Math 210C. Smoothness of inversion

The aim of this handout is to prove that smoothness of inversion can be dropped from the definition of a Lie group (i.e.., it is a consequence of the other conditions in the definition).

Theorem 0.1. *Let G be a C^∞-manifold and suppose it is equipped with a group structure such that the composition law $m : G \times G \to G$ is C^∞. Then inversion $G \to G$ is C^∞ (in particular, continuous!).*

This will amount to an exercise in the Chain Rule and the behavior of tangent spaces on product manifolds. The most important lesson from the proof is that the tangent map $\mathrm{d}m(e,e) : \mathrm{T}_e(G) \oplus \mathrm{T}_e(G) \to \mathrm{T}_e(G)$ is $(v,w) \mapsto v + w$ (so near the identity point, to first order *all* C^∞ group laws look like vector addition).

The proof will freely use that in general with product manifolds, the *linear* identification

$$\mathrm{T}_{(x_0,y_0)}(X \times Y) \simeq \mathrm{T}_{x_0}(X) \oplus \mathrm{T}_{y_0}(Y)$$

is defined in both directions as follows. Going from left to right, the first component $\mathrm{T}_{(x_0,y_0)}(X \times Y) \to \mathrm{T}_{x_0}(X)$ of this identification is $\mathrm{d}(\mathrm{pr}_1)(x_0,y_0)$ where $\mathrm{pr}_1 : X \times Y \to X$ is the projection $(x,y) \mapsto x$, and similarly for the second component (using $\mathrm{pr}_2 : X \times Y \to Y$). Going from right to left, the first factor inclusion

$$\mathrm{T}_{x_0}(X) \hookrightarrow \mathrm{T}_{x_0}(X) \oplus \mathrm{T}_{y_0}(Y) \simeq \mathrm{T}_{(x_0,y_0)}(X \times Y)$$

(using $v \mapsto (v,0)$ for the initial step) is $\mathrm{d}(i_{y_0})(x_0,y_0)$ where $i_{y_0} : X \to X \times Y$ is the "right slice" map defined by $x \mapsto (x, y_0)$, and the second factor inclusion of $\mathrm{T}_{y_0}(Y)$ into $\mathrm{T}_{(x_0,y_0)}(X \times Y)$ is similar using the "left slice" map $j_{x_0} : Y \to X \times Y$ defined by $y \mapsto (x_0, y)$.

Proof. (of Theorem) Consider the "shearing transformation"

$$\Sigma : G \times G \to G \times G$$

defined by $\Sigma(g,h) = (g, gh)$. This is bijective since we are using a group law, and it is C^∞ since the composition law m is assumed to be C^∞. (Recall that if M, M', M'' are C^∞ manifolds, a map $M \to M' \times M''$ is C^∞ if and only if its component maps $M \to M'$ and $M \to M''$ are C^∞, due to the nature of product manifold structures.)

We claim that Σ is a diffeomorphism (i.e., its inverse is C^∞). Granting this,

$$G = \{e\} \times G \to G \times G \stackrel{\Sigma^{-1}}{\to} G \times G$$

is C^∞, but explicitly this composite map is $g \mapsto (g, g^{-1})$, so its second component $g \mapsto g^{-1}$ is C^∞ as desired. Since Σ is a C^∞ bijection, the C^∞ property for its inverse is equivalent to Σ being a *local isomorphism* (i.e., each point in its source has an open neighborhood carried diffeomorphically onto an open neighborhood in the target). By the Inverse Function Theorem, this is equivalent to the isomorphism property for the tangent map

$$\mathrm{d}\Sigma(g,h) : \mathrm{T}_g(G) \oplus \mathrm{T}_h(G) = \mathrm{T}_{(g,h)}(G \times G) \to \mathrm{T}_{(g,gh)}(G \times G) = \mathrm{T}_g(G) \oplus \mathrm{T}_{gh}(G)$$

for all $g, h \in G$ (where $\mathrm{d}f(x) : \mathrm{T}_x(X) \to \mathrm{T}_{f(x)}(Y)$ is the tangent map for $f : X \to Y$).

We shall now use left and right translations to reduce this latter "linear" problem to the special case $g = h = e$, and in that special case we will be able to compute the tangent map explicitly and see the isomorphism property by inspection. For any $g \in G$, let $\ell_g : G \to G$ be the left translation map $x \mapsto gx$; this is C^∞ with inverse $\ell_{g^{-1}}$. Likewise, let $r_g : G \to G$ be the right translation map $x \mapsto xg$, which is also a diffeomorphism (with inverse $r_{g^{-1}}$). Note that $\ell_g \circ r_h = r_h \circ \ell_g$ (check!), and for any $g, h \in G$,

$$(\ell_g \times (\ell_g \circ r_h)) \circ \Sigma = \Sigma \circ (\ell_g \times r_h)$$

since evaluating both sides at any $(g', h') \in G \times G$ yields $(gg', g(g'h')h) = (gg', (gg')(h'h))$. But $\ell_g \times r_h$ is a diffeomorphism carrying (e, e) to (g, h), so the Chain Rule applied to the tangent maps induced by both sides of this identity at (e, e) yields that $\mathrm{d}\Sigma(e, e)$ is an isomorphism if and only if $\mathrm{d}\Sigma(g, h)$ is an isomorphism. Hence, indeed it suffices to treat the tangential isomorphism property only at (e, e).

Let $\mathfrak{g} := \mathrm{T}_e(G)$, so we wish to describe the map

$$\mathrm{d}\Sigma(e, e) : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}.$$

It suffices to show that this map is $(v, w) \mapsto (v, v + w)$, the "linear" version of the shearing map construction (since it is clearly an isomorphism). This says that the map $\mathrm{d}\Sigma(e, e)$ has as its first and second components the respective maps $(v, w) \mapsto v$ and $(v, w) \mapsto v + w$.

By the Chain Rule and the *definition* of the direct sum decomposition for tangent spaces on product manifolds (reviewed at the start), these components are the respective differentials at (e, e) of the components $\mathrm{pr}_1 \circ \Sigma = \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ \Sigma = m$ of Σ. Since tangent maps are *linear* and $(v, w) = (v, 0) + (0, w)$, our problem is to show that the compositions of $\mathrm{d}(\mathrm{pr}_1)(e, e)$ with the inclusions of $\mathfrak{g}$ into the respective first and second factors of $\mathfrak{g} \oplus \mathfrak{g}$ are $v \mapsto v$ and $w \mapsto 0$ and the compositions of $\mathrm{d}m(e, e)$ with those inclusions are $v \mapsto v$ and $w \mapsto w$ respectively.

Since factor inclusions at the level of tangent spaces are differentials of the associated slice maps at the level of manifolds, by the Chain Rule we want to show that the maps

$$\mathrm{pr}_1 \circ i_e, \mathrm{pr}_1 \circ j_e, m \circ i_e, m \circ j_e$$

from G to G carrying e to e have associated differentials $\mathfrak{g} \to \mathfrak{g}$ at e respectively equal to

$$v \mapsto v, w \mapsto 0, v \mapsto v, w \mapsto w.$$

But clearly $\mathrm{pr}_1 \circ i_e : g \mapsto \mathrm{pr}_1(g, e) = g$ is the identity map of G (so its differential on every tangent space is the identity map), and by the group law axioms $m \circ i_e$ and $m \circ j_e$ are also the identity map of G (and hence also have differential equal to the identity map on every tangent space). Finally, the map $\mathrm{pr}_1 \circ j_e$ is the map $g \mapsto \mathrm{pr}_1(e, g) = e$ that is the *constant map* to e, so its differential at every point vanishes. ■

Since $m \circ (\mathrm{id}_G, \mathrm{inv}) : G \to G \times G \to G$ is the *constant map* to e (i.e., $m(g, g^{-1}) = e$), its differential vanishes on all tangent spaces. Computing at the point e on the source, it follows via the Chain Rule and our computation of $\mathrm{d}m(e, e)$ as ordinary addition that the map

$$\mathrm{d}(\mathrm{id}_G)(e) + \mathrm{d}(\mathrm{inv})(e) : \mathfrak{g} \to \mathfrak{g}$$

vanishes. But $\mathrm{d}(\mathrm{id}_G)(g) : \mathrm{T}_g(G) \to \mathrm{T}_g(G)$ is the identity map for any $g \in G$, so by setting $g = e$ we conclude that $v + (\mathrm{d}(\mathrm{inv})(e))(v) = 0$ for all $v \in \mathfrak{g}$. In other words:

Corollary 0.2. *The tangent map* $\mathrm{d}(\mathrm{inv})(e) : \mathfrak{g} \to \mathfrak{g}$ *is negation.*

This says that near the identity point, to first order the inversion in every Lie group looks like vector negation.