(n)$) or when k is separably closed (recall $\mathrm{char}(k) \neq 2$). Our failure for more general settings can be overcome in an appropriate sense by introducing ideas in the theory of linear algebraic groups that lie beyond the level of this course.

The key to constructing an interesting exact sequence

$$1 \to \mathbf{Z}/2\mathbf{Z} \to \mathrm{Spin}(q) \to \mathrm{SO}(q) \to 1$$

is to introduce a certain associative k-algebra $\mathrm{C}(q) = \mathrm{C}(V, q)$ that contains V and has dimension 2^n as a k-vector space; its group of units will contain $\mathrm{Spin}(q)$. Such algebras are called *Clifford algebras*, and they will be defined and studied in the next section.

P.2. **Clifford algebras.** For our initial construction of the Clifford algebra associated to (V,q) we make no non-degeneracy hypothesis; the best properties occur only when (V,q) is non-degenerate, but for the purpose of some early examples the case $q=0$ (with $V\neq 0$!) is worth keeping in mind.

The problem that Clifford algebras universally solve is that of finding a k-algebra containing V in which q looks like squaring. More specifically, consider pairs (A,j) where A is an associative k-algebra and $j: V\to A$ is a k-linear map (not necessarily injective) such that $j(v)^2 = q(v)\cdot 1_A$ in A for all $v\in V$. Since the tensor algebra

$$T(V) = \bigoplus_{n\geq 0} V^{\otimes n} = k\oplus V\oplus V^{\otimes 2}\oplus\dots$$

equipped with its evident map $V\to T(V)$ is the *initial* assocative k-algebra equipped with a k-linear map from V, we can construct a pair (A,j) by imposing the relation $v\otimes v = q(v)$ on the tensor algebra. That is:

Definition P.1. The *Clifford algebra* of (V,q) is the associated k-algebra

$$\mathrm{C}(V,q) = \mathrm{C}(q) := T(V)/\langle v\otimes v - q(v)\rangle$$

equipped with the k-linear map $V\to\mathrm{C}(q)$ induced by the natural inclusion $V\hookrightarrow T(V)$; the quotient is by the 2-sided ideal in $T(V)$ generated by elements of the form $v\otimes v - q(v)$.

In view of the universal property of the tensor algebra, it is easy to see (check!) that $\mathrm{C}(q)$ equipped with its canonical k-linear map from V is the initial pair (A,j) as above: for any such pair there is a unique k-algebra map $\mathrm{C}(q)\to A$ compatible with the given k-linear maps from V into each. Also, if k'/k is an extension field then it is easy to see that the scalar extension $V_{k'}\to\mathrm{C}(q)_{k'}$ uniquely factors through $V_{k'}\to\mathrm{C}(q_{k'})$ via a k'-algebra map $\mathrm{C}(q_{k'})\to\mathrm{C}(q)_{k'}$ that is an isomorphism. Thus, the formation of $\mathrm{C}(q)$ commutes with ground field extension. We will soon show that the canonical map $V\to\mathrm{C}(q)$ is injective.

Example P.2.2. As an instance of the universal property, if $f:(V,q)\to(V',q')$ is a map of quadratic spaces (i.e., $f: V\to V'$ is k-linear and $q'\circ f = q$) then there is a unique map of k-algebras $\mathrm{C}(q)\to\mathrm{C}(q')$ compatible with $f: V\to V'$ via the natural maps from V and V' into their respective Clifford algebras. In other words, the formation of the Clifford algebra is functorial in the quadratic space.

Note that upon imposing the relation $v\otimes v = q(v)$ for some $v\in V$, the relation $v'\otimes v' = q(v')$ automatically holds for any $v'\in kv$. Thus, we only need to impose the relation $v\otimes v = q(v)$ for a single v on each line in V. In fact, since

$$(v_1+v_2)\otimes(v_1+v_2) = v_1\otimes v_1 + v_2\otimes v_2 + (v_1\otimes v_2 + v_2\otimes v_1)$$

and

$$q(v_1+v_2) = q(v_1)+q(v_2)+B_q(v_1,v_2),$$

if $\{e_i\}$ is a basis of V then imposing the necessary relations

$$e_i\otimes e_i = q(e_i),\ \ e_i\otimes e_j + e_j\otimes e_i = B_q(e_i,e_j) \tag{P.2.1}$$

for all i,j implies that $v\otimes v = q(v)$ for all $v\in V$. Thus, the 2-sided ideal used to define $\mathrm{C}(q)$ as a quotient of $T(V)$ is generated (as a 2-sided ideal!) by the relations (P.2.1) with the basis vectors.

Example P.2.3. Suppose $V = k$ and $q(x) = cx^2$ for some $c \in k$ (allowing $c = 0$!). In this case the tensor algebra $T(V)$ is the *commutative* 1-variable polynomial ring $k[t]$ (with t corresponding to the element $1 \in V \subset T(V)$) and $\mathrm{C}(q) = k[t]/(t^2 - c)$ since $1 \in k = V$ is a basis (ensuring that the single relation $1 \otimes 1 = q(1) = c$, which is to say $t^2 = c$, is all we need to impose).

For example, if $V = k = \mathbf{R}$ and $q(x) = -x^2$ then $\mathrm{C}(V) = \mathbf{C} = \mathbf{R} \oplus i\mathbf{R}$.

Example P.2.4. If $q = 0$ then $\mathrm{C}(V, q) = \wedge^\bullet(V) = \bigoplus_{n \geq 0} \wedge^n(V)$ is exactly the exterior algebra of V. This has dimension $2^{\dim(V)}$.

By construction, if $\{e_i\}$ is a basis of V then $\mathrm{C}(q)$ is spanned over k by the products $e_{i_1} \cdots e_{i_r}$ for indices $i_1, \dots, i_r$. Systematic use of the relation

$$e_i e_j = -e_j e_i + B_q(e_i, e_j)$$

with $B_q(e_i, e_j) \in k$ allows us to "rearrange terms" in these basis products at the cost of introducing the scaling factor $B_q(e_i, e_j) \in k$ into some k-coefficients in the k-linear combination, so we may arrange that $i_1 \leq \cdots \leq i_r$. We can then use the relation $e_i^2 = q(e_i) \in k$ to eliminate the appearance of any repeated indices (by absorbing such repetition into a k-multiplier coefficient). Thus, exactly as with exterior algebras (the case $q = 0$), a spanning set is given by the products $e_{i_1} \cdots e_{i_r}$ with $1 \leq i_1 < \cdots < i_r \leq n := \dim(V)$. Hence, $\dim \mathrm{C}(q) \leq 2^n$, with equality if and only if these products are linearly independent inside $\mathrm{C}(q)$. When such equality holds, the e_i's (singleton products) are k-linearly independent inside $\mathrm{C}(q)$, which is to say that the canonical map $V \to \mathrm{C}(q)$ is *injective*.

Example P.2.5. Suppose $k = \mathbf{R}$, $V = \mathbf{R}^2$ (with standard basis e_1, e_2) and $q(x_1, x_2) = -x_1^2 - x_2^2$, Then $\mathrm{C}(q)$ is generated over $\mathbf{R}$ by elements e_1, e_2, and $e_1 e_2$ with $e_1^2 = e_2^2 = -1$, $e_1 e_2 = -e_2 e_1$ (since e_1 is B_q-orthogonal to e_2, as q is diagonal with respect to the $\{e_1, e_2\}$ basis) and $(e_1 e_2)^2 = -1$. Thus, there is a well-defined ring map $\mathrm{C}(q) \to \mathbf{H}$ via $e_1 \mapsto j$ and $e_2 \mapsto k$ (so $e_1 e_2 \mapsto i$), and it is surjective, so the inequality $\dim \mathrm{C}(q) \leq 4$ is an equality; i.e., $\mathrm{C}(q) = \mathbf{H}$.

In order to show that the "expected dimension" $2^{\dim V}$ of $\mathrm{C}(q)$ is always the actual dimension, we need to understand how the formation of the Clifford algebra interacts with orthogonal direct sums of quadratic spaces. To that end, and for other purposes, we now digress to discuss a natural grading on $\mathrm{C}(V)$.

Note that $T(V)$ is a $\mathbf{Z}$-graded k-algebra: it is a direct sum of terms $\oplus_{n \in \mathbf{Z}} A_n$ as a k-vector space ($A_n = V^{\otimes n}$ with $n \geq 0$ and $A_n = 0$ for $n < 0$) such that $A_n A_m \subset A_{n+m}$ and $1 \in A_0$. The elements of a single A_n are called *homogeneous*. The relations that we impose in the construction of the quotient $\mathrm{C}(q)$ are generated by expressions $v \otimes v - q(v)$ that straddle two separate degrees, namely degree 2 (for $v \otimes v$) and degree 0 (for $-q(v)$). Thus, the $\mathbf{Z}$-grading is *not* inherited in a well-defined manner by the quotient $\mathrm{C}(q)$ of $T(V)$, but the *parity* of the $\mathbf{Z}$-grading on $T(V)$ survives passage to the quotient. (You should check this, by considering left and right multiples of $v \otimes v - q(v)$ against homogeneous elements in $T(V)$, and using that every element in $T(V)$ is a finite sum of homogeneous terms.)

We may therefore speak of the "even part" $\mathrm{C}(q)_0$ and the "odd part" $\mathrm{C}(q)_1$ of $\mathrm{C}(q)$, which are the respective images of the even and odd parts of the tensor algebra $T(V)$. It is clear that $\mathrm{C}(q)_0$ is a k-subalgebra of $\mathrm{C}(q)$ and that $\mathrm{C}(q)_1$ is a 2-sided $\mathrm{C}(q)_0$-module inside $\mathrm{C}(q)$.

Example P.2.6. For $V = k$ and $q(x) = cx^2$ we have $\mathrm{C}(q) = k[t]/(t^2 - c)$ with $\mathrm{C}(q)_0 = k$ and $\mathrm{C}(q)_1$ is the k-line spanned by the residue class of t. As a special case, if $k = \mathbf{R}$ and $V = k$ with $q(x) = -x^2$ then $\mathrm{C}(q) = \mathbf{C}$ with even part $\mathbf{R}$ and odd part $i\mathbf{R}$,

Likewise, if $k = \mathbf{R}$ and $V = \mathbf{R}^2$ with $q(x_1, x_2) = -x_1^2 - x_2^2$ then we have seen that $\mathrm{C}(q) = \mathbf{H}$. The even part is $\mathbf{C} = \mathbf{R} \oplus \mathbf{R}i$ and the odd part $\mathbf{C}j = \mathbf{R}j \oplus \mathbf{R}k$, with $j = e_1$, $k = e_2$, and $i = e_1e_2$.

The decomposition

$$\mathrm{C}(q) = \mathrm{C}(q)_0 \oplus \mathrm{C}(q)_1$$

thereby makes $\mathrm{C}(q)$ into a $\mathbf{Z}/2\mathbf{Z}$-graded algebra with $\mathrm{C}(q)_0$ as the "degree 0" part and $\mathrm{C}(q)_1$ as the "degree 1" part. To be precise, a *$\mathbf{Z}/2\mathbf{Z}$-graded k-algebra* is an associative k-algebra A equipped with a direct sum decomposition $A = A_0 \oplus A_1$ as k-vector spaces such that $A_iA_j \subset A_{i+j}$ (for indices $i, j \in \mathbf{Z}/2\mathbf{Z}$) and $1 \in A_0$. An element of A is *homogeneous* when it lies in either A_0 or A_1.

Example P.2.7. The $\mathbf{Z}/2\mathbf{Z}$-graded algebra property for $\mathrm{C}(q)$ is easily checked by computing in the tensor algebra $T(V)$ to see that $\mathrm{C}(q)_1 \cdot \mathrm{C}(q)_1 \subset \mathrm{C}(q)_0$, etc.

In the special case $q = 0$ on V, the grading on the exterior algebra $\mathrm{C}(V, 0) = \wedge^\bullet(V)$ has even part $\oplus_{j \geq 0} \wedge^{2j}(V)$ and odd part $\oplus_{j \geq 0} \wedge^{2j+1}(V)$.

If $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ are two $\mathbf{Z}/2\mathbf{Z}$-graded k-algebras then we define the *$\mathbf{Z}/2\mathbf{Z}$-graded tensor product $A \widehat{\otimes} B$* to have as its underlying k-vector space exactly the usual tensor product

$$A \otimes_k B = ((A_0 \otimes_k B_0) \oplus (A_1 \otimes_k B_1)) \oplus ((A_1 \otimes_k B_0) \oplus (A_0 \otimes B_1))$$

on which we declare $A_i \otimes_k B_j$ to lie in the graded piece for $i + j$ mod 2 and we impose the skew-commutative k-algebra structure (as for exterior algebras)

$$b_j \cdot a_i = (-1)^{ij} a_i \otimes b_j$$

for $a_i \in A_i$ and $b_j \in B_j$. In other words, we define

$$(a_i \otimes b_j)(a'_{i'} \otimes b'_{j'}) = (-1)^{i'j} a_i a'_{i'} \otimes b_j b'_{j'}.$$

This is not generally isomorphic *as a k-algebra* to the usual tensor product of algebras $A \otimes_k B$ in which A and B are made to commute with each other; sometimes the k-algebra $A \widehat{\otimes} B$ is called the *super tensor product* to distinguish it from the usual one.

For our purposes, the main reason for interest in the k-algebra construction $A \widehat{\otimes} B$ is its role in expressing how Clifford algebras interact with orthogonal direct sums of quadratic spaces, as we shall now explain. As a preliminary observation, here is the "universal property" for our $\mathbf{Z}/2\mathbf{Z}$-graded tensor product. If $f : A \to C$ and $g : B \to C$ are maps of $\mathbf{Z}/2\mathbf{Z}$-graded algebras such that $f(a_i)g(b_j) = (-1)^{ij} g(b_j) f(a_i)$ for all $a_i \in A_i$ and $b_j \in B_j$ then the k-bilinear map

$$A \otimes_k B \to C$$

defined by $a \otimes b \mapsto f(a)g(b)$ is a map of $\mathbf{Z}/2\mathbf{Z}$-graded k-algebras $h : A\widehat{\otimes}B \to C$. Indeed, by k-bilinearity it suffices to check that for $a_i \in A_i$, $a'_{i'} \in A_{i'}$, $b_j \in B_j$, and $b'_{j'} \in B_{j'}$ we have

$$h(a_i \otimes b_j)h(a'_{i'} \otimes b'_{j'}) = h((a_i \otimes b_j)(a'_{i'} \otimes b'_{j'})).$$

Since

$$h : (a_i \otimes b_j)(a'_{i'} \otimes b'_{j'}) = (-1)^{i'j}(a_i a'_{i'} \otimes b_j b'_{j'}) \mapsto (-1)^{i'j} f(a_i a'_{i'})g(b_j b'_{j'}),$$

the problem is to show that

$$f(a_i)g(b_j)f(a'_{i'})g(b'_{j'}) = (-1)^{i'j} f(a_i a'_{i'})g(b_j b'_{j'}).$$

This reduces to the identity $g(b_j)f(a'_{i'}) = (-1)^{i'j} f(a'_{i'})g(b_j)$, which is exactly our initial hypothesis on f and g.

Lemma P.2.8. *Consider an orthogonal direct sum $(V, q) = (V_1, q_1)\widehat{\oplus}(V_2, q_2)$ of quadratic spaces (i.e., $V = V_1 \oplus V_2$ and $q(v_1, v_2) = q_1(v_1) + q_2(v_2)$, so V_1 is B_q-orthogonal to V_2). The pair of k-algebra maps*

$$\mathrm{C}(q_1), \mathrm{C}(q_2) \rightrightarrows \mathrm{C}(q)$$

satisfies the required skew-commutativity to combine them to define a map

$$\mathrm{C}(q_1)\widehat{\otimes}\mathrm{C}(q_2) \to \mathrm{C}(q)$$

of $\mathbf{Z}/2\mathbf{Z}$-graded k-algebras. This latter map is an isomorphism. In particular, $\dim \mathrm{C}(q) = \dim \mathrm{C}(q_1) \cdot \dim \mathrm{C}(q_2)$.

Proof. The desired skew-commutativity says that for homogeneous elements x_i in $\mathrm{C}(q_1)$ of degree $i \in \mathbf{Z}/2\mathbf{Z}$ and $y_j \in \mathrm{C}(q_2)$ of degree $j \in \mathbf{Z}/2\mathbf{Z}$, $x_i y_j = (-1)^{ij} y_j x_i$ inside $\mathrm{C}(q)$. By k-bilinearity, to check this property we may assume x_i and y_j are products of vectors in V_1 and V_2 respectively in number having the same parity as i and j. By induction on the number of such vectors in the products, we're reduced to the case $x_i = v \in V_1$ and $y_j = v' \in V_2$. We want to show that $vv' = -v'v$ in $\mathrm{C}(q)$. By the laws of Clifford algebras,

$$vv' + v'v = (v + v')^2 - v^2 - v'^2 = B_q(v, v'),$$

and $B_q(v, v') = 0$ precisely because (V, q) is an orthogonal direct sum of (V_1, q_1) and (V_2, q_2).

Having constructed the desired k-algebra map, it remains to prove that it is an isomorphism. We will construct an inverse. Let $j_1 : V_1 \to \mathrm{C}(q_1)$ and $j_2 : V_2 \to \mathrm{C}(q_2)$ be the natural k-linear maps. Thus, the k-linear map

$$j : V = V_1 \oplus V_2 \to \mathrm{C}(q_1)\widehat{\otimes}\mathrm{C}(q_2)$$

given by $v_1 + v_2 \mapsto j_1(v_1) \otimes 1 + 1 \otimes j_2(v_2)$ satisfies

$$(j_1(v_1) \otimes 1 + 1 \otimes j_2(v_2))^2 = q(v_1 + v_2)$$

since

$$q(v_1 + v_2) = q_1(v_1) + q_2(v_2) = j_1(v_1)^2 + j_2(v_2)^2$$

and

$$(1 \otimes j_2(v_2))(j_1(v_1) \otimes 1) = -j_1(v_1) \otimes j_2(v_2)$$

due to the laws of Clifford multiplication and the B_q-orthgonality of v_1, v_2 in V. It follows from the universal property of Clifford algebras that we obtain a unique k-algebra map

$$\mathrm{C}(q) \to \mathrm{C}(q_1)\widehat{\otimes}\mathrm{C}(q_2)$$

extending the map j above. By computing on algebra generators coming from V_1 and V_2, this is readily checked to be an inverse to the map we have built in the opposite direction. $\square$

Proposition P.2.9. *For $n = \dim(V) > 0$, $\dim \mathrm{C}(q) = 2^n$ and $\dim \mathrm{C}(q)_0 = \dim \mathrm{C}(q)_1 = 2^{n-1}$.*

Proof. By diagonalizing q we obtain an orthogonal direct sum decomposition of (V, q) into quadratic spaces (V_i, q_i) of dimension 1. Since the preceding lemma ensures that the formation of Clifford algebras turns orthogonal direct sums of quadratic spaces into $\mathbf{Z}/2\mathbf{Z}$-graded tensor products, we have

$$\dim \mathrm{C}(q) = \prod \dim \mathrm{C}(q_i)$$

for n quadratic spaces (V_i, q_i) with $\dim V_i = 1$. Hence, to establish the formula for $\dim \mathrm{C}(q)$ it suffices to show that if $\dim V = 1$ then $\dim \mathrm{C}(V) = 2$. This follows from Example P.2.3.

To show that the even and odd parts of $\mathrm{C}(q)$ each have dimension $2^{n-1} = (1/2)2^n$, it suffices to show that their dimensions coincide (as the sum of their dimensions is 2^n). It is clear from the basis description that the even and odd parts of $\mathrm{C}(q)$ have the same dimension as in the case $q = 0$, which is to say the exterior algebra. The difference between the dimensions of its even and odd parts is given by a difference of sums of binomial coefficients that is readily seen to concide with the binomial exapansion of $(1-1)^n = 0$. $\square$

Example P.2.10. The Clifford algebra $\mathrm{C}(\mathbf{R}^n, -\sum x_i^2)$ is generated by elements $e_1, \dots, e_n$ subject to the relations $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ when $i \neq j$. It has as an $\mathbf{R}$-basis the 2^n products $e_{i_1} \cdots e_{i_r}$ for $1 \leq i_1 < \cdots < i_r \leq n$ ($r \geq 0$). These were the Clifford algebras introduced by Clifford, and in an evident sense they generalize the construction of quaternions. The description of this Clifford algebra as an n-fold $\mathbf{Z}/2\mathbf{Z}$-graded tensor product (in the $\widehat{\otimes}$-sense) of copies of $\mathrm{C}(\mathbf{R}, -x^2) = \mathbf{C}$ should not be confused with the usual n-fold tensor power of $\mathbf{C}$ as a commutative $\mathbf{R}$-algebra; the former is highly non-commutative!

Remark P.2.11. A deeper study of the structure of Clifford algebras reveals that when (V, q) is *non-degenerate* the Clifford algebra has very nice properties, depending on the parity of $n := \dim(V)$. If n is even then $\mathrm{C}(q)$ has center k and is a central simple k-algebra; i.e., a "twisted form" of the matrix algebra $\mathrm{Mat}_{2^{n/2}}(k)$ in the sense that $\mathrm{C}(q)$ and $\mathrm{Mat}_{2^{n/2}}(k)$ become isomorphic after a scalar extension to some finite Galois extension of k. Also, in such cases $\mathrm{C}(q)_0$ has center Z that is either a quadratic field extension or $k \times k$ and $\mathrm{C}(q)_0$ is a "twisted form" of $\mathrm{Mat}_{2^{(n-2)/2}}(Z)$ (i.e., they become isomorphic after scalar extension to a finite Galois extension of k). Moreover, $\mathrm{C}(q)_0$ is the centralizer of Z inside $\mathrm{C}(q)$.

Explicitly, if n is even and $\{e_i\}$ is a basis of V diagonalizing q then the center Z of $\mathrm{C}(q)_0$ has k-basis $\{1, z\}$ where $z = e_1 \cdots e_n$ (central in $\mathrm{C}(q)_0$, *not* in $\mathrm{C}(q)$: $ze_i = -e_i z$ for all i) and $z^2 = (-1)^{n(n-1)/2} \prod q(e_i) \in k^\times$. For instance, if $k = \mathbf{R}$ and $V = \mathbf{R}^n$ with $q = -\sum x_i^2$ then $z^2 = (-1)^{n(n+1)/2}$, so $\mathrm{C}(q)_0$ has center $Z = \mathbf{C}$ if $n \equiv 2 \bmod 4$ and $Z = \mathbf{R} \times \mathbf{R}$ if $n \equiv 0 \bmod 4$.

Suppose instead that n is odd. In these cases the behavior is somewhat "opposite" that for even n in the sense that it is $\mathrm{C}(q)_0$ that has center k and $\mathrm{C}(q)$ that has larger center. More precisely, $\mathrm{C}(q)$ has center Z of dimension 2 over k that is either a quadratic field extension or $k \times k$ and it inherits a $\mathbf{Z}/2\mathbf{Z}$-grading from $\mathrm{C}(q)$ with $Z_0 = k$ and $Z_1 \otimes_k Z_1 \simeq Z_0 = k$ via multiplication. (That is, $Z = k[t]/(t^2 - c)$ for some $c \in k^\times$ with Z_1 the line spanned by the residue class of t.) The subalgebra $\mathrm{C}(q)_0$ is a "twisted form" of $\mathrm{Mat}_{2^{(n-1)/2}}(k)$ and the natural map

$$\mathrm{C}(q)_0 \otimes_k Z \to \mathrm{C}(q)$$

of $\mathbf{Z}/2\mathbf{Z}$-graded algebras is an isomorphism. (In particular, $\mathrm{C}(q)_1$ is free of rank 1 as a left or right $\mathrm{C}(q)_0$-module with basis given by a k-basis of Z_1.)

Explicitly, if q diagonalizes with respect to a basis $\{e_i\}$ of V (so $e_ie_j = -e_je_i$ for all $i \neq j$) then $Z = k \oplus Z_1$ where the k-line Z_1 is spanned by the element $z = e_1 \cdots e_n \in \mathrm{C}(q)_1$ satisfying $z^2 = (-1)^{n(n-1)/2} \prod q(e_i) \in k^\times$. For instance, if $k = \mathbf{R}$ and $V = \mathbf{R}^n$ with $q = -\sum x_i^2$ then $z^2 = (-1)^{n(n+1)/2}$, so $Z = \mathbf{C}$ if $n \equiv 1 \bmod 4$ and $Z = \mathbf{R} \times \mathbf{R}$ if $n \equiv 3 \bmod 4$.

Example P.2.12. Consider a 3-dimensional quadratic space (V, q). In this case $\mathrm{C}(q)$ has dimension 8 and so its subalgebra $\mathrm{C}(q)_0$ has dimension 4. If (V, q) is non-degenerate, $\mathrm{C}(q)_0$ is either a quaternion division algebra over k or it is isomorphic to $\mathrm{Mat}_2(k)$. For example, if $V = \mathbf{R}^3$ and $q = -x_1^2 - x_2^2 - x_3^2$ then $\mathrm{C}(q)_0 \simeq \mathbf{H}$ via $i \mapsto e_1e_2$, $j \mapsto e_2e_3$, and $k \mapsto e_3e_1$.

Definition P.13. The *classical Clifford algebras* are $\mathrm{C}_n = \mathrm{C}(\mathbf{R}^n, -\sum x_i^2)$ for $n \geq 1$.

The non-degenerate quadratic forms on $\mathbf{R}^n$ up to isomorphism are

$$q_{r,n-r} = \sum_{i=1}^{r} x_i^2 - \sum_{j=r+1}^{n} x_j^2,$$

and the corresponding Clifford algebra $\mathrm{C}(\mathbf{R}^n, q_{r,n-r})$ is sometimes denoted $\mathrm{Cl}_{r,s}$ (so $\mathrm{C}_n = \mathrm{Cl}_{0,n}$). As we have noted above, $\mathrm{C}_1 = \mathbf{C}$ and $\mathrm{C}_2 = \mathbf{H}$. One can show that as $\mathbf{R}$-algebras $\mathrm{C}_3 = \mathbf{H} \times \mathbf{H}$, $\mathrm{C}_4 = \mathrm{Mat}_2(\mathbf{H})$, $\mathrm{C}_5 = \mathrm{Mat}_4(\mathbf{C})$, $\mathrm{C}_6 = \mathrm{Mat}_8(\mathbf{R})$, $\mathrm{C}_7 = \mathrm{Mat}_8(\mathbf{R}) \times \mathrm{Mat}_8(\mathbf{R})$, $\mathrm{C}_8 = \mathrm{Mat}_{16}(\mathbf{R})$, and then remarkably a periodicity property applies: as $\mathbf{R}$-algebras $\mathrm{C}_{n+8} = \mathrm{Mat}_{16}(\mathrm{C}_n) = \mathrm{Mat}_{16}(\mathbf{R}) \otimes_{\mathbf{R}} \mathrm{C}_n$ for $n > 0$. This is closely related to "Bott periodicity" in topology.

P.3. **Clifford groups.** Now assume that (V, q) is *non-degenerate*. Since V naturally occurs as a subspace of $\mathrm{C}(q)$, it makes sense to consider the units $u \in \mathrm{C}(q)^\times$ such that u-conjugation on V inside $\mathrm{C}(q)$ preserves V. Let's define the *naive Clifford group*

$$\Gamma(q)' = \{u \in \mathrm{C}(q)^\times \mid uVu^{-1} = V\}.$$

(This really is a group, since if $uVu^{-1} = V$ then multiplying suitably on the left and right gives $V = u^{-1}Vu$.) Clearly there is a natural representation $\Gamma(q)' \to \mathrm{GL}(V)$ via

$$u \mapsto (v \mapsto uvu^{-1}),$$

and it lands inside $\mathrm{O}(q)$ because inside $\mathrm{C}(q)$ we may do the computation

$$q(uvu^{-1}) = (uvu^{-1})^2 = uv^2u^{-1} = q(v).$$

This representation has kernel that contains the multiplicative group of the center of $\mathrm{C}(q)$, so we'd like to cut it down to size in order to have a chance at making a degree-2 "cover" of $\mathrm{O}(q)$ (which we can then restrict over $\mathrm{SO}(q)$ and hope to get something connected for the case of $(\mathbf{R}^n, -\sum x_i^2)$).

What we call the "naive" Clifford group is called the "Clifford group" in the classic book by Chevalley on Clifford algebras and spinors. It does give rise to the correct notion of "spin group" inside the unit groups of the classical Clifford algebras over $\mathbf{R}$ (following a procedure explained below), but it is *not* the definition used in [BtD] (introduced below, to be called the "Clifford group"). The two definitions give *distinct* subgroups of $\mathrm{C}(q)^\times$ when n is odd; the advantage of the definition in [BtD] is that it avoids a problem with $\Gamma(q)'$ for odd n that we now explain.

The difficulty with the naive Clifford group can be seen by noticing that there is a nuisance lurking in the above natural-looking representation of $\Gamma(q)'$ on V via conjugation inside the Clifford algebra:

Example P.3.1. A basic source of elements in $\Gamma(q)'$ is the q-isotropic vectors $u \in V$; i.e., those u for which $q(u) \neq 0$. These u are certainly units in $\mathrm{C}(q)$ since $u^2 = q(u) \in k^\times$ (so $u^{-1} = u/q(u)$), and since $uv + vu = B_q(u,v)$ for $v \in V$ we have

$$uvu^{-1} = -v + B_q(u,v)u^{-1} = -v + (B_q(u,v)/q(u))u = -(v - (B_q(u,v)/q(u))u) \in V.$$

In other words, $u \in \Gamma(q)'$ and its conjugation action on V is the *negative* of the q-orthogonal reflection r_u through u. Since $-r_u$ leaves u fixed and negates the hyperplane B_q-orthogonal to u, its determinant $(-1)^{\dim(V)-1}$ depends on $\dim(V)$ whereas $\det r_u = -1$.

In order to fix the glitch in the preceding example, we'd like to negate the effect of such u under the representation, and extend this correction *homomorphically* to the entire group $\Gamma(q)' \subset \mathrm{C}(q)^\times$. The natural idea is to modify the action of any *homogeneous* element $u \in \Gamma(q)' \cap \mathrm{C}(q)_j$ via the sign $(-1)^j$ since in the case of q-isotropic vectors $u \in V$ we have homogeneity with $j = 1$. But then we are faced with a puzzle: are all elements of $\Gamma(q)'$ necessary homogeneous inside the $\mathbf{Z}/2\mathbf{Z}$-graded $\mathrm{C}(q)$? The answer is affirmative when $n := \dim(V)$ is even (via a calculation that we omit) but *negative* when n is odd.

The failure for odd n is obvious when $n = 1$: $\mathrm{C}(q) = k[t]/(t^2 - c)$ for some $c \in k^\times$, so this algebra is commutative and hence $\Gamma(q)' = \mathrm{C}(q)^\times$, which has plenty of non-homogeneous elements ($a + bt$ with $a, b \in k^\times$ such that $a^2 - cb^2 \in k^\times$). The failure for general odd n is also caused by the structure of the center Z of $\mathrm{C}(q)$: as we noted (without proof) in Remark P.2.11 that as a $\mathbf{Z}/2\mathbf{Z}$-graded algebra $Z = k[t]/(t^2 - c)$ for some $c \in k^\times$ with Z_1 spanned by the class of t, so $Z^\times$ contains a lot of non-homogeneous elements (as for the case $n = 1$) and these all lie inside the naive Clifford group $\Gamma(q)'$.

Remark P.3.2. A closer study reveals that the inhomogeneity in the center Z of the Clifford algebra for odd n is the entire source of the problem, in the sense that for odd n the group $\Gamma(q)'$ is generated by its subgroup of homogeneous elements and the central subgroup $Z^\times$ (whose homogeneous part is generated by the degree-0 part $k^\times$ and an additional element in degree 1 whose square lies in $k^\times$, but which has lots of non-homogeneous elements).

One way to fix this problem is to manually insert a homogeneity condition into the definition of $\Gamma(q)'$: define the *Clifford group* $\Gamma(q)$ to be the group of *homogeneous* units u in the Clifford algebra such that $uVu^{-1} = V$ (any homogeneous unit has inverse that is also homogeneous). For even n this turns out to recover the naive Clifford group $\Gamma(q)'$, and personally I find this alternative procedure to be the most elegant one. We can then define the representation $\Gamma(q) \to \mathrm{O}(q)$ by letting $u \in \Gamma(q)$ act on V via $v \mapsto (-1)^{i_u} uvu^{-1}$ where $i_u \in \mathbf{Z}/2\mathbf{Z}$ is the degree of the homogeneous $u \in \Gamma(q)$. This "works" (e.g., isotropic $u \in V$ acts as r_u) but it is not the procedure used in [BtD] (or in many other references, which generally follow an idea of Atiyah, Bott, and Shapiro that we now explain).

Following [BtD], we shall base our development on a different-looking definition that yields the same Clifford group $\Gamma(q)$. It rests on useful involutions of the Clifford algebra:

Definition P.3. Let $\alpha : \mathrm{C}(q) \simeq \mathrm{C}(q)$ be the automorphism induced by the negation automorphism of (V, q). In other words, α acts as the identity on the even part and negation on the odd part. Let $t : \mathrm{C}(q) \simeq \mathrm{C}(q)$ by the anti-automorphism induced by $v_1 \otimes \cdots \otimes v_j \mapsto v_j \otimes \cdots \otimes v_1$ on the tensor algebra (swapping the order of multiplication in the tensor algebra and preserving the relations $v \otimes v - q(v)$, hence passing to the Clifford algebra quotient as an anti-automorphism).

It is clear that t commutes with α, and our main interest will be in the anti-automorphism $\alpha \circ t : \mathrm{C}(q) \simeq \mathrm{C}(q)$, which we call *conjugation* and denote as $x \mapsto \overline{x}$. For $v_1, \dots, v_j \in V$, the effect of conjugation is $v_1 \cdots v_j \mapsto (-1)^j v_j \cdots v_1$. Thus, on the quaternions $\mathbf{H} = \mathrm{C}_2$ (see Example P.2.12) this is the usual conjugation, and likewise on $\mathbf{C} = \mathrm{C}_1$. In general

$$\overline{xy} = \alpha(t(xy)) = \alpha(t(y)t(x)) = \overline{y}\,\overline{x}, \;\; \overline{\overline{x}} = \alpha(t(\alpha(t(x)))) = \alpha(\alpha(t(t(x)))) = x.$$

The effect of both α and conjugation on V is negation, but since α is a homomorphism whereas conjugation is an anti-homomorphism we need to use α as the "generalized negation" in the following definition that fixes the glitch with the naive Clifford group noted above.

Definition P.4. The *Clifford group* $\Gamma(V, q) = \Gamma(q)$ is the group of units $u \in \mathrm{C}(q)^\times$ such that $\alpha(u)Vu^{-1} = V$ inside $\mathrm{C}(q)$. Its *natural representation* on V is the homomorphism $\rho : \Gamma(q) \to \mathrm{GL}(V)$ defined by $\rho(u) : v \mapsto \alpha(u)vu^{-1}$. The *classical Clifford group* is $\Gamma_n = \Gamma(\mathbf{R}^n, -\sum x_i^2) \subset \mathrm{C}_n^\times$.

The map α restricts to $(-1)^j$ on $\mathrm{C}(q)_j$, so $\Gamma(q)$ meets the homogeneous parts of $\mathrm{C}(q)$ exactly where the naive Clifford group $\Gamma(q)'$ does. In particular, $k^\times$ is a central subgroup of $\Gamma(q)$ and any discrepancy between $\Gamma(q)$ and $\Gamma(q)'$ is in the non-homogeneous aspect. (In Corollary P.3.8 we will see that all elements of $\Gamma(q)$ are homogeneous.)

Since α restricts to negation on V, an element $u \in V$ lies in $\Gamma(q)$ precisely when u is a unit in $\mathrm{C}(q)$ and $uVu^{-1} = V$; i.e., precisely when u lies in the naive Clifford group. But $u^2 = q(u) \in k$, so $u \in \mathrm{C}(q)^\times$ precisely when u is isotropic (i.e., $q(u) \neq 0$). Hence, $\Gamma(q) \cap V$ consists of exactly the isotropic vectors $u \in V$, and for such u the automorphism $\rho(u)$ of V is reflection through u in the quadratic space (V, q) precisely because $\alpha(u) = -u$ (this bypasses the glitch which affects the natural representation on V by the naive Clifford group). The proof of [BtD, Ch. I, Lemma 6.9] applies verbatim in our generality to show that the automorphism α and anti-automorphism t of $\mathrm{C}(q)$ carry the Clifford group $\Gamma(q)$ isomorphically back onto itself. Hence, conjugation on the Clifford algebra also restricts to an anti-automorphism of the Clifford group.

On the classical Clifford algebras $\mathrm{C}_1 = \mathbf{C}$ and $\mathrm{C}_2 = \mathbf{H}$ over $k = \mathbf{R}$, since the conjugation anti-automorphism is the usual one, we see that $\overline{c}$ commutes with c and $c\overline{c} \in k$, so $c \mapsto c\overline{c}$ is a multiplicative map from the Clifford algebra into k in such cases. This holds whenever $\dim(V) \leq 2$. To see this we may assume k is algebraically closed, so with a suitable basis (V, q) is (k, x^2) or (k, xy). In the first case $\mathrm{C}(q) = k \times k$ and conjugation is the "flip" automorphism. In the second case $\mathrm{C}(q) \simeq \mathrm{Mat}_2(k)$ via universal property of $\mathrm{C}(k^2, xy)$ applied to the k-linear map $k^2 \to \mathrm{End}(k \oplus ke_1) = \mathrm{Mat}_2(k)$ carrying (a, b) to $\left(\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right)$, and conjugation is thereby the map $\left(\begin{smallmatrix} r & s \\ t & z \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} z & -s \\ -t & r \end{smallmatrix}\right)$ that is $M \mapsto wM^\top w^{-1}$ with $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ the standard "Weyl element" of $\mathrm{SL}_2(k)$. Thus, on the Zariski-open subset $\mathrm{C}(q)^\times$ of units in $\mathrm{C}(q)$ we have $\overline{u} = \det(u)u^{-1}$, making it evident that $\overline{u}$ commutes with u and that $u\overline{u}$ is a scalar; we also directly compute that $c\overline{c} = \overline{c}c = \det(c)$ for all $c \in \mathrm{C}(q) = \mathrm{Mat}_2(k)$.

These properties break down in higher dimensions. If $\dim V = 3$ then c commutes with $\overline{c}$ but the map $c \mapsto c\overline{c}$ has image equal to the center Z that is larger than k. Explicitly, for algebraically closed k we have $\mathrm{C}(q) \simeq \mathrm{Mat}_2(k) \times \mathrm{Mat}_2(k)$ with conjugation given as above on each $\mathrm{Mat}_2(k)$ factor. If $\dim(V) \geq 4$ then in general $\overline{c}$ does not commute with c and the map $c \mapsto c\overline{c}$ is not even valued in the center. For example, if $k = \overline{k}$ and $\dim V = 4$ then $\mathrm{C}(q) \simeq \mathrm{Mat}_4(k)$ with conjugation given by

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \mapsto \begin{pmatrix} X^\top & -Z^\top \\ -Y^\top & W^\top \end{pmatrix}$$

for $X, Y, Z, W \in \mathrm{Mat}_2(k)$.

Restricting conjugation to the Clifford group $\Gamma(q)$ (recall that conjugation preserves the Clifford group), the above problems with conjugation and the map $c \mapsto c\overline{c}$ on the entire Clifford algebra disappear. The proof rests on a lemma of independent interest:

Lemma P.3.5. *The kernel of ρ is equal to the central subgroup $k^\times$.*

Proof. We know that $k^\times$ is a central subgroup of $\mathrm{C}(q)^\times$ on which α is the identity, so certainly it lies inside $\ker \rho$. To show that any $u \in \ker \rho$ lies in $k^\times$, we adapt the proof of [BtD, Ch. I, 6.11] (which treats the classical Clifford groups over $k = \mathbf{R}$) by computing relative to a diagonalizing basis of V for q.

Let $\{e_i\}$ be a basis of V such that $q(\sum x_i e_i) = \sum q(e_i) x_i^2$ (so all $q(e_i) \neq 0$ by non-degeneracy). Thus, $\mathrm{C}(q)$ has a basis given by products of the e_i's with strictly increasing indices, and $e_i e_j = -e_j e_i$ for $i \neq j$.

By hypothesis $vu = \alpha(u)v$ for all $v \in V$, so if we write $u = u_0 + u_1$ as a sum of homogeneous parts in the Clifford algebra then we wish to show that $u_1 = 0$ and $u_0 \in k$. Since $\alpha(u) = u_0 - u_1$, comparing homogeneous parts on both sides of the equality $vu = \alpha(u)v$ gives

$$vu_0 = u_0 v, \;\; vu_1 = -u_1 v$$

for all $v \in V$. We will show that $u_0 \in k$ by proving that its expression in terms of the basis of products of an even number of e_i's involves no term with any e_i's (leaving only the option of the empty product, which is to say k). Likewise, to prove that the odd u_1 vanishes, it suffices to show that its basis expansion involves no appearance of any e_i.

Fix an index i, so we may uniquely write

$$u_0 = y_0 + e_i y_1$$

with even y_0 and odd y_1 involving no appearances of e_i in their basis expansion inside the Clifford algebra. Thus, the equality $vu_0 = u_0 v$ with $v = e_i$ gives

$$e_i y_0 + q(e_i) y_1 = y_0 e_i + e_i y_1 e_i = e_i y_0 - q(e_i) y_1.$$

(Here we have used that y_0 and y_1 do not involve e_i, and that e_i anti-commutes with e_j for any $j \neq i$.) Thus, $q(e_i)y_1 = -q(e_i)y_1$, and since $q(e_i) \neq 0$ it follows that $y_1 = -y_1$, so $y_1 = 0$ since $\mathrm{char}(k) \neq 2$. This shows that u_0 doesn't involve e_i, and since i was arbitrary we may conclude (as indicated already) that $u_0 \in k$.

Now we analyze u_1 using that $vu_1 = -u_1 v$ for all $v \in V$. Choose an index i, so uniquely

$$u_1 = w_1 + e_i w_0$$

for even w_0 and odd w_1 that involve no appearance of e_i in their basis expansion inside the Clifford algebra. Hence, the equation $vu_1 = -u_1 v$ with $v = e_i$ gives

$$e_i w_1 + q(e_i) w_0 = -w_1 e_i - e_i w_0 e_i = e_i w_1 - q(e_i) w_0.$$

But $q(e_i) \in k^\times$, so $w_0 = 0$ and $u_1 = w_1$ does not involve e_i. Varying i, we get $u_1 = 0$. $\square$

Proposition P.3.6. *For $u \in \Gamma(q)$, the element $u\overline{u} \in \Gamma(q)$ lies in $k^\times$ and is equal to $\overline{u}u$. In particular, $\mathrm{N} : u \mapsto u\overline{u}$ is a multiplicative map from $\Gamma(q)$ into $k^\times$ that extends squaring on $k^\times$. Moreover, its restriction to the subset of isotropic vectors in V is $-q$.*

We call $\mathrm{N} : \Gamma(q) \to k^\times$ the *Clifford norm*. The multiplicativity properties of N are obvious on elements $u = v_1 \cdots v_r$ for isotropic v_j (with $\mathrm{N}(u) = \prod q(v_j)$ in such cases). We will see in Corollary P.3.8 that all elements in $\Gamma(q)$ are of this type, and in particular are homogeneous in $\mathrm{C}(q)$, but the proof rests on the multiplicativity properties of the Clifford norm.

Proof. For any $v \in V$, $v\overline{v} = -v^2 = -q(v)$. For $u \in \Gamma(q)$, provided that $u\overline{u} = c \in k^\times$ it follows that $\overline{u} = cu^{-1}$, so $\overline{u}u = c$ as well. To show that $u\overline{u} \in k^\times$, the key idea is to show that $\rho(u\overline{u}) = 1$, so we can apply Lemma P.3.5 to conclude. The calculation $\rho(u\overline{u}) = 1$ is carried out in the proof of [BtD, Ch. I, 6.12] (the computation given there, which involves working with both α and t, carries over without change to the general case). $\square$

Although it was obvious that the naive Clifford group $\Gamma(q)'$ acts on V via elements of $\mathrm{O}(q)$, this is not quite as obvious for the Clifford group $\Gamma(q)$ due to the intervention of α (since we have not addressed homogeneity properties of elements of $\Gamma(q)$!), but it is true. The proof requires applying α, t, and conjugation to elements of the Clifford group, and gives more:

Proposition P.3.7. *The image $\rho(\Gamma(q))$ is equal to $\mathrm{O}(q)$.*

Proof. We have seen that every reflection in V through an isotropic vector u lies in $\rho(\Gamma(q))$, as such u lie in $\Gamma(q)$ and the associated reflection literally is $\rho(u)$. Thus, to show that $\mathrm{O}(q)$ is contained in $\rho(\Gamma(q))$ it suffices to prove that every element in $\mathrm{O}(q)$ is a product of finitely many (in fact, at most $\dim(V)$) such reflections. This is explained in the classical case by an elementary induction argument in the proof of [BtD, Ch. I, 6.15]. That method works in general using a diagonalizing basis for q *provided* that for any vectors $v, v' \in V$ satisfying $q(v) = q(v')$ there exists $g \in \mathrm{O}(q)$ such that $g(v) = v'$. (For $q = -\sum x_i^2$ on $V = \mathbf{R}^n$ with $k = \mathbf{R}$ it is proved by using Gramm-Schmidt with orthonormal bases.) This is a special case of Witt's Extension theorem in the theory of quadratic forms: the orthogonal group of a non-degenerate quadratic space acts transitively on the set of embeddings of a fixed quadratic space (such as embeddings of (k, cx^2) with $c = q(v) = q(v')$).

To prove that $\rho(\Gamma(q)) \subset \mathrm{O}(q)$, we have to carry out a more sophisticated calculation than in the case of the naive Clifford group. For $u \in \Gamma(q)$, we want to prove that $q(\alpha(u)vu^{-1}) =$

$q(v)$ for any $v \in V$. Recall that the function $\mathrm{N} : x \mapsto x\overline{x}$ on the Clifford algebra restricts to $-q$ on V. Hence, it is equivalent to show that $\mathrm{N}(\alpha(u)vu^{-1}) = \mathrm{N}(v)$ for all $v \in V$. This proposed equality for fixed u and varying v is an equality of polynomial functions on V. But in general N only has good multiplicative properties on the Clifford group $\Gamma(q)$, not on the entire Clifford algebra. Thus, we want to restrict to attention to v that is q-isotropic (as then $v \in \Gamma(q)$). Note that the restriction to isotropic v is harmless in the classical cases with $(\mathbf{R}^n, -\sum x_i^2)$, as there all nonzero v are isotropic.

To reduce to treating isotropic v in general, we wish to appeal to a density argument with the Zariski topology. For that purpose, we want to work over an algebraically closed field. The problem of proving $q = q \circ \rho$ is visibly sufficient to check after applying an extension of the ground field, so we may and do now assume that k is algebraically closed. Thus, the locus of isotropic vectors is the Zariski-open subset of the affine space V complementary to the hypersurface $q = 0$. This is a proper hypersurface (i.e., q isn't identically zero), so the open complement is non-empty (and hence is Zariski-dense). An equality of polynomial functions on V holds everywhere if it does so on a Zariski-dense subset. In other words, it suffices to prove $\mathrm{N}(\alpha(u)vu^{-1}) = \mathrm{N}(v)$ for $v \in V$ that are isotropic for q, so $v \in \Gamma(q)$.

The multiplicativity of $\mathrm{N} : \Gamma(q) \to k^\times$ gives

$$\mathrm{N}(\alpha(u)vu^{-1}) = \mathrm{N}(\alpha(u))\mathrm{N}(v)\mathrm{N}(u)^{-1}$$

in $k^\times$, so it suffices to show that $\mathrm{N} \circ \alpha = \mathrm{N}$ on $\Gamma(q)$. Since conjugation is the composition of the commuting operations t and α, it follows that conjugation commutes with α, so for $u \in \Gamma(q)$ we have $\mathrm{N}(\alpha(u)) = \alpha(u)\overline{\alpha(u)} = \alpha(u)\alpha(\overline{u}) = \alpha(u\overline{u}) = u\overline{u}$ since $u\overline{u} \in k$ for $u \in \Gamma(q)$ and α is the identity on k. $\square$

Corollary P.3.8. *Every element in $\Gamma(q)$ is a $k^\times$-multiple of a product of at most* $\dim(V)$ *isotropic vectors. In partcular, each $u \in \Gamma(q)$ is homogeneous in* $\mathrm{C}(q)$ *and* $\det(\rho(u)) = (-1)^{\deg(u)}$*, where* $\deg(u) \in \mathbf{Z}/2\mathbf{Z}$ *is the degree of u in* $\mathrm{C}(q)$.

Proof. As we noted in the preceding proof, every element in $\mathrm{O}(q)$ is a product of at most $\dim(V)$ reflections in isotropic vectors. Writing

$$\rho(u) = r_{v_1} \cdots r_{v_m} = \rho(v_1) \cdots \rho(v_m) = \rho(v_1 \cdots v_m)$$

for isotropic v_j's, we see that $u(v_1 \cdots v_m)^{-1} \in \ker \rho$. But our work with the multiplicative Clifford norm on $\Gamma(q)$ showed that $\ker \rho = k^\times$. Since reflections in isotropic vectors have determinant -1, we're done. $\square$

P.4. **Pin and Spin.** Let (V, q) be a non-degenerate quadratic space over k. The homomorphism $\rho : \Gamma(q) \twoheadrightarrow \mathrm{O}(q)$ has kernel $k^\times$, and we'd like to cut this kernel down to the subgroup $\mu_2 = \{\pm 1\}$ of order 2. To do this, we note that the Clifford norm $\mathrm{N} : \Gamma(q) \to k^\times$ restricts to squaring on $k^\times$, so its kernel meets $k^\times$ in $\{\pm 1\}$. This suggests trying the subgroup $\ker \mathrm{N}$ as a candidate for a "natural" double cover of $\mathrm{O}(q)$. But does $\ker \mathrm{N}$ maps onto $\mathrm{O}(q)$?

In view of how we proved the surjectivity of ρ onto $\mathrm{O}(q)$ using reflections, it would suffice to show that if $r \in \mathrm{O}(q)$ is a reflection through an isotropic vector u then we can find a $k^\times$-multiple u' so that $\mathrm{N}(u') = 1$ (as such a u' has the same associated reflection as u). The Clifford norm on $\Gamma(q)$ restricts to $-q$ on the set $\Gamma(q) \cap V$ of isotropic vectors, so we seek a $k^\times$-multiple u' of u such that $-q(u') = 1$. Writing $u' = cu$ with $c \in k^\times$ to be determined, this amounts to asking that $-c^2q(u) = 1$, so we need to be able to extract a square root of $-q(u)$ in k. This may not be possible (depending on k and q)! Of course, it could be that our strategy is simply too naive, and that if we used deeper properties of $\mathrm{O}(q)$ then we might be able to prove surjectivity.

In fact there is a genuine obstruction called the *spinor norm* which is nontrivial even for indefinite non-degenerate (V, q) over $\mathbf{R}$. These matters for general (V, q) are best understood via the theory of linear algebraic groups (exact sequences of algebraic groups, Galois cohomology, etc.). To avoid this, *now* we finally specialize our attention to the classical case $(\mathbf{R}^n, -\sum x_i^2)$ over $k = \mathbf{R}$ and the representation $\rho : \Gamma_n \to \mathrm{O}(n)$. In this case $-q = \sum x_i^2$ is positive-definite, so all nonzero $v \in V$ are isotropic and admit a $k^\times$-multiple on which $-q$ takes the value 1 (i.e., a unit-vector multiple). This brings us to:

Definition P.1. The *Pin group* $\mathrm{Pin}(n)$ is the kernel of the Clifford norm $\mathrm{N} : \Gamma_n \to \mathbf{R}^\times$.

Note that $\mathrm{C}_n^\times$ is naturally a Lie group, being the units of a finite-dimensional associative $\mathbf{R}$-algebra, and Γ_n is visibly a closed subgroup (check!), so it inherits Lie group structure from $\mathrm{C}_n^\times$. Also, from the construction inside the Clifford algebra it is straightforward to check (do it!) that the representation $\rho : \Gamma_n \twoheadrightarrow \mathrm{O}(n) \subset \mathrm{GL}_n(\mathbf{R})$ and Clifford norm $\mathrm{N} : \Gamma_n \to \mathbf{R}^\times$ are C^∞ (equivalently: continuous), so $\mathrm{Pin}(n)$ is a closed Lie subgroup and the restriction $\rho_n : \mathrm{Pin}(n) \to \mathrm{O}(n)$ of ρ is C^∞. We have seen that ρ_n is surjective (as $\mathbf{R}^\times_{>0}$ admits square roots in $\mathbf{R}^\times$) and it fits into an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Pin}(n) \to \mathrm{O}(n) \to 1.$$

In particular, $\mathrm{Pin}(n)$ is *compact* since it is a degree-2 covering space of the compact $\mathrm{O}(n)$.

The preimage $\mathrm{Spin}(n) = \rho_n^{-1}(\mathrm{SO}(n)) \subset \mathrm{Pin}(n)$ is an open and closed subgroup of index 2 (exactly as for $\mathrm{SO}(n)$ inside $\mathrm{O}(n)$); it consists of the even elements of $\mathrm{Pin}(n)$, by Corollary P.3.8. Thus, $\mathrm{Spin}(n)$ is a compact Lie group fitting into an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$$

that realizes $\mathrm{Spin}(n)$ as a degree-2 cover of $\mathrm{SO}(n)$. We have to make sure that $\mathrm{Spin}(n)$ is *connected* (e.g., to know we haven't wound up with $\mathrm{SO}(n) \times \{\pm 1\}$ by another name).

The surjective Lie group map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is necessarily surjective on Lie algebras (as for any surjective map of Lie groups), so by the usual arguments with the submersion theorem we know via the connectedness of $\mathrm{SO}(n)$ that $\mathrm{Spin}(n)^0 \to \mathrm{SO}(n)$ is surjective. Hence, $\mathrm{Spin}(n)$ is generated by its identity component $\mathrm{Spin}(n)^0$ together with the central kernel $\{\pm 1\}$ of order 2. Consequently, to prove that $\mathrm{Spin}(n)$ is connected, it suffices to show that this kernel lies in the identity component. More specifically, it suffices to find a path in $\mathrm{Spin}(n)$ that links the identity to the nontrivial element of the kernel. Such a path is written down explicitly in the proof of 6.17 in Chapter I by working inside the Clifford algebra C_n.

Remark P.4.2. In Chevalley's book on Clifford algebras and spinors, specialized to the classical case $(\mathbf{R}^n, -\sum x_i^2)$, he calls the preimage of $\mathrm{SO}(n)$ in Γ_n the *special Clifford group* and he calls its intersection $\mathrm{Spin}(n)$ with the kernel of the Clifford norm the *reduced Clifford group*. He gives no name to $\mathrm{Pin}(n)$. The name for $\mathrm{Pin}(n)$ was coined in the early 1960's: much as the notation $\mathrm{O}(n)$ is obtained typographically from $\mathrm{SO}(n)$ by removing the initial letter "S", due to the fortuitous coincidence that the word "Spin" begins with the letter "S" we can apply the same procedure to arrive at the name $\mathrm{Pin}(n)$ from $\mathrm{Spin}(n)$.

APPENDIX Q. THE REMARKABLE SU(2)

Let G be a non-commutative connected compact Lie group, and assume that its rank (i.e., dimension of maximal tori) is 1; equivalently, G is a compact connected Lie group of rank 1 that has dimension > 1. In class we have seen a natural way to make such G, namely $G = Z_H(T_a)/T_a$ for a non-commutative connected compact Lie group H, a maximal torus T in H, a root $a \in \Phi(H, T)$, and the codimension-1 subtorus $T_a := (\ker a)^0 \subset T$; this G has maximal torus T/T_a. (If $\dim T = 1$ then $T_a = 1$.)

There are two examples of such G that we have seen: SO(3) and its connected double cover SU(2). These Lie groups are not homeomorphic, as their fundamental groups are distinct. Also, by inspecting the adjoint action of a maximal torus, SU(2) has center $\{\pm 1\}$ of order 2 whereas SO(3) has trivial center (see HW7, Exercise 1(iii)), so they are not isomorphic as abstract groups.

The main aim of this appendix is to prove that there are *no other examples*. Once that is proved, we use it to describe the structure of $Z_G(T_a)$ in the general case (without a rank-1 assumption). This is a crucial building block in the structure theory of general G. In [BtD, Ch. V, pp. 186–188] you'll find two proofs that any rank-1 non-commutative G is isomorphic to SO(3) or SU(2): a topological proof using higher homotopy groups ($\pi_2(S^m) = 1$ for $m > 2$) and an algebraic proof that looks a bit "unmotivated" (for a beginner). Our approach is also algebraic, using the representation theory of $\mathfrak{sl}_2(\mathbf{C})$ to replace hard group-theoretic problems with easier Lie algebra problems.

Q.1. **Rank** 1. Fix a maximal torus T in G and an isomorphism $T \simeq S^1$. Consider the representation of T on $\mathfrak{g}$ via Ad_G. By Appendix H we know that the subspace $\mathfrak{g}^T$ of T-invariants is $\mathrm{Lie}(Z_G(T))$, and this is $\mathfrak{t}$ since $Z_G(T) = T$ (due to the maximality of T in G). On HW7 Exercise 5, you show that the (continuous) representation theory of compact Lie groups on finite-dimensional $\mathbf{R}$-vector spaces is completely reducible, and in particular that the non-trivial irreducible representations of $T = S^1$ over $\mathbf{R}$ are all 2-dimensional and indexed by integers $n \geq 1$: these are $\rho_n : S^1 = \mathbf{R}/\mathbf{Z} \to \mathrm{GL}_2(\mathbf{R})$ via n-fold counterclockwise rotation: $\rho_n(\theta) = r_{2\pi n\theta}$. (This makes sense for $n < 0$ via clockwise $|n|$-fold rotation, and $\rho_{-n} \simeq \rho_n$ by choosing an orthonormal basis with the opposite orientation.) Note that $(\rho_n)_{\mathbf{C}} = \chi^n \oplus \chi^{-n}$ where $\chi : S^1 \to \mathbf{C}^\times$ is the standard embedding.

As $\mathbf{R}$-linear T-representations,

$$\mathfrak{g} = \mathfrak{t} \oplus (\oplus_{n \geq 1} \mathfrak{g}(n))$$

where $\mathfrak{g}(n)$ denotes the ρ_n-isotypic subspace. In particular, each $\mathfrak{g}(n)$ is even-dimensional and so has dimension at least 2 if it is nonzero. Passing to the complexfication $\mathfrak{g}_{\mathbf{C}}$ and using the decomposition of $(\rho_n)_{\mathbf{C}}$ as a direct sum of *reciprocal* characters with weights n and $-n$, as a $\mathbf{C}$-linear representation of T we have

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus (\oplus_{a \in \Phi} (\mathfrak{g}_{\mathbf{C}})_a)$$

where the set $\Phi \subset \mathrm{X}(T) = \mathrm{X}(S^1) = \mathbf{Z}$ of nontrivial T-weights is *stable under negation* and $\dim(\mathfrak{g}_{\mathbf{C}})_a = \dim(\mathfrak{g}_{\mathbf{C}})_{-a}$ (this common dimension being $(1/2)\dim_{\mathbf{R}} \mathfrak{g}(n)$ if $a : t \mapsto t^n$).

Since the action of T on $(\mathfrak{g}_{\mathbf{C}})_a$ via Ad_G is given by the character $a : S^1 = T \to S^1 \subset \mathbf{C}^\times$ (some power map), the associated action $X \mapsto [X, \cdot]_{\mathfrak{g}_{\mathbf{C}}}$ of $\mathfrak{t} = \mathbf{R} \cdot \partial_\theta$ on $(\mathfrak{g}_{\mathbf{C}})_a$ via $\mathrm{Lie}(\mathrm{Ad}_G) = (\mathrm{ad}_{\mathfrak{g}})_{\mathbf{C}} = \mathrm{ad}_{\mathfrak{g}_{\mathbf{C}}}$ is given by multiplication against $\mathrm{Lie}(a)(\partial_\theta|_{\theta=1}) \in \mathbf{Z} \subset \mathbf{R}$.

Thus, the $\mathfrak{t}_{\mathbf{C}}$-action on $(\mathfrak{g}_{\mathbf{C}})_a$ via the Lie bracket on $\mathfrak{g}_{\mathbf{C}}$ is via multiplication by the same integer. This visibly scales by $c \in \mathbf{R}^\times$ if we replace ∂_θ with $c\partial_\theta$. Hence, we obtain:

Lemma Q.1.1. *Let H be a nonzero element in the line $\mathfrak{t}$. The action of $\mathrm{ad}(H) = [H, \cdot]$ on $\mathfrak{g}_{\mathbf{C}}$ has as its nontrivial weight spaces exactly the subspaces $(\mathfrak{g}_{\mathbf{C}})_a$, with eigenvalue $(\mathrm{Lie}(a))(H)$.*

Since $\Phi(G, T) \subset \mathrm{X}(T) - \{0\} = \mathbf{Z} - \{0\}$ is a non-empty subset stable under negation, it contains a unique highest element, say $a \in \mathbf{Z}_{>0}$. The stability of $\Phi(G, T)$ under negation implies that $-a$ is the unique lowest weight. For any $b, b' \in \Phi(G, T)$ and $v \in (\mathfrak{g}_{\mathbf{C}})_b$, $v' \in (\mathfrak{g}_{\mathbf{C}})_{b'}$ we have

$$[v, v'] \subset (\mathfrak{g}_{\mathbf{C}})_{b+b'}$$

since applying $\mathrm{Ad}_G(t)$ to $[v, v']$ carries it to $[t^b v, t^b v'] = t^{b+b'}[v, v']$. (We allow the case $b + b' = 0$, the 0-weight space being $\mathfrak{t}_{\mathbf{C}}$.) In particular, since $(\mathfrak{g}_{\mathbf{C}})_{\pm 2a} = 0$ (due to the nonzero a and $-a$ being the respective highest and lowest T-weights for the T-action on $\mathfrak{g}_{\mathbf{C}}$), each $(\mathfrak{g}_{\mathbf{C}})_{\pm a}$ is a *commutative Lie subalgebra* of $\mathfrak{g}_{\mathbf{C}}$ and

$$[(\mathfrak{g}_{\mathbf{C}})_a, (\mathfrak{g}_{\mathbf{C}})_{-a}] \subseteq \mathfrak{t}_{\mathbf{C}}.$$

Hence, this latter bracket pairing is either 0 or exhausts the 1-dimensional $\mathfrak{t}_{\mathbf{C}}$.

Lemma Q.1.2. *For any $a \in \Phi(G, T)$, $[(\mathfrak{g}_{\mathbf{C}})_a, (\mathfrak{g}_{\mathbf{C}})_{-a}] = \mathfrak{t}_{\mathbf{C}}$.*

Proof. Suppose otherwise, so this bracket vanishes. Hence, $V_a := (\mathfrak{g}_{\mathbf{C}})_a \oplus (\mathfrak{g}_{\mathbf{C}})_{-a} = V_{-a}$ is a *commutative* Lie subalgebra of $\mathfrak{g}_{\mathbf{C}}$. By viewing a as an element of $\mathrm{X}(T) = \mathbf{Z}$, we see that $V_a \simeq (\rho_a)_{\mathbf{C}}^{\oplus d}$ as $\mathbf{C}$-linear T-representations, where d is the common $\mathbf{C}$-dimension of $(\mathfrak{g}_{\mathbf{C}})_{\pm a}$. Clearly $V_a = \mathfrak{g}(a)_{\mathbf{C}}$ inside $\mathfrak{g}_{\mathbf{C}}$, where $\mathfrak{g}(a)$ denotes the ρ_a-isotypic part of $\mathfrak{g}$ as an $\mathbf{R}$-linear representation of T, so $\mathfrak{g}(a)$ is a *commutative* Lie subalgebra of $\mathfrak{g}$ with dimension at least 2.

Choose $\mathbf{R}$-linearly independent $X, Y \in \mathfrak{g}(a)$, so $\alpha_X(\mathbf{R})$-conjugation on G leaves the map $\alpha_Y : \mathbf{R} \to G$ *invariant* since by connectedness of $\mathbf{R}$ such invariance may be checked on the map $\mathrm{Lie}(\alpha_Y) = \alpha_Y'(0) : \mathbf{R} \to \mathfrak{g}$ sending $c \in \mathbf{R}$ to $c\alpha_Y'(0) = cY$ by using $\mathrm{Ad}_G(\alpha_X(\mathbf{R}))$ (and noting that $\mathrm{Lie}(\mathrm{Ad}_G \circ \alpha_X) = \mathrm{ad}_{\mathfrak{g}}(\alpha_X'(0)) = [X, \cdot]$ and $[X, Y] = 0$). Hence, the closure

$$\overline{\alpha_X(\mathbf{R}) \cdot \alpha_Y(\mathbf{R})}$$

is a connected commutative closed subgroup of G whose Lie algebra contains $\alpha_X'(0) = X$ and $\alpha_Y'(0) = Y$, so its dimension is at least 2. But connected compact commutative Lie groups are necessarily tori, and by hypothesis G has no tori of dimension larger than 1! $\square$

Now we may choose $X_\pm \in (\mathfrak{g}_{\mathbf{C}})_{\pm a}$ such that the element $H := [X_+, X_-] \in \mathfrak{t}_{\mathbf{C}}$ is *nonzero*. Clearly $[H, X_\pm] = \mathrm{Lie}(\pm a)(H)X_\pm = \pm\mathrm{Lie}(a)(H)X_\pm$ with $\mathrm{Lie}(a)(H) \in \mathbf{C}^\times$. If we replace X_+ with cX_+ for $c \in \mathbf{C}^\times$ then H is replaced with $H' := cH$, and $[H', cX_+] = \mathrm{Lie}(a)(H')(cX_+)$, $[H', X_-] = -\mathrm{Lie}(a)(H')X_-$ with $\mathrm{Lie}(a)(H') = c\,\mathrm{Lie}(a)(H)$. Hence, using such scaling with a suitable c allows us to arrange that $\mathrm{Lie}(a)(H) = 2$, so $\{X_+, X_-, H\}$ span an $\mathfrak{sl}_2(\mathbf{C})$ as a Lie subalgebra of $\mathfrak{g}_{\mathbf{C}}$. The restriction of $\mathrm{ad}_{\mathfrak{g}_{\mathbf{C}}}$ to this copy of $\mathfrak{sl}_2(\mathbf{C})$ makes $\mathfrak{g}_{\mathbf{C}}$ into a $\mathbf{C}$-linear representation space for $\mathfrak{sl}_2(\mathbf{C})$ such that the H-weight spaces for this $\mathfrak{sl}_2(\mathbf{C})$-representation are $\mathfrak{t}_{\mathbf{C}}$ for the trivial weight and each $(\mathfrak{g}_{\mathbf{C}})_b$ (on which H acts as scaling by $\mathrm{Lie}(b)(H) \in \mathbf{C}$).

The highest weight for $\mathfrak{g}_{\mathbf{C}}$ as an $\mathfrak{sl}_2(\mathbf{C})$-representation is $\mathrm{Lie}(a)(H) = 2$ (due to how a was chosen!), so the only other possible weights are $\pm 1, 0, -2$, and the entire weight-0 space for the action of H is a single $\mathbf{C}$-line $\mathfrak{t}_{\mathbf{C}}$. Our knowledge of the finite-dimensional representation theory of $\mathfrak{sl}_2(\mathbf{C})$ (e.g., its complete reducibility, and the determination of each irreducible representation via its highest weight) shows that the adjoint representation of $\mathfrak{sl}_2(\mathbf{C})$ on itself is the *unique* irreducible representation with highest weight 2. This representation contains a line with H-weight 0, so as an $\mathfrak{sl}_2(\mathbf{C})$-representation we see that the highest weight of 2 cannot occur in $\mathfrak{g}_{\mathbf{C}}$ with multiplicity larger than 1 (as otherwise $\mathfrak{g}_{\mathbf{C}}$ would contain multiple independent copies of the adjoint representation $\mathfrak{sl}_2(\mathbf{C})$, contradicting that the weight-0 space for the H-action on $\mathfrak{g}_{\mathbf{C}}$ is only 1-dimensional).

It follows that $\mathfrak{g}_{\mathbf{C}}$ is $\mathfrak{sl}_2(\mathbf{C})$-equivariantly isomorphic to a direct sum of $\mathfrak{sl}_2(\mathbf{C})$ and copies of the "standard 2-dimensional representation" (the only irreducible option with highest weight < 2 that doesn't introduce an additional weight-0 space for H). We conclude that $\Phi(G,T)$ is either $\{\pm a\}$ or $\{\pm a, \pm a/2\}$ and that $\dim(\mathfrak{g}_{\mathbf{C}})_{\pm a} = 1$.

It remains to rule out that possibility that the weights $\pm a/2$ also occur as T-weights on $\mathfrak{g}_{\mathbf{C}}$. Let's suppose these weights do occur, so $a \in 2\mathrm{X}(T)$, and let $b = a/2$ for typographical simplicity. Choose a nonzero $X \in (\mathfrak{g}_{\mathbf{C}})_b$, and let $\overline{X}$ be the complex conjugate of X inside the complex conjugate $(\mathfrak{g}_{\mathbf{C}})_{-b}$ of $(\mathfrak{g}_{\mathbf{C}})_b$ (this "makes sense" since the T-action on $\mathfrak{g}_{\mathbf{C}}$ respects the $\mathbf{R}$-structure $\mathfrak{g}$, and the complex conjugate of t^b is t^{-b} for $t \in T = S^1$). Although we cannot argue that $(\mathfrak{g}_{\mathbf{C}})_{\pm b}$ is commutative, since b isn't the highest weight, we can nonetheless use:

Lemma Q.1.3. *The element $[X, \overline{X}]$ is nonzero.*

Proof. The elements $v := X + \overline{X}$ and $v := i(X - \overline{X})$ in $(\mathfrak{g}_{\mathbf{C}})_b \oplus (\mathfrak{g}_{\mathbf{C}})_{-b} \subset \mathfrak{g}_{\mathbf{C}}$ are visibly nonzero (why?) and linearly indepedent over $\mathbf{R}$, and they are invariant under complex conjugation on $\mathfrak{g}_{\mathbf{C}}$, so v and v' lie in $\mathfrak{g}$. If $[X, \overline{X}] = 0$ then clearly $[v, v'] = 0$, so v and v' would span a *two-dimensional* commutative Lie subalgebra of $\mathfrak{g}$. But we saw above via 1-parameter subgroups that such a Lie subalgebra creates a torus inside G with dimension at least 2, contradicting the assumption that G has rank 1. $\square$

Since $H' := [X, \overline{X}] \in \mathfrak{t}_{\mathbf{C}}$, we can use X, $\overline{X}$, and H' (after preliminary $\mathbf{C}^{\times}$-scalings) to generate *another* Lie subalgebra inclusion $\mathfrak{sl}_2(\mathbf{C}) \hookrightarrow \mathfrak{g}_{\mathbf{C}}$ such that the diagonal subalgebra of $\mathfrak{sl}_2(\mathbf{C})$ is $\mathfrak{t}_{\mathbf{C}}$. Let H_0 be the "standard" diagonal element of $\mathfrak{sl}_2(\mathbf{C})$, so our new embedding of $\mathfrak{sl}_2(\mathbf{C})$ into $\mathfrak{g}_{\mathbf{C}}$ identifies $\mathrm{Lie}(b)$ with the weight 2 for the action of H_0. Since $a = 2b$ we see that $\mathfrak{g}_{\mathbf{C}}$ as an $\mathfrak{sl}_2(\mathbf{C})$-representation has H_0-weights 0, ± 2, ± 4, with the weight-0 space just the line $\mathfrak{t}_{\mathbf{C}}$ and the highest-weight line (for weight 4) equal to $(\mathfrak{g}_{\mathbf{C}})_a$ that we have already seen is 1-dimensional. Consequently, as an $\mathfrak{sl}_2(\mathbf{C})$-representation, $\mathfrak{g}_{\mathbf{C}}$ must contain a copy of the 5-dimensional irreducible representation V_4 with highest weight 4.

If $W \subset \mathfrak{g}_{\mathbf{C}}$ is any $\mathfrak{sl}_2(\mathbf{C})$-subrepresentation that contains the weight-0 line then $W \cap V_4$ is nonzero and thus coincdes with V_4 (since V_4 is irreducible); i.e., $V_4 \subset W$. This V_4 contains a weight-0 line that must be $\mathfrak{t}_{\mathbf{C}}$, and it also exhausts the lines $(\mathfrak{g}_{\mathbf{C}})_{\pm a}$ with weights ± 4. But the new copy of $\mathfrak{sl}_2(\mathbf{C})$ that we have built as a Lie subalgebra of $\mathfrak{g}_{\mathbf{C}}$ is a V_2, which has a weight-0 line and thus contains the unique copy of V_4, an absurdity.

This contadiction shows that the case $\Phi(G,T) = \{\pm a, \pm a/2\}$ cannot occur, so in the rank-1 non-commutative case we have shown $\dim G = 3$, as desired. (In class we deduced from the condition $\dim G = 3$ that G is isomorphic to either $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$.)

Q.2. **Centralizers in higher rank.** Now consider a pair (G,T) with maximal torus $T \neq G$ (equivalently, $\Phi := \Phi(G,T) \neq \emptyset$), and $\dim T > 0$ arbitrary. Choose $a \in \Phi$, so $Z_G(T_a)/T_a$ is either $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$. Hence, $Z_G(T_a)$ sits in the middle of a short exact sequence

$$1 \to T_a \to Z_G(T_a) \to H \to 1$$

with H non-commutative of rank 1. In this section, we shall explicit describe *all* Lie group extensions of such an H by a torus. This provides information about the structure of the group $Z_G(T_a)$ that we shall later use in our proof that the commutator subgroup $G' = [G, G]$ of G is closed and perfect (i.e., $(G')' = G'$) in general.

Lemma Q.2.1. *Let $q : \widetilde{H} \to H$ be an isogeny between connected Lie groups with $\pi_1(\widetilde{H}) = 1$. For any connected Lie group G and isogeny $f : G \to H$, there is a unique Lie group homomorphism $F : \widetilde{H} \to G$ over H; i.e., a unique way to fill in a commutative diagram*

$$\begin{tikzcd} \widetilde{H} \arrow[r, "F"] \arrow[dr, "q"'] & G \arrow[d, "f"] \\ & H \end{tikzcd}$$

Moreover, F is an isogeny.

For our immediate purposes, the main case of interest is the degree-2 isogeny q : $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Later we will show that if H is a connected compact Lie group with finite center then such a q always exists with $\widetilde{H}$ compact too. The lemma then says that such an $\widetilde{H}$ *uniquely* sits "on top" of all isogenous connected covers of H.

Proof. Consider the pullback $\widetilde{f} : \widetilde{G} := G \times_H \widetilde{H} \to \widetilde{H}$ of the isogeny f along q, as developed in HW7 Exercise 4, so $\widetilde{f}$ is surjective with kernel $\ker f$ that 0-dimensional and central in $\widetilde{G}$. In particular, $\mathrm{Lie}(\widetilde{f})$ is surjective with kernel $\mathrm{Lie}(\ker f) = 0$, so $\mathrm{Lie}(\widetilde{f})$ is an isomorphism. Hence, $\widetilde{f}^0 : \widetilde{G}^0 \to \widetilde{H}$ is a map between *connected* Lie groups that is an isomorphism on Lie algebras, so it is surjective and its kernel Γ is also 0-dimensional and central. Thus, as we saw on HW5 Exercise 4(iii), there is a surjective homomorphism $1 = \pi_1(\widetilde{H}) \twoheadrightarrow \Gamma$, so $\Gamma = 1$ and hence $\widetilde{f}^0$ is an isomorphism. Composing its inverse with $\mathrm{pr}_1 : \widetilde{G} \to G$ defines an F fitting into the desired commutative diagram, and $\mathrm{Lie}(F)$ is an isomorphism since f and q are isogenies. Thus, F is also an isogeny.

It remains to prove the uniqueness of an F fitting into such a commutative diagram. If $F' : \widetilde{H} \to G$ is a map fitting into the commutative diagram then the equality $f \circ F' = f \circ F$ implies that for all $\widetilde{h} \in \widetilde{H}$ we have $F'(\widetilde{h}) = \phi(\widetilde{h})F(\widetilde{h})$ for a unique $\phi(\widetilde{h}) \in \ker f$. Since $\ker f$ is central in G, it follows that $\phi : \widetilde{H} \to \ker f$ is a homomorphism since F and F' are, and ϕ is continuous since F' and F are continuous (and G is a topological group). But $\ker f$ is discrete and $\widetilde{H}$ is connected, so ϕ must be constant and therefore trivial; i.e., $F' = F$. □

Proposition Q.2.2. *Consider an exact sequence of compact connected Lie groups $1 \to S \to G \to H \to 1$ with S a central torus in G and H non-commutative of rank 1.*

1. *The commutator subgroup G' is closed in G and $G' \to H$ is an isogeny, with the given exact sequence split group-theoretically if and only if $G' \to H$ is an isomorphism, in which case there is a spliting via $S \times H = S \times G' \to G$ (using multiplication).*
2. *The isogeny $G' \to H$ is an isomorphism if $H \simeq \mathrm{SU}(2)$ or $G' \simeq \mathrm{SO}(3)$, and otherwise $S \cap G'$ coincides with the order-2 center μ of $G' \simeq \mathrm{SU}(2)$ and $G = (S \times G')/\mu$*

Proof. We know that H is isomorphic to either $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$. First we treat the case $H = \mathrm{SU}(2)$, and then we treat the case $H = \mathrm{SO}(3)$ via a pullback argument to reduce to the case of $\mathrm{SU}(2)$. Assuming $H = \mathrm{SU}(2)$, we get an exact sequence of Lie algebras

$$0 \to \mathfrak{s} \to \mathfrak{g} \to \mathfrak{su}(2) \to 0 \tag{Q.2.1}$$

with $\mathfrak{s}$ in the Lie-algebra center $\ker \mathrm{ad}_{\mathfrak{g}}$ of $\mathfrak{g}$ since $\mathrm{ad}_{\mathfrak{g}} = \mathrm{Lie}(\mathrm{Ad}_G)$ and S is central in G.

We shall prove that this is uniquely split as a sequence of Lie algebras This says exactly that there is a unique Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ such that $\mathfrak{g}' \to \mathfrak{h}$ is an isomorphism, as then the vanishing of the bracket between $\mathfrak{s}$ and $\mathfrak{g}'$ implies that $\mathfrak{g} = \mathfrak{s} \times \mathfrak{g}'$ as Lie algebras, providing the unique splitting of the sequence (Q.2.1) using the isomorphism of $\mathfrak{g}'$ onto $\mathfrak{h}$.

Since $H \simeq \mathrm{SU}(2)$, it follows that $\mathfrak{h} \simeq \mathfrak{su}(2)$ is its own commutator subalgebra (see HW4 Exercise 3(ii)). Hence, if there is to be a splitting of (Q.2.1) as a direct product of $\mathfrak{s}$ and a Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ mapping isomorphically onto $\mathfrak{h}$ then *necessarily* $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. Our splitting assertion for Lie algebras therefore amounts to the claim that the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ maps isomorphically onto $\mathfrak{h}$. This isomorphism property is something which is necessary and sufficient to check after extension of scalars to $\mathbf{C}$, so via the isomorphism $\mathfrak{su}(2)_{\mathbf{C}} \simeq \mathfrak{sl}_2(\mathbf{C})$ it is sufficient to show that over a field k of characteristic 0 (such as $\mathbf{C}$) *any* "central extension" of Lie algebras

$$0 \to \mathfrak{s} \to \mathfrak{g} \to \mathfrak{sl}_2 \to 0 \tag{Q.2.2}$$

(i.e., $\mathfrak{s}$ is killed by $\mathrm{ad}_{\mathfrak{g}}$) is split, as once again this is precisely the property that $[\mathfrak{g}, \mathfrak{g}]$ maps isomorphically onto $\mathfrak{sl}_2$ (which is its own commutator subalgebra, by HW4 Exercise 3(iii)).

Now we bring in the representation theory of $\mathfrak{sl}_2$. Consider the Lie algebra representation

$$\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) := \mathrm{End}(\mathfrak{g}).$$

This factors through the quotient $\mathfrak{g}/\mathfrak{s} = \mathfrak{sl}_2$ since $\mathfrak{s}$ is central in $\mathfrak{g}$, and so as such defines a representation of $\mathfrak{sl}_2$ on $\mathfrak{g}$ lifting the adjoint representation of $\mathfrak{sl}_2$ on itself (check!). Note that since $\mathfrak{s}$ is in the Lie-algebra center of $\mathfrak{g}$, it is a direct sum of copies of the trivial representation (the action via "zero") of $\mathfrak{sl}_2$, so (Q.2.2) presents the $\mathfrak{sl}_2$-representation $\mathfrak{g}$ as an extension of the irreducible adjoint representation by a direct sum of copies of the trivial representation.

By the complete reducibility of the finite-dimensional representation theory of $\mathfrak{sl}_2$, we conclude that the given central extension of Lie algebras admits an $\mathfrak{sl}_2$-equivariant splitting as a representation space, which is to say that there is a k-linear subspace $V \subset \mathfrak{g}$ stable under the $\mathfrak{sl}_2$-action that is a linear complement to $\mathfrak{s}$. We need to show that V a Lie subalgebra of $\mathfrak{g}$. By the very definition of the $\mathfrak{sl}_2$-action on $\mathfrak{g}$ via the *central quotient* presentation $\mathfrak{g}/\mathfrak{s} \simeq \mathfrak{sl}_2$, it follows that V is stable under the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$ of $\mathfrak{g}$ on itself, so certainly V is stable under Lie bracket against itself (let alone against the entirety of $\mathfrak{g}$).

Now returning to (Q.2.1) that we have split, we have found a copy of $\mathfrak{h} = \mathfrak{su}(2)$ inside $\mathfrak{g}$ lifting the quotient $\mathfrak{h}$ of $\mathfrak{g}$. But H is *connected* and $\pi_1(H) = 1$, so by Appendix H this inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ "integrates" to a Lie group homomorphism $H \to G$, and its composition $H \to H$ with the quotient map $G \to H$ is the identity map (as that holds on the level of Lie algebras). Thus, we have built a Lie group section $H \to G$ to the quotient map, and since S is central in G it follows that the multiplication map $S \times H \to G$ is a Lie group homomorphism. This latter map is visibly bijective, so it is an isomorphism of Lie groups. This provides a splitting when $H = \mathrm{SU}(2)$, and in such cases the commutator subgroup G' of G is visibly this direct factor H (forcing uniqueness of the splitting) since S is central and $\mathrm{SU}(2)$ *is its own commutator subgroup* (HW7, Exercise 3(i)). In particular, G' is closed in G.

For the remainder of the proof we may assume $H = \mathrm{SO}(3)$. Let $\widetilde{H} = \mathrm{SU}(2)$ equipped with the degree-2 isogeny $q : \widetilde{H} \to H$ in the usual manner. Now form the pullback central extension of Lie groups (see HW7 Exercise 4 for this pullback construction)

$$\begin{tikzcd} 1 \arrow[r] & S \arrow[r] \arrow[d, "\mathrm{id}"'] & E \arrow[r] \arrow[d] & \widetilde{H} \arrow[r] \arrow[d, "q"] & 1 \\ 1 \arrow[r] & S \arrow[r] & G \arrow[r] & H \arrow[r] & 1 \end{tikzcd}$$

We may apply the preceding considerations to the top exact sequence as long as E is compact and connected. Since E is an S-fiber bundle over $\widetilde{H}$, E inherits connectedness from $\widetilde{H}$ and S, and likewise for compactness. We conclude that the top exact sequence is *uniquely split*. In particular, the commutator subgroup E' is closed in E and maps isomorphically onto $\widetilde{H} = \mathrm{SU}(2)$.

The compact image of E' in G is visibly the commutator subgroup, so G' is closed with the surjective $E' \to G'$ sandwiching G' in the middle of the degree-2 covering $\widetilde{H} \to H$! Thus, the two maps $\widetilde{H} = E' \to G'$ and $G' \to H$ are isogenies whose degrees have product equal to the degree 2 of $\widetilde{H}$ over H. A degree-1 isogeny is an isomorphism, so either $G' \to H$ is an isomorphism or $G' \to H$ is identified with $\widetilde{H} \to H$ (and in this latter case the identification "over H" is unique, by Lemma Q.2.1).

Since $G' \to H = \mathrm{SO}(3)$ is an isogeny, the maximal tori of G' map isogenously onto those of H and so have dimension 1. Hence, abstractly G' is isomorphic to $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. In the latter case G' has trivial center (as $\mathrm{SO}(3)$ has trivial center; see HW7 Exercise 1(iii)), so the isogeny $G' \to H$ cannot have nontrivial kernel and so must be an isomorphism. In other words, G' is isomorphic to $\mathrm{SO}(3)$ abstractly as Lie groups if and only if the isogeny $G' \to H$ is an isomorphism. In such cases, the multiplication homomorphism of Lie groups $S \times G' \to G$ is visibly bijective, hence a Lie group isomorphism, so the given exact sequence is split as Lie groups (and the splitting is unique since $G = \mathrm{SO}(3)$ is its own commutator subgroup).

Now we may and do suppose G' is abstractly isomorphic to $\mathrm{SU}(2)$ as Lie groups, so the isogeny $G' \to H \simeq \mathrm{SO}(3)$ must have nontrivial central kernel, yet $\mathrm{SU}(2)$ has center $\{\pm 1\}$ of order 2. Hence, the map $G' \to H$ provides a specific identification of H with the quotient $\mathrm{SU}(2)/\{\pm 1\} = \mathrm{SO}(3)$ of $\mathrm{SU}(2)$ modulo its center. In these cases the given exact sequence cannot be split even group-theoretically, since a group-theoretic splitting of G as a direct product of the commutative S and the perfect H would force the commutator subgroup of G to coincide with this copy of the centerless H, contradicting that in the present circumstances $G' = \mathrm{SU}(2)$ has nontrivial center.

Consider the multiplication map $S \times G' \to G$ that is not an isomorphism of groups (since we have seen that there is no group-theoretic splitting of the given exact sequence). This is surjective since $G' \to H = G/S$ is surjective, so its kernel must be nontrivial. But the kernel is $S \cap G'$ anti-diagonally embedded via $s \mapsto (s, 1/s)$, and this has to be a nontrivial *central* subgroup of $G' = \mathrm{SU}(2)$. The only such subgroup is the order-2 center (on which inversion has no effect), so we are done. $\square$

APPENDIX R. EXISTENCE OF THE COROOT

R.1. **Motivation.** Let G be a non-commutative connected compact Lie group, T a maximal torus in G, and $a \in \Phi(G,T)$ a root. Let $T_a = (\ker a)^0$ be the codimension-1 subtorus of T killed by $a : T \twoheadrightarrow S^1$, so $Z_G(T_a)$ is a connected closed subgroup containing T such that $\Phi(Z_G(T_a),T) = \{\pm a\}$.

We have seen in class that $Z_G(T_a)$ is the almost direct product of its maximal central torus T_a and its closed commutator subgroup $G_a := Z_G(T_a)'$ that is either SU(2) or SO(3) and has as a 1-dimensional maximal torus $T'_a := T \cap G_a$. Any element of G_a centralizes T_a and so if it normalizes T'_a then it normalizes $T_aT'_a = T$. Thus, $N_{G_a}(T'_a) \subset N_G(T)$, and since all elements of G_a centralize T_a it follows that

$$N_{G_a}(T'_a) \cap T = N_{G_a}(T'_a) \cap Z_G(T) = N_{G_a}(T'_a) \cap Z_G(T'_a) = Z_{G_a}(T'_a) = T'_a.$$

In other words, we have an inclusion $j : W(G_a, T'_a) \hookrightarrow W(Z_G(T_a), T)$.

The action of $W(Z_G(T_a),T)$ on $\mathrm{X}(T)_{\mathbf{Q}} = \mathrm{X}(T_a)_{\mathbf{Q}} \oplus \mathrm{X}(T'_a)_{\mathbf{Q}}$ respects the direct sum decomposition since all elements of $Z_G(T_a)$ normalizing T certainly centralize T_a and normalize $T \cap Z_G(T_a)' =: T'_a$. Thus, the action of $W(Z_G(T_a),T)$ on $\mathrm{X}(T)$ preserves $\mathrm{X}(T'_a)$ and is determined by its effect on this $\mathbf{Z}$-line, so we have an injection

$$W(Z_G(T_a),T) \hookrightarrow \mathrm{GL}(\mathrm{X}(T'_a)) = \mathrm{GL}(\mathbf{Z}) = \mathbf{Z}^\times = \{\pm 1\}$$

and it is straightforward to check (do it!) that this is compatible via j with the natural action $W(G_a, T'_a) \hookrightarrow \mathrm{GL}(\mathrm{X}(T'_a)) = \mathbf{Z}^\times$. In other words, we have compatibly

$$W(G_a, T'_a) \subset W(Z_G(T_a),T) \subset \{\pm 1\}.$$

But by case-checking for $G_a = \mathrm{SU}(2)$ and $G_a = \mathrm{SO}(3)$ with specific maximal tori, we saw in class that $W(G_a, T'_a)$ has order 2. Hence, by squeezing, $W(Z_G(T_a),T)$ also has order 2. Explicitly, $W(Z_G(T_a),T) = \{1, r_a\}$ for an element r_a of order 2 represented by $n_a \in N_G(T)$ that centralizes the codimension-1 torus T_a and induces inversion on its 1-dimensional isogeny-complement T'_a.

As we explained in class, the effect of the reflection r_a on $\mathrm{X}(T)_{\mathbf{Q}} = \mathrm{X}(T_a)_{\mathbf{Q}} \oplus \mathrm{X}(T'_a)_{\mathbf{Q}}$ is the identity on the hyperplane $\mathrm{X}(T_a)_{\mathbf{Q}}$ and is negation on the line $\mathrm{X}(T'_a)_{\mathbf{Q}}$ that contains a (as a is trivial on $T_a = (\ker a)^0$, so a has vanishing component along the hyperplane factor $\mathrm{X}(T_a)_{\mathbf{Q}}$ of $\mathrm{X}(T)_{\mathbf{Q}}$). Hence, there is a unique linear form $\ell_a : \mathrm{X}(T)_{\mathbf{Q}} \to \mathbf{Q}$ such that $r_a(x) = x - \ell_a(x)a$. The $\mathbf{Z}$-dual of $\mathrm{X}(T)$ is the cocharacter lattice $\mathrm{X}_*(T)$ via the perfect pairing

$$\langle \cdot, \cdot \rangle : \mathrm{X}(T) \times \mathrm{X}_*(T) \to \mathrm{End}(S^1) = \mathbf{Z}$$

defined by $\langle \chi, \lambda \rangle = \chi \circ \lambda$, so we can thereby identify the $\mathbf{Q}$-dual of $\mathrm{X}(T)_{\mathbf{Q}}$ with $\mathrm{X}_*(T)_{\mathbf{Q}}$.

R.2. **Main result.** The key point in the story is that $\ell_a : \mathrm{X}(T)_{\mathbf{Q}} \to \mathbf{Q}$ carries $\mathrm{X}(T)$ into $\mathbf{Z}$, which is to say that it lies in the $\mathbf{Z}$-dual $\mathrm{X}_*(T)$ of $\mathrm{X}(T)$ inside the $\mathbf{Q}$-dual of $\mathrm{X}(T)_{\mathbf{Q}}$. In other words:

Proposition R.2.1. *There exists a unique $a^\vee : S^1 \to T$ in $\mathrm{X}_*(T)$ such that $r_a(x) = x - \langle x, a^\vee \rangle a$ for all $x \in \mathrm{X}(T)$. Moreover, $a^\vee$ is valued in the 1-dimensional subtorus T'_a.*

We call $a^\vee$ the *coroot* attached to a.

Proof. The uniqueness is clear, since the content of the existence of $a^\vee$ is precisely that $\ell_a \in \mathrm{X}_*(T)_{\mathbf{Q}}$ happens to lie inside $\mathrm{X}_*(T)$ (and as such is then renamed as $a^\vee$). To prove existence, we shall give two proofs: one abstract and one by computing with $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ (depending on what $G_a = Z_G(T_a)'$ is).

For the computational proof, let $a' = a|_{T'_a}$. Note that r_a is the identity on the hyperplane $\mathrm{X}(T_a)_{\mathbf{Q}}$ and its restriction to the line $\mathrm{X}(T'_a)$ is exactly through the reflection $r_{a'} \in W(G_a, T'_a)$ (via how we made r_a using the equality $W(G_a, T'_a) = W(Z_G(T_a), T)$!), For any $\lambda \in \mathrm{X}_*(T'_a) \subset \mathrm{X}_*(T)$ certainly $x \mapsto x - \langle x, \lambda \rangle_T a$ is the identity on the hyperplane $\mathrm{X}(T_a)_{\mathbf{Q}}$ and has restriction to $\mathrm{X}(T'_a)_{\mathbf{Q}}$ given by $x \mapsto x - \langle x, \lambda \rangle_{T'_a} a'$, so a solution in $\mathrm{X}_*(T'_a)$ for $r_{a'}$ is also a solution in $\mathrm{X}_*(T)$ for r_a. Hence, we may replace (G, T, a) with (G_a, T'_a, a') to reduce to the case when G is $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$, so $\dim T = 1$ and $\Phi = \{\pm a\}$.

It is harmless to replace a with $-a$ for our purposes (negate the result to handle the other root). It is also sufficient to treat a single maximal torus (due to the Conjugacy Theorem!). The desired formula $r_a(x) = x - \langle x, a^\vee \rangle a$ on $\mathrm{X}(T)_{\mathbf{Q}}$ for some $a^\vee \in \mathrm{X}_*(T)$ is a comparison of linear endomorphisms of a 1-dimensional $\mathbf{Q}$-vector space. Thus, it suffices to verify such a formula on a single nonzero element. We may take a to be that element, and since r_a is negation on $\mathrm{X}(T)_{\mathbf{Q}}$, our task comes down to finding $a^\vee : S^1 \to T$ such that $\langle a, a^\vee \rangle = 2$. But $T \simeq S^1$, so $a : T \to S^1$ can be identified with a nonzero endomorphism of S^1 (so $\ker a = \mu_n$ for some $n \geq 1$). Our task is to show squaring on S^1 factors through a; equivalently, $\ker a$ has order 1 or 2. By inspection, for $\mathrm{SU}(2)$ the roots have kernel of order 2. The quotient map $\mathrm{SU}(2) \to \mathrm{SO}(3)$ induces an isomorphism on Lie algebras and a degree-2 isogeny $T \to \overline{T}$ between maximal tori, so the roots $\overline{T} \to S^1$ for $\mathrm{SO}(3)$ are isomorphisms (!).

Now we give a conceptual proof. The key idea is to consider not just the group $Z_G(T_a) = Z_G((\ker a)^0)$ but also the centralizer $Z_G(\ker a)$ of the entire kernel, or rather its identity component $Z_G(\ker a)^0$. This contains T as a maximal torus too, and T_a as a central subtorus (therefore maximal as such), so we have a closed subgroup inclusion

$$Z_G(\ker a)^0 / T_a \subset Z_G(T_a)/T_a$$

between rank-1 connected compact subgroups whose maximal torus T/T_a is *not* central (as $\mathrm{Lie}(Z_G(\ker a)^0/T_a) = \mathrm{Lie}(Z_G(\ker a))/\mathrm{Lie}(T_a)$ supports the $\pm a$-weight spaces after complexification, since the possibly disconnected $\ker a$ certainly acts trivially on those weight spaces!). These *connected* groups have dimension 3, so the inclusion between them is an *equality*.

For any $n \in N_{Z_G(T_a)}(T)$, its conjugation action on T is unaffected by changing it by multiplication against an element of T, such as against an element of T_a. Hence, the equality of 3-dimensional groups implies that r_a can be represented by an element $n_a \in N_G(T)$ that centralizes the entirety of $\ker a$! Hence, the endomorphism $f : T \to T$ defined by $t \mapsto t/r_a(t)$ kills the entire $\ker a$, so it factors through the quotient map $a : T \twoheadrightarrow T/(\ker a) = S^1$. In other words, we obtain a Lie group map $a^\vee : S^1 \to T$ such that $f = a^\vee \circ a$. This says that for all $t \in T$, $t/r_a(t) = a^\vee(t^a)$; i.e., $r_a(t) = t/a^\vee(t^a)$. Applying a character $x \in \mathrm{X}(T)$, we get

$$x(r_a(t)) = \frac{x(t)}{x(a^\vee(t^a))} = \frac{x(t)}{(t^a)^{\langle x, a^\vee \rangle}} = \frac{x(t)}{t^{\langle x, a^\vee \rangle a}}.$$

In other words, $x \circ r_a = x - \langle x, a^\vee \rangle a$ in $\mathrm{X}(T)$. By definition of the action of $W(G,T)$ on $\mathrm{X}(T) = \mathrm{Hom}(T, S^1)$ through inner composition, $x \circ r_a$ is the action on x by $r_a^{-1} = r_a \in W(G,T)$, so $r_a(x) = x - \langle x, a^\vee \rangle a$ for all $x \in \mathrm{X}(T)$. □

APPENDIX S. THE DUAL ROOT SYSTEM AND THE **Q**-STRUCTURE ON ROOT SYSTEMS

By our definition, for a root system (V, Φ) we assume V is a finite-dimensional vector space over some field k of characteristic 0. In practice, the only cases of interest are $k = \mathbf{Q}$ (for "algebraic" aspects) and $k = \mathbf{R}$ (for geometric arguments with Weyl chambers later, as well as for applications to compact Lie groups). In this appendix, we explain how the general case reduces to the case with $k = \mathbf{Q}$. Along the way, we introduce and use the notion of the *dual root system*.

Let $\langle \cdot, \cdot \rangle : V \times V^* \to k$ be the evaluation pairing. For each $a \in \Phi$, the uniquely determined reflection $r_a : V \simeq V$ has the form

$$r_a(v) = v - \langle v, a^\vee \rangle a$$

for a unique $a^\vee \in V^*$ (the *coroot* associated to a) that is required to satisfy the integrality condition $a^\vee(\Phi) \subset \mathbf{Z} \subset k$. The condition $r_a(a) = -a$ forces $\langle a, a^\vee \rangle = 2$; in particular, $a^\vee \neq 0$. We saw in class that $a^\vee$ uniquely determines a, so the set $\Phi^\vee \subset V^* - \{0\}$ of coroots is in bijection with Φ via $a \mapsto a^\vee$. We define the reflection $r_{a^\vee} = (r_a)^* : V^* \simeq V^*$ to be dual to r_a (i.e., $r_{a^\vee}(\ell) = \ell \circ r_a$); this *is* a reflection since it is dual to a reflection. More specifically:

$$r_{a^\vee}(v^*) = v^* - \langle a, v^* \rangle a^\vee$$

since evaluating the left side on $v' \in V$ gives

$$v^*(r_a(v')) = v^*(v' - \langle v', a^\vee \rangle a) = v^*(v') - \langle v', a^\vee v^*(a) = v^*(v') - \langle a, v^* \rangle a^\vee(v'),$$

which is the right side evaluated on v'.

We aim to show that $(V^*, \Phi^\vee)$ equipped with these dual reflections is a root system (called the *dual root system*). This requires establishing two properties: $\Phi^\vee$ spans V^* over k, and $r_{a^\vee}(\Phi^\vee) = \Phi^\vee$ for all $a \in \Phi$. For this latter equality, we will actually prove the more precise result that $r_{a^\vee}(b^\vee) = r_a(b)^\vee$ for all $a, b \in \Phi$. The spanning property turns out to lie a bit deeper for general k, and is tied up with proving that root systems have canonical **Q**-structures.

S.1. **Coroot reflections and spanning over Q.**

Let's first show that $r_{a^\vee}(\Phi^\vee) = \Phi^\vee$ for all $a \in \Phi$, or more precisely:

Proposition S.1.1. *For all $a, b \in \Phi$, $r_{a^\vee}(b^\vee) = r_a(b)^\vee$.*

Proof. By the unique characterization of the coroot associated to a root, we want to show that the linear form $r_{a^\vee}(b^\vee) \in V^*$ satisfies the condition that

$$r_{r_a(b)}(x) = x - \langle x, r_{a^\vee}(b^\vee) \rangle r_a(b)$$

for all $x \in V$. To do this, we seek another expression for $r_{r_a(b)}$.

Let Γ be the *finite* subgroup of elements of $\mathrm{GL}(V)$ that preserve the finite spanning set Φ, so all reflections r_c lie in Γ ($c \in \Phi$). Inside Γ there is at most one reflection negating a given line (since Γ is finite and k has characteristic 0), so since $r_b \in \Gamma$ is uniquely determined by the property that it is a reflection negating the line kb, it follows that $r_a r_b r_a^{-1} \in \Gamma$ is uniquely determined as being a reflection that negates $r_a(kb) = kr_a(b)$. But $r_{r_a(b)} \in \Gamma$ is also such an element, so we conclude that

$$r_{r_a(b)} = r_a r_b r_a^{-1} = r_a r_b r_a.$$

Evaluating on $v \in V$, we get that

$$v - \langle v, r_a(b)^\vee \rangle r_a(b) = r_a(r_b(r_a(v))).$$

Applying $r_a = r_a^{-1}$ to both sides, we get

$$r_a(v) - \langle v, r_a(b)^\vee \rangle b = r_b(r_a(v)) = r_a(v) - \langle r_a(v), b^\vee \rangle b.$$

Hence, $\langle v, r_a(b)^\vee \rangle = \langle r_a(v), b^\vee \rangle = \langle v, r_{a^\vee}(b^\vee) \rangle$ by definition of the dual reflection $r_{a^\vee} := (r_a)^*$. This holds for all $v \in V$, so $r_{a^\vee}(b^\vee) = r_a(b)^\vee$. □

To prove that $\Phi^\vee$ spans V^*, we will first give an argument that works when $k = \mathbf{Q}$, and then we will bootstrap that to the general case.

Proposition S.1.2. *If $k = \mathbf{Q}$ then $\Phi^\vee$ spans V^*. In particular, $(V^*, \Phi^\vee)$ is a root system when $k = \mathbf{Q}$.*

The proof we give works verbatim over $\mathbf{R}$, or any ordered field at all. (The special roles of $\mathbf{Q}$ and $\mathbf{R}$ is that they admit unique order structures as fields.)

Proof. Choose a positive-definite quadratic form $q : V \to \mathbf{Q}$, and by averaging this over the finite Weyl group $W = W(\Phi)$ we arrive at a positive-definite q that is W-invariant. Hence, the associated symmetric bilinear form $B = B_q$ is W-invariant in the sense that $B_q(w.v, w.v') = B_q(v, v')$ for all $v, v' \in V$ and $w \in W$, and it is *non-degenerate* since $B_q(v, v) = 2q(v) > 0$ for $v \neq 0$. This bilinear form defines a W-equivariant isomorphism $V \simeq V^*$ via $v \mapsto B_q(v, \cdot) = B_q(\cdot, v)$.

For each root a, the reflection $r_a : V \simeq V$ induces negation on the line L spanned by a, so it restricts to an automorphism of the B_q-orthogonal hyperplane $H = L^\perp$. But $L \cap L^\perp = 0$ since q is positive-definite, so addition $L \oplus L^\perp \to V$ is an isomorphism. Since $L^\perp$ is characterized in terms of L and B_q, and W leaves B_q invariant, so r_a leaves B_q invariant, the r_a-stability of L implies the same for $L^\perp$. But the eigenvalue -1 for r_a is already accounted for on L, so the finite-order automorphism of $L^\perp$ arising from r_a has only 1 as an eigenvalue, and hence $r_a|_{L^\perp}$ must be the identity. Writing $v \in V$ as $v = v' + ca$ for $v' \in L^\perp$ and a scalar c,

$$r_a(v) = v' - ca = (v' + ca) - 2ca = v - 2ca$$

and $B_q(v, a) = B_q(v', a) + cB_q(a, a) = cB_q(a, a)$ with $B_q(a, a) \neq 0$.

We conclude that $c = B_q(v, a)/B_q(a, a)$, so

$$r_a(v) = v - 2ca = v - \frac{2B_q(v, a)}{B_q(a, a)} a = v - B_q(v, a')a$$

where $a' := 2a/B_q(a, a)$. In other words, the identification of V^* with V via B_q identifies $a^\vee \in V^*$ with $a' = 2a/B_q(a, a) \in V$. This is traditionally written as:

$$a^\vee = \frac{2a}{(a|a)}$$

with $(\cdot|\cdot)$ denoting a positive-definite symmetric bilinear form on V that is W-invariant (and the role of this *choice* of such positive-definite form in the identification of V^* with V has to be remembered when using that formula!).

Now we're ready to show $\Phi^\vee$ spans V^*. If not, its span is contained in some hyperplane in V^*, and a hyperplane in V^* is nothing more or less than the set of linear forms that kill a specified nonzero $v \in V$. Hence, there would exist some nonzero $v \in V$ such that $\langle v, a^\vee \rangle = 0$ for all $a \in \Phi$. The identification of V^* with V via B_q carries the evaluation pairing between V^* and V over to the symmetric bilinear form B_q, so the coroot $a^\vee$ is brought to $a' = 2a/B_q(a,a)$. Thus,

$$0 = \langle v, a^\vee \rangle = B_q(v, a') = \frac{2B_q(v,a)}{B_q(a,a)}$$

for all $a \in \Phi$. In other words, v is B_q-orthogonal to all $a \in \Phi$. But Φ spans V (!), so v is B_q-orthogonal to the entirety of V, a contradiction since $v \neq 0$ and B_q is non-degenerate. $\square$

S.2. **The spanning property over general** k**.** The verification of the root system properties for $(V^*, \Phi^\vee)$ when k is general shall now be deduced from the settled case $k = \mathbf{Q}$. The trick is to introduce an auxiliary $\mathbf{Q}$-structure, apply the result over $\mathbf{Q}$ there, and then return to the situation over k. To that end, let $V_0 = \mathbf{Q}\Phi$ denote the $\mathbf{Q}$-span of Φ inside V, and write a_0 to denote a viewed inside V_0. Also write $\Phi_0 \subset V_0$ to denote Φ viewed inside V_0.

Note that since $r_a(\Phi) = \Phi$ for all a, we see that $r_a(V_0) = V_0$ for all a. Likewise, by the integrality hypothesis, $a^\vee(\Phi) \subset \mathbf{Z} \subset \mathbf{Q}$ for all a, so $a^\vee(V_0) \subset \mathbf{Q}$ for all a. Hence, we get $\mathbf{Q}$-linear form $a_0^\vee : V_0 \to \mathbf{Q}$ that is the restriction of $a^\vee$, and for all $v_0 \in V_0$ we have

$$r_a(v_0) = v_0 - \langle v_0, a^\vee \rangle a_0 = v_0 - \langle v_0, a_0^\vee \rangle a_0.$$

Thus, (V_0, Φ_0) is a root system over $\mathbf{Q}$ with associated reflections $r_{a_0} = r_a|_{V_0}$ for all $a_0 \in \Phi_0 = \Phi$, so the associated coroot is $a_0^\vee$. It follows from the settled case over $\mathbf{Q}$ that we have a dual root system $(V_0^*, \Phi_0^\vee)$ where V_0^* denotes the $\mathbf{Q}$-dual of V_0 and $\Phi_0^\vee$ is the set of $\mathbf{Q}$-linear forms $a_0^\vee$. In particular, the elements $a_0^\vee \in V_0^*$ are a spanning set over $\mathbf{Q}$ by the settled case over $\mathbf{Q}$!

Consider the natural k-linear map $f : k \otimes_{\mathbf{Q}} V_0 \to V$. This carries $1 \otimes a_0$ to a for all $a \in \Phi$, so it is surjective (since Φ spans V over k). Moreover, the k-linear form $a^\vee : V \to k$ is compatible with the scalar extension of $a_0^\vee : V_0 \to \mathbf{Q}$ under this surjection since we can compare against the sets Φ_0 and Φ that compatibly span V over k and V_0 over $\mathbf{Q}$ respectively. Once we show that f is also injective, it follows that $a^\vee$ is identified with the scalar extension of $a_0^\vee$, so in fact the initial root system (V, Φ) is obtained by scalar extension from the root system (V_0, Φ_0) over $\mathbf{Q}$. (The notion of *scalar extension* for root systems is defined in an evident manner.) In this sense, *every root system will have a canonical* $\mathbf{Q}$*-structure*. This is why the case $k = \mathbf{Q}$ is essentially the "general" case (though it is very convenient to perform certain later arguments after scalar extension from $\mathbf{Q}$ to $\mathbf{R}$).

Why is f also injective? It is equivalent to show that the dual k-linear map $f^* : V^* \to k \otimes_{\mathbf{Q}} V_0^*$ is surjective. In other words, we seek a spanning set in V^* over k that is carried to a spanning set for $k \otimes_{\mathbf{Q}} V_0^*$ over k. Well, $f^*(a^\vee) = a^\vee \circ f = 1 \otimes a_0^\vee$ (the compatibility of $a^\vee$ and $a_0^\vee$ that has already been noted), so it remains to recall that the coroots $a_0^\vee$ in V_0^* are a spanning set over $\mathbf{Q}$!

Appendix T. Calculation of some root systems

T.1. **Introduction.** The *classical groups* are certain non-trivial compact connected Lie groups with finite center: $\mathrm{SU}(n)$ for $n \geq 2$, $\mathrm{SO}(2m+1)$ for $m \geq 1$, $\mathrm{Sp}(n)$ for $n \geq 1$, and $\mathrm{SO}(2m)$ for $m \geq 2$. For the even special orthogonal groups one sometimes requires $m \geq 3$ (because in HW6 we showed $\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2$, so it is isogenous to a direct product of smaller such groups, whereas in all other cases the group is (almost) simple in the sense that its only proper normal closed subgroups are the subgroups of the finite center).

For each of these groups G, in Appendix N we have specified a "standard" maximal torus T (for the purpose of doing computations) and we explicitly computed $W(G,T) = N_G(T)/T$ as a finite group equipped with its action on $\mathrm{X}(T)$ (or on $\mathrm{X}(T)_{\mathbf{Q}}$ if one wishes to not worry about isogeny issues). For example, in $G = \mathrm{SU}(n)$ we have $T = \{(z_1, \dots, z_n) \in (S^1)^n \mid \prod z_j = 1\}$ and $\mathrm{X}(T) = \mathbf{Z}^n/\Delta(\mathbf{Z})$ (quotient by the diagonally embedded $\mathbf{Z}$) with $W(G,T) = S_n$ acting through coordinate permutation in the usual way.

Remark T.1.1. Working rationally, we can identify $\mathrm{X}(T)_{\mathbf{Q}}$ canonically as a direct summand of the rationalized character lattice $\mathbf{Q}^n$ of the diagonal maximal torus $(S^1)^n$ of $\mathrm{U}(n)$, namely $\mathrm{X}(T)_{\mathbf{Q}} = \{\vec{x} \in \mathbf{Q}^n \mid \sum x_j = 0\}$. This hyperplane maps isomorphically onto the quotient of $\mathbf{Q}^n$ modulo its diagonal copy of $\mathbf{Q}$, but the same is not true for the $\mathbf{Z}$-analogue: the map

$$\{\vec{x} \in \mathbf{Z}^n \mid \sum x_j = 0\} \to \mathbf{Z}^n/\Delta(\mathbf{Z}) = \mathrm{X}(T)$$

is injective with cokernel $\mathbf{Z}/n\mathbf{Z}$ of size n. (This corresponds to the quotient torus $T \twoheadrightarrow T/\mu_n$ modulo the center μ_n of $\mathrm{SU}(n)$; this quotient is a maximal torus of the centerless quotient $\mathrm{SU}(n)/Z_{\mathrm{SU}(n)}$.) But upon dualizing, the $\mathbf{Z}$-hyperplane $\mathrm{X}_*(T) \subset \mathbf{Z}^n$ is the locus defined by $\sum x_j = 0$ (check!).

We have also computed $\Phi(G,T) \subset \mathrm{X}(T)$ and the coroot $a^\vee \in \mathrm{X}_*(T) = \mathrm{Hom}(\mathrm{X}(T), \mathbf{Z})$ associated to each root $a \in \Phi(G,T)$ for $G = \mathrm{SU}(n)$. Explicitly, the roots are the classes modulo $\Delta(\mathbf{Z})$ of the differences $a_i - a_j$ among the standard basis vectors $\{a_1, \dots, a_n\}$ of $\mathbf{Z}^n$ (which conveniently do lie in the $\mathbf{Z}$-hyperplane $\sum x_j = 0$ in $\mathbf{Z}^n$). For $a = a_i - a_j \bmod \Delta(\mathbf{Z})$ the coroot $a^\vee \in \mathrm{X}_*(T) \subset \mathbf{Z}^n$ is $a_i^* - a_j^*$. The standard quadratic form $q = \sum x_i^2$ on $\mathbf{Q}^n$ is invariant under the standard action of S_n, so restricting this to the hyperplane $\sum x_i = 0$ gives an explicit positive-definite quadratic form that is Weyl-invariant. Under this quadratic form the roots have squared-length equal to 2, so the resulting identification $a^\vee = 2a/q(a) = a$ works out very cleanly. In particular, the root system is self-dual. For $n \geq 2$, the root system $\Phi(\mathrm{SU}(n), T)$ of rank $n-1$ is called A_{n-1} (or "type A" if the rank is not essential to the discussion).

In this appendix, we work out analogous computations for the other classical groups, beginning with the case of $\mathrm{SO}(2m)$ ($m \geq 2$). Then we move on to $\mathrm{SO}(2m+1)$ ($m \geq 2$), which is only a tiny bit more work beyond the case of $\mathrm{SO}(2m)$. (We may ignore $\mathrm{SO}(3)$ since it is the central quotient $\mathrm{SU}(2)/\{\pm 1\}$ and we have already handled $\mathrm{SU}(2)$; recall that central isogeny has no effect on the root system $\Phi = \Phi(G,T)$ by Exercise 1 in HW7 – and so it also does not affect $W(\Phi)$, which we shall soon prove is equal to $W(G,T)$ – though it affects the character lattice of the maximal torus as a lattice inside the $\mathbf{Q}$-vector space of the root system.) We finish by treating $\mathrm{Sp}(n)$ for $n \geq 2$; we can ignore $\mathrm{Sp}(1)$ since it is $\mathrm{SU}(2)$ by another name.

Remark T.1.2. The viewpoint of Dynkin diagrams will eventually explain why treating $\mathrm{SO}(3)$ separately from all other odd special orthogonal groups and treating $\mathrm{Sp}(1)$ separately from all other compact symplectic groups is a very natural thing to do. It is not just a matter of bookkeeping convenience.

T.2. **Even special orthogonal groups.** Let $G = \mathrm{SO}(2m)$ with $m \geq 2$. This has Lie algebra $\mathfrak{so}(2m)$ consisting of the skew-symmetric matrices in $\mathfrak{gl}_{2m}(\mathbf{R})$, of dimension $(2m)(2m-1)/2 = m(2m-1)$. We write the quadratic form underlying the definition of G as

$$\sum_{j=1}^{m}(x_j^2 + x_{-j}^2)$$

and view $\mathbf{R}^{2m}$ as having ordered basis denoted $\{e_1, e_{-1}, e_2, e_{-2}, \dots, e_m, e_{-m}\}$. We take the standard maximal torus T to consist of the block matrix consisting of 2×2 rotation matrices r_{θ_j} on the plane $\mathbf{R}e_j \oplus \mathbf{R}e_{-j}$ for $1 \leq j \leq m$. This is a direct product of m circles in an evident manner, so $\mathrm{X}(T) = \mathbf{Z}^m$ on which the index-2 subgroup $W(G,T) \subset (\mathbf{Z}/2\mathbf{Z})^m \rtimes S_m$ acts via S_m-action in the standard way and $(\mathbf{Z}/2\mathbf{Z})^m$ acting through negation on the m coordinate lines.

To compute $\Phi(G,T) \subset \mathbf{Z}^m$ and $\Phi(G,T)^\vee \subset \mathrm{X}_*(T) = \mathrm{X}(T)^* = \mathbf{Z}^m$ we seek to find the weight spaces for the T-action on $\mathfrak{so}(2m)_{\mathbf{C}} \subset \mathfrak{gl}_{2m}(\mathbf{C})$ with nontrivial weights. The total number of roots is $m(2m-1) - m = 2m^2 - 2m$, so once we find that many distinct nontrivial weights we will have found all of the roots.

Let's regard a $2m \times 2m$ matrix as an $m \times m$ matrix in 2×2 blocks, so in this way T naturally sits as a subgroup of the m such blocks along the "diagonal" of $\mathrm{SO}(2m)$, and so likewise for $\mathfrak{t}$ inside the space $\mathfrak{so}(2m)$ of skew-symmetric matrices. We focus our attention on the 2×2 blocks "above" that diagonal region, since the corresponding transposed block below the diagonal is entirely determined via negation-transpose.

For $1 \leq j < j' \leq m$ consider the 2×2 block $\mathfrak{gl}_2(\mathbf{R})$ in the jj'-position within the $2m \times 2m$ matrix. (Such pairs (j, j') exist precisely because $m \geq 2$.) Working inside $\mathfrak{so}(2m) \subset \mathfrak{gl}_{2m}(\mathbf{R})$, the effect of conjugation by

$$(r_{\theta_1}, \dots, r_{\theta_m}) \in (S^1)^m = T$$

on the 2×2 block $\mathfrak{gl}_2(\mathbf{R})$ is readily computed to be

$$M \mapsto [r_{\theta_j}] M [r_{\theta_{j'}}]^{-1}$$

where $[r_\theta] \in \mathrm{GL}_2(\mathbf{R})$ is the standard matrix for counterclockwise rotation by θ on $\mathbf{R}^2$.

Working over $\mathbf{C}$, to find the weight spaces we recall the standard diagonalization of rotation matrices over $\mathbf{C}$:

$$[r_\theta] = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}$$

and note that the elementary calculation

$$\mathrm{diag}(\lambda, 1/\lambda) \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mathrm{diag}(1/\mu, \mu) = \begin{pmatrix} (\lambda/\mu)x & (\lambda\mu)y \\ (\lambda\mu)^{-1}z & (\mu/\lambda)w \end{pmatrix}$$

identifies the weight spaces as the images under $\left(\begin{smallmatrix} 1 & 1 \\ -i & i \end{smallmatrix}\right)$-conjugation of the "matrix entry" lines. In this way, we see that the four nontrivial characters $T \to S^1$ given by $t \mapsto$

$t_j/t_{j'}, t_{j'}/t_j, t_jt_{j'}, (t_jt_{j'})^{-1}$ are roots with respective weight spaces in $\mathfrak{so}(2m)_{\mathbf{C}} \subset \mathfrak{gl}_{2m}(\mathbf{C})$ given inside the 2×2 block in the jj'-position by the respective 1-dimensional $\mathbf{C}$-spans

$$\mathbf{C}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},\ \mathbf{C}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},\ \mathbf{C}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},\ \mathbf{C}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

Varying across all such pairs $j < j'$, we arrive at $4m(m-1)/2 = 2m^2 - 2m$ roots. But this is the number that we were looking for, so

$$\Phi(G,T) = \{\pm a_j \pm a_{j'} \mid 1 \le j < j' \le m\} \subset \mathbf{Z}^m$$

(with all 4 sign options allowed for each (j,j')). Since the standard quadratic form $q = \sum z_j^2$ on $\mathbf{Z}^m$ is invariant under the action of the explicitly determined $W(G,T)$, we can use the formula $a^\vee = 2a/q(a)$ to compute the coroots in the dual lattice $\mathrm{X}_*(T) = \mathbf{Z}^m$: each root $\pm a_j \pm a_{j'}$ has squared-length 2, so $a^\vee = a$ under the self-duality, whence $(\pm a_j \pm a_{j'})^\vee = \pm a_j^* \pm a_{j'}^*$. This determines $\Phi(G,T)^\vee$, and makes evident that the root system is self-dual. This root system of rank m is called D_m (or "type D" if the rank is not essential to the discussion).

Since $\mathrm{SO}(4)$ is an isogenous quotient of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, we have $\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$ (for the evident notion of direct product of root systems: direct sum of vector spaces and disjoint union of roots embedded in their own factor spaces, consistent with direct product of pairs (G,T)). Likewise, $\mathrm{D}_3 = \mathrm{A}_3$ by inspection, encoding an exceptional isomorphism $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ that we might discuss later if time permits. Thus, the most interesting cases for even special orthogonal groups $\mathrm{SO}(2m)$ are $m \ge 4$; the case $m = 4$, which is to say $\mathrm{SO}(8)$, is especially remarkable (because the Dynkin diagram for type D_4 possesses more symmetry than any other Dynkin diagram; see Remark T.2.1).

Inside $\mathrm{X}(T) = \mathbf{Z}^m$ it is easy to check that the $\mathbf{Z}$-span $\mathbf{Z}\Phi$ is the index-2 subgroup $\{\vec{x} \mid \sum x_j \equiv 0 \bmod 2\}$; this index of 2 corresponding to the size of the center μ_2 of $\mathrm{SO}(2m)$. The $\mathbf{Z}$-dual $(\mathbf{Z}\Phi^\vee)' \subset \mathbf{Q}^m$ is $\mathrm{X}(T) + (1/2)(1,1,\dots,1)$, which contains $\mathrm{X}(T)$ with index 2; this index of 2 corresponds to the fact that the simply connected cover $\mathrm{Spin}(2m)$ of $\mathrm{SO}(2m)$ is a degree-2 covering. A very interesting question now arises: the quotient $\Pi := (\mathbf{Z}\Phi^\vee)'/(\mathbf{Z}\Phi)$ has order 4, but is it cyclic of order 4 or not? This amounts to asking if it is 2-torsion or not, which amounts to asking if twice $(1/2)(1,1,\dots,1)$, which is to say $(1,1,\dots,1)$, lies in $\mathbf{Z}\Phi$. This latter vector has coordinate sum m that vanishes mod 2 precisely for m even.

We conclude that Π is cyclic of order 4 when m is odd and is $(\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$ when m is even. By Exercise 1(iii) in HW7, this says that $\mathrm{Spin}(2m)$ has center that is μ_4 (cyclic) when m is odd and is $\mu_2 \times \mu_2$ when m is even. (For example, when $m = 2$ we recall from HW6 that $\mathrm{SO}(4)$ is the quotient of the simply connected $\mathrm{SU}(2) \times \mathrm{SU}(2)$ modulo the diagonal μ_2, so $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ with center visibly $\mu_2 \times \mu_2$. The exceptional isomorphism $\mathrm{SU}(4) \simeq \mathrm{Spin}(6)$ – which we might discuss later in the course if time permits – gives a concrete way to see that for $m = 3$ the center of $\mathrm{Spin}(2m)$ is μ_4.)

It turns that among *all* "simple" compact connected Lie groups with finite center (not just the classical ones), the groups $\mathrm{Spin}(2m)$ for even $m \ge 4$ are the *only* ones whose center is non-cyclic. For any $m \ge 2$, the proper isogenous quotients of $\mathrm{Spin}(2m)$ larger than the centerless quotient $\mathrm{Spin}(2m)/Z_{\mathrm{Spin}(2m)} = \mathrm{SO}(2m)/\langle -1 \rangle$ are the quotients of $\mathrm{Spin}(2m)$ by the proper nontrivial subgroups of $Z_{\mathrm{Spin}(2m)}$, so $\mathrm{SO}(2m)$ is the *unique* such intermediate quotient when m is odd but there are *two additional* intermediate quotients when m is even. This dichotomy in the structure of the center sometimes causes properties of $\mathrm{SO}(2m)$ or $\mathfrak{so}(2m)$ to be sensitive to the parity of m (i.e., $\mathrm{SO}(n)$ can exhibit varying behavior depending on n mod 4, not just n mod 2).

Remark T.2.1. For any connected Lie group G, its universal cover (equipped with a base point $\widetilde{e}$ over the identity e of G) admits a unique compatible structure of Lie group, denoted $\widetilde{G}$; this is called the *simply connected cover* of G and will be discussed in Exercise 3 on HW9 (e.g., if $G = \mathrm{SO}(n)$ then $\widetilde{G} = \mathrm{Spin}(n)$ for $n \geq 3$). It will be shown in that exercise that every homomorphism $G \to H$ between connected Lie groups uniquely lifts to a homomorphism $\widetilde{G} \to \widetilde{H}$ between their universal covers. In particular, $\mathrm{Aut}(G)$ is a naturally a subgroup of $\mathrm{Aut}(\widetilde{G})$ containing the normal subgroup $\widetilde{G}/Z_{\widetilde{G}} = G/Z_G$ of inner automorphisms. The image of $\mathrm{Aut}(G)$ in $\mathrm{Aut}(\widetilde{G})$ consists of the automorphisms of $\widetilde{G}$ that preserve the central discrete subgroup $\ker(\widetilde{G} \twoheadrightarrow G)$ (and so descend to automorphisms of the quotient G).

Understanding the gap between $\mathrm{Aut}(\widetilde{G})$ and $\mathrm{Aut}(G)$ amounts to understanding the effect of the outer automorphism group $\mathrm{Out}(\widetilde{G}) := \mathrm{Aut}(\widetilde{G})/(\widetilde{G}/Z_{\widetilde{G}}) = \mathrm{Aut}(\widetilde{G})/(G/Z_G)$ on the set of subgroups of $Z_{\widetilde{G}}$, especially on the central subgroup $\ker(\widetilde{G} \twoheadrightarrow G)$. For example, when $Z_{\widetilde{G}}$ is finite cyclic then the outer automorphisms must preserve each such subgroup (since a finite cyclic group has each subgroup uniquely determined by its size) and hence $\mathrm{Aut}(G) = \mathrm{Aut}(\widetilde{G})$ for all quotients G of $\widetilde{G}$ by central subgroups in such cases. For example, $\mathrm{Aut}(\mathrm{SO}(2m)) = \mathrm{Aut}(\mathrm{Spin}(2m))$ for odd $m \geq 2$.

A closer analysis is required for even $m \geq 2$ since (as we saw above) $Z_{\mathrm{Spin}(2m)}$ is non-cyclic for even m, but the equality of automorphism groups remains true as for odd m *except* when $m = 4$. The key point (which lies beyond the level of this course) is that the outer automorphism group of any simply connected compact connected Lie group with finite center is naturally isomorphic to the automorphism group of its Dynkin diagram, and in this way one can show that $\mathrm{Out}(\mathrm{Spin}(2m))$ has order 2 for all $m \geq 2$ *except* for $m = 4$ (when it is S_3 of order 6). Using this determination of the outer automorphism group of $\mathrm{Spin}(2m)$, it follows that if $m \geq 2$ with $m \neq 4$ then inside $\mathrm{Aut}(\mathrm{Spin}(2m))$ the subgroup $\mathrm{Spin}(2m)/Z_{\mathrm{Spin}(2m)} = \mathrm{SO}(2m)/Z_{\mathrm{SO}(2m)}$ of inner automorphisms has index 2, whence $\mathrm{Aut}(\mathrm{SO}(2m)) = \mathrm{Aut}(\mathrm{Spin}(2m))$ provided that $\mathrm{SO}(2m)$ has *some* non-inner automorphism.

But there is an easy source of a non-inner automorphism of $\mathrm{SO}(2m)$ for *any* $m \geq 1$: conjugation by the non-identity component of $\mathrm{O}(2m)$! Indeed, if $g \in \mathrm{O}(2m) - \mathrm{SO}(2m)$ and g-conjugation on $\mathrm{SO}(2m)$ is inner then replacing g with an $\mathrm{SO}(2m)$-translate would provide such g that centralizes $\mathrm{SO}(2m)$. A direct inspection shows that for the standard maximal torus T, its centralizer in $\mathrm{O}(2m)$ is still equal to T (the analogue for $\mathrm{O}(2m+1)$ is false since $\mathrm{O}(2m+1) = \mathrm{SO}(2m+1) \times \langle -1 \rangle$), so $\mathrm{Out}(\mathrm{SO}(2m))$ is always nontrivial and hence $\mathrm{Aut}(\mathrm{SO}(2m)) = \mathrm{Aut}(\mathrm{Spin}(2m))$ for all $m \geq 2$ with $m \neq 4$. In contrast, additional work shows that $\mathrm{Aut}(\mathrm{Spin}(8))$ contains $\mathrm{Aut}(\mathrm{SO}(8))$ as a non-normal subgroup of index 3 (underlying the phenomenon called *triality*).

T.3. **Odd special orthogonal groups.** Consider $G = \mathrm{SO}(2m+1)$ with $m \geq 2$, in which a maximal torus T is given by the "standard" maximal torus of $\mathrm{SO}(2m)$ used above. We regard G as associated to the quadratic form $x_0^2 + \sum_{j=1}^{m}(x_j^2 + x_{-j}^2)$ in which $\mathbf{R}^{2m+1}$ has ordered basis

$$\{e_1, e_{-1}, e_2, e_{-2}, \ldots, e_m, e_{-m}, e_0\}.$$

The dimension of G is $(2m+1)(2m)/2 = m(2m+1)$, so we are searching for $2m^2 + m - m = 2m^2$ root spaces inside $\mathfrak{so}(2m+1)_{\mathbf{C}} \subset \mathfrak{gl}_{2m+1}(\mathbf{C})$.

Inside the upper left $2m \times 2m$ submatrix we get $\mathfrak{so}(2m)_{\mathbf{C}}$ in which we have already found $2m^2 - 2m$ roots. We seek $2m$ additional roots, and we will find them inside $\mathfrak{gl}_{2m+1}(\mathbf{C}) = \mathrm{Mat}_{2m+1}(\mathbf{C})$ by looking at the right column and bottom row viewed as vectors in $\mathbf{C}^{2m+1}$. Taking into account the skew-symmetry of $\mathfrak{so}(2m+1) \subset \mathfrak{gl}_{2m+1}(\mathbf{R})$, we may focus on the right column. Since T viewed inside $\mathrm{SO}(2m+1) \subset \mathrm{GL}_{2m+1}(\mathbf{R})$ has lower-right entry equal to 1, the conjugation action of $T \subset \mathfrak{gl}_{2m}(\mathbf{R}) \subset \mathfrak{gl}_{2m+1}(\mathbf{R})$ preserves the right column with trivial action along the bottom entry and action on the remaining part of the right column in accordance accordance with how $\mathrm{GL}_{2m}(\mathbf{R})$ acts on $\mathbf{R}^{2m}$. This provides a direct sum of planes $\mathbf{R}e_j \oplus \mathbf{R}e_{-j}$ on which $t \in (S^1)^m = T$ acts through the standard rotation representation of S^1 on $\mathbf{R}^2$ via the jth component projection $t \mapsto t_j \in S^1$.

To summarize, $\Phi(\mathrm{SO}(2m+1), T)$ is the union of $\Phi(\mathrm{SO}(2m), T)$ along with the additional roots $t \mapsto t^{\pm a_j} := t_j^{\pm 1}$ (the standard basis vectors of $\mathrm{X}(T) = \mathbf{Z}^m$ and their negatives), and the corresponding root spaces are given by the eigenlines $\mathbf{C}(e_j \pm ie_{-j})$ for the standard rotation action of S^1 on $\mathbf{R}e_j \oplus \mathbf{R}e_{-j}$. Explicitly, $e_j \mp ie_{-j}$ spans the root space for the root $\pm a_j : t \mapsto t_j^{\pm 1}$, the dichotomy between i and $-i$ arising from how we view S^1 inside $\mathbf{C}^\times$ for the purpose of regarding a character $T \to S^1$ as giving a homomorphism of T into $\mathbf{C}^\times$.

To compute the additional coroots in $\mathrm{X}_*(T)$, we observe that the standard positive-definite quadratic form $q = \sum z_j^2$ is invariant under $W(\mathrm{SO}(2m+1), T)$ since this Weyl group is the full $(\mathbf{Z}/2\mathbf{Z})^m \rtimes S_m$ that we have already noted (in our treatment of even special orthogonal groups) preserves this quadratic form. Hence, we can use the formula $a^\vee = 2a/q(a)$ to compute the coroots associated to this new roots.

Now we meet a new phenomenon: the additional roots (beyond $\Phi(\mathrm{SO}(2m), T)$) have q-length 1 rather than q-length $\sqrt{2}$ as for all roots in $\Phi(\mathrm{SO}(2m), T)$, so we have two distinct root lengths. The coroot $a^\vee$ is equal to $2a$ in this way for each additional root, or in other words $(\pm a_j)^\vee = \pm 2a_j^*$ in the dual $\mathbf{Z}^m$. Thus, $\Phi(\mathrm{SO}(2m+1), T)$ is *not* a self-dual root system, due to the distinct root lengths: there are $2m$ short roots and $2m^2 - 2m$ long roots. (Strictly speaking, for this to be a valid proof of non-self-duality we have to show that in this case the $\mathbf{Q}$-vector space of Weyl-invariant quadratic forms on $\mathrm{X}(T)_{\mathbf{Q}}$ is 1-dimensional, so the notion of "ratio of root lengths" is intrinsic. This will be addressed in Exercise 1(ii) on HW9.)

The rank-m root system $\Phi(\mathrm{SO}(2m+1), T)$ is called B_m (or "type B" if the rank is not essential to the discussion). For $m = 2$, it recovers the rank-2 example called by the name B_2 in class.

Remark T.3.1. The additional roots $\pm a_j$ ensure that the inclusion $\mathbf{Z}\Phi \subset \mathrm{X}(T)$ is an *equality*, in contrast with $\mathrm{SO}(2m)$. By Exercise 1(iii) in HW7, this implies that $\mathrm{SO}(2m+1)$ has trivial center. (The element $-1 \in \mathrm{O}(2m+1)$ is not in $\mathrm{SO}(2m+1)$ since it has determinant $(-1)^{2m+1} = -1$, so there is no "obvious guess" for nontrivial elements in the center, and the equality of root lattice and character lattice ensures that the center really is trivial.)

The $\mathbf{Z}$-dual $(\mathbf{Z}\Phi^\vee)' \subset \mathrm{X}(T)_{\mathbf{Q}} = \mathbf{Q}^m$ to the coroot lattice coincides with the lattice

$$\mathbf{Z}^m + (1/2)(1,1,\ldots,1)$$

computed for $\mathrm{SO}(2m)$ since the additional coroots $\pm 2a_j^*$ lie in the $\mathbf{Z}$-span of the coroots for $(\mathrm{SO}(2m), T)$. Hence, $(\mathbf{Z}\Phi^\vee)'$ contains $\mathrm{X}(T)$ with index 2, corresponding to the fact that the simply connected $\mathrm{Spin}(2m+1)$ is a degree-2 cover of $\mathrm{SO}(2m+1)$.

T.4. **Symplectic groups.** Finally, we treat $\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{GL}_n(\mathbf{H})$ for $n \geq 2$. Viewed inside $\mathrm{U}(2n) \subset \mathrm{GL}_{2n}(\mathbf{C})$, this is the group of matrices

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \in \mathrm{U}(2n),$$

and the standard maximal torus $T = (S^1)^n$ of G consists of diagonal matrices

$$\mathrm{diag}(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n)$$

for $z_j \in S^1$. The Lie algebra $\mathfrak{sp}(n)$ consists of the matrices

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

with $A, B \in \mathfrak{gl}_n(\mathbf{C})$ satisfying ${}^\top(\overline{A}) = -A$ and ${}^\top B = B$. In particular, it has $\mathbf{R}$-dimension $2n^2 + n$. Since $\dim T = n$, we are looking for $2n^2$ roots.

Working with blocks of $n \times n$ matrices over $\mathbf{C}$, the adjoint action of $t = (z, \overline{z}) = (z, z^{-1}) \in T \subset \mathrm{GL}_{2n}(\mathbf{C})$ (for $z \in (S^1)^n$) on $\mathfrak{sp}(n) \subset \mathfrak{gl}_{2n}(\mathbf{C})$ is given by

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \mapsto \begin{pmatrix} zAz^{-1} & zBz \\ -z^{-1}\overline{B}z^{-1} & z^{-1}\overline{A}z \end{pmatrix}.$$

Beware that in this formula we use the Lie algebra $\mathfrak{sp}(n)$ over $\mathbf{R}$ and not its complexification. This is relevant to the appearance of complex conjugation in the matrix expression, since the $\mathbf{R}$-linear inclusions $\mathbf{C} \to \mathbf{C} \times \mathbf{C}$ via $u \mapsto (u, \pm\overline{u})$ define 2-dimensional $\mathbf{R}$-subspaces and hence after complexification provide 2-dimensional $\mathbf{C}$-subspaces of $\mathbf{C} \otimes_{\mathbf{R}} (\mathbf{C} \times \mathbf{C})$. Thus, the roles of B and $-\overline{B}$, as well as of A and $\overline{A}$, provide twice as much $\mathbf{C}$-dimension in the complexification as one might have naively expected based on reasoning solely inside $\mathfrak{gl}_{2n}(\mathbf{C})$ (where one has to recognize that the maps $u \mapsto (u, \pm\overline{u})$ at the level of certain matrix entries are *not* $\mathbf{C}$-linear).

Consider the B-part. The weight space decomposition there is for the action $(z, B) \mapsto z.B := zBz$ of $(S^1)^n$ on the space of symmetric $n \times n$ matrices over $\mathbf{C}$ *viewed as an* $\mathbf{R}$*-vector space*. This has each "off-diagonal matrix entry" paired with its transposed identical matrix entry as T-stable $\mathbf{R}$-subspace that is a copy of $\mathbf{C}$; the jj'-entry scaled by the character $z \mapsto z_j z_{j'}$ for $1 \leq j \neq j' \leq n$ (regardless of which of j or j' is larger). This is at the $\mathbf{R}$-linear level, and passing to the complexification entails using the $\mathbf{C}$-linear isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathbf{C} \times \mathbf{C}$ defined by $u \otimes u' \mapsto (uu', u\overline{u}')$ (where $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ is a $\mathbf{C}$-vector space via the left tensor factor and $\mathbf{C} \times \mathbf{C}$ is a $\mathbf{C}$-vector space via the diagonal scaling action) which turns the $\mathbf{R}$-linear scaling $u' \mapsto zu'$ on the right tensor factor for $z \in S^1 \subset \mathbf{C}^\times$ into $(y_1, y_2) \mapsto (zy_1, \overline{z}y_2) = (zy_1, (1/z)y_2)$. Hence, each off-diagonal entry of the symmetric B (as a 2-dimensional $\mathbf{R}$-vector space) contributes two root spaces in the complexification $\mathfrak{sp}(n)_{\mathbf{C}}$, with roots $z \mapsto z_j z_{j'}$ and $z \mapsto 1/(z_j z_{j'})$ for $1 \leq j < j' \leq n$. That is $2(n(n-1)/2) = n^2 - n$

roots. The jj-entries of B provide two additional roots in $\mathfrak{sp}(n)_{\mathbf{C}}$, namely $z \mapsto z_j^{\pm 2}$. That provides $2n$ additional roots, for a total of $n^2 + n$ roots.

For the A-part, we are considering the standard conjugation action of $(S^1)^n$ on $\mathfrak{gl}_n(\mathbf{C})$ restricted to the $\mathbf{R}$-subspace of skew-hermitian matrices. We can focus our attention on the strictly upper triangular part, since the diagonal vanishes and the strictly lower triangular part is determined by the strictly upper triangular part via the $\mathbf{R}$-linear negated conjugation operation. This is exactly the content of the computation of the root system for $\mathrm{SU}(n)$ via the identification $\mathfrak{su}(n)_{\mathbf{C}} = \mathfrak{sl}_n(\mathbf{C})$ except that once again we are viewing this as an $\mathbf{R}$-vector space representation of T, so complexifying turns each $\mathbf{C}$-linear eigenline (an $\mathbf{R}$-plane!) for the $\mathbf{C}$-linear action of T on $\mathfrak{sl}_n(\mathbf{C})$ into a pair of root lines in $\mathfrak{sp}(n)_{\mathbf{C}}$ having opposite roots. That is, for $1 \leq j < j' \leq n$ the jj'-entry of A contributes root lines for the weights $z \mapsto z_j/z_{j'}, z_{j'}/z_j$ in the complexfication of $\mathfrak{sp}(n)$. This provides $2((n^2 - n)/2) = n^2 - n$ additional roots. Together with the ones we already found, we have found all $2n^2$ roots.

To summarize, inside $\mathrm{X}(T) = \mathbf{Z}^n$ the set of roots is

$$\Phi(G, T) = \{\pm(a_j + a_{j'}) \,|\, 1 \leq j \leq j' \leq n\} \cup \{\pm(a_j - a_{j'}) \,|\, 1 \leq j < j' \leq n\} \subset \mathbf{Z}^n.$$

The Weyl group is $(\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n$ acting on $\mathrm{X}(T) = \mathbf{Z}^n$ exactly as for $\mathrm{SO}(2n+1)$, so once again the standard quadratic form $q = \sum x_j^2$ on $\mathbf{Z}^m$ is Weyl-invariant and thereby enables us to compute the coroots $a^\vee = 2a/q(a)$ via the induced standard self-duality of $\mathrm{X}(T)_{\mathbf{Q}} = \mathbf{Q}^m$. We conclude that for $1 \leq j \leq n$ and $j < j' \leq n$,

$$(\pm(a_j + a_{j'}))^\vee = \pm(a_j^* + a_{j'}^*), \;\; (\pm(a_j - a_{j'}))^\vee = \pm(a_j^* - a_{j'}^*), \;\; (\pm 2a_j)^\vee = \pm a_j^*$$

inside the dual lattice $\mathrm{X}_*(T) = \mathrm{X}(T)^* = \mathbf{Z}^n$. There are once again two root lengths, as for $\Phi(\mathrm{SO}(2n+1), T)$, but now there are $2n$ long roots and $2n^2 - 2n$ short roots. A bit of inspection reveals that $\Phi(\mathrm{Sp}(n), T)$ is the dual root system to $\Phi(\mathrm{SO}(2n+1), T)$! We call this rank-$n$ root system C_n (or "type C" if the rank is not essential to the discussion).

Remark T.4.1. There is a remarkably special feature of the C_n root systems: there are roots (the long ones) that are nontrivially divisible (in fact, by 2) in $\mathrm{X}(T)$. Among *all* irreducible (and reduced) root systems (V, Φ) over $\mathbf{Q}$, not just the classical types, the only cases in which there is a $\mathbf{Z}$-submodule X intermediate between the root lattice $Q := \mathbf{Z}\Phi$ and the so-called weight lattice $P := (\mathbf{Z}\Phi^\vee)'$ such that some elements of Φ are nontrivially divisible in X is the case of type C_n (allowing $n = 1$, via the convention that $\mathrm{C}_1 = \mathrm{A}_1$ that is reasonable since $\mathrm{Sp}(1) = \mathrm{SU}(2)$) with $\mathrm{X} = P$. This corresponds precisely to the compact symplectic groups $\mathrm{Sp}(n)$ ($n \geq 1$) among *all* "simple" compact connected Lie groups with finite center.

T.5. **Computation of a weight lattice.** Later we will see the topological significance of the *weight lattice* $P = (\mathbf{Z}\Phi^\vee)' \subset \mathrm{X}(T)_{\mathbf{Q}}$ for the root system $(\mathrm{X}(T)_{\mathbf{Q}}, \Phi)$ attached to a connected compact Lie group G with finite center. (Recall that if V is a finite-dimensional $\mathbf{Q}$-vector space and $L \subset V^*$ is a lattice in the dual space then the *dual lattice* $L' \subset V$ consists of those $v \in V$ such that $\ell(v) \in \mathbf{Z}$ for all $\ell \in L$.) We shall see later that P is the character lattice for the simply connected universal cover $\widetilde{G}$ of G (that will be proved to be a compact finite-degree cover of G).

In this final section, for the root system Φ of type $\mathrm{B}_2 = \mathrm{C}_2$ (corresponding to the double cover $\mathrm{Spin}(5)$ of $\mathrm{SO}(5)$) we compute P. Explicitly, Φ viewed inside $\mathbf{R}^2$ equipped with the standard inner product (identifying $\mathbf{R}^2$ with its own dual) consists of the vertices and edge midpoints for a square centered at the origin with sides parallel to the coordinate axes; a is an edge midpoint and b is the vertex counterclockwise around by the angle $3\pi/4$. We have $\|b\|/\|a\| = \sqrt{2}$, $\langle b, a^\vee\rangle = -2$, and $\langle a, b^\vee\rangle = -1$.

Let's scale so that $\|a\| = \sqrt{2}$, as then $a^\vee = 2a/\|a\|^2 = a$ (and likewise for the coroots associated to all short roots). Now $\|b\| = 2$, so $b^\vee = 2b/\|b\|^2 = b/2$ (and likewise for the coroots associated to all long roots). Using the standard dot product, we have $a \cdot b = -2$ (reality check: $-2 = \langle b, a^\vee\rangle = 2(b\cdot a)/\|a\|^2 = b \cdot a$).

The description of the coroots associated to short and long roots (all viewed inside $\mathbf{R}^2$ via the self-duality arising from the standard dot product) gives by inspection that $\mathbf{Z}\Phi^\vee = \mathbf{Z}a + \mathbf{Z}(b/2)$. Hence, the dual lattice $P = (\mathbf{Z}\Phi^\vee)' \subset \mathbf{R}^2$ is the set of vectors $xa + yb$ ($x, y \in \mathbf{R}$) such that $(xa + yb) \cdot a \in \mathbf{Z}$ and $(xa + yb) \cdot (b/2) \in \mathbf{Z}$. This says $2x - 2y =: n \in \mathbf{Z}$ and $-x + 2y := m \in \mathbf{Z}$, or in other words $x = n + m, y = m + n/2$ for $m, n \in \mathbf{Z}$. Equivalently,

$$P = \{(n+m)a + (m + n/2)b \,|\, m, n \in \mathbf{Z}\} = \mathbf{Z}(a + b/2) + \mathbf{Z}(a + b) = \mathbf{Z}a + \mathbf{Z}(b/2).$$

This lattice contains the root lattice $\mathbf{Z}\Phi$ with index 2, by inspection.

Remark T.5.1. The determination of P from Φ can be done by more "combinatorial" means without reference to Euclidean structures or $\mathbf{R}$, using only the relations $\langle b, a^\vee\rangle = -2$ and $\langle a, b^\vee\rangle = -1$. The key point is that since $\{a, b\}$ rationally span the $\mathbf{Q}$-structure for the root system, elements of the dual lattice $P = (\mathbf{Z}\Phi^\vee)'$ are precisely $xa + yb$ for $x, y \in \mathbf{Q}$ such that $n := \langle xa + yb, a^\vee\rangle \in \mathbf{Z}$ and $m := \langle xa + yb, b^\vee\rangle \in \mathbf{Z}$. Since the definitions give $n = 2x - 2y$ and $m = -x + 2y$ respectively, we have arrived at exactly the same equations obtained in the Euclidean considerations above (and solved via elementary algebraic manipulations).

In terms of the visualization with a square, if Φ "corresponds" to a given square S with an edge midpoint a and a vertex b as above (where a is short of length $\sqrt{2}$ and b is long of length 2) then via the self-duality of the ambient plane as used above we have that $\Phi^\vee$ "corresponds" to a square S' centered inside S that is tilted by $\pi/4$ with size $1/\sqrt{2}$ that of S (so half the area of S), where the edge midpoint $a = a^\vee$ is a vertex b' of S' and an edge midpoint a' of S' is given by the vector $b/2$ halfway along b. If we try to iterate this again then we don't get a yet smaller square S'' half the size of S since for the preceding geometric considerations with self-duality of $\mathbf{R}^2$ it was crucial that S has side length $2\sqrt{2}$ (so $a^\vee = a$, etc.); this is consistent with that $(\Phi^\vee)^\vee = \Phi$. More specifically,

$$a'^\vee = 2a'/\|a'\|^2 = 2(b/2)/1^2 = b, \;\; b'^\vee = 2b'/\|b'\|^2 = 2a/(\sqrt{2})^2 = a.$$

Appendix U. Irreducible decomposition of root systems

U.1. **Introduction.** Let (V, Φ) be a root system over a field k of characteristic 0, with $V \neq 0$ (so $\Phi \neq \emptyset$). If it is reducible, then by definition there is a decomposition $\Phi = \Phi_1 \coprod \Phi_2$ so that $V = V_1 \oplus V_2$ with V_i the k-span of Φ_i and each (V_i, Φ_i) a root system. Continuing in this way, eventually we arrive at an expression for (V, Φ) as a direct sum of finitely many *irreducible* root systems. Also, in view of how the canonical $\mathbf{Q}$-structure of a root system is made, it is clear that irreducibility or not of a root system is insensitive to ground field extension and can be checked on the $\mathbf{Q}$-structure.

A (nonzero) root system with connected Dynkin diagram is certainly irreducible (as the diagram of a direct sum of root systems is easily seen to be the disjoint union of the diagrams of the factors). The converse is also true:

Lemma U.1.1. *An irreducible root system has connected Dynkin diagram.*

This connectedness is an ingredient in the classification of possible Dynkin diagrams for irreducible root systems.

Proof. We consider (V, Φ) a root system with disconnected Dynkin diagram. We seek to show that (V, Φ) is reducible. Let B be a basis (which yields the diagram, as does any basis), let B_1 be the subset of B corresponding to the vertices in one connected component of the diagram, and let $B_2 = B - B_1 \neq \emptyset$. Upon choosing a $W(\Phi)$-invariant positive-definite quadratic form $q : V \to \mathbf{Q}$, the elements of B_1 are q-orthogonal to the elements of B_2, so their respective spans V_i are q-orthogonal. Thus, V_i is stable under the reflections on B_i and for $a_i \in B_i$ the reflections in a_1 and a_2 commute. But $W(\Phi)$ is generated by the reflections in B and $\Phi = W(\Phi).B$. It follows that $\Phi = \Phi_1 \coprod \Phi_2$ with $\Phi_i = \Phi \cap V_i$, so (V_i, Φ_i) is a root system (Exercise 3(iii), HW8) and $(V_1, \Phi_1) \times (V_2, \Phi_2) \simeq (V, \Phi)$. $\square$

U.2. **Uniqueness result.** Our main aim is to prove a strong uniqueness result concerning the "irreducible decomposition" of a nonzero root system. Unlike representations of finite groups (in characteristic zero), for which the irreducible subrepresentations are not uniquely determined as subspaces in case of multiplicities, the subspaces will be unique in the case of irreducible decomposition of a root system. More precisely:

Theorem U.2.1. *Let $\{(V_i, \Phi_i)\}_{i \in I}$ be a finite collection of irreducible root systems, and let (V, Φ) be their direct sum (i.e., $V = \oplus V_i$ with $\Phi = \coprod \Phi_i$). If $\{(W_j, \Psi_j)\}_{j \in J}$ is a finite collection of irreducible root sytems and*

$$f : \prod (W_j, \Psi_j) \simeq (V, \Phi)$$

is an isomorphism then there must be a bijection $\sigma : J \simeq I$ and isomorphisms $f_j : (W_j, \Psi_j) \simeq (V_{\sigma(j)}, \Phi_{\sigma(j)})$ whose direct sum coincides with f.

Proof. Let B_j be a basis of (W_j, Ψ_j), so $\coprod B_j$ is carried by f onto a basis B for (V, Φ). Since f is an isomorphism of root systems, so it induces an isomorphism between the associated *finite* Weyl groups, the uniqueness of reflections inside a finite group in terms of the line that is negated implies that f is compatible with the reflections associated to the elements in the bases on both sides. That is, if $a \in B_j$ then f intertwines r_a and $r_{f(a)}$. These reflections determined the Cartan integers among pairs in the bases on both sides, so the connectedness of the Dynkin diagrams for (W_j, Ψ_j) implies that the elements in $f(B_j)$ are "orthogonal" (in the sense of Cartan integers) to those in $f(B_{j'})$ for any $j' \neq j$ whereas any two elements $a, a' \in f(B_j)$ can be linked to each other in finitely many steps $a = a_0, a_1, \dots, a_m = a'$ using elements of $f(B_j)$ in such a way that the Cartan integers n_{a_r, a_s} are nonzero.

As we vary j, it follows that the sets $f(B_j)$ constitute connected components of the diagram for (V, Φ, B). Since the decomposition $\prod (V_i, \Phi_i)$ of (V, Φ) provides a basis for (V, Φ) as a disjoint union of bases for the (V_i, Φ_i)'s and identifies $W(\Phi)$ with $\prod W(\Phi_i)$, we see from the Weyl-transitivity on the set of bases that *every* basis of (V, Φ) is a disjoint union of bases of the (V_i, Φ_i)'s. Thus, B has such a disjoint union decomposition: $B = \coprod B_i$ for a basis B_i of (V_i, Φ_i). The constituents B_i of this decomposition of B underlie the connected components of the diagram for (V, Φ, B) (in view of how the diagram is defined using Cartan integers), yet $\{f(B_j)\}$ is another such decomposition. Hence, the uniqueness of connected component decomposition for a Dynkin diagram implies that $f(B_j) = B_{\sigma(j)}$ for a bijection $\sigma : J \to I$, so f carries the span W_j of B_j onto the span $V_{\sigma(j)}$ of $B_{\sigma(j)}$.

For $a_j \in B_j$ and the associated root $f(a_j) \in B_{\sigma(j)}$, the reflections r_{a_j} and $r_{f(a_j)}$ must be intertwined by f. As we vary j, these reflections respectively generate $W(\Psi_j)$ and $W(\Phi_{\sigma(j)})$. But $W(\Psi_j).B_j = \Psi_j$ and $W(\Phi_{\sigma(j)}).B_{\sigma(j)} = \Phi_{\sigma(j)}$, so $f(\Psi_j) = \Phi_{\sigma(j)}$. In other words, f carries (W_j, Ψ_j) isomorphically onto $(V_{\sigma(j)}, \Phi_{\sigma(j)})$ for all j. $\square$

APPENDIX V. SIZE OF FUNDAMENTAL GROUP

V.1. **Introduction.** Let (V, Φ) be a nonzero root system. Let G be a connected compact Lie group that is semisimple (equivalently, Z_G is finite, or $G = G'$; see Exercise 4(ii) in HW9) and for which the associated root system is isomorphic to (V, Φ) (so $G \neq 1$). More specifically, we pick a maximal torus T and suppose an isomorphism $(V, \Phi) \simeq (\mathrm{X}(T)_{\mathbf{Q}}, \Phi(G, T))$ is chosen. Where can $\mathrm{X}(T)$ be located inside V? It certainly has to contain the *root lattice* $Q := \mathbf{Z}\Phi$ determined by the root system (V, Φ). Exercise 1(iii) in HW7 computes the index of this inclusion of lattices: $\mathrm{X}(T)/Q$ is dual to the finite abelian group Z_G. The root lattice is a "lower bound" on where the lattice $\mathrm{X}(T)$ can sit inside V.

This appendix concerns the deeper "upper bound" on $\mathrm{X}(T)$ provided by the $\mathbf{Z}$-dual

$$P := (\mathbf{Z}\Phi^\vee)' \subset V^{**} = V$$

of the coroot lattice $\mathbf{Z}\Phi^\vee$ (inside the dual root system $(V^*, \Phi^\vee)$). This lattice P is called the *weight lattice* (the French word for "weight" is "poids") because of its relation with the representation theory of any possibility for G (as will be seen in our discussion of the Weyl character formula).

To explain the significance of P, recall from HW9 that semisimplicity of G implies $\pi_1(G)$ is a finite abelian group, and more specifically if $f : \widetilde{G} \to G$ is the finite-degree universal cover and $\widetilde{T} := f^{-1}(T) \subset \widetilde{G}$ (a maximal torus of $\widetilde{G}$) then

$$\pi_1(G) = \ker f = \ker(f : \widetilde{T} \to T) = \ker(\mathrm{X}_*(\widetilde{T}) \to \mathrm{X}_*(T))$$

is dual to the inclusion $\mathrm{X}(T) \hookrightarrow \mathrm{X}(\widetilde{T})$ inside V. Moreover, by HW8, the equality $V = \mathrm{X}(T)_{\mathbf{Q}} = \mathrm{X}(\widetilde{T})_{\mathbf{Q}}$ identifies $\mathrm{X}(\widetilde{T})$ with an intermediate lattice between $\mathrm{X}(T)$ and the weight lattice P. Thus, $\#\pi_1(G) | [P : \mathrm{X}(T)]$, with equality if and only if the inclusion $\mathrm{X}(\widetilde{T}) \subseteq P$ is an equality.

Our aim in this appendix is to prove that $\mathrm{X}(\widetilde{T})$ is "as large as possible":

Theorem V.1.1. *For any (G, T) as above, $\mathrm{X}(\widetilde{T}) = P$.*

This amounts to saying that the $\mathbf{Z}$-dual $\mathbf{Z}\Phi^\vee$ of P is equal to the $\mathbf{Z}$-dual $\mathrm{X}_*(\widetilde{T})$ of the character lattice of $\widetilde{T}$. In [BtD, Ch.,V, §7] you will find a proof of this result which has the same mathematical content as what follows, but is presented in a different style; see [BtD, Ch. V, Prop. 7.16(ii)].

Since $\mathbf{Z}\Phi^\vee$ has as a $\mathbf{Z}$-basis the set $B^\vee = \{a^\vee \,|\, a \in B\}$ of "simple positive coroots" associated to a choice of basis B of (V, Φ), it would follow that the map $\prod_{a \in B} S^1 \to$

$\widetilde{T}$ defined by $(z_a) \mapsto \prod_{a \in B} a^\vee(z_a)$ is an isomorphism. This is "dual" to the fact that $\mathrm{X}(T/Z_G) = Q = \mathbf{Z}\Phi$, which says the map $T/Z_G \to \prod_{a \in B} S^1$ defined by $t \mapsto (t^a)_{a \in B}$ is an isomorphism.

To summarize, upon choosing a basis B for the root system, the set B of simple positve roots is a basis for the character lattice of the "adjoint torus" T/Z_G whereas the set $B^\vee$ of simple positive coroots is a basis for the cocharacter lattice of the torus $\widetilde{T}$. This gives a very hands-on description for the maximal torus at the two extremes (simply connected or centerless).

In view of the containments $Q \subset \mathrm{X}(T) \subset \mathrm{X}(\widetilde{T}) \subset P$ where the first two inclusions have respective indices $\#Z_G$ and $\#\pi_1(G)$, yet another way to express that $\mathrm{X}(\widetilde{T}) \stackrel{?}{=} P$ is

$$\#Z_G \cdot \#\pi_1(G) \stackrel{?}{=} [P : Q]$$

for all semisimple G (or even just one G) with root system (V, Φ). Informally, this says that there is always a balancing act between the size of the center and the size of the fundamental group for semisimple connected compact Lie groups with a specified root system. (Note that $[P : Q]$ is instrisic to the root system (V, Φ), without reference to the lattice $\mathrm{X}(T)$!)

V.2. **An application to isomorphism classes.** This section will not be used later in this appendix or anywhere in the course, but it is sufficiently important in practice and interesting for its own sake that we now digress on it before taking up the proof of Theorem V.1.1.

Note that every sublattice of $\mathrm{X}(\widetilde{T})$ containing $\mathrm{X}(\widetilde{T}/Z_{\widetilde{G}}) = \mathrm{X}(T/Z_G) = Q$ has the form $\mathrm{X}(\widetilde{T}/Z)$ for a unique subgroup $Z \subset Z_{\widetilde{G}}$, and all subgroups Z of the finite group $Z_{\widetilde{G}}$ obviously arise in this way. Thus, once we show that $\mathrm{X}(\widetilde{T}) = P$, so the set of intermediate lattices between $\mathrm{X}(\widetilde{T}) = P$ and Q corresponds to the set of subgroups of P/Q (a set that is intrinsic to the root system), it follows that the set of such subgroups corresponds bijectively to the set of isogenous quotients of $\widetilde{G}$, or equivalently to the set of connected compact Lie groups with a specified universal cover $\widetilde{G}$.

The structure theory of semisimple Lie algebras over $\mathbf{R}$ and $\mathbf{C}$ (beyond the level of this course) implies that the Lie algebra of a semisimple connected compact Lie group is determined up to isomorphism by the root system, so since a connected Lie group that is *simply connected* is determined up to isomorphism by its Lie algebra (due to the Frobenius theorem and the link between π_1 and covering spaces in Exercise 3 of HW9), it follows that $\widetilde{G}$ is uniquely determined up to isomorphism by (V, Φ). Thus, the set of subgroups of the finite group P/Q associated to the root system (V, Φ) parameterizes the set of *all* possibilities for G with root system (V, Φ) without reference to a specific $\widetilde{G}$. (Explicitly, to any G we associate the subgroup $\mathrm{X}(T)/Q \subset P/Q$.)

There is a subtlety lurking in this final "parameterization". The argument shows that the possibilities for (G, T) with a given root system (V, Φ) are labeled by the finite subgroups of $P/Q = \mathrm{Hom}(Z_{\widetilde{G}}, S^1)$, which is to say that as we vary through the subgroups Z of $Z_{\widetilde{G}} \simeq \mathrm{Hom}(P/Q, S^1)$ the quotients $\widetilde{G}/Z$ exhaust all possibilities for G. However, this description rests on a *choice* of how to identify the universal cover of G with a fixed simply connected semisimple compact Lie group $\mathbf{G}$ having a given (V, Φ) as its root system; if $\mathbf{G}$ admits nontrivial automorphisms them in principle there might be several ways to associate a subgroup of P/Q to a possibility for G (via several isogenies $\mathbf{G} \to G$ not related through automorphisms of G)

More specifically, we have not accounted for the possibility that distinct $Z_1 \neq Z_2$ inside $Z_{\widetilde{G}}$ might yield quotients $G_i = \widetilde{G}/Z_i$ that are abstractly isomorphic. Since an isomorphism between connected Lie groups uniquely lifts to an isomorphism between their universal covers (combine parts (iii) and (iv) in Exercise 3 of HW9), the situation we need to look for is an *automorphism* f of $\widetilde{G}$ that descends to an isomorphism $G_1 \simeq G_2$, which is to say $f(Z_1) = Z_2$.

To summarize, the subtlety we have passed over in silence is the possibility that $\mathrm{Aut}(\widetilde{G})$ might act nontrivially on the set of subgroups of the finite center $Z_{\widetilde{G}}$. Obviously the normal subgroup $\mathrm{Inn}(\widetilde{G}) := \widetilde{G}/Z_{\widetilde{G}}$ of inner automorphisms acts trivially on the center, so it is really the effect on $Z_{\widetilde{G}}$ by the outer automorphism group

$$\mathrm{Out}(\widetilde{G}) := \mathrm{Aut}(\widetilde{G})/\mathrm{Inn}(\widetilde{G})$$

that we need to understand. This outer automorphism group for simply connected groups turns out to coincide with the automorphism group of the Dynkin diagram Δ (a fact whose proof requires relating Lie algebra automorphisms to diagram automorphisms, and so lies beyond the level of this course). Since P has a $\mathbf{Z}$-basis that is $\mathbf{Z}$-dual to $B^\vee$, and Q has B as a $\mathbf{Z}$-basis with B the set of vertices of Δ, there is an evident action of $\mathrm{Aut}(\Delta)$ on P/Q which computes exactly the dual of the action of $\mathrm{Out}(G)$ on $Z_{\widetilde{G}}$.

Thus, the uniform answer to the problem of parameterizing the set of isomorphism classes of semisimple compact connected G with a given root system (V, Φ) is the set of $\mathrm{Aut}(\Delta)$-orbits in the set of subgroups of P/Q. This is messy to make explicit when (V, Φ) is reducible: by Exercise 5 on HW9, it corresponds to the case $\widetilde{G} = \prod \widetilde{G}_i$ for several "simple factors" $\widetilde{G}_i$ (all of which are simply connected), and thereby gets mired in unraveling the effect on the set of subgroups of $\prod Z_{\widetilde{G}_i}$ by factor-permutation according to isomorphic $\widetilde{G}_i$'s.

The case of *irreducible* (V, Φ) has a very elegant description, as follows. Note that whenever $Z_{\widetilde{G}}$ is cyclic, each of its subgroups is determined by the size and hence is preserved under any automorphism of $Z_{\widetilde{G}}$, so in such cases there is no overlooked subtlety. It is a convenient miracle of the classification of irreducible root systems that the only cases for which P/Q is not cyclic are type D_{2m} for $m \geq 2$, which is to say $\widetilde{G} = \mathrm{Spin}(4m)$ with $m \geq 2$. In this case there are three proper nontrivial subgroups of $Z_{\widetilde{G}} = \mu_2 \times \mu_2$, corresponding to $\mathrm{SO}(4m)$ and two other degree-2 quotients of $\mathrm{Spin}(4m)$. For $m \neq 2$ the diagram has only a single order-2 symmetry, and as an automorphism of $Z_{\mathrm{Spin}(4m)}$ this turns out to preserve $\ker(\mathrm{Spin}(4m) \to \mathrm{SO}(4m))$ and swap the other two order-2 subgroups of $Z_{\mathrm{Spin}(4m)}$. In contrast, the D_4 diagram has automorphism group S_3 that transitively permutes all order-2 subgroups of $Z_{\mathrm{Spin}(8)}$, so in this case all three degree-2 quotients of $\mathrm{Spin}(8)$ are abstractly isomorphic!

V.3. **The regular locus.** As we have already noted above, Theorem V.1.1 is equivalent to the assertion that the inequality $\#\pi_1(G) \leq [P : \mathrm{X}(T)]$ is an equality. By passing to $\mathbf{Z}$-dual lattices, this index is equal to that of the coroot lattice $\mathbf{Z}\Phi^\vee$ in the cocharacter lattice $\mathrm{X}_*(T)$. Thus, it suffices to show $\pi_1(G) \geq [\mathrm{X}_*(T) : \mathbf{Z}\Phi^\vee]$. We will achieve this by constructing a connected covering space of degree $[\mathrm{X}_*(T) : \mathbf{Z}\Phi^\vee]$ over an open subset of G whose complement is sufficiently thin. In this section, we introduce and study this open subset.

In Exercise 2 of HW10, we define $g \in G$ to be *regular* if g lies in a unique maximal torus of G, and it is shown there that the locus G^{reg} of regular elements of G is always open and non-empty. Although G^{reg} does not rest on a choice of maximal torus of G, it has a nice description in terms of such a choice:

Proposition V.3.1. *The map $q : (G/T) \times T \to G$ defined by $(g \bmod T, t) \mapsto gtg^{-1}$ has all finite fibers of size $\#W$, and the fiber-size $\#W$ occurs precisely over G^{reg}. The restriction $q^{-1}(G^{\text{reg}}) \to G^{\text{reg}}$ is a (possibly disconnected) covering space with all fibers of size $\#W$.*

Here and below, $W = W(G,T)$.

Proof. For any $g_0 \in G$, if we let $f_{g_0} : (G/T) \times T \simeq (G/T) \times T$ be defined by left g_0-translation on G/T and the identity map on T then $q \circ f_{g_0} = c_{g_0} \circ q$ where $c_{g_0} : G \simeq G$ is g_0-conjugation. Thus, the q-fiber over $g \in G$ is identified via f_{g_0} with the q-fiber of the g_0-conjugate of g, so for the purpose of studying fiber-size and relating it to G^{reg} we may restrict our attention to $g \in T$ (as every element of G has a conjugate in T, by the Conjugacy Theorem).

The q-fiber over $t_0 \in T$ consists of points $(g \bmod T, t)$ such that $gtg^{-1} = t_0$. We saw earlier in the course that $T/W \to \text{Conj}(G)$ is injective (even bijective), or more specifically that elements of T are conjugate in G if and only if they are in the same orbit under the natural action of $W = N_G(T)/T$ on T (via $N_G(T)$-conjugation). Thus, for any such (g,t), necessarily $t = w.t_0$ for some $w \in W$. Hence, $q^{-1}(t_0)$ consists of points $(g \bmod T, w.t_0)$ for $w \in W$ and $g \in G/T$ such that $gn_w \in Z_G(t_0)$ (where $n_w \in N_G(T)$ is a fixed representative of w. In other words, $g \bmod T \in (Z_G(t_0)/T).w$ via the natural right W-action on G/T using right-translation by representatives in $N_G(T)$.

We have shown that $\#q^{-1}(t_0) \geq \#W$ with equality if and only if $(Z_G(t_0)/T).W = W$ inside G/T, which is to say $Z_G(t_0) \subset N_G(T)$. Hence, we need to show that T is the unique maximal torus containing t_0 if and only if $Z_G(t_0) \subset N_G(T)$ (and then in all other cases $\dim Z_G(t_0) > \dim T$, so $q^{-1}(t_0)$ is infinite). If T is the only maximal torus containing t_0 then for any $g \in Z_G(t_0)$ the maximal torus gTg^{-1} containing $gt_0g^{-1} = t_0$ must equal T, which is to say $g \in N_G(T)$. Conversely, if $Z_G(t_0) \subset N_G(T)$ and T' is a maximal torus of G containing t_0 then T and T' are both maximal tori of the connected compact Lie group $Z_G(t_0)^0$, so they are conjugate under $Z_G(t_0)^0$. But any such conjugation preserves T since $Z_G(t_0) \subset N_G(T)$ by hypothesis, so $T' = T$.

It remains to prove that the map $q^{-1}(G^{\text{reg}}) \to G^{\text{reg}}$ whose fibers all have size $\#W$ is a covering map. In our proof of the Conjugacy Theorem, we showed rather generally that for any $(g \bmod T, t) \in (G/T) \times T$ and suitable bases of tangent spaces, the map $\mathrm{d}q(g \bmod T, t)$ has determinant $\det(\text{Ad}_{G/T}(t^{-1}) - 1)$, where $\text{Ad}_{G/T}$ is the T-action on $\text{Tan}_e(G/T)$ induced by T-conjugation on G. Explicitly, we saw by considering the complexified tangent space that $\text{Ad}_{G/T}(t^{-1}) - 1$ diagonalizes over $\mathbf{C}$ with eigenlines given by the root spaces in $\mathfrak{g}_{\mathbf{C}}$, on which the eigenvalues are the numbers $t^a - 1$. Hence,

$$\det(\text{Ad}_{G/T}(t^{-1}) - 1) = \prod_{a \in \Phi(G,T)} (t^a - 1),$$

and this is nonzero precisely when $t^a \neq 1$ for all roots $a \in \Phi(G,T)$. By Exercise 2 of HW10, this is precisely the condition that $t \in T \cap G^{\text{reg}}$, and that in turn is exactly the condition that the point $gtg^{-1} = q(g \bmod T, t)$ lies in G^{reg}. Hence, on $q^{-1}(G^{\text{reg}})$ the map q is a local diffeomorphism. The following lemma concludes the proof. $\square$

Lemma V.3.2. *If $q : X \to Y$ is a continuous map between Hausdorff topological spaces and it is a local homeomorphism with finite constant fiber size $d > 0$ then it is a covering map.*

Proof. Choose $y \in Y$ and let $\{x_1, \dots, x_d\} = q^{-1}(y)$. By the local homeomorphism property, each x_i admits an open set U_i mapping homeomorphically onto an open set $V_i \subset Y$ around y. By the Hausdorff property, if we shrink the U_i's enough (and then the V_i's correspondingly) we can arrange that they are pairwise disjoint. Let $V = \cap V_i$, and let $U_i' = U_i \cap q^{-1}(V)$. Since V is an open subset of V_i containing y, U_i' is an open subset of U_i containing x_i such that q carries U_i' homeomorphically onto V. But the U_i''s are pairwise disjoint (since the U_i's are), so $\coprod U_i'$ is an open subset of $q^{-1}(V)$ with each U_i' carried homeomorphically onto V.

All q-fibers have the same size d by hypothesis, and each of the d pairwise disjoint U_i''s contributes a point to each q-fiber over V, so this must exhaust all such fibers. That is, necessarily $\coprod U_i' = q^{-1}(V)$. In other words, we have identified the entire preimage $q^{-1}(V)$ with a disjoint union of d topological spaces U_i' that are each carried homeomorphically onto V by q, so q is a covering map. □

We will show G^{reg} is connected and $\pi_1(G^{\text{reg}}) \to \pi_1(G)$ is an isomorphism, and build a *connected* covering space of G^{reg} with degree $[\mathrm{X}_*(T) : \mathbf{Z}\Phi^\vee]$. The link between covering spaces and π_1 that was developed in Exercise 3 of HW9 then implies $\#\pi_1(G^{\text{reg}}) \geq [\mathrm{X}_*(T) : \mathbf{Z}\Phi^\vee]$, so we would be done.

To show G^{reg} is connected and $\pi_1(G^{\text{reg}}) \to \pi_1(G)$ is an isomorphism, we have to understand the effect on connectedness and on π_1 upon removing a closed subset from a connected manifold (as G^{reg} is obtained from G by removing the closed subset $G - G^{\text{reg}}$ whose elements are called *singular*, as "singular" is an archaic word for "special"). If we remove a point from $\mathbf{R}^2$ then we retain connectedness (in contrast with removing a point from $\mathbf{R}$) but we increase the fundamental group. If $n \geq 3$ and we remove a point from $\mathbf{R}^n$ then the fundamental group does not change. More generally, if V is a subspace of $\mathbf{R}^n$ with codimension c then $\mathbf{R}^n - V$ is connected if $c \geq 2$ and $\pi_1(\mathbf{R}^n - V)$ is trivial if $c \geq 3$: we can change linear coordinates to arrange that $V = \{0\} \times \mathbf{R}^{n-c}$, so $\mathbf{R}^n - V = (\mathbf{R}^c - \{0\}) \times \mathbf{R}^{n-c}$, which retracts onto $\mathbf{R}^c - \{0\}$, whic is connected if $c \geq 2$ and has trivial π_1 if $c \geq 3$. Roughly speaking, connectedness is insensitive to closed subset of "codimension ≥ 2" and π_1 is insensitive to closed subsets of "codimension ≥ 3". To exploit this, we need to understand how thin $G - G^{\text{reg}}$ is inside G.

Lemma V.3.3. *The set $G - G^{\text{reg}}$ is the image under a C^∞ map $M \to G$ for a non-empty compact C^∞ manifold M of dimension* $\dim G - 3$.

We know $\dim G \geq 3$ since $Z_G(T_a)/T_a$ is SU(2) or SO(3) for any $a \in \Phi(G, T)$ (and G has root system (V, Φ) that was assumed to be non-zero; i.e., $\Phi \neq \emptyset$).

Proof. Fix a maximal torus T in G, so by the Conjugacy Theorem every $g \in G$ is conjugate to an element of T. For each $a \in \Phi(G, T)$ the kernel $K_a = \ker(a : T \to S^1)$ is a subgroup whose identity component is the codimension-1 torus T_a killed by a, and the singular set meets T in exactly the union of the K_a's (by Exercise 2 in HW10). Hence, $G - G^{\text{reg}}$ is the union of the G-conjugates of the K_a's for varying $a \in \Phi(G, T)$.

In other words, $G - G^{\text{reg}}$ is the union of the images of the the C^∞ conjugation maps $c_a : (G/Z_G(T_a)) \times K_a \to G$ defined by $(g \bmod Z_G(T_a), k) \mapsto gkg^{-1}$. For $r := \dim T$, we have $\dim K_a = \dim T_a = r - 1$ and (by complexified Lie algebra considerations) $\dim Z_G(T_a) =$

$3 + (r - 1) = r + 2$, so $\dim G/Z_G(T_a) = \dim G - r - 2$. Thus, $(G/Z_G(T_a)) \times K_a$ has dimension $\dim G - 3$, so we can take M to be the union of the finitely many compact manifolds $(G/Z_G(T_a)) \times K_a$. $\square$

Corollary V.3.4. *The regular locus G^{reg} inside G is connected and $\pi_1(G^{\mathrm{reg}}) \to \pi_1(G)$ is an isomorphism.*

Proof. This is part of [BtD, Ch. V, Lemma 7.3], and it amounts to some general facts concerning deformations of C^∞ maps to C^∞ manifolds, for which references are provided in [BtD]. Here we sketch the idea.

Consider a C^∞ map $f : Y \to X$ between non-empty C^∞ compact manifolds with X *connected*, Y possibly disconnected but with all connected components of the dimension at most some $d \leq \dim X - 2$. (We have in mind the example $M \to G$ from Lemma V.3.3.) Although $f(Y)$ may be rather nasty inside X, informally we visualize it as having "codimension" at least $\dim X - d \geq 2$, even though we have not made a rigorous definition of codimension for general closed subsets of X. (It is crucial that f is C^∞ – or at least differentiable – and not merely continuous, in view of space-filling curves. We also need the compactness of Y to avoid situations like a densely-wrapped line in $S^1 \times S^1$.)

Consider points $x, x' \in X - f(Y)$. They can be joined by a path in X, since X is path-connected. The first key thing to show, using the "codimension-2" property for $f(Y)$, is that any path $\sigma : [0,1] \to X$ with endpoints away from $f(Y)$ can be continuously deformed without moving the endpoints so that it entirely avoids $f(Y)$. The method goes as follows. By compactness of $[0,1]$, we can deform σ to be C^∞ without moving the endpoints, so suppose σ is C^∞. Intuitively, $\sigma([0,1])$ and $f(Y)$ are closed subsets of X whose "dimensions" add up to $< \dim X$, and σ carries the boundary $\{0,1\}$ of $[0,1]$ away from $f(Y)$, so a small perturbation of σ *leaving it fixed at the boundary* should provide a smooth path $[0,1] \to X - f(Y)$ entirely avoiding $f(Y)$ and having the same endpoints as σ. (As motivation, consider a pair of *affine linear* subspaces V, V' in $\mathbf{R}^n$ with $\dim V + \dim V' < n$. It is clear that by applying a tiny translation to V, we make these affine spaces disjoint, such as for a pair of lines in $\mathbf{R}^3$.) It follows that $X - f(Y)$ is (path-)connected, and likewise that $\pi_1(X - f(Y)) \to \pi_1(X)$ is surjective (using a base point in $X - f(Y)$).

Now consider the injectivity question for π_1's, assuming $d \leq \dim X - 3$. For a path $\gamma : [0,1] \to X - f(Y)$ carrying 0 and 1 to points $x_0, x_1 \notin f(Y)$, suppose γ is homotopic to another path γ' linking x_0 to x_1, with the homotopy leaving the endpoints fixed. (The relevant case for π_1's is $x_0 = x_1$.) This homotopy is a continuous map $F : [0,1] \times [0,1] \to X$ carrying the boundary square S into $X - f(Y)$. By compactness of $[0,1] \times [0,1]$, we can deform F to be C^∞ without moving $F(0,t) = x_0$ or $F(1,t) = x_1$ and keeping $F(S)$ away from the closed $f(Y)$. Informally, $F([0,1] \times [0,1])$ and $f(Y)$ have "dimensions" adding up to $< \dim X$, and $F(S)$ avoids $f(Y)$, so a small perturbation of F *leaving it fixed on the boundary* should be a smooth map $H : [0,1] \times [0,1] \to X - f(Y)$ entirely avoiding $f(Y)$ and agreeing with F on S. Then $H(0,t) = F(0,t) = x_0$, $H(1,t) = F(1,t) = x_1$, $H(\cdot, 0) = F(\cdot, 0) = \gamma$, and $H(\cdot, 1) = F(\cdot, 1) = \gamma'$, so H is a homotopy from γ to γ' entirely within $X - f(Y)$. $\square$

Now our problem is reduced to constructing a connected covering space of G^{reg} with degree $[\mathrm{X}_*(T) : \mathbf{Z}\Phi^\vee]$.

V.4. **Construction of covering spaces.** Let's revisit the covering space $q^{-1}(G^{\mathrm{reg}}) \to G^{\mathrm{reg}}$ built above. We saw that this has fibers of size $\#W$. But this fiber-count can be refined in a useful way by bringing in group actions. Note that $W = N_G(T)/T$ acts on $(G/T) \times T$ on the right via the formula

$$(g \bmod T, t).w = (gn_w \bmod T, w^{-1}.t)$$

using the natural left W-action on T (pre-composed with inversion on W) and the right action of W on G/T via right $N_G(T)$-translation on G. By design, this action leaves q invariant: $q(x.w) = q(x)$ for all $x \in (G/T) \times T$ and $w \in W$. Hence, the W-action on $(G/T) \times T$ preserves each q-fiber, so in particular W acts on $q^{-1}(G^{\text{reg}})$ permuting all q-fibers in here.

For $t \in T \cap G^{\text{reg}}$, the q-fiber $q^{-1}(t)$ consists of the points $(w^{-1}, w.t)$ for $w \in W$, and these are the points in the W-orbit of $(1, t)$. Thus, by counting we see that W acts simply transitivity on the q-fiber over each point of $T \cap G^{\text{reg}}$, and so likewise on the entirety of G^{reg} since G-conjugation on G^{reg} brings any point into $T \cap G^{\text{reg}}$ (as we recall that $c_{g_0} \circ q = q \circ f_{g_0}$ by a suitable automorphism f_{g_0} of $(G/T) \times T$).

To summarize, not only is $q^{-1}(G^{\text{reg}}) \to G^{\text{reg}}$ a covering space with fibers of size $\#W$, but there is a W-action on $q^{-1}(G^{\text{reg}})$ leaving q invariant and acting simply transitively on all fibers. This leads us to a useful notion for *locally connected* topological spaces (i.e., topological spaces for which every point admits a base of open neighborhoods that are connected, such as any topological manifold):

Definition V.1. Let Γ be a group, and X a locally connected non-empty topological space. A Γ-*space* over X is a covering space $f : E \to X$ equipped with a right Γ-action on E leaving f invariant (i.e., $f(\gamma.e) = f(e)$ for all $e \in E$ and $\gamma \in \Gamma$) such that Γ acts simply transitively on all fibers.

In practice, Γ-spaces tend to be disconnected.

Proposition V.4.2. *Let X be non-empty and locally connected, $E \to X$ a covering space equipped with a right action over X by a group Γ, and Γ' a normal subgroup of Γ.*

1. *If E is a Γ-space then E/Γ' with the quotient topology is a Γ/Γ'-space over X.*
2. *If the Γ'-action makes E a Γ'-space over some Y then $E/\Gamma' \simeq Y$, making $Y \to X$ a covering space. If moreover the resulting Γ/Γ'-action on Y makes it a Γ/Γ'-space over X then E is a Γ-space over X.*
3. *If X is a connected and Γ' acts transitively on the set of connected components of X then E/Γ' is connected.*

Proof. The first assertion is of local nature over X, which is to say that it suffices to check it over the constituents of an open covering of X. Thus, we choose open sets over which the covering space splits, which is to say that we may assume $E = \coprod_{i \in I} U_i$ for pairwise disjoint open subsets U_i carried homeomorphically onto X. By shrinking some more we can arrange that X is (non-empty and) connected, so every U_i is connected. In other words, the U_i's have an intrinsic meaning relative to E having nothing to do with the covering space map down to X: they are the connected components of E.

Consequently, the action on E over X by each $\gamma \in \Gamma$ must permute the U_i's. Moreover, since Γ acts simply transitively on each fiber, and each fiber meets a given U_i in exactly one point (!), it follows that Γ acts simply transitively on the set of U_i's. That is, if we choose some U_{i_0} then for each $i \in I$ there is a unique $\gamma_i \in \Gamma$ such that γ_i carries U_i onto U_{i_0}, and moreover $i \mapsto \gamma_i$ is a bijection from I to Γ. In other words, the natural map $U_{i_0} \times \Gamma \to E$

over X defined by $(u, \gamma) \mapsto u.\gamma$ is a homeomorphism (giving Γ the discrete topology and $U_{i_0} \times \Gamma$ the product topology).

Identifying U_{i_0} with X via the covering map $E \to X$, we can summarize our conclusion as saying that when X is connected and the covering space is split, a Γ-space over X is simply $X \times \Gamma$ (mapping to X via pr_1) equipped with its evident right Γ-action; this is the *trivial* Γ-space. Now it is clear that passing to the quotient by Γ' yields a Γ/Γ'-equivariant homeomorphism $E/\Gamma' \simeq X \times (\Gamma/\Gamma')$, so by inspecting the right side we see that we have a Γ/Γ'-space. This completes the proof of (1).

For the proof of (2), we note that in (1) the case $\Gamma' = \Gamma$ recovers the base space as the quotient. Hence, in the setting of (2), we immediately get that $Y = E/\Gamma'$. Thus, there is a natural map $Y = E/\Gamma' \to X$, and by hypothesis this is a Γ/Γ'-space. We want to show that $E \to X$ is a Γ-space. This is of local nature on X, so we can shrink X to be connected with $E/\Gamma' = X \times (\Gamma/\Gamma')$. Since $E \to E/\Gamma'$ is a Γ'-space, any point of E/Γ' has a base of connected open neighborhoods over which the preimage in E is a trivial Γ'-space. The topological description of E/Γ' as $X \times (\Gamma/\Gamma')$ provides a copy of $X \times \{1\}$ inside E/Γ' and by working on sufficiently small connected open sets in there we can arrange by shrinking X that the preimage of $X \times \{1\}$ under $E \to E/\Gamma'$ is a disjoint union E' of copies of X permutted simply transitively by Γ'. But then the Γ-action on E permutes a single connected component E'_0 of E' simply transitively to all connected components of E (the ones over $X \times \{1\} \subset E/\Gamma'$ constitute a Γ'-orbit, and the ones over the other Γ/Γ'-translates of $X \times \{1\}$ inside E/Γ' exhaust the set of components without repetition). Hence, $E \to X$ is identified with $E'_0 \times \Gamma$ where E'_0 maps homeomorphically onto X, so $E = X \times \Gamma$. This establishes the Γ-space property, and so proves (2).

Now we prove (3). Since X is locally connected by hypothesis, so is its covering space E, so the connected components of E (which are always closed, as for any topological space) are also *open*. In other words, $E = \coprod E_i$ for the set $\{E_i\}$ of connected components of E. Fix some i_0, so if Γ' acts transitively on the set of connected components of E then each E_i is a Γ'-translate of E_{i_0}. Hence, the natural continuous map $E_{i_0} \to E/\Gamma'$ is surjective, so the target inherits connectedness from E_{i_0}. $\square$

We will construct a (highly disconnected) $W \ltimes \mathrm{X}_*(T)$-space $E \to G^{\mathrm{reg}}$ (with W acting on $\mathrm{X}_*(T)$ on the right by composing its usual left action with inversion on W) for which the normal subgroup $W \ltimes \mathbf{Z}\Phi^\vee$ acts transitively on the set of connected components of E. By the preceding proposition, this then yields a *connected* covering space over G^{reg} that is a $\mathrm{X}_*(T)/\mathbf{Z}\Phi^\vee$-space, so it has degree equal to the size of this finite quotient group. That would establish the desired lower bound on the size of $\pi_1(G^{\mathrm{reg}}) = \pi_1(G)$.

The construction of E uses the exponential map for T as follows. By construction, $q^{-1}(G^{\mathrm{reg}}) = (G/T) \times T^{\mathrm{reg}}$ inside $(G/T) \times T$, where $T^{\mathrm{reg}} = T \times G^{\mathrm{reg}}$ is the set of $t \in T$ such that $t^a \neq 1$ for all $a \in \Phi(G, T)$. The exponential map $\exp_T : \mathfrak{t} \to T$ is an $\mathrm{X}_*(T)$-space (check!), so

$$\mathrm{id} \times \exp_T : (G/T) \times \mathfrak{t} \to (G/T) \times T$$

is also an $\mathrm{X}_*(T)$-space. Its restriction over the connected open $q^{-1}(G^{\mathrm{reg}}) = (G/T) \times T^{\mathrm{reg}}$ is then an $\mathrm{X}_*(T)$-space over $q^{-1}(G^{\mathrm{reg}})$. Explicitly, this $\mathrm{X}_*(T)$-space is $E := (G/T) \times \mathfrak{t}^{\mathrm{reg}}$ where $\mathfrak{t}^{\mathrm{reg}}$ is the set of points $v \in \mathfrak{t} = \mathrm{X}_*(T)_{\mathbf{R}}$ such that $(\exp_T(v))^a \neq 1$ for all $a \in \Phi(G, T)$. Each root $a : T \to S^1 = \mathbf{R}/\mathbf{Z}$ is obtained by exponentiating its Lie algebra map $\mathrm{Lie}(a) : \mathfrak{t} \to \mathbf{R}$, which is to say $\exp_T(v)^a = e^{2\pi i \mathrm{Lie}(a)(v)}$. Hence, $\exp_T(v)^a \neq 1$ if and only if

$\mathrm{Lie}(a)(v) \notin \mathbf{Z}$. Thus,

$$\mathfrak{t}^{\mathrm{reg}} = \mathfrak{t} - \bigcup_{a\in\Phi, n\in\mathbf{Z}} H_{a,n} \tag{V.4.1}$$

where $H_{a,n}$ is the affine hyperplane in $\mathfrak{t}$ defined by the equation $\mathrm{Lie}(a)(v) = n$. The affine hyperplane $H_{a,n}$ is a translate of $H_{a,0} = \mathrm{Lie}(T_a) \subset \mathfrak{t}$. Since Φ is finite, it is easy to see that the collection $\{H_{a,n}\}$ of affine hyperplanes inside $\mathfrak{t}$ is locally finite.

Composing with the W-space map $q^{-1}(G^{\mathrm{reg}}) \to G^{\mathrm{reg}}$ yields a covering space map

$$E = (G/T) \times \mathfrak{t}^{\mathrm{reg}} \to G^{\mathrm{reg}}.$$

The source has a natural action by $\mathrm{X}_*(T)$ via translation on $\mathfrak{t} = \mathrm{X}_*(T)_{\mathbf{R}}$, and it also has a natural right action by $W = N_G(T)/T$ via right translation on G/T and via pre-composition of inversion on W with the natural left action on $\mathfrak{t}$ (akin to the W-action used to make $q^{-1}(G^{\mathrm{reg}}) \to G^{\mathrm{reg}}$ into a W-space). The W-action and $\mathrm{X}_*(T)$-action are compatible via the fact that the W-action on $\mathfrak{t}$ preserves the lattice $\mathrm{X}_*(T)$.

In this way we get an action by $W \ltimes \mathrm{X}_*(T)$ on $(G/T) \times \mathfrak{t}^{\mathrm{reg}}$ extending the $\mathrm{X}_*(T)$-space structure over $q^{-1}(G^{\mathrm{reg}})$ and inducing upon

$$q^{-1}(G^{\mathrm{reg}}) = E/\mathrm{X}_*(T)$$

its W-space structure as built above. Consequently, by Proposition V.4.2(2), E has been equipped with a structure of $W \ltimes \mathrm{X}_*(T)$-space over G^{reg}. Since G^{reg} is a connected manifold, by Proposition V.4.2(3) we just need to show that $W \ltimes (\mathbf{Z}\Phi^\vee)$ acts transitively on the set of connected components of $E = (G/T) \times \mathfrak{t}^{\mathrm{reg}}$.

There is an evident continuous left G-action on E through the first factor G/T, and this *commutes* with the right action of $W \ltimes (\mathbf{Z}\Phi^\vee)$ since this right action affects G/T through right $N_G(T)$-translation by G (and right translations commute with left translations on any group). The connectedness of G forces its action on E to preserve each connected component of E, so we may focus our attention on the action of $W \ltimes (\mathbf{Z}\Phi^\vee)$ on the quotient space $G\backslash E$. But the natural map $\mathfrak{t}^{\mathrm{reg}} \to G\backslash E$ is a homeomorphism under which the right $W \ltimes (\mathbf{Z}\Phi^\vee)$-action on E induces upon $\mathfrak{t}^{\mathrm{reg}}$ the right action given by $\mathbf{Z}\Phi^\vee$-translation and pre-composition of inversion on W with the usual left action by W (check this W-aspect!).

To summarize, upon looking back at (V.4.1), our task has been reduced to showing:

Proposition V.4.3. *In the space $\mathfrak{t}^{\mathrm{reg}}$ obtained by removing from $\mathfrak{t}$ the locally finite collection of affine hyperplanes $H_{a,n}$, the action by $W \ltimes \mathbf{Z}\Phi^\vee$ is transitive on the set of connected components.*

These connected components are called *alcoves*; they are intrinsic to the root system (over $\mathbf{R}$). Further study of root systems shows that each Weyl chamber K contains a single alcove with 0 in its closure, and that alcove has an additional wall beyond the hyperplanes defined by $B(K)$, using the unique "highest root" relative to the basis $B(K)$. For example, in type A_3 these are open triangles. (See [BtD, Ch. V, §7, Figure 24] for pictures of the alcoves for several semisimple groups of low rank.) In general $W \ltimes \mathbf{Z}\Phi^\vee$ acts simply transitively on the set of alcoves, and it is called the *affine Weyl group* of the root system.

Proof. Recall that the proof of transitivity of the action of $W(\Phi)$ on the set of Weyl chambers proved more generally (with help from Exercise 1 in HW8) that for *any* locally finite set $\{H_i\}$ of affine hyperplanes in a finite-dimensional inner product space V over $\mathbf{R}$, the subgroup of affine transformations in $\mathrm{Aff}(V)$ generated by the orthogonal reflections in the H_i's acts transitively on the set of connected components of $V - (\cup H_i)$. Thus, equipping $\mathfrak{t}$ with a W-invariant inner product, it suffices check that group of affine-linear transformations of $\mathfrak{t}$ provided by $W \ltimes \mathbf{Z}\Phi^\vee$ (which permutes the set of affine hyperplanes $H_{a,n}$, as does even the bigger group $W \ltimes \mathrm{X}_*(T)$) contains the subgroup of $\mathrm{Aff}(\mathfrak{t})$ generated by the orthogonal reflections in the $H_{a,n}$'s.

Since $W = W(\Phi)$ (!) and we use a W-invariant inner product, the W-action on $\mathfrak{t}$ provides exactly the subgroup of $\mathrm{Aff}(\mathfrak{t})$ generated by orthogonal reflections in the hyperplanes $H_{a,0} = \mathrm{Lie}(T_a)$ through 0. We have to show that by bringing in translations by elements of the coroot lattice, we obtain the reflections in the $H_{a,n}$'s for any $n \in \mathbf{Z}$.

For ease of notation, let $a' = \mathrm{Lie}(a) \in \mathfrak{t}^*$. The orthogonal reflection $r_{a,n}$ in $H_{a,n}$ is given as follows. We know $r_{a,0}(\lambda) = \lambda - \langle \lambda, a \rangle a^\vee$, and the inclusion $\mathrm{X}_*(T) \hookrightarrow \mathrm{X}_*(T)_{\mathbf{R}} \simeq \mathfrak{t}$ makes $a'(b^\vee) = \langle a, b^\vee \rangle$ (apply the Chain Rule to a composition $S^1 \to T \to S^1$), so the vector $(n/2)a^\vee$ lies in $H_{a,n}$ and lies in the line through 0 orthogonal to this affine hyperplane since $H_{a,n}$ is a translate of $H_{a,0}$ and the W-invariant inner product identifies a' with a scalar multiple of dot product against $a^\vee$. Thus, to compute orthogonal reflection about $H_{a,n}$ we can translate by $-(n/2)a^\vee$ to move to $H_{a,0}$, then apply $r_{a,0}$, and finally translate by $(n/2)a^\vee$. This gives the formula

$$r_{a,n}(\lambda) = r_{a,0}(\lambda - (n/2)a^\vee) + (n/2)a^\vee = r_{a,0}(\lambda) + na^\vee,$$

so $r_{a,n} = r_{a,0} + na^\vee$. $\square$

Appendix W. A non-closed commutator subgroup

W.1. **Introduction.** In Exercise 3(i) of HW7 we saw that every element of $\mathrm{SU}(2)$ is a commutator (i.e., has the form $xyx^{-1}y^{-1}$ for $x, y \in \mathrm{SU}(2)$), so the same holds for its quotient $\mathrm{SO}(3)$. We have seen in Exercise 2 of HW8 that if G is a connected compact Lie group then its commutator subgroup G' is closed.

This can be pushed a bit further. The arguments in Exercise 2 of HW8 show there is a finite set of subgroups G_i of the form $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$ such that the multiplication map of manifolds $G_1 \times \cdots \times G_n \to G'$ is surjective on tangent spaces at the identity, and hence (by the submersion theorem) has image containing an open neighborhood U of e in G'. Since U generates G' algebraically (as for an open neighborhood of the identity in any connected Lie group), G' is covered by open sets of the form $Uu_1 \dots u_r$ for finite sequences $\{u_1, \dots, u_r\}$ in U. But any open cover of G' has a finite subcover since G' is compact, so there is a *finite* upper bound on the number of commutators needed to express any element of G'.

It is natural to wonder about the non-compact case. Suppose G is an arbitrary connected Lie group and $H_1, H_2 \subset G$ are normal closed connected Lie subgroups. Is the commutator subgroup (H_1, H_2) closed? It is an important theorem in the theory of linear algebraic groups that if G is a Zariski-closed subgroup of some $\mathrm{GL}_n(\mathbf{C})$ with $H_1, H_2 \subset G$ also Zariski-closed then the answer is always affirmative, with the commutator even Zariski-closed. But counterexamples occur for the non-compact $G = \mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{R})$! Before discussing such counterexamples, we discuss important cases in which the answer is affirmative.

W.2. **The compact case and the simply connected case.** For compact G, we know from HW9 that G' is closed in G. Let's push that a bit further before we explore the non-compact case.

Proposition W.2.1. *If G is a compact connected Lie group and H_1, H_2 are normal closed connected Lie subgroups then (H_1, H_2) is a semisimple compact subgroup.*

In Corollary W.2.6, we will see that if G is semisimple then the closedness hypothesis can be dropped: any normal connected Lie subgroup of G is automatically closed.

Proof. If $Z_i \subset H_i$ is the maximal central torus in H_i then it is normal in G since G-conjugation on H_i is through automorphisms and clearly Z_i is carried onto itself under any automorphism of H_i. But a normal torus T in a connected compact Lie group is always central since its n-torsion $T[n]$ subgroups are collectively dense and each is a *finite* normal subgroup of G (so central in G by discreteness). Thus, each Z_i is central in G.

By Exercise 4(ii) in HW9, $H_i = Z_i \cdot H_i'$ with H_i' a semisimple compact connected Lie group. The central Z_i in G is wiped out in commutators, so $(H_1, H_2) = (H_1', H_2')$. Thus, we can replace H_i with H_i' so that each H_i is semisimple. Now $H_i \subset G'$, so we can replace G with G' to arrange that G is semisimple. If $\{G_1, \dots, G_n\}$ is the set of pairwise commuting simple factors of G in the sense of Theorem X.1.1 then each H_i is a generated by some of the G_j's due to normality, so the isogeny $\prod G_j \to G$ implies that (H_1, H_2) is generated by the G_j's contained in H_1 and H_2. □

Our main aim is to prove the following result, which addresses general connected Lie groups and highlights a special feature of the simply connected case. It rests on an important Theorem of Ado in the theory of Lie algebras that lies beyond the level of this course and will never be used in our study of compact Lie groups.

Theorem W.2.2. *Suppose G is a connected Lie group, and H_1, H_2 are normal connected Lie subgroups. The subgroup (H_1, H_2) is a connected Lie subgroup with Lie algebra $[\mathfrak{h}_1, \mathfrak{h}_2]$, and if G is simply connected then the H_i and (H_1, H_2) are closed in G.*

In particular, $G = G'$ if and only if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

To prove this theorem, we first check that each $\mathfrak{h}_i$ is a "Lie ideal" in $\mathfrak{g}$ (i.e., $[\mathfrak{g}, \mathfrak{h}_i] \subset \mathfrak{h}_i$). More generally:

Lemma W.2.3. *If $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{g}$ and N is the corresponding connected Lie subgroup of G then N is normal in G if and only if $\mathfrak{n}$ is a Lie ideal in $\mathfrak{g}$.*

Proof. Normality of N is precisely the assertion that for $g \in G$, conjugation $c_g : G \to G$ carries N into itself. This is equivalent to $\mathrm{Lie}(c_g) = \mathrm{Ad}_G(g)$ carries $\mathfrak{n}$ into itself for all g. In other words, N is normal if and only if $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ lands inside the closed subgroup of linear automorphisms preserving the subspace $\mathfrak{n}$. Since G is connected, it is equivalent that the analogue hold on Lie algebras. But $\mathrm{Lie}(\mathrm{Ad}_G) = \mathrm{ad}_{\mathfrak{g}}$, so the Lie ideal hypothesis on $\mathfrak{n}$ is precisely this condition. □

Returning to our initial setup, since each $\mathfrak{h}_i$ is a Lie ideal in $\mathfrak{g}$, it follows from the Jacobi identity (check!) that the $\mathbf{R}$-span $[\mathfrak{h}_1, \mathfrak{h}_2]$ consisting of commutators $[X_1, X_2]$ for $X_i \in \mathfrak{h}_i$ is a Lie ideal in $\mathfrak{g}$. Thus, by the lemma, it corresponds to a normal connected Lie subgroup N in G. We want to prove that $N = (H_1, H_2)$, with N closed if G is simply connected.

Let $\widetilde{G} \to G$ be the universal cover (as in Exericse 3(iv) in HW9), and let $\widetilde{H}_i$ be the connected Lie subgroup of $\widetilde{G}$ corresponding to the Lie ideal $\mathfrak{h}_i$ inside $\mathfrak{g} = \widetilde{\mathfrak{g}}$. The map $\widetilde{H}_i \to G$ factors through the connected Lie subgroup H_i via a Lie algebra isomorphism since this can be checked on Lie algebras, so $\widetilde{H}_i \to H_i$ is a covering space.

Remark W.2.4. Beware that $\widetilde{H}_i$ might *not* be the entire preimage of H_i in $\widetilde{G}$. Thus, although it follows that (H_1, H_2) is the image of $(\widetilde{H}_1, \widetilde{H}_2)$, this latter commutator subgroup of $\widetilde{G}$ might not be the full preimage of (H_1, H_2). In particular, closedness of $(\widetilde{H}_1, \widetilde{H}_2)$ will not imply closedness of (H_1, H_2) (and it really cannot, since we will exhibit counterexamples to such closedness later when G is not simply connected).

Letting $\widetilde{N} \subset \widetilde{G}$ be the normal connected Lie subgroup corresponding to the Lie ideal to $[\mathfrak{h}_1, \mathfrak{h}_2] \subset \widetilde{\mathfrak{g}}$, it likewise follows that $\widetilde{N} \to G$ factors through a covering space map onto N, so if $\widetilde{N} = (\widetilde{H}_1, \widetilde{H}_2)$ then $N = (H_1, H_2)$. Hence, it suffices to show that if G is simply connected then the H_i and N are closed in G and $N = (H_1, H_2)$. For the rest of the argument, we therefore may and do *assume G is simply connected*.

Lemma W.2.5. *For simply connected G and any Lie ideal $\mathfrak{n}$ in $\mathfrak{g}$, the corresponding normal connected Lie subgroup N in G is closed.*

This is false without the simply connected hypothesis; consider a line with "irrational angle" in the Lie algebra of $S^1 \times S^1$ (contrasted with the analogue for its universal cover $\mathbf{R}^2$!).

Proof. There is an important Theorem of Ado (proved as the last result in [Bou1, Ch. I]), according to which *every* finite-dimensional Lie algebra over a field F is a Lie subalgebra of $\mathfrak{gl}_n(F)$; we only need the case when F has characteristic 0 (more specifically $F = \mathbf{R}$). Thus, every finite-dimensional Lie algebra over $\mathbf{R}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbf{R})$, and so occurs as a (not necessarily closed!) connected Lie subgroup of $\mathrm{GL}_n(\mathbf{R})$. In particular, all finite-dimensional Lie algebras occur as Lie algebras of connected Lie groups! Hence, there is a connected Lie group Q such that $\mathrm{Lie}(Q) \simeq \mathfrak{g}/\mathfrak{n}$.

Since G is *simply connected*, so it has no nontrivial connected covering spaces (by HW9, Exercise 3(ii)!), the discussion near the end of §H.4 ensures that the Lie algebra map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n} = \mathrm{Lie}(Q)$ arises from a Lie group homomorphism $\varphi : G \to Q$. Consider $\ker \varphi$. By Proposition H.4.4, this is a closed Lie subgroup of G and $\mathrm{Lie}(\ker \varphi)$ is identified with the kernel of the map $\mathrm{Lie}(\varphi)$ that is (by design) the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$. In other words, $\ker \varphi$ is a closed Lie subgroup of G whose Lie subalgebra of $\mathfrak{g}$ is $\mathfrak{n}$, so the same also holds for its identity component $(\ker \varphi)^0$. Thus, we have produced a *closed* connected subgroup of G whose Lie subalgebra coincides with a given Lie ideal $\mathfrak{n}$. It must therefore coincide with N by uniqueness, so N is closed. $\square$

Corollary W.2.6. *If G is a semisimple compact connected Lie group and N is a normal connected Lie subgroup then N is closed.*

Proof. The case of simply connected G is already settled. In general, compactness ensures that the universal cover $\widetilde{G} \to G$ is a finite-degree covering space. The normal connected Lie subgroup $\widetilde{N} \subset \widetilde{G}$ corresponding to the Lie ideal $\mathfrak{n} \subset \mathfrak{g} = \widetilde{\mathfrak{g}}$ is therefore closed, yet $\widetilde{N} \to G$ must factor through N via a covering space map, so N is the image of the compact $\widetilde{N}$ and hence it is compact. Thus, N is closed. $\square$

We have constructed a normal closed connected Lie subgroup N in G such that $\mathfrak{n} = [\mathfrak{h}_1, \mathfrak{h}_2]$. It remains to prove that $N = (H_1, H_2)$. First we show that $(H_1, H_2) \subset N$. Since $\mathfrak{n}$ is a Lie ideal, we know by the preceding arguments that N is normal in G. Thus, the quotient map $G \to \overline{G} := G/N$ makes sense as a homomorphism of Lie groups, and

$\mathrm{Lie}(\overline{G}) = \mathfrak{g}/\mathfrak{n} =: \overline{\mathfrak{g}}$. Let $\overline{\mathfrak{h}}_i \subset \overline{\mathfrak{g}}$ denote the image of $\mathfrak{h}_i$, so it arises from a unique connected (perhaps not closed) Lie subgroup $\overline{H}_i \hookrightarrow \overline{G}$. The map $H_i \to \overline{G}$ factors through $\overline{H}_i$ since this holds on Lie algebras, so to prove that $(H_1, H_2) \subset N$ it suffices to show that $\overline{H}_1$ and $\overline{H}_2$ commute inside $\overline{G}$.

By design, $[\overline{\mathfrak{h}}_1, \overline{\mathfrak{h}}_2] = 0$ inside $\overline{\mathfrak{g}}$ since $[\mathfrak{h}_1, \mathfrak{h}_2] = \mathfrak{n}$, so we will prove that in general a pair of connected Lie subgroups $\overline{H}_1$ and $\overline{H}_2$ inside a connected Lie group $\overline{G}$ commute with each other if their Lie algebras satisfy $[\overline{\mathfrak{h}}_1, \overline{\mathfrak{h}}_2] = 0$ inside $\overline{\mathfrak{g}}$. For any $\overline{h}_1 \in \overline{H}_1$, we want to prove that $c_{\overline{h}_1}$ on $\overline{G}$ carries $\overline{H}_2$ into itself via the identity map. It suffices to prove that $\mathrm{Ad}_{\overline{G}}(\overline{h}_1)$ on $\overline{\mathfrak{g}}$ is the identity on $\overline{\mathfrak{h}}_2$.

In other words, we claim that $\mathrm{Ad}_{\overline{G}} : \overline{H}_1 \to \mathrm{GL}(\overline{\mathfrak{g}})$ lands inside the subgroup of automorphisms that restrict to the identity on the subspace $\overline{\mathfrak{h}}_2$. By connectedness of $\overline{H}_1$, this holds if it does on Lie algebras. But $\mathrm{Lie}(\mathrm{Ad}_{\overline{G}}) = \mathrm{ad}_{\overline{\mathfrak{g}}}$, so we're reduced to checking that $\mathrm{ad}_{\overline{\mathfrak{g}}}$ on $\overline{\mathfrak{h}}_1$ restricts to 0 on $\overline{\mathfrak{h}}_2$, and that is precisely the vanishing hypothesis on Lie brackets between $\overline{H}_1$ and $\overline{H}_2$. This completes the proof that $(H_1, H_2) \subset N$.

To prove that the subgroup (H_1, H_2) exhausts the connected Lie group N, it suffices to prove that it contains an open neighborhood of the identity in N. By design, $\mathfrak{n} = [\mathfrak{h}_1, \mathfrak{h}_2]$ is spanned by finitely many elements of the form $[X_i, Y_i]$ with $X_i \in \mathfrak{h}_1$ and $Y_i \in \mathfrak{h}_2$ ($1 \leq i \leq n$). Consider the C^∞ map $f : \mathbf{R}^{2n} \to G$ given by

$$f(s_1, t_1, \ldots, s_n, t_n) = \prod_{i=1}^{n} (\exp_G(s_i X_i), \exp_G(t_i Y_i)) = \prod_{i=1}^{n} (\alpha_{X_i}(s_i), \alpha_{Y_i}(t_i)).$$

This map is valued inside $(H_1, H_2) \subset N$ and so by the Frobenius theorem (applied to the integral manifold N in G to a subbundle of the tangent bundle of G) it factors smoothly through N.

Rather generally, for any $X, Y \in \mathfrak{g}$, consider the C^∞ map $F : \mathbf{R}^2 \to G$ defined by

$$F(s,t) = (\exp_G(sX), \exp_G(tY)) = \exp_G(\mathrm{Ad}_G(\alpha_X(s))(tY)) \exp_G(-tY).$$

For fixed s_0, the parametric curve $F(s_0, t)$ passes through e at $t = 0$ with velocity vector $\mathrm{Ad}_G(\alpha_X(s_0))(Y) - Y$. Hence, if $s_0 \neq 0$ then $t \mapsto F(s_0, t/s_0)$ sends 0 to e with velocity vector at $t = 0$ given by

$$(1/s_0)(\mathrm{Ad}_G(\alpha_X(s_0))(Y) - Y).$$

But

$$\mathrm{Ad}_G(\alpha_X(s_0)) = \mathrm{Ad}_G(\exp_G(s_0 X)) = e^{\mathrm{ad}_{\mathfrak{g}}(s_0 X)}$$

as linear automorphisms of $\mathfrak{g}$, and since $\mathrm{ad}_{\mathfrak{g}}(s_0 X) = s_0 \mathrm{ad}_{\mathfrak{g}}(X)$, its exponential is $1 + s_0[X, \cdot] + O(s_0^2)$ (with implicit constant depending only on X), so

$$\mathrm{Ad}_G(\alpha_X(s_0))(Y) - Y = s_0[X, Y] + O(s_0^2).$$

Hence, $F(s_0, t/s_0)$ has velocity $[X, Y] + O(s_0)$ at $t = 0$ (where $O(s_0)$ has implied constant depending just on X and Y).

For fixed $s_i \neq 0$, we conclude that the map $\mathbf{R}^n \to (H_1, H_2) \subset N$ given by

$$(t_1, \ldots, t_n) \mapsto f(s_1, t_1/s_1, \ldots, s_n, t_n/s_n)$$

carries $(0, \ldots, 0)$ to e with derivative there carrying $\partial_{t_i}|_0$ to $[X_i, Y_i] + O(s_i) \in \mathfrak{n}$ with implied constant depending just on X_i and Y_i. The elements $[X_i, Y_i] \in \mathfrak{n}$ form a spanning set, so likewise for elements $[X_i, Y_i] + O(s_i)$ if we take the s_i's sufficiently close to (but distinct from) 0. Hence, we have produced a map $\mathbf{R}^n \to N$ valued in (H_1, H_2) carrying 0 to e with derivative that is *surjective*. Thus, by the submersion theorem this map is open image near e, which gives exactly that (H_1, H_2) contains an open neighborhood of e inside N, as desired. This completes the proof of Theorem W.2.2.

W.3. **Non-closed examples.** We give two examples of non-closed commutators. One is an example with (G, G) not closed in G, but in this example G is not built as a "matrix group" over $\mathbf{R}$ (i.e., a subgroup of some $\mathrm{GL}_n(\mathbf{R})$ that is a zero locus of polynomials over $\mathbf{R}$ in the matrix entries). The second example will use the matrix group $G = \mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{R})$ and provide closed connected Lie subgroups H_1 and H_2 for which (H_1, H_2) is not closed.

Such examples are "best possible" in the sense that (by basic results in the theory of linear algebraic groups over general fields) if G is a matrix group over $\mathbf{R}$ then relative to any chosen realization of G as a zero locus of polynomials in some $\mathrm{GL}_n(\mathbf{R})$, if H_1, H_2 are any two subgroups *Zariski-closed* in G relative to such a matrix realization of G then (H_1, H_2) is closed in G, in fact closed of finite index in a possibly disconnected Zariski-closed subgroup of G (the Zariski topology may be too coarse to detect distinct connected components of a Lie group, as for $\mathbf{R}^\times_{>0} \subset \mathbf{R}^\times$ via the matrix realization as $\mathrm{GL}_1(\mathbf{R})$). In particular, for any such matrix group G, (G, G) is always closed in G.

Example W.3.1. Let $H \to \mathrm{SL}_2(\mathbf{R})$ be the universal cover, so it fits into an exact sequence

$$1 \to \mathbf{Z} \xrightarrow{\iota} H \to \mathrm{SL}_2(\mathbf{R}) \to 1.$$

Since $\mathfrak{h} = \mathfrak{sl}_2(\mathbf{R})$ is its own commutator subalgebra, we have $(H, H) = H$.

Choose an injective homomorphism $j : \mathbf{Z} \to S^1$, so $j(n) = z^n$ for $z \in S^1$ that is not a root of unity. Then $\mathbf{Z}$ is naturally a closed discrete subgroup of $S^1 \times H$ via $n \mapsto (j(-n), \iota(n))$, so we can form the quotient

$$G = (S^1 \times H)/\mathbf{Z}.$$

There is an evident connected Lie subgroup inclusion $H \to G$. The commutator subgroup (G, G) is equal to $(H, H) = H$, and we claim that this is not closed in G.

Suppose to the contrary that H were closed in G. By normality we could then form the Lie group quotient G/H that is a quotient of S^1 in which the image of z is trivial. But the kernel of $f : S^1 \twoheadrightarrow G/H$ would have to be a closed subgroup, so containment of z would imply that it contains the closure of the subgroup generated by z, which is dense in S^1. In other words, $\ker f = S^1$, which forces $G/H = 1$, contradicting that obviously $H \neq G$.

Example W.3.2. The following example is an explicit version of a suggestion made by Tom Goodwillie on Math Overflow (as I discovered via Google search; try "Commutator of closed subgroups" to find it). Let $G = \mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{R})$. Let

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

inside the space $\mathfrak{sl}_2(\mathbf{R})$ of traceless 2×2 matrices. Obviously $\alpha_X(\mathbf{R})$ is the closed subgroup of diagonal elements in $\mathrm{SL}_2(\mathbf{R})$ with positive entries (a copy of $\mathbf{R}^\times_{>0}$), and $\alpha_Y(\mathbf{R})$ is a conjugate of this since $Y = gYg^{-1}$ for $g = \left(\begin{smallmatrix} 1 & -1/2 \\ 1 & 1/2 \end{smallmatrix}\right)$. But $[X, Y] = 2\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ has pure imaginary eigenvalues, so its associated 1-parameter subgroup is a circle.

Note that $(X, 0)$ and $(0, X)$ commute inside $\mathfrak{g}$, so they span of subspace of $\mathfrak{g}$ that exponentiates to a closed subgroup isomorphic to $\mathbf{R}^2$. In particular, (X, sX) exponentiates to a closed subgroup of G isomorphic to $\mathbf{R}$ for any $s \in \mathbf{R}$. We choose $s = 1$ to define H_1. Likewise,any (Y, cY) exponentiates to a closed subgroup of G isomorphic to $\mathbf{R}$, and we choose $c \in \mathbf{R}$ to be irrational to define H_2. Letting $Z = [X, Y]$, we likewise have that $(Z, 0)$ and $(0, Z)$ span a subspace of $\mathfrak{g}$ that exponentiates to a closed subgroup isomorphic to $S^1 \times S^1$. The commutator $[\mathfrak{h}_1, \mathfrak{h}_2]$ is (Z, cZ), so by the irrationality of c this exponentiates to a densely wrapped line L in the torus $S^1 \times S^1$. Hence, $(H_1, H_2) = L$ is not closed in G.

APPENDIX X. SIMPLE FACTORS

X.1. **Introduction.** Let G be a nontrivial connected compact Lie group, and let $T \subset G$ be a maximal torus. Assume Z_G is finite (so $G = G'$, and the converse holds by Exercise 4(ii) in HW9). Define $\Phi = \Phi(G, T)$ and $V = \mathrm{X}(T)_{\mathbf{Q}}$, so (V, Φ) is a nonzero root system.

By Appendix U, (V, Φ) is *uniquely* a direct sum of irreducible components $\{(V_i, \Phi_i)\}$ in the following sense. There exists a unique collection of nonzero $\mathbf{Q}$-subspaces $V_i \subset V$ such that for $\Phi_i := \Phi \cap V_i$ the following hold: each pair (V_i, Φ_i) is an irreducible root system, $\oplus V_i = V$, and $\coprod \Phi_i = \Phi$.

Our aim is to use the unique irreducible decomposition of the root system (which involves the choice of T, unique up to G-conjugation) and some Exercises in HW9 to prove:

Theorem X.1.1. *Let $\{G_j\}_{j \in J}$ be the set of minimal non-trivial connected closed normal subgroups of G.*

(i) The set J is finite, the G_j's pairwise commute, and the multiplication homomorphism
$$\prod G_j \to G$$
is an isogeny.

(ii) If $Z_G = 1$ or $\pi_1(G) = 1$ then $\prod G_i \to G$ is an isomorphism (so $Z_{G_j} = 1$ for all j or $\pi_1(G_j) = 1$ for all j respectively).

(iii) Each connected closed normal subgroup $N \subset G$ has finite center, and $J' \mapsto G_{J'} = \langle G_j \rangle_{j \in J'}$ is a bijection between the set of subsets of J and the set of connected closed normal subgroups of G.

(iv) The set $\{G_j\}$ is in natural bijection with the set $\{\Phi_i\}$ via $j \mapsto i(j)$ defined by the condition $T \cap G_j = T_{i(j)}$. Moreover, $T_{i(j)}$ is a maximal torus in G_j and $\Phi_{i(j)} = \Phi(G_j, T_{i(j)})$.

We emphasize that the collection of G_j's does *not* involve a choice of T (and though T is only unique up to conjugacy, by normality each G_j is preserved by G-conjugation). Parts (i) and (iv) (along with Exercise 4(ii) in HW9) are the reason that in the study of general connected compact Lie groups H, by far the most important case is that of semisimple H with an irreducible root system.

Note that by (i) and (iii), if N is a connected closed normal subgroup of G then every connected closed normal subgroup of N is normalized by *every* G_j and hence is normal in G! Thus, normality is *transitive* for connected closed subgroups of a semisimple connected compact Lie group (with all such subgroups also semisimple, by (iii)). In particular, if $J' \subset J$ corresponds to N as in (iii) then $\{G_j\}_{j \in J'}$ is also the set of minimal nontrivial connected closed normal subgroups of N. As a special case, each G_j has no nontrivial connected closed normal subgroups; for this reason, one often says that each G_j is "almost simple" and the G_j are called the "simple factors" of G (even though they are generally not direct factors of G except when $Z_G = 1$ or $\pi_1(G) = 1$, by (ii)).

Remark X.1.2. By Corollary W.2.6, every normal connected Lie subgroup of a semisimple connected compact Lie group is *automatically* closed (so closedness can be dropped as a hypothesis on N above). The proof of that rests on Ado's theorem in the theory of Lie algebras, which in turn lies rather beyond the level of this course. Thus, in this appendix we never make use of that closedness result.

It is nonetheless of interest to record one consequence: G is almost simple if and only if the non-commutative Lie algebra $\mathfrak{g}$ is simple over $\mathbf{R}$ (i.e., is non-commutative and has no nonzero proper Lie ideal). Indeed, by Corollary H.4.6 we know that Lie ideals in $\mathfrak{g}$ are in bijective correspondence with connected Lie subgroups $H \subset G$ such that $\mathrm{Lie}(N_G(H)) = \mathfrak{g}$, or equivalently (by dimension reasons and the connectedness of G) that $N_G(H) = G$, and this equality is precisely the normality of H in G. Thus, $\mathfrak{g}$ is simple if and only if G has no nontrivial connected normal proper Lie subgroups, and that is the "almost simple" property as defined above since every connected normal Lie subgroup of the semisimple compact connected G is *closed*.

By using the theory of root systems for semisimple Lie algebras over $\mathbf{C}$, one can go further and deduce that when $\mathfrak{g} = \mathrm{Lie}(G)$ is simple over $\mathbf{R}$ then $\mathfrak{g}_{\mathbf{C}}$ is simple over $\mathbf{C}$. Such "absolute simplicity" for $\mathfrak{g}$ is a special feature of the simple Lie algebras arising from *compact* (connected semisimple) Lie groups. That is, if H is a general connected Lie group for which $\mathfrak{h}$ is simple over $\mathbf{R}$ then $\mathfrak{h}_{\mathbf{C}}$ is generally not simple over $\mathbf{C}$; examples of such (non-compact) H are provided by the underlying real Lie group of many important connected matrix groups over $\mathbf{C}$ such as $\mathrm{SL}_n(\mathbf{C})$ ($n \geq 2$) and $\mathrm{Sp}_{2n}(\mathbf{C})$ ($n \geq 1$).

Part (iv) in Theorem X.1.1 makes precise the connection between almost-simple factors of G and the irreducible components of Φ, and yields that G is almost simple if and only if Φ is irreducible.

The proof of Theorem X.1.1 will involve initially constructing the G_j's in a manner very different from their initial definition, using the Φ_i's. It will not be apparent at the outset that the G_j's built in this alternative way are normal (or even independent of T!), but the construction in terms of the Φ_i's will be essential for showing that these subgroups pairwise commute and generate G (from which we will eventually deduce that they do recover the G_j's as defined at the start of Theorem X.1.1, so are normal and independent of T).

X.2. **Constructions via roots.** We begin by using the $\mathbf{Q}$-subspaces $V_i \subset V = \mathrm{X}(T)_{\mathbf{Q}}$ to build subtori $T_i \subset T$. This might initially seem surprising since the character-lattice functor is contravariant, but keep in mind that we have not just a single V_i but an entire collection of V_i's giving a direct-sum decomposition of V, so we also have projections $V \twoheadrightarrow V_i$:

Lemma X.2.1. *There are unique subtori $T_i \subset T$ such that $\prod T_i \to T$ is an isogeny and the resulting isomorphism $\prod \mathrm{X}(T_i)_{\mathbf{Q}} \simeq \mathrm{X}(T)_{\mathbf{Q}} = V$ identifies $\mathrm{X}(T_i)_{\mathbf{Q}}$ with V_i.*

Proof. A subtorus $S \subset T$ is "the same" as a short exact sequence of tori

$$1 \to S \to T \to \overline{T} \to 1$$

and so is "the same" as a quotient map $\mathrm{X}(T) \twoheadrightarrow L = \mathrm{X}(S)$ onto a finite free $\mathbf{Z}$-module. But such quotient maps correspond to saturated subgroups $\Lambda \subset \mathrm{X}(T)$ (i.e., subgroups with torsion-free cokernel), and as such are in bijective correspondence with subspaces $W \subset V$ (via the inverse operations $\Lambda \mapsto \Lambda_{\mathbf{Q}}$ and $W \mapsto W \cap \mathrm{X}(T)$). This defines a bijective correspondence between the sets of subtori $S \subset T$ and quotient maps $\pi : V \twoheadrightarrow \overline{V}$. (Explicitly, from π we form the saturated subgroup $\Lambda = \mathrm{X}(T) \cap \ker(\pi)$ of $\mathrm{X}(T)$, and by saturatedness the common kernel S of all characters in Λ is connected; i.e., a subtorus of T. Conversely, to a subtorus $S \subset T$ we associate the quotient $\mathrm{X}(S)_{\mathbf{Q}}$ of $V = \mathrm{X}(T)_{\mathbf{Q}}$.)

Thus, to each projection $V = \bigoplus_i V_i \twoheadrightarrow V_{i_0}$ we get an associated subtorus $T_{i_0} \subset T$ for which $\mathrm{X}(T_{i_0})_{\mathbf{Q}}$ is identified with V_{i_0} as a quotient of $\mathrm{X}(T)_{\mathbf{Q}} = V$. The resulting multiplication map $f : \prod T_i \to T$ induces a map of character lattices $\mathrm{X}(T) \to \prod \mathrm{X}(T_i)$ whose rationalization is precisely the sum $V \to \oplus V_i$ of the projection maps (why?), and this sum is an isomorphism by design (why?). Hence, $\mathrm{X}(f)_{\mathbf{Q}}$ is an isomorphism, so f is an isogeny. This shows that the T_i's as just built do the job. The method of construction also yields the desired uniqueness of the collection of T_i's (check!). $\square$

For each $i \in I$, let $S_i \subset T$ be the subtorus generated by the T_k's for $k \neq i$. Keeping in mind the desired goal in Theorem X.1.1(iv), for each $i \in I$ we are motivated to define

$$G_i = Z_G(S_i)'.$$

By Exercise 4(ii) in HW9 applied to the connected compact Lie group $H = Z_G(S_i)$, we see that each G_i is its own derived group and more specifically has finite center.

Beware that at this stage the subgroups $G_i \subset G$ might depend on T, and it is not at all apparent if they are normal in G. In particular, we do not yet know if these groups coincide with the ones as defined at the start of Theorem X.1.1! We will prove that the G_i's as just defined satisfy all of the desired properties in the first 3 parts of Theorem X.1.1, from which we will deduce that they do recover the G_j's as at the start of Theorem X.1.1 (in accordance with the recipe in part (iv)).

The first step in the analysis of the G_i's is to link them to the Φ_i's in a direct manner:

Lemma X.2.2. *For each i, $T_i \subset G_i$ and T_i is maximal as a torus in G_i (so $T \cap G_i = T_i$). Moreover, $\Phi(G_i, T_i) = \Phi_i$ inside $\mathrm{X}(T_i)_{\mathbf{Q}} = V_i$ and G_i is generated by the 3-dimensional subgroups G_a associated to pairs of opposite roots in $\Phi_i \subset \Phi$.*

Proof. The Lie algebra $\mathfrak{h}_i$ of $H_i := Z_G(S_i)$ inside $\mathfrak{g}$ has complexification $\mathfrak{g}_{\mathbf{C}}^{S_i}$. This is the span of $\mathfrak{t}_{\mathbf{C}}$ and the root lines for roots trivial on the subtorus $S_i \subset T$ generated by the T_k's for $k \neq i$.

The design of the T_k's implies that a root $a \in \Phi$ is trivial on T_k if and only if $a \notin \Phi_k$ (check!). Thus, $a|_{S_i} = 1$ if and only if $a \in \Phi_i$. Thus, $(\mathfrak{h}_i)_{\mathbf{C}}$ is spanned by $\mathfrak{t}_{\mathbf{C}}$ and the root lines for roots in Φ_i. In particular, for each $a \in \Phi_i$ the 3-dimensional connected Lie subgroup $G_a \subset G$ associated to $\pm a$ is contained in H_i because $\mathrm{Lie}(G_a) \subset \mathfrak{h}_i$ (as we may check after complexification, with $\mathrm{Lie}(G_a)_{\mathbf{C}}$ generated as a Lie algebra by the $\pm a$-root lines). Each G_a is perfect (being $\mathrm{SU}(2)$ or $\mathrm{SO}(3) = \mathrm{SU}(2)/\langle -1 \rangle$) and has maximal torus $a^\vee(S^1)$.

We conclude that the derived group G_i of H_i contains every such G_a, so the coroot $a^\vee : S^1 \to T$ for each $a \in \Phi_i$ factors through H_i. Such coroots generate T_i inside T: it is equivalent (why?) to show that they rationally span $\mathrm{X}_*(T_i)_{\mathbf{Q}}$ inside $\mathrm{X}_*(T)_{\mathbf{Q}} = V^*$, and the dual space $V_i^* = \mathrm{X}_*(T_i)_{\mathbf{Q}}$ for the root system dual to Φ_i is spanned by the associated coroots (as for dual root systems in general). This proves that $T_i \subset G_i$ for every i. But G_i has finite center, so its root system has rank equal to the common dimension of its maximal tori. The roots for $G_i = (H_i)'$ relative to the maximal torus $T \cap G_i$ coincides with the T-roots of H_i, and we have seen above that $\Phi(H_i, T) = \Phi_i$ inside $\mathrm{X}(T)_{\mathbf{Q}}$. This has $\mathbf{Q}$-span $V_i = \mathrm{X}(T_i)_{\mathbf{Q}}$ of dimension $\dim T_i$, so the maximal tori of G_i have dimension $\dim T_i$.

Hence, the torus $T_i \subset G_i$ is maximal! This argument also identifies Φ_i with $\Phi(G_i, T_i)$ in the desired manner.

Every semisimple connected compact Lie group is generated by the 3-dimensional subgroups associated to pairs of opposite roots. Such subgroups for G_i are precisely the G_a's from G for $a \in \Phi_i$ since the T_i-root lines in $(\mathfrak{g}_i)_{\mathbf{C}}$ are precisely the T-root lines in $\mathfrak{g}_{\mathbf{C}}$ for the T-roots in Φ_i. $\square$

We now use the preceding lemma to deduce that the G_i's pairwise commute, so there is then a multiplication homomorphism $\prod G_i \to G$. A criterion for connected Lie subgroups to commute is provided by:

Lemma X.2.3. *Connected Lie subgroups H_1 and H_2 of a Lie group G commute if and only if their Lie algebras $\mathfrak{h}_i$ commute inside $\mathfrak{g}$.*

Note that we do not assume closedness of the H_j's in G.

Proof. First assume H_1 and H_2 commute. For any $h_1 \in H_1$, the conjugation c_{h_1} of h_1 on G is trivial on H_2, so $\mathrm{Ad}_G(h_1)$ is trivial on $\mathfrak{h}_2$. That is, the representation $\mathrm{Ad}_G|_{H_1}$ of H_1 on $\mathfrak{g}$ is trivial on the subspace $\mathfrak{h}_2$. By visualizing this in terms of matrices via a basis of $\mathfrak{g}$ extending a basis of $\mathfrak{h}_2$, differentiating gives that $\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{h}_1}$ vanishes on $\mathfrak{h}_2$. This says that $[X_1, \cdot]$ kills $\mathfrak{h}_2$ for all $X_1 \in \mathfrak{h}_1$, which is the desired conclusion that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ commute.

Now assume conversely that $[X_1, X_2] = 0$ for all $X_i \in \mathfrak{h}_i$. We want to deduce that H_1 and H_2 commute. Since H_i is generated by any open neighborhood of the identity due to connectedness, it suffices to check neighborhoods of the identity in each H_i commute with each other. Thus, letting exp denote $\exp_G$, it suffices to show that for any $X_i \in \mathfrak{h}_i$ we have

$$\exp(X_1)\exp(tX_2)\exp(-X_1) = \exp(tX_2)$$

for all $t \in \mathbf{R}$ (as then by setting $t = 1$ we conclude by choosing X_i in an open neighborhood of 0 in $\mathfrak{h}_i$ on which $\exp_{H_i} = \exp_G|_{H_i}$ is an isomorphism onto an open image in H_i).

Both sides of this desired identity are 1-parameter subgroups $\mathbf{R} \to G$, so they agree if and only if their velocities at $t = 0$ are the same. The right side has velocity X_2 at $t = 0$, and by the Chain Rule (check!) the left side has velocity

$$(\mathrm{Ad}_G(\exp(X_1)))(X_2).$$

But $\mathrm{Ad}_G(\exp(X)) \in \mathrm{GL}(\mathfrak{g})$ coincides with $e^{[X,\cdot]}$ for any $X \in \mathfrak{g}$ (proof: replace X with tX for $t \in \mathbf{R}$ to make this a comparison of 1-parameter subgroups $\mathbf{R} \to \mathrm{GL}(\mathfrak{g})$, and compare velocities at $t = 0$ using that $\mathrm{Lie}(\mathrm{Ad}_G) = \mathrm{ad}_{\mathfrak{g}}$), so

$$(\mathrm{Ad}_G(\exp(X_1)))(X_2) = (e^{[X_1,\cdot]})(X_2).$$

Since $[X_1, \cdot]$ kills X_2 by hypothesis, the final expression on the right side collapses to X_2 (due to the expansion $e^T = \mathrm{id} + T + T^2/2! + T^3/3! + \dots$ for any linear endomorphism T of a finite-dimensional $\mathbf{R}$-vector space). $\square$

Let's now check that the $\mathfrak{g}_i$'s pairwise commute inside $\mathfrak{g}$. It suffces to check this after extending scalars to $\mathbf{C}$. The advantage of such scalar extension is that $(\mathfrak{g}_i)_{\mathbf{C}}$ is generated as a Lie algebra by the T_i-root lines (recall that by considerations with $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$, any pair of opposite root lines have Lie bracket that is a nonzero vector in the corresponding coroot line over $\mathbf{C}$), so to check the pairwise-commuting property it suffices to check that the root lines for roots $a \in \Phi_i$ and $b \in \Phi_{i'}$ commute with each other for $i \neq i'$. By

T-weight considerations, the bracket $[(\mathfrak{g}_{\mathbf{C}})_a, (\mathfrak{g}_{\mathbf{C}})_b]$ is contained in the T-weight space for the character $a+b$. But $a+b \neq 0$ and $a+b$ is not a root since a, b lie in *distinct* irreducible components of Φ. This completes the proof that the G_i's pairwise commute.

Now consider the multiplication homomorphism

$$m : \prod G_i \to G.$$

This restriction to the multiplication map $\prod T_i \to T$ between maximal tori that is an isogeny by design of the T_i's, so by dimension considerations with maximal tori we see that the connected normal subgroup $(\ker m)^0$ contains no nontrivial torus (why not?). Hence, the connected compact Lie group $(\ker m)^0$ is trivial, which is to say $\ker m$ is finite. But $\mathrm{Lie}(m)$ is also surjective since its complexification hits every root line in $\mathfrak{g}_{\mathbf{C}}$ (as well as hits the entirety of $\mathfrak{t}_{\mathbf{C}}$), so $\mathrm{Lie}(m)$ is an isomorphism. Thus, m is an isogeny.

Since m is surjective, it follows that every $g \in G$ is a commuting product of some elements $g_i \in G_i$ (one per i). But the effect of conjugation on G by $\prod g_i$ clearly preserves each G_{i_0} (as G_{i_0} commutes with G_i for all $i \neq i_0$), so each G_i is *normal* in G. Applying g-conjugation carries the entire construction resting on T to the one resting on gTg^{-1}, so normality of the G_i's and the conjugacy of maximal tori in G thereby implies that the collection of subgroups $\{G_i\}$ is *independent* of T.

We have not completed the proof of Theorem X.1.1(i) because we have not yet proved that $\{G_i\}$ coincides with the set of minimal nontrivial connected closed normal subgroups of G. But conditional on that, (i) is proved and (ii) is then immediate: if $\pi_1(G) = 1$ then the connected covering space $\prod G_i \to G$ is an isomorphism (see the end of Exercise 3(ii) in HW9), and if $Z_G = 1$ then each $Z_{G_i} = 1$ (as the surjectivity of $\prod G_i \to G$ forces $Z_{G_i} \subset Z_G$) and so the central kernel of the isogeny $\prod G_i \to G$ is trivial.

X.3. **Normal subgroups.** We next prove that the G_i's are built above fulfill the requirements in (iii), from which we will deduce that they are precisely the minimal nontrivial connected closed normal subgroups of G. This will finally bring us in contact with the G_j's as defined at the start of Theorem X.1.1, and complete the proofs of all 4 parts of the theorem.

Let N be a connected closed normal subgroup of G. The key task is to show that if $N \neq 1$ then N contains *some* G_i. Once this is proved, we can pass to $N/G_i \subset G/G_i$ (with root system $\oplus_{i' \neq i} \Phi_{i'}$ relative to the maximal torus T/T_i) to then conclude by dimension induction that in general N is generated by some of the G_i's (so in particular Z_N is finite, so $N = N'$!).

Let $\widetilde{N} = m^{-1}(N)^0$. Since N is connected and m is surjective, the natural map $\widetilde{N} \to N$ is surjective (so $\widetilde{N} \neq 1$) and $\widetilde{N}$ is a connected closed *normal* subgroup of $\prod G_i$. Thus, to prove that the subgroup $N \subset G$ contains some G_i it suffices to prove the same for the subgroup $\widetilde{N} \subset \prod G_i$. In other words, for this step we may rename $\prod G_i$ as G (as $\prod T_i$ as T) so that $G = \prod G_i$ and $T = \prod T_i$.

Let $p_i : G \to G_i$ be the projection. By nontriviality of N, some $p_i(N)$ is nontrivial. But $p_i(N)$ is certainly a connected closed normal subgroup of G_i. Thus, *provided that* G_i has no nontrivial proper connected closed normal subgroups, it would follows that $p_i(N) = G_i$, so since $G_i = G_i'$ and $p_i|_{G_i}$ is the identity map on G_i we see that p_i carries (N, G_i) *onto* G_i. But this commutator subgroup is contained in each of N and G_i since both N and G_i are normal in G, so the identity map $p_i : G_i \to G_i$ identifies (N, G_i) with G_i. We conclude that $G_i = (N, G_i) \subset N$, as desired, once we prove:

Lemma X.3.1. *Each G_i contains no nontrivial proper connected closed normal subgroup.*

Proof. Since G_i has finite center and an irreducible root system, we may rename G_i as G to reduce to the case when Φ is irreducible. In this case we want to show that a nontrivial connected closed normal subgroup $N \subset G$ must coincide with G.

Let $S \subset N$ be a maximal torus, so $S \neq 1$ since $N \neq 1$, and let $T \subset G$ be a maximal torus containing S. Define $\Phi = \Phi(G, T)$; by hypothesis, Φ is irreducible. Since $T \cap N = S$ (since $Z_N(S) = S$), the action of the normalizer $N_G(T)$ on N preserves S. Hence, $\mathrm{X}(S)_{\mathbf{Q}}$ is a nonzero quotient representation of $\mathrm{X}(T)_{\mathbf{Q}}$ relative to the action of the Weyl group $W(G, T) = W(\Phi)$.

Finiteness of Z_G ensures that $\mathrm{X}(T)_{\mathbf{Q}}$ is the $\mathbf{Q}$-span of Φ, so by Exercise 1(ii) in HW9 and the irreducibility of Φ the action of $W(\Phi)$ on $\mathrm{X}(T)_{\mathbf{Q}}$ is (absolutely) irreducible! Thus, the quotient representation $\mathrm{X}(S)_{\mathbf{Q}}$ of $\mathrm{X}(T)_{\mathbf{Q}}$ must be full, which is to say $S = T$ (for dimension reasons). Hence, N contains some maximal torus T of G. But every element of G lies in a conjugate of T, and so by normality of N in G we conclude that $N = G$. $\square$

We have proved that every connected closed subgroup N of G has the form

$$N = G_{I'} := \langle G_{i'} \rangle_{i' \in I'}$$

for some subset $I' \subset I$. Since the multiplication map $m : \prod G_i \to G$ is an isogeny, it is clear that $N \times G_{I-I'} \to G$ is an isogeny for Lie algebra (or other) reasons. Hence, $N \cap G_{I-I'}$ is finite, so I' is uniquely characterized in terms of N as the set of $i \in I$ such that $G_i \subset N$. In particular, the G_i's are precisely the minimal nontrivial connected closed normal subgroups of G! Thus, the G_i's are exactly the G_j's as defined at the start of Theorem X.1.1, and we have proved all of the desired properties for this collection of subgroups.

Appendix Y. Centers of simply connected semisimple compact groups

As we have discussed in class, if G is a connected compact Lie group that is semisimple and simply connected and T is a maximal torus then for a basis B of $(X(T)_{\mathbf{Q}}, \Phi(G, T))$ and the corresponding basis $B^\vee$ of the dual root system $(X_*(T)_{\mathbf{Q}}, \Phi(G, T)^\vee)$, we have a canonical description of T as a direct product of circles:

$$\prod_{b \in B} S^1 \simeq T$$

defined by $(x_b) \mapsto \prod_{b \in B} b^\vee(x_b)$. The center Z_G lies inside T, and is dual to the cokernel P/Q of the weight lattice $P = (\mathbf{Z}\Phi^\vee)'$ modulo the root lattice $Q = \mathbf{Z}\Phi$. Moreover, the *simply connected* G is determined up to isomorphism by its root system (this is part of the Isomorphism Theorem who proof requires work with Lie groups and Lie algebras over $\mathbf{C}$, so beyond the scope of this course).

Hence, it is natural to seek an explicit description of the resulting inclusion

$$\mathrm{Hom}(P/Q, S^1) = Z_G \hookrightarrow T = \mathrm{Hom}(P, S^1) = \prod_{b \in B} b^\vee(S^1)$$

in terms of the "coroot parameterization" of T (the final equality using that $B^\vee$ is a $\mathbf{Z}$-basis of $\mathbf{Z}\Phi^\vee$). From the tables at the end of [Bou2] one can read off this information (essentially the inclusion of Q into P in terms of a Cartan matrix relative to B and $B^\vee$).

Below we tabulate all of this information for all of the irreducible (and reduced) root systems, using the labeling of vertices in the diagram by the elements of B; the notation next to a vertex b indicates what is evaluated inside $b^\vee : S^1 \to T$. (The same works for split connected semisimple linear algebraic groups over any field, once one learns that theory and its relation to root systems.) We are very grateful to Jason Starr for his assistance in rendering a hand-written table in modern typography.

Here are some general comments to help with reading the tables:

(1) For the group $\mathrm{SU}(n+1)$ the description below indicates that the center is the group μ_{n+1} of $(n+1)$th roots of unity, and that if we label the vertices in the Dynkin diagram from left to right as $b_1, \dots, b_n$ then an inclusion $\mu_{n+1} \hookrightarrow \prod_{j=1}^n b_j^\vee(S^1) = T$ is given by

$$t \mapsto b_1^\vee(t) b_2^\vee(t^2) \cdots b_n^\vee(t^n).$$

Let's see that this really does encode the usual way of describing the center of $\mathrm{SU}(n+1)$ as diagonal scalar matrices with a common $(n+1)$th root of unity along the diagonal.

Let D be the diagonal torus of $\mathrm{U}(n+1)$ (with the usual bases $\{e_j\}_{1 \le j \le n+1}$ and $\{e_j^*\}_{1 \le j \le n+1}$ of its respective character and cocharacter lattices, corresponding to its $n+1$ entries in order from upper left to lower right). Let $D' = D \cap \mathrm{SU}(n+1)$ be the corresponding diagonal maximal torus of $\mathrm{SU}(n+1)$. A basis of the associated root system $\Phi(\mathrm{SU}(n+1), D') = \Phi(\mathrm{U}(n+1), D)$ is given by characters $b_j = e_j - e_{j+1}$ $(1 \le j \le n)$ which send a diagonal $d' \in D' \subset D$ to the ratio $d'_j/d'_{j+1} \in S^1$, and the corresponding collection of coroots consists of the cocharacters

$$b_j^\vee = e_j^* - e_{j+1}^* : S^1 \to D' \subset D \ (1 \le j \le n)$$

sending z to the diagonal matrix whose jth entry is z and $(j+1)$th entry is $1/z$.

Hence, for $t \in \mu_{n+1}$ we have

$$\begin{aligned}
\prod_{j=1}^n b_j^\vee(t^j) = \prod_{j=1}^n (e_j^*(t^j)/e_{j+1}^*(t^j)) &= e_1^*(t) \left(\prod_{j=2}^n e_j^*(t^j)/e_j^*(t^{j-1}) \right) (1/e_{n+1}^*(t^n)) \\
&= e_1^*(t) \left(\prod_{j=2}^n e_j^*(t) \right) e_{n+1}^*(1/t^n) \\
&= \prod_{j=1}^{n+1} e_j^*(t),
\end{aligned}$$

where the final equality uses that $1/t^n = t$ since $t^{n+1} = 1$. But this final product is exactly the scalar diagonal matrix with t as the common entry along the diagonal, so we have indeed recovered the usual identification of the center of $\mathrm{SU}(n+1)$ with μ_{n+1}.

(2) Here are some comments for the compact special orthogonal groups, or rather their spin double covers. For the root system B_n with $n \ge 2$ (i.e., the group $\mathrm{Spin}(2n+1)$) the table says that the center is identified with μ_2 sitting inside the direct factor of $T = \prod_{b \in B} b^\vee(S^1)$ corresponding to the coroot associated to the unique short root in the basis.

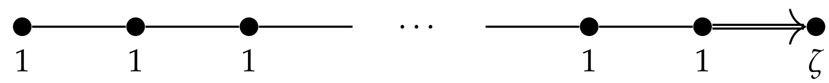
Turning to type D_n (i.e., the group $\mathrm{Spin}(2n)$), the center for type D_n depends on the parity of n (this corresponds to the fact that the group structure of P/Q for type D_n depends on the parity of n, being $\mathbf{Z}/(4)$ for odd n and $(\mathbf{Z}/(2)) \times (\mathbf{Z}/(2))$ for even n). In the case of D_n for odd n, we write "$\zeta^2 = -1$" to indicate that the formula as written is for ζ a primitive 4th root of unity (though it really works as written for all 4th roots on unity: the coroot contributions of ζ^2 are trivial when $\zeta \in \mu_2 \subset \mu_4$).

(3) At the end, we provide a short list of exceptional isomorphisms arising from the coincidence of equality of very low-rank root systems in the various infinite families. (It is reasonable to consider D_2 to be $A_1 \times A_1$ if you stare at the diagram for type D, and likewise it is reasonable to regard B_1 as being A_1 since $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$.)

SU$(n+1)$ $\quad$ A_n

$t^{n+1} = 1$

Spin$(2n+1)$ $\downarrow$ **SO**$(2n+1)$ $\quad$ B_n $(n \geq 2)$ $\quad$ $B_1 = A_1$

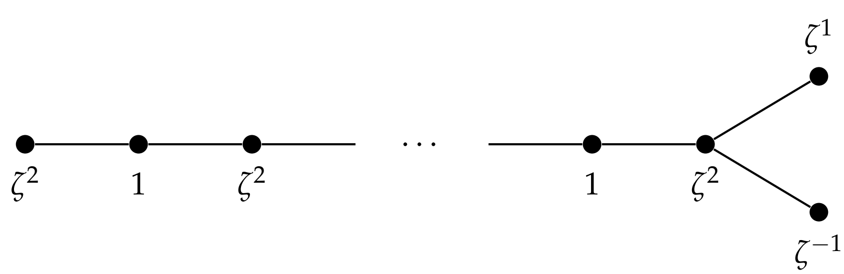


$\zeta^2 = 1$

Sp(n) $\quad$ C_n $(n \geq 2)$ $\quad$ $C_1 = A_1$ $\quad$ $C_2 = B_2$

$\zeta^2 = 1$

Spin$(2n)$ $\downarrow$ **SO**$(2n)$ not adjoint $\quad$ D_n $(n \geq 4)$ $\quad$ (n odd) $\quad$ $D_3 = A_3$

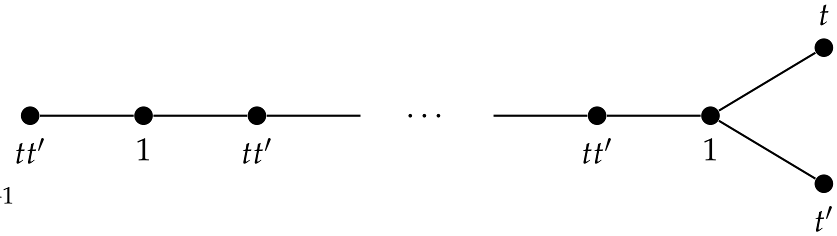


$\zeta^4 = 1$

"$\zeta^2 = -1$"

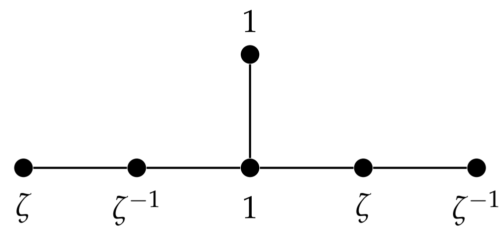
Spin$(2n)$
$\downarrow$
SO$(2n)$
not adjoint

D_n $(n \geq 4)$
(n even)
$\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$



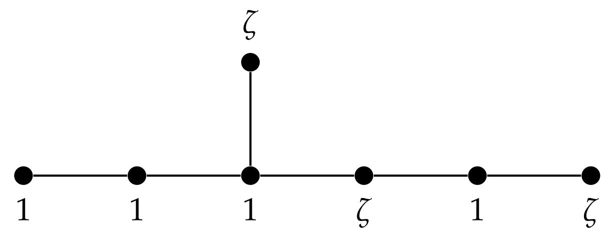
$(t, t') \in \mu_2 \times \mu_2$

E_6



$\zeta^3 = 1$

E_7



$\zeta^2 = 1$

$$\mathrm{E}_8,\ \mathrm{F}_4,\ \mathrm{G}_2 : \text{trivial center}$$

$$\mathbf{SO}(6) \simeq \mathbf{SU}(4)/\mu_2 \quad (\ \mathrm{D}_3 = \mathrm{A}_3\ )$$

$$\mathbf{SO}(5) \simeq \mathbf{Sp}(2)/\text{center} \quad (\ \mathrm{B}_2 = \mathrm{C}_2\ )$$

$$\mathbf{SO}(4) \simeq (\mathbf{SU}(2) \times \mathbf{SU}(2))/\Delta(\mu_2) \quad (\ \mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1\ )$$

$$\mathbf{SO}(3) \simeq \mathbf{SU}(2)/\mu_2 \quad (\ \mathrm{B}_1 = \mathrm{A}_1\ )$$

$$\mathbf{SO}(2) \simeq S^1$$

Appendix Z. Representation ring and algebraicity of compact Lie groups

Z.1. **Introduction.** Let G be a nontrivial connected compact Lie group that is semisimple and simply connected (e.g., $\mathrm{SU}(n)$ for $n \geq 2$, $\mathrm{Sp}(n)$ for $n \geq 1$, or $\mathrm{Spin}(n)$ for $n \geq 3$). Let T be a maximal torus, with $r = \dim T > 0$, and let $\Phi = \Phi(G, T)$ the associated root system.

Let $B = \{a_i\}$ be a basis of Φ, so $B^\vee := \{a_i^\vee\}$ is a basis for $\Phi^\vee$. In particular, $B^\vee$ is a $\mathbf{Z}$-basis for the coroot lattice $\mathbf{Z}\Phi^\vee$ that is dual to the weight lattice P. Since G is simply connected, $\mathbf{Z}\Phi^\vee = \mathrm{X}_*(T)$. Passing to $\mathbf{Z}$-duals, we have $\mathrm{X}(T) = P$. Since $B^\vee$ is a $\mathbf{Z}$-basis for $\mathbf{Z}\Phi^\vee = \mathrm{X}_*(T)$, the dual lattice $\mathrm{X}(T)$ has a corresponding dual basis $\{\varpi_i\}$ characterized by the condition $\langle \varpi_i, a_j^\vee \rangle = \delta_{ij}$. We call these ϖ_i's the *fundamental weights* with respect to (G, T, B).

The closure $\overline{K(B)}$ of the Weyl chamber $K(B)$ in $\mathrm{X}(T)_{\mathbf{R}}$ corresponding to B is characterized by the property of having non-negative pairing against each $a_j^\vee$, so

$$\mathrm{X}(T) \cap \overline{K(B)} = P \cap \overline{K(B)} = \sum_i \mathbf{Z}_{\geq 0} \cdot \varpi_i;$$

these are called the *dominant weights* with respect to B.

By the Theorem of the Highest Weight, any irreducible (continuous finite-dimensional $\mathbf{C}$-linear) representation V of G has a unique "highest weight" χ that is dominant, the weight space for which is 1-dimensional, and every χ arises in this way from a unique such V, denoted as V_χ. The representations $V_{\varpi_1}, \dots, V_{\varpi_r}$ are the *fundamental representations* for G.

Example Z.1.1. Consider $G = \mathrm{SU}(n)$ with maximal torus T given by the diagonal. Then $\mathrm{X}(T) = \mathbf{Z}^{\oplus n}/\Delta$ (where Δ is the diagonal copy of $\mathbf{Z}$), and its dual $\mathrm{X}_*(T) \subset \mathbf{Z}^{\oplus n}$ is the "hyperplane" defined by $\sum x_j = 0$. Letting $\{e_i\}$ denote the standard basis of $\mathbf{Z}^{\oplus n}$, and $\overline{e}_i$ the image of e_i in the quotient $\mathrm{X}(T)$ of $\mathbf{Z}^{\oplus n}$, we have $\Phi = \{\overline{e}_i - \overline{e}_j \mid i \neq j\}$ with a basis

$$B = \{a_i = \overline{e}_i - \overline{e}_{i+1} \mid 1 \leq i \leq n-1\}.$$

The reader can check that $B^\vee = \{a_i^\vee = e_i^* - e_{i+1}^*\}$, where $\{e_i^*\}$ is the dual basis in $\mathbf{Z}^{\oplus n}$ to the standard basis $\{e_i\}$ of $\mathbf{Z}^{\oplus n}$, and $\overline{K(B)} = \{\sum x_i \overline{e}_i \mid x_i \in \mathbf{R}, x_1 \geq \dots \geq x_n\}$. From this it follows that

$$\varpi_i = \overline{e}_1 + \dots + \overline{e}_i$$

for $1 \leq i \leq n-1$. In HW10 Exercise 4 one finds that the corresponding representation V_{ϖ_i} is $\wedge^i(\rho_{\mathrm{std}})$ where ρ_{std} is the standard n-dimensional representation of G.

Example Z.1.2. For the group $\mathrm{Spin}(2n)$ with $n \geq 4$, the diagram of type D_n has two "short legs" whose extremal vertices (as elements of a basis of the root system) have corresponding fundamental weights ϖ_{n-1} and ϖ_n whose associated fundamental representations of $\mathrm{Spin}(2n)$ called the *half-spin* representations. These do *not* factor through the central quotient $\mathrm{SO}(2n)$, and they are constructed explicitly via Clifford-algebra methods.

For any dominant weight $\lambda = \sum_i n_i \varpi_i$ (with integers $n_1, \dots, n_r \geq 0$), we saw in class that the irreducible representation V_λ of G with highest weight λ occurs exactly once as a subrepresentation of

$$V(n_1, \dots, n_r) := V_{\varpi_1}^{\otimes n_1} \otimes \dots \otimes V_{\varpi_r}^{\otimes n_r}$$

due to the Theorem of the Highest Weight.

Recall that the commutative *representation ring*

$$R(G) = \bigoplus \mathbf{Z}[\rho]$$

is the free abelian group generated by the irreducible representations ρ of G and it has the ring structure given by $[\rho] \cdot [\rho'] = \sum c_\sigma[\sigma]$ where $\rho \otimes \rho' = \bigoplus \sigma^{\oplus c_\sigma}$. The ring $R(G)$ is a subring of the ring of $\mathbf{C}$-valued class functions on G via the formation of characters (e.g., $[\rho] \mapsto \chi_\rho$).

We saw long ago via Weyl's theorems on maximal tori that for *any* connected compact Lie group G (not necessarily semisimple or simply connected) and maximal torus $T \subset G$, restriction of class functions to T (or of G-representations to T-representations) makes $R(G)$ naturally a subring of $R(T)^W$, where $W := W(G,T) = N_G(T)/T$ acts in the natural way on $R(T)$. The aim of this appendix is to use the Theorem of the Highest Weight and fundamental representations to prove that $R(G) = R(T)^W$ in general (using reduction to the simply connected semisimple case, where deeper structural results are available such as the Theorem of the Highest Weight), and to discuss some further results in the representation theory of compact Lie groups.

Remark Z.1.3. For each finite-dimensional representation V of G, the associated character χ_V is an element of $R(G)$. The elements obtained in this way constitute the subset $R_{\text{eff}}(G) \subset R(G)$ of "effective" elements: the $\mathbf{Z}_{\geq 0}$-linear combinations of $[\rho]$'s for irreducible ρ. (If $V \simeq \bigoplus_\sigma \sigma^{e_\sigma}$ for pairwise distinct irreducible σ then $\chi_V = \sum e_\sigma \chi_\sigma$, with $\chi_\sigma = [\sigma]$ in $R(G)$.)

The subset $R_{\text{eff}}(T) \subset R(T)$ is W-stable and $R_{\text{eff}}(G) \subset R_{\text{eff}}(T)^W$. Our proof of the equality $R(G) = R(T)^W$ will involve much use of subtraction, so the proof certainly does not address how much bigger $R_{\text{eff}}(T)^W$ is than $R_{\text{eff}}(G)$. In general $R_{\text{eff}}(T)^W$ is much larger than $R_{\text{eff}}(G)$, so it is not obvious how to determine in terms of the language of $R(T)^W$ whether a given element of $R(G)$ comes from an actual representation of G or not.

For example, if $G = \mathrm{SU}(2)$ and T is the diagonal torus then $R(T)$ is equal to the Laurent polynomial ring $\mathbf{Z}[t, t^{-1}]$ on which the nontrivial element of $W = \mathfrak{S}_2$ acts through swapping t and t^{-1}, so

$$R(T)^W = \mathbf{Z}[t+t^{-1}] = \mathbf{Z} + \sum_{n>0} \mathbf{Z}(t^n + t^{-n}) \supset \mathbf{Z}_{\geq 0} + \sum_{n>0} \mathbf{Z}_{\geq 0}(t^n + t^{-n}) = R_{\text{eff}}(T)^W.$$

But HW6 Exercise 4(ii) gives $R(G) = \mathbf{Z}[t+t^{-1}]$, and inside here $R_{\text{eff}}(G) = \mathbf{Z}_{\geq 0}[t+t^{-1}]$. The latter does not contain $t^m + t^{-m}$ for any $m > 1$ since such containment would yield an identity of Laurent polynomials

$$t^m + t^{-m} = c_0 + \sum_{1 \leq i \leq m} c_i (t+t^{-1})^i$$

with integers $c_i \geq 0$, and by the positivity of binomial coefficients it is clear that no such identity is possible.

The upshot is that using $R(T)^W$ to describe $R(G)$ does not easily keep track of even very basic questions related to identifying characters of actual (rather than just virtual) representations of G.

Z.2. **The equality** $R(G) = R(T)^W$**.** As a warm-up to showing that $R(G)$ exhausts $R(T)^W$, we first show that if G is semisimple and simply connected then $R(G)$ is a polynomial ring on the fundamental representations:

Proposition Z.2.1. *Assume G is semisimple and simply connected. The map $\mathbf{Z}[Y_1, \dots, Y_r] \to R(G)$ defined by $Y_j \mapsto [V_{\varpi_j}]$ is an isomorphism.*

Proof. First we prove surjectivity, by showing $[\rho]$ is hit for each irreducible ρ. The case of trivial ρ is obvious (as $[1]$ is the identity element of $R(G)$), so suppose ρ is nontrivial. It is given by V_λ for $\lambda = \sum n_i \varpi_i$ with $n_i \geq 0$ not all 0. Note that $V := \bigotimes V_{\varpi_i}^{\otimes n_i}$ is typically not irreducible, but it contains ρ as an irreducible subrepresentation *with multiplicity one* and all other dominant T-weights occurring in V are $< \lambda$ with respect to the lexicographical order on $\mathrm{X}(T) = \bigoplus \mathbf{Z}\varpi_i$ (by the very meaning of λ being the "highest weight" of ρ).

By the Theorem of the Highest Weight, $\prod [V_{\varpi_i}]^{n_i}$ differs from $[\rho]$ by the $\mathbf{Z}$-span of "lower-weight" irreducible representations (i.e., those whose highest weights are $< \lambda$), so induction with respect to the lexicographical order on $\mathrm{X}(T) = \bigoplus \mathbf{Z}\varpi_i$ yields surjectivity. (In effect, we build up an expression for $[\rho]$ as a $\mathbf{Z}$-linear combination of monomials in the Y_j's.) This completes the proof of surjectivity.

To prove injectivity, consider a hypothetical non-trivial $\mathbf{Z}$-linear dependence relation in $R(G)$ on pairwise distinct monomials in the $[V_{\varpi_i}]$'s. In this relation, some unique monomial term is maximal for the lexicographical ordering. Rewrite the dependence relation as an equality with non-negative coefficients on both sides:

$$c \prod [V_{\varpi_i}]^{n_i} + \cdots = \ldots$$

in $R(G)$, where c is a positive integer and the omitted terms involve monomials in the V_{ϖ_i}'s which are strictly smaller in the lexicographical order. Such an equality of elements of $R(G)$ with non-negative coefficients corresponds to an *isomorphism of representations* since it expresses an equality of the corresponding characters as class functions on G, so we get an isomorphism of representations

$$V(n_1, \ldots, n_r)^{\oplus c} \oplus V' \simeq V''$$

where the irreducible representation $\rho := V_{\sum n_i \varpi_i}$ occurs once in $V(n_1, \ldots, n_r)$ (by the Theorem of the Highest Weight) and not at all in V' or V'' (since all of the irreducible constituents of V' and V'' have highest weights $< \sum n_i \varpi_i$ by design). Comparing complete reducibility on both sides, ρ occurs with multiplicity c on the left side and not at all on the right side, a contradiction. $\square$

The preceding result for semisimple G that is simply connected yields a result in general:

Corollary Z.2.2. *For any connected compact Lie group G, the inclusion $R(G) \subset R(T)^W$ is an equality.*

Proof. First we treat the case when G is semisimple and simply connected. Fix a basis B of $\Phi = \Phi(G, T)$ to define a notion of "dominant weight". The inclusion $\mathrm{X}(T) \subset P$ into the weight lattice P of the root system $\Phi(G, T)$ is an equality since G is simply connected, so the representation ring $R(T) = \mathbf{Z}[\mathrm{X}(T)]$ is equal to the group ring $\mathbf{Z}[P]$ compatibly with the natural action of $W = W(G, T) = W(\Phi)$ throughout. Hence, viewing $R(T)$ as the free $\mathbf{Z}$-module on the set P, $R(T)^W$ is a free $\mathbf{Z}$-module with basis given by the sums (without multiplicity!) along each W-orbit in P (why?).

Since W acts simply transitively on the set of Weyl chambers, and $\mathrm{X}(T)_{\mathbf{R}}$ is covered by the closures of the Weyl chambers, $\mathrm{X}(T)_{\mathbf{R}}$ is covered by the members of the W-orbit of $\overline{K(B)}$, where $K(B)$ is the Weyl chamber corresponding to the basis B of Φ. Thus, each W-orbit of an element of $P = \mathrm{X}(T)$ meets $\overline{K(B)}$. But $\overline{K(B)} \cap P = \bigoplus \mathbf{Z}_{\geq 0} \cdot \varpi_i$, so each W-orbit in P contains a dominant weight.

If V is a (finite-dimensional continuous $\mathbf{C}$-linear) representation of G then a 1-dimensional character of T has multiplicity in the T-restriction $V|_T$ equal to that of any member of its W-orbit in $\mathrm{X}(T)$ since $W = N_G(T)/T$. In particular, $V_{\varpi_i}|_T$ viewed in $R(T)^W$ involves the sum (without multiplicity!) of the W-orbit of ϖ_i *exactly once*, due to the multiplicity aspect of the Theorem of the Highest Weight, and all other W-orbits occurring in $V_{\varpi_i}|_T$ (if any arise) are for dominant weights *strictly smaller* than ϖ_i.

We conclude that under the composite map

$$\mathbf{Z}[Y_1, \dots, Y_r] \simeq R(G) \hookrightarrow R(T)^W,$$

if we arrange W-orbits in P according to the biggest dominant weight that occurs in each (there might be more than dominant weight in an orbit, for weights on the boundary of the Weyl chamber $K(B)$) then for a given element of $R(T)^W$ we can subtract off a unique $\mathbf{Z}$-multiple of a unique monomial in the Y_j's to reduce the unique biggest weight that remains under the W-orbit decomposition (or arrive at 0). In this manner, we see by descending inductive considerations that $\mathbf{Z}[Y_1, \dots, Y_r] \to R(T)^W$ is surjective, forcing $R(G) = R(T)^W$. That settles the case when G is semisimple and simply connected.

The rest of the argument is an exercise in the general structure theory of compact Lie groups via tori and semisimple groups, as follows. The case $G = T$ (so $W = 1$) is trivial. In general, if $H \to G'$ is the finite-degree universal cover of the semisimple derived group of G and Z is the maximal central torus of G then

$$G = (Z \times H)/\mu$$

for a finite central closed subgroup $\mu \subset Z \times H$.

If G_1 and G_2 are (possibly disconnected) compact Lie groups then the natural map

$$R(G_1) \otimes_{\mathbf{Z}} R(G_2) \to R(G_1 \times G_2)$$

is an isomorphism. Indeed, this expresses the fact that irreducible representations of $G_1 \times G_2$ are precisely $(g_1, g_2) \mapsto \rho_1(g_1) \otimes \rho_2(g_2)$ for uniquely determined irreducible representations (V_i, ρ_i) of G_i, a result that is well-known for finite groups and carries over without change to compact Lie groups since the latter have a similarly robust character theory. Since the maximal tori of $G_1 \times G_2$ are exactly $T = T_1 \times T_2$ for maximal tori $T_i \subset G_i$, and the corresponding Weyl group $W = W(G, T)$ is naturally $W_1 \times W_2$ for $W_i = W(G_i, T_i)$, we have

$$R(T_1 \times T_2)^W = (R(T_1) \otimes_{\mathbf{Z}} R(T_2))^{W_1 \times W_2} = R(T_1)^{W_1} \otimes_{\mathbf{Z}} R(T_2)^{W_2}$$

(the final equality using that representation rings are torsion-free and hence $\mathbf{Z}$-flat). Hence, if the equality "$R(G) = R(T)^W$" holds for a pair of connected compact Lie groups then it holds for their direct product. In particular, the desired equality holds for $Z \times H$ by the settled cases of tori and simply connected semisimple compact Lie groups.

It remains to show that if $G = \mathscr{G}/\mu$ is a central quotient of a connected compact Lie group $\mathscr{G}$, so $T = \mathscr{T}/\mu$ for a unique maximal torus $\mathscr{T} \subset \mathscr{G}$ and the natural map

$$W(\mathscr{G}, \mathscr{T}) = N_{\mathscr{G}}(\mathscr{T})/\mathscr{T} \to N_G(T)/T = W(G, T)$$

is an equality (easy from the definitions, as we saw in class), then the equality $R(\mathscr{G}) = R(\mathscr{T})^W$ implies that $R(G) = R(T)^W$. In other words, consider the commutative diagram

$$\begin{tikzcd} R(\mathscr{G}) \arrow[r] & R(\mathscr{T})^W \\ R(G) \arrow[r] \arrow[u] & R(T)^W \arrow[u] \end{tikzcd}$$

whose vertical maps are natural inclusions (composing representations of G and T with the quotient maps $\mathscr{G} \twoheadrightarrow G$ and $\mathscr{T} \twoheadrightarrow T$ respectively) and horizontal maps are natural inclusions (restricting $\mathscr{G}$-representations to $\mathscr{T}$-representations and restricting G-representations to T-representations). We claim that if the top is an equality then so is the bottom.

Thinking in terms of class functions, it is clear via respective $\mathbf{Z}$-bases of characters of irreducible representations that the subset $R(T) \subset R(\mathscr{T})$ is the $\mathbf{Z}$-span of the irreducible characters trivial on Z, and likewise (using Schur's Lemma!) for $R(G) \subset R(\mathscr{G})$. Hence, $R(T)$ consists of the elements of $R(\mathscr{T})$ that as class functions on $\mathscr{T}$ are invariant under Z-translation on $\mathscr{T}$, and likewise for $R(G)$ in relation to $R(\mathscr{G})$. (Keep in mind that Z is central in $\mathscr{G}$, so composing with translation by any $z \in Z$ carries class functions to class functions.) Thus, passing to W-invariants, $R(T)^W$ consists of elements of $R(\mathscr{T})^W = R(\mathscr{G})$ that as class functions on $\mathscr{T}$ are invariant under Z-translation on $\mathscr{T}$. So it remains to check that if a class function f on $\mathscr{G}$ has $\mathscr{T}$-restriction that is invariant under Z-translation on $\mathscr{G}$ then f is also Z-invariant on $\mathscr{G}$. But a class function on $\mathscr{G}$ is *uniquely determined* by its $\mathscr{T}$-restriction (why?), and $Z \subset \mathscr{T}$, so indeed the Z-invariance of a class function on G can be checked on its restriction to $\mathscr{T}$. $\square$

Z.3. **Further refinements.** In [BtD, Ch. III], functional analysis is used to obtain some important results concerning Hilbert-space representations of compact Lie groups G (i.e., continuous homomorphism $G \to \mathrm{Aut}(H)$ into the topological group of bounded linear automorphisms of a Hilbert space H). This provides the inspiration for many considerations in the more subtle non-compact case. The following two big theorems are proved in Chapter III:

Theorem Z.3.1. *Every irreducible Hilbert representation H of a compact Lie group G (i.e., no nonzero proper closed G-stable subspaces) is finite-dimensional.*

This result explains the central importance of the finite-dimensional case in the representation theory of compact groups. A natural infinite-dimensional Hilbert representation of G is $L^2(G, \mathbf{C})$ with the right regular action $g.f = f \circ r_g$, where $r_g : x \mapsto xg$ (this is a left action on $L^2(G, \mathbf{C})$!). For finite G this is just the group ring $\mathbf{C}[G]$ on which G acts through right multiplication composed with inversion on G (as the point mass $[g] \in \mathbf{C}[G]$ is carried to the point mass $[gh^{-1}]$ under the right-regular action of $h \in G$). Generalizing the well-known decomposition of $\mathbf{C}[G]$ as a G-representation for finite G, one has the much deeper:

Theorem Z.3.2 (Peter–Weyl)**.** *Let G be a compact Lie group. Equip the space $C^0(G)$ of continuous $\mathbf{C}$-valued functions on G with a hermitian structure via $\langle f_1, f_2 \rangle = \int_G f_1(g)\overline{f}_2(g)\mathrm{d}g$ for the volume-1 Haar measure $\mathrm{d}g$ on G, having completion $L^2(G, \mathbf{C})$.*

- *(i) Every irreducible finite-dimensional continuous $\mathbf{C}$-linear representation V of G occurs inside $L^2(G, \mathbf{C})$ with finite multiplicity $\dim(V)$.*

  Upon equipping each such V with its G-invariant inner product that is unique up to a scaling factor (by irreducibility), $L^2(G, \mathbf{C})$ is the Hilbert direct sum of the representations $V^{\oplus \dim(V)}$.
- *(ii) The group G admits a faithful finite-dimensional representation over $\mathbf{C}$; i.e., there is an injective continuous linear representation $G \to \mathrm{GL}_n(\mathbf{C})$ for some n, or equivalently G arises as a compact subgroup of some $\mathrm{GL}_n(\mathbf{C})$.*

(iii) The pairwise-orthogonal characters χ_V of the irreducible representations of G span a dense subspace of the Hilbert space of continuous $\mathbf{C}$-valued class functions on G. In particular, a continuous class function $f : G \to \mathbf{C}$ that is nonzero must satisfy $\langle f, \chi_V \rangle \neq 0$ for some irreducible V.

Moreover, the $\mathbf{C}$-subalgebra of $C^0(G)$ generated by the finitely many "matrix coefficients" $a_{ij} : G \to \mathbf{C}$ for a single faithful continuous representation $\rho : G \to \mathrm{GL}_n(\mathbf{C})$ is dense.

The proof of the Weyl character formula in [BtD], based on analytic methods, uses (iii). There is a purely algebraic proof of the Weyl character formula (usually expressed in terms of semisimple Lie algebras over $\mathbf{C}$, and given in books on Lie algebras such as by Humphreys), but that involves infinite-dimensional algebraic tools (such as Verma modules).

We conclude our discussion by highlighting the use of "matrix coefficients" from (iii) to algebraize the theory of compact Lie groups. The starting point is the following interesting property of the collection of matrix coefficients $a_{ij} : G \to \mathbf{C}$ for a single representation $\rho : G \to \mathrm{GL}_n(\mathbf{C})$: the identity $\rho(gh) = \rho(g)\rho(h)$ for $g, h \in G$ says $a_{ij}(gh) = \sum_k a_{ik}(g)a_{kj}(h)$ for all i, j, or equivalently

$$a_{ij} \circ r_h = \sum_k a_{kj}(h) \cdot a_{ik}$$

for all $h \in G$ and i, j. In other words, under the right-regular representation of G on $C^0(G)$ we have

$$g.a_{ij} = \sum_k a_{kj}(g) a_{ik}$$

for all $g \in G$ and all i, j. Hence, inside the representation $C^0(G)$ of G, the vectors $a_{ij} \in C^0(G)$ are "G-finite" in the sense that the G-orbit of a_{ij} is contained in the finite-dimensional $\mathbf{C}$-vector space $\sum_k \mathbf{C} \cdot a_{ik}$.

It is clear that sums and products of G-finite vectors in $C^0(G)$ are G-finite, so by (iii) above we see that the subspace of G-finite vectors in $L^2(G)$ is *dense*. Note that the functions a_{ij} are given by $g \mapsto e_i^*(g.e_j)$ where $\{e_1, \dots, e_n\}$ is the standard basis of the representation space $\mathbf{C}^n$ for ρ and $\{e_1^*, \dots, e_n^*\}$ is the dual basis. In [BtD, Ch. III, Prop. 1.2], one finds a proof of the remarkable converse result that *every* G-finite vector in $C^0(G)$ is a matrix coefficient $g \mapsto \ell(g.v)$ for some finite-dimensional continuous $\mathbf{C}$-linear representation V of G, some vector $v \in V$, and some linear form $\ell \in V^*$.

This brings us to the core of the algebraization of the theory, *Tannaka–Krein Duality* (proved in [BtD, Ch. III]). Consider the $\mathbf{R}$-subspace $A(G) \subset C^0(G, \mathbf{R})$ consisting of the G-finite vectors. This is an $\mathbf{R}$-subalgebra (why?), and the existence of a faithful finite-dimensional representation of G over $\mathbf{C}$ can be used to prove that $A(G)$ is finitely generated as an $\mathbf{R}$-algebra. Moreover, in [BtD, Ch. III] it is proved that the natural map

$$A(G_1) \otimes_{\mathbf{R}} A(G_2) \to A(G_1 \times G_2)$$

an isomorphism for compact Lie groups G_1 and G_2. Thus, the map of topological spaces $m : G \times G \to G$ give by multiplication induces a map of $\mathbf{R}$-algebras

$$m^* : A(G) \to A(G \times G) = A(G) \otimes_{\mathbf{R}} A(G)$$

such that the resulting composition law on the topological space

$$G^{\mathrm{alg}} = \mathrm{Hom}_{\mathbf{R}\text{-alg}}(A(G), \mathbf{R})$$

via

$$G^{\text{alg}} \times G^{\text{alg}} = \text{Hom}_{\mathbf{R}\text{-alg}}(A(G) \otimes_{\mathbf{R}} A(G), \mathbf{R}) \xrightarrow{m^*} G^{\text{alg}}$$

is a continuous group law. Most remarkably of all, G^{alg} can be identified with a (smooth) Zariski-closed subgroup of some $\text{GL}_n(\mathbf{R})$ (so it is a "matrix Lie group" over $\mathbf{R}$) with the natural map $G \to G^{\text{alg}}$ carrying g to evaluation ev_g at g an isomorphism of Lie groups. In this sense, G is recovered from the $\mathbf{R}$-algebra $A(G)$ consisting of matrix coefficients for representations of G over $\mathbf{R}$.

Example Z.3.3. The $\mathbf{R}$-algebra $A(\text{SO}(n))$ is $\mathbf{R}[x_{ij}]/(XX^\top = 1)$.

The following amazing theorem, beyond the level of the course, expresses the precise link between compact Lie groups and purely algebro-geometric notions over $\mathbf{R}$.

Theorem Z.3.4. *The functor $G \rightsquigarrow \text{Spec}(A(G))$ is an equivalence of categories from the category of compact Lie groups to the category of smooth affine group schemes $\mathscr{G}$ over $\mathbf{R}$ such that $\mathscr{G}$ does not contain GL_1 as a Zariski-closed $\mathbf{R}$-subgroup and $\mathscr{G}(\mathbf{R})$ meets every Zariski-connected component of $\mathscr{G}$. The inverse functor is $\mathscr{G} \rightsquigarrow \mathscr{G}(\mathbf{R})$, and G^0 goes over to $\mathscr{G}^0$.*

This explains why all examples of compact Lie groups that we have seen in this course, *and* all Lie group homomorphisms between them, are given by polynomial constructions in matrix entries (possibly decomposed into real and imaginary parts when the compact Lie groups are presented as closed subgroups of $\text{GL}_n(\mathbf{C})$'s).

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