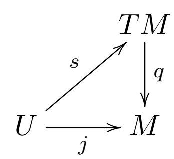
Math 210C. Local and global Frobenius theorems

In this handout, we explain how to relate Lie subalgebras to connected Lie subgroups by using a higher-dimensional version of the theory of integral curves to vector fields. This rests on some serious work in differential geometry, for the core parts of which (beyond our earlier work on integral curves to vector fields and the associated global flow) we refer to a well-written book for a good self-contained account.

1. Subbundles

For ease of discussion we want to use the language of "vector bundles", but we don't need much about this beyond a few basic things of "algebraic" nature in the specific setting of smooth vector fields on a smooth manifold. Thus, we now introduce some notions with an eye towards what is relevant later.

In this handout, the *tangent bundle* TM of a smooth manifold is shorthand for the assignment to each open set $U \subseteq M$ of the $C^\infty(U)$-module $\mathrm{Vec}_M(U)$ of smooth vector fields on U; this may also be denoted as $(TM)(U)$. In differential geometry there is geometric object given the same name, so for the sake of completeness let's briefly describe how that goes (which we will never use in what follows). The geometers build a smooth manifold TM equipped with a submersion $q : TM \to M$ for which several properties are satisfied. Firstly, there is an identification $q^{-1}(m) = \mathrm{T}_m(M)$ for all $m \in M$. Second, for every open set $U \subseteq M$ the set $\Gamma(U, TM)$ of *smooth* cross-sections $s : U \to TM$ to q (i.e., $s(u) \in q^{-1}(u) = \mathrm{T}_u(M)$ for all $u \in U$) is identified with $\mathrm{Vec}_M(U)$ *compatibly with* restriction to open subsets $U' \subseteq U$. Visually, such an s can be viewed as a map filling in a commutative diagram:



where $j : U \hookrightarrow M$ is the natural inclusion. Finally, there is a compatibility between these two kinds of identifications (pointwise and over open subsets of M): for any smooth cross-section $s \in \Gamma(U, TM)$ and its associated smooth vector field $X \in \mathrm{Vec}_M(U)$ the identification $q^{-1}(u) = \mathrm{T}_u(M)$ carries $s(u)$ over to the tangent vector $X(u) \in \mathrm{T}_u(M)$ for all $u \in U$. The interplay of pointwise notions and "over varying open sets" notions is very important.

Definition 1.1. A *subbundle* E of TM is a choice of subspace $E(m) \subseteq \mathrm{T}_m(M)$ for each $m \in M$ that "varies smoothly in m" in the following precise sense: M is covered by open sets U_α over which there are *smooth* vector fields $X_1^{(\alpha)}, \dots, X_{r_\alpha}^{(\alpha)} \in \mathrm{Vec}_M(U_\alpha)$ such that the vectors $X_i^{(\alpha)}(m) \in \mathrm{T}_m(M)$ are a basis of $E(m)$ for all $m \in U_\alpha$. Such $\{U_\alpha\}$ is called a *trivializing cover* of M for E, and $\{X_i^\alpha\}_i$ is called a *trivializing frame* for E over U_α.

Informally, the subspaces $E(m)$ for $m \in U_\alpha$ admit a "common" basis $\{X_i^\alpha(m)\}_i$, and the "smooth variation" of the subspaces $E(m) \subset \mathrm{T}_m(M)$ for varying $m \in U_\alpha$ is encoded in the fact that the X_i^α are *smooth*.

Example 1.2. If $\{x_1, \dots, x_n\}$ is a C^∞ coordinate system on an open subset V of M then the vector fields ∂_{x_i} constitute a trivializing frame for TM over V.

To think about the notion of trivializing frame in a more algebraic manner, modeled on the notion of a basis of a module, it is convenient to introduce some notation: for any open set $U \subseteq M$ we define $E(U)$ to be the set of *smooth* vector fields $X \in \mathrm{Vec}_M(U)$ satisfying $X(m) \in E(m)$ for all

$m \in U$. (For example, if $E = TM$ then $E(U)$ is just $\mathrm{Vec}_M(U)$ by another name.) Clearly $E(U)$ is a $C^\infty(U)$-module. The key point is this:

Lemma 1.3. *Let $\{U_\alpha\}$ be a trivializing cover of M for E, and $\{X_i^\alpha\}$ a trivializing frame for E over U_α. For any open subset $U \subseteq U_\alpha$, a set-theoretic vector field X on U (i.e., an assignment $u \mapsto X(u) \in \mathrm{T}_u(M)$ for all $u \in U$) lies in the subset $E(U) \subseteq \mathrm{Vec}_M(U)$ precisely when $X = \sum a_i X_i^{(\alpha)}|_U$ for smooth functions a_i. In other words, $E(U)$ is a free $C^\infty(U)$-module with basis $\{X_i^\alpha|_U\}$.*

We refer to $\{X_i^{(\alpha)}\}$ $(1 \leq i \leq r_\alpha)$ as a "trivializing frame" for E over U_α: the restrictions X_i^α over *any* open $U \subset U_\alpha$ are a basis for $E(U)$ as a module over $C^\infty(U)$, not just for $U = U_\alpha$!

Proof. Since $X_i^\alpha(m) \in E(m)$ for all $m \in U$, clearly any $C^\infty(U)$-linear combination of the $X_i^\alpha|_U$'s lies in $E(U)$. The real content is to prove the converse: if X is a *smooth* vector field on U and $X(m) \in E(m)$ for all $m \in U$ then X is a $C^\infty(U)$-linear combination of the vector fields $X_i^\alpha|_U$. It is harmless to rename U as M, so we can rephrase the problem as follows. Consider smooth vector fields $X_1, \dots, X_r$ on M that are *pointwise* linearly independent, and a set-theoretic vector field X that is pointwise a linear combination of the X_i's (much as $X(m)$ above is in the span $E(m)$ of the linearly independent set $\{X_i^\alpha(m)\}_i$ for each $m \in U \subseteq U_\alpha$). For the unique pointwise expansions

$$X(m) = \sum a_i(m) X_i(m),$$

we claim that the functions $a_i : M \to \mathbf{R}$ are *smooth* when X is smooth.

Our problem is local on M, so it suffices to work on individual open coordinate domains that cover M. In other words, we can assume that M is an open subset of some $\mathbf{R}^n$, say with coordinates $x_1, \dots, x_n$. The idea, roughly speaking, is to reduce to the case when (after rearranging coordinates) $X_i = \partial_{x_i}$ for $1 \leq i \leq r$, in which case the smoothness of the coefficient functions a_i follows from the *definition* of smoothness for the vector field X (which may be checked using any *single* coordinate system). This isn't quite what we will do, but it gives the right idea.

Consider the expansions

$$X_i = \sum c_{ij} \partial_{x_j}$$

where the coefficient functions $c_{ij} : M \to \mathbf{R}$ are smooth by the smoothness hypothesis on each X_i. For each $m \in M$, the $r \times n$ matrix $(c_{ij}(m))$ has rows that are linearly independent: this expresses that the r vectors $X_i(m) \in \mathrm{T}_m(M)$ are linearly independent for all m. By the equality of "row rank" and "column rank", there are r linearly independent columns. In other words, there is an $r \times r$ submatrix that is invertible. The specific choices of r columns that give an invertible $r \times r$ submatrix of $(c_{ij}(m))$ may vary as we change m, but whatever such choice of columns gives an invertible submatrix at a specific $m_0 \in M$ also works at nearby m since the associated $r \times r$ determinant is nonzero at m_0 and hence is nonzero nearby (since each c_{ij} is continuous). Thus, we can *cover* M by open subsets on each of which a fixed set of r columns gives a pointwise invertible $r \times r$ submatrix (with the specific r columns depending on the open subset).

Since our smoothness assertion for the a_i's is of local nature on M, by passing to such individual open subsets we may now assume that there are r specific columns for which the associated $r \times r$ submatrix is *everywhere* invertible. By rearranging the coordinates, we may assume this is the leftmost $r \times r$ submatrix. Now consider the expansion

$$X(m) = \sum_i a_i(m) X_i(m) = \sum_i \sum_j a_i(m) c_{ij}(m) \partial_{x_j} = \sum_j (\sum_i a_i(m) c_{ij}(m)) \partial_{x_j}.$$

The smoothness hypothesis on X means that the coefficient functions $\sum_i a_i c_{ij}$ are smooth on M for each j. Taking $1 \leq j \leq r$ and letting A be the commutative ring of all $\mathbf{R}$-valued functions on M (no smoothness conditions!), we see that the matrix $C := (c_{ij})_{1\leq i,j\leq r}$ over A carries $(a_1, \dots, a_r) \in A^r$ into $C^\infty(M)^r$. But C is a matrix with entries in the subring $C^\infty(M)$ and its determinant is in $C^\infty(M)^\times$, so by Cramer's Formula its inverse C^{-1} exists over $C^\infty(M)$. Hence, any $(a_1, \dots, a_r) \in C^{-1}(C^\infty(M)^r) = C^\infty(M)^r$ inside A^r. ■

2. Integrable subbundles

For smooth vector fields on M we have the notion of an integral curve, and through any point $m \in M$ there exists a unique "maximal" integral curve $c : I \to M$ to X at m with $I \subseteq \mathbf{R}$ an open interval around 0 (i.e., $c(0) = m$ and $c'(t) = X(c(t))$ for all $t \in I$). There is a higher-dimensional generalization of this notion for subbundles E of TM. Consider a *submanifold* N of M, by which we mean an injective immersion $j : N \to M$. This might not be a topological embedding (consider a densely wrapped line on a torus), but one knows from basic manifold theory (essentially, the Immersion Theorem) that it satisfies a good mapping property: if M' is a smooth manifold and a C^∞-map $M' \to M$ from a smooth manifold M' factors through j *continuously* (i.e., M' lands inside $j(N)$ and the associated map $M' \to N$ is continuous, as is automatic if j is topological embedding) then the resulting map $M' \to N$ is actually C^∞. Hence, a submanifold works as nicely as a subset from the viewpoint of C^∞ maps *provided* we keep track of continuity aspects of maps. This continuity condition cannot be dropped:

Example 2.1. Let $M = \mathbf{R}^2$ and let $N = (-2\pi, 2\pi)$ with $i : N \hookrightarrow M$ an injective immersion whose image is a "figure 8", where $i((-2\pi, 0))$ and $i((0, 2\pi))$ are the two "open halves" of the figure 8 complementary to the crossing point $m_0 = i(0)$, so for small $c > 0$ the restriction $i|_{(-c,c)}$ traces out *one* of the two short intervals in the figure 8 passing through m_0. For an explicit example, we can define

$$i(t) = \begin{cases} (-1 + \cos(t), \sin(t)) \text{ if } t \leq 0, \\ (1 - \cos(t), \sin(t)) \text{ if } t \geq 0. \end{cases}$$

Note that i is *not* a topological embedding (i.e., not a homeomorphism onto its image with the subspace topology) since it carries points very near to $\pm 2\pi$ arbitrarily close to $i(0)$.

Let $M' = (-c, c)$ for small $c > 0$ and let $f : M' \to M$ be an injective immersion that is a homeomorphism onto the *other* short interval in the figure 8 passing through m_0. Then $f(M') \subset i(N)$ but the resulting set-theoretic factorization $M' \to N$ of f through i isn't even continuous (let alone C^∞), due to the same reasoning given for why i isn't a topological embedding.

Definition 2.2. A connected submanifold $j : N \to M$ is an *integral manifold* to a subbundle E of TM if $E(j(n)) = \mathrm{d}j(n)(\mathrm{T}_n(N))$ inside $\mathrm{T}_{j(n)}(M)$ for all $n \in N$; informally, $E(j(n)) = \mathrm{T}_n(N)$ inside $\mathrm{T}_n(M)$ for all $n \in N$. We say such an N is a *maximal* integral manifold to E if it is not contained in a strictly larger one.

The basic question we wish to address is this: when does a given E admit an integral submanifold through any point of M, and when do such integral submanifolds lie in uniquely determined maximal ones? In the special case that the subspaces $E(m) \subseteq \mathrm{T}_m(M)$ are lines, this generalizes a weakened form of the theory of integral curves to smooth vector fields (and their associated "maximal" integral curves): we're considering a situation similar to that of integral curves to *nowhere-vanishing* smooth vector fields, and in fact we're *ignoring* the specific vectors in the vector field and focusing only on the lines spanned by each vector. In particular, we are throwing away the information of the parameterization of the curve and focusing on its actual image inside M.

In contrast with the case of vector fields, for which integral curves *always* exist (at least for short time) through any point, when the vector spaces $E(m)$ have dimension larger than 1 it turns out to be a highly nontrivial condition for integral manifolds to exist through any point. The local and global Frobenius theorems will identify a sufficient condition for such existence, including the refinement of maximal integral manifolds to E through any point. This sufficient condition (which is also necessary, by an elementary argument that we omit) involves the bracket operation $[X, Y] = X \circ Y - Y \circ X$ on $(TM)(U) = \mathrm{Vec}_M(U)$ that is $\mathbf{R}$-bilinear (but *not* $C^\infty(U)$-bilinear!):

Definition 2.3. A subbundle E of the tangent bundle TM is *integrable* when for all open sets $U \subseteq M$ and smooth vector fields $X, Y \in E(U)$ over U lying in the subbundle, we have $[X, Y] \in E(U)$.

This notion is only of interest when $E(m)$ has dimension larger than 1 for $m \in M$: if the $E(m)$'s are lines then locally any element of $E(U)$ is a $C^\infty(U)$-multiple of a single non-vanishing smooth vector field and hence the integrability condition is automatic: $[fX, gX] = (fX(g) - gX(f))X$ for smooth functions f and g on a common open set in M.

Remark 2.4. The local and global Frobenius theorems will make precise the sense in which integrability of E is sufficient for E to be "integrated" in the sense of admitting an integral manifold through any point of m, and in fact a unique maximal one.

A particularly interesting example of an integrable subbundle is the following. Let $M = G$ be a Lie group and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$. In this case, we have a *global* trivialization of TG via the construction of left-invariant vector fields. More specifically, to any $v \in \mathfrak{g}$ we have the associated left-invariant smooth vector field $\widetilde{v}$ on G, and for a basis $\{v_i\}$ of $\mathfrak{g}$ the resulting collection $\{\widetilde{v}_i\}$ is a trivializing frame for TG over the entirety of G. In particular, for *any* open U in M we have an isomorphism

$$C^\infty(U) \otimes_{\mathbf{R}} \mathfrak{g} = \mathrm{Vec}_G(U) =: (TG)(U).$$

We define $\mathfrak{h}(m) \subseteq \mathrm{T}_m(M)$ to be $\mathrm{d}\ell_g(e)(\mathfrak{h})$ and define

$$\widetilde{\mathfrak{h}}(U) := C^\infty(U) \otimes_{\mathbf{R}} \mathfrak{h} \subseteq C^\infty(U) \otimes_{\mathbf{R}} \mathfrak{g} = \mathrm{Vec}_G(U).$$

Explicitly, $\widetilde{\mathfrak{h}}(U)$ is the set of $C^\infty(U)$-linear combinations of the U-restrictions of the left-invariant vector fields arising from $\mathfrak{h}$.

To justify this notation, let us check that a smooth vector field $X \in \mathrm{Vec}_M(U)$ lies in $\widetilde{\mathfrak{h}}(U)$ if and only if $X(u) \in \widetilde{\mathfrak{h}}(u)$ for all $u \in U$. Choose a basis $\{v_1, \dots, v_n\}$ of $\mathfrak{g}$ extending a basis $\{v_1, \dots, v_r\}$ of $\mathfrak{h}$. Clearly $\widetilde{\mathfrak{h}}(U)$ is a free $C^\infty(U)$-module on the basis $\widetilde{v}_1|_U, \dots, \widetilde{v}_r|_U$, whereas $\mathrm{Vec}_G(U)$ is a free $C^\infty(U)$-module on the basis of all $\widetilde{v}_i|_U$, so the condition of membership in $\widetilde{\mathfrak{h}}(U)$ for a general smooth vector field X on U is that the unique $C^\infty(U)$-linear expansion $X = \sum a_i \widetilde{v}_i$ has $a_i = 0$ in $C^\infty(U)$ for all $i > r$. But such vanishing is equivalent to $a_i(u) = 0$ for all $i > r$ and $u \in U$, which in turn says precisely that $X(u) \in \widetilde{\mathfrak{h}}(u)$ inside $\mathrm{T}_u(M)$ for all $u \in U$.

So far we have not used that $\mathfrak{h}$ is a Lie subalgebra, just that it is a linear subspace of $\mathfrak{g}$. The Lie subalgebra property underlies:

Proposition 2.5. *The subbundle $\widetilde{\mathfrak{h}}$ of TG is integrable.*

Proof. First observe that by construction if $v \in \mathfrak{h} \subseteq \mathfrak{g}$ then the associated left-invariant vector field $\widetilde{v}$ on G lies in $\widetilde{\mathfrak{h}}(G) \subseteq \mathrm{Vec}_G(G)$ since this can be checked pointwise on G. If $v_1, \dots, v_n$ is a basis of $\mathfrak{h}$ then $\widetilde{v}_1, \dots, \widetilde{v}_n$ is a global trivializing frame for $\widetilde{\mathfrak{h}}$.

In general, to prove integrability of a subbundle of the tangent bundle it suffices to prove that the bracket operation applied to members of a trivializing frame over the constituents of an open covering of the base space yields output that is a section of the subbundle. (Why?) In our case there is the global trivializing frame $\widetilde{v}_1, \dots, \widetilde{v}_n$ of $\widetilde{\mathfrak{h}}$, so to prove integrability of $\widetilde{\mathfrak{h}} \subseteq TG$ we just have to prove that $[\widetilde{v}_i, \widetilde{v}_j] \in \widetilde{\mathfrak{h}}(G)$ inside of $(TG)(G) = \mathrm{Vec}_G(G)$. But $\widetilde{v}_i$ and $\widetilde{v}_j$ are left-invariant vector fields on G, and by the very definition of the Lie algebra structure on $\mathfrak{g} = \mathrm{T}_e(G)$ in terms of the commutator operation on global left-invariant vector fields we have $[\widetilde{v}_i, \widetilde{v}_j] = [v_i, v_j]^{\sim}$.

That is, the bracket of $\widetilde{v}_i$ and $\widetilde{v}_j$ is equal to the left-invariant vector field associated to the tangent vector $[v_i, v_j] \in \mathfrak{g}$. But $v_i, v_j \in \mathfrak{h}$ and by hypothesis $\mathfrak{h}$ *is a Lie subalgebra of* $\mathfrak{g}$. Hence, $[v_i, v_j] \in \mathfrak{h}$, so by construction when this is propagated to a left-invariant vector field on G the resulting global vector field lies in $\widetilde{\mathfrak{h}}(G)$ inside $\mathrm{Vec}_G(G)$ (as may be checked pointwise on G). ■

We shall see later in this handout that a maximal integral submanifold H in G through e to the integrable subbundle $\widetilde{\mathfrak{h}}$ is a connected *Lie subgroup* of G (by which we mean an injective immersion of Lie groups $i : H \to G$ that respects the group structures), and that its associated Lie subalgebra $\mathrm{Lie}(H) \subseteq \mathfrak{g}$ is the initial choice of Lie subalgebra $\mathfrak{h}$.

In what follows we shall state the local and global Frobenius theorems, discuss some aspects of the proofs, and work out the application to the existence and uniqueness of a *connected* Lie subgroup H of a Lie group G such that $\mathrm{Lie}(H) \subseteq \mathrm{Lie}(G)$ coincides with a given Lie subalgebra $\mathfrak{h} \subseteq \mathrm{Lie}(G)$.

3. Statement of main results

If E is a subbundle of TM then the existence of local trivializing frames implies that $\dim E(m)$ is locally constant in m, and so is constant on each connected component of M. In practice one often focuses on E of constant rank (i.e., $\dim E(m)$ is the same for all $m \in M$), though passage to that case is also achieved by passing to separate connected components of M.

Here is the local Frobenius theorem.

Theorem 3.1 (Frobenius)**.** *Let E be an integrable subbundle of TM with $\dim E(m) = r$ for all $m \in M$ with some $r > 0$. There exists a covering of M by C^∞ charts (U, φ) with $\varphi = \{x_1, \dots, x_n\}$ a C^∞ coordinate system satisfying $\varphi(U) = \prod(a_i, b_i) \subseteq \mathbf{R}^n$ such that the embedded r-dimensional slice submanifolds $\{x_i = c_i\}_{i>r}$ for $(c_{r+1}, \dots, c_n) \in \prod_{i>r}(a_i, b_i)$ are integral manifolds for E. Moreover, all* (*connected!*) *integral manifolds for E in U lie in a unique such slice set-theoretically, and hence lie in these slices as C^∞ submanifolds of U.*

Geometrically, the local coordinates in the theorem have the property that E is the subbundle spanned by the vector fields $\partial_{x_1}, \dots, \partial_{x_r}$. The *proof* of this local theorem proceeds by induction on the rank r of E, and to get the induction started in the case $r = 1$ it is necessary to prove a local theorem concerning a non-vanishing vector field (chosen to locally trivialize the line subbundle E in TM):

Theorem 3.2. *For any non-vanishing smooth vector field on an open subset of a smooth manifold, there are local coordinate systems in which the vector field is ∂_{x_1}.*

We give a complete proof of this theorem in §5, using the technique of vector flow from the theory of integral curves. This base case for the inductive proof of the local Frobenius theorem uses the entire force of the theory of ODE's, especially smooth dependence of solutions on varying initial conditions. Given such a local coordinate system as in Theorem 3.2, it is clear from the existence and uniqueness of integral curves for vector fields that the x_1-coordinate lines in a coordinate box (all other coordinates held fixed) *do* satisfy the requirements of the local Frobenius integrability theorem in the case of rank 1. That is, Theorem 3.2 does settle the rank 1 case of the local Frobenius theorem. The general geometric inductive proof of the local Frobenius theorem, building on the special case for rank 1, is given in section 1.60 in Warner's book "Foundations of differentiable manifolds and Lie groups".

We now turn to the statement of the global Frobenius theorem (see sections 1.62 and 1.64 in Warner's book). We state it in a slightly stronger form than in Warner's book (but his proof yields the stronger form, as we will explain), and it is certainly also stronger than the version in many other references (which is why we prefer to reference Warner's book for the proof):

Theorem 3.3 (Frobenius)**.** *Let E be an integrable subbundle of TM.*

(1) *For all $m \in M$, there is a (unique) maximal integral submanifold $i : N \hookrightarrow M$ through m_0.*
(2) *For any C^∞ mapping $M' \to M$ landing in $i(N)$ set-theoretically, the unique factorization $M' \to N$ is continuous and hence smooth.*
(3) *Any* connected *submanifold $i' : N' \hookrightarrow M$ satisfying $\mathrm{T}_{n'}(N') \subseteq E(i'(n'))$ for all $n' \in N'$ lies in a maximal integral submanifold for E.*

Note that in (3), we allow for the possibility that N' might be "low-dimensional" with respect to the rank of E, and so it is a definite strengthening of the property of maximal integral submanifolds for E in M (which are only required to be maximal with respect to other integral submanifolds for E in M, not with respect to connected submanifolds whose tangent spaces are pointwise just contained in – rather than actually equal to – the corresponding fiber of E).

To appreciate how special the automatic continuity is in (2) even if i is *not* a topological embedding (as often occurs even in Lie-theoretic settings as in Theorem 4.3), see Example 2.1. Also, in (2) we do not require that the map from M' to M be injective. The deduction of smoothness from continuity in (2) follows from the fact that the only obstruction to smoothness for a C^∞ map factoring set-theoretically through an injective immersion is topological (i.e., once the first step of the factorization is known to be continuous, the immersion theorem can be used *locally on the source* to infer its smoothness).

In Warner's book, the above global theorem is proved except that he omits (3). However, his proof of the "maximal integral submanifold" property in (1) does not use the "maximal dimension" condition on the connected submanifold source, and so it actually proves (3). We will use (3) at one step below.

4. Applications to Lie subgroups and homomorphisms

Before we turn to the task of proving Theorem 3.2, let us explain how to use the global Frobenius theorem to prove a striking result on the existence of connected Lie subgroups realizing a given Lie subalgebra as its Lie algebra. First, a definition:

Definition 4.1. A *Lie subgroup* of a Lie group G is a subgroup $H \subseteq G$ equipped with a C^∞ submanifold structure that makes it a Lie group.

In other words, a Lie subgroup "is" (up to unique isomorphism) an injective immersion $i : H \to G$ of Lie groups with i a group homomorphism. The example of the real line densely wrapped around the torus by the mapping $i : \mathbf{R} \to S^1 \times S^1$ defined by $t \mapsto ((\cos t, \sin t), (\cos(bt), \sin(bt)))$ for b not a rational multiple of π is a Lie subgroup that is *not* an embedded submanifold.

As we have seen in class, if $f : H \to G$ is a map of Lie groups (i.e., smooth map of manifolds that is also a group homomorphism) then $(\mathrm{d}f)(e_H) : \mathrm{T}_{e_H}(H) \to \mathrm{T}_{e_G}(G)$ respects the brackets on both sides (i.e., it is a "Lie algebra" map). Hence, in the immersion case we get $\mathrm{Lie}(H)$ as a Lie subalgebra of $\mathrm{Lie}(G)$.

Remark 4.2. The passage between Lie subalgebras and Lie subgroups pervades many arguments in the theory of (non-compact) Lie groups. In particular, as we will make precise in the next theorem, any connected Lie subgroup $i : H \to G$ is uniquely determined (with its topology and C^∞

structure!) by the associated Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Beware that in general it is very hard to tell in terms of $\mathfrak{h}$ (in an abstract situation) whether or not H is an embedded submanifold (i.e., has the subspace topology), in which case it turns out to be necessarily a *closed* submanifold. However, there are some convenient criteria on a Lie subalgebra $\mathfrak{h}$ in $\mathrm{Lie}(G)$ that are sufficient to ensure closedness. For example, if G is closed in some $\mathrm{GL}_n(\mathbf{R})$ and the subspace in $\mathfrak{h}$ spanned by all "brackets" $[x, y]$ with $x, y \in \mathfrak{h}$ is equal to $\mathfrak{h}$ then closedness is automatic (this implication is not at all obvious). It may seem that this criterion for closedness is a peculiar one, but it is actually a rather natural one from the perspective of the general structure theory of semisimple Lie algebras. Moreover, in practice it is a very mild condition.

Theorem 4.3. *Let G be a Lie group, with Lie algebra $\mathfrak{g}$. For every Lie subalgebra $\mathfrak{h}$ there exists a unique connected Lie subgroup H in G with Lie algebra $\mathfrak{h}$ inside of $\mathfrak{g}$. Moreover, if H and H' are connected Lie subgroups then $\mathrm{Lie}(H) \subseteq \mathrm{Lie}(H')$ if and only if $H \subseteq H'$ as subsets of G, in which case the inclusion is C^∞.*

Before we explain the proof of this theorem (using the Frobenius theorems), we make some comments. The connectivity is crucial in the theorem. For example, the closed subgroup $\mathrm{O}_n(\mathbf{R})$ of orthogonal matrices in $\mathrm{GL}_n(\mathbf{R})$ for the standard inner product is a Lie subgroup (even a closed submanifold), but it is disconnected with identity component given by the index-2 open subgroup $\mathrm{SO}_n(\mathbf{R})$ of orthogonal matrices with determinant 1. Both $\mathrm{O}_n(\mathbf{R})$ and $\mathrm{SO}_n(\mathbf{R})$ agree near the identity inside of $\mathrm{GL}_n(\mathbf{R})$, so they give the same Lie subalgebra of $\mathfrak{gl}_n(\mathbf{R})$ (consisting of the skew-symmetric matrices in $\mathfrak{gl}_n(\mathbf{R}) = \mathrm{Mat}_{n\times n}(\mathbf{R})$). Beware that there are non-injective immersions of Lie groups, such as $\mathrm{SL}_2(\mathbf{R}) \to \mathrm{SL}_2(\mathbf{R})/\langle -1\rangle$ that induce isomorphisms of Lie algebras. Hence, the passage between the isomorphism problem for connected Lie groups and for Lie algebras is a little subtle and we will not get into it here. The moral of the story is that a good understanding of the structure of $\mathrm{Lie}(G)$ as a Lie algebra does encode a lot of information about the Lie group G. In this way, the structure theory of finite-dimensional Lie algebras over $\mathbf{R}$ (which is a purely algebraic theory that makes sense over any field, though is best behaved in characteristic 0) plays a fundamental role in the theory of Lie groups.

Proof. Let $i : H \to G$ be an arbitrary connected Lie subgroup. Since the inclusion $i : H \to G$ is a group homomorphism and hence is compatible with left translations by elements of H, it follows that for $h \in H$ the mapping $(\mathrm{d}\ell_{i(h)})(e_G)$ carries $\mathrm{T}_{e_H}(H) \subseteq \mathrm{T}_{e_G}(G)$ (inclusion via $(\mathrm{d}i)(e_H)$) over to the subspace $\mathrm{T}_h(H) \subseteq \mathrm{T}_{i(h)}(G)$ (inclusion via $(\mathrm{d}i)(h)$). In other words (since $\dim(\mathrm{Lie}(H)) = \dim H$), the connected submanifold H is an integral manifold for the integrable subbundle of $TG \simeq G \times \mathfrak{g}$ given by $G\times \mathrm{Lie}(H)$. It therefore follows that the integral manifold H must factor smoothly through the maximal integral submanifold through e_G for the subbundle $G \times \mathrm{Lie}(H)$ in TG. In particular, once we know that H agrees with this maximal integral submanifold we will get the assertion that one Lie subgroup factors smoothly through another if and only if there is a corresponding inclusion of their Lie algebras inside of $\mathfrak{g}$ (as an inclusion of such Lie subalgebras forces a corresponding inclusion of subbundles of TG, and hence a smooth inclusion of maximal integral submanifolds through e_G by the *third part* of the global Frobenius theorem). This gives the uniqueness (including the manifold structure!) for a connected Lie subgroup of G with a specified Lie algebra inside of $\mathfrak{g}$.

Our problem is now reduced to: given a Lie subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ we seek to prove that the maximal integral submanifold H for the integrable subbundle $\widetilde{\mathfrak{h}}$ of TG is the unique connected Lie subgroup of G with $\mathfrak{h}$ as Lie algebra. First, we prove that this maximal integral submanifold H is in fact a Lie subgroup. That is, we must prove that H is algebraically a subgroup of G and then that the induced group law and inversion mappings are smooth for the manifold structure on H (and on

$H \times H$). The stability of H under the group law and inversion will use the maximality, and the uniqueness will use a trick for connected groups.

Pick $h \in H$. We want $hH \subseteq H$. In other words, if $i : H \to G$ is the inclusion for H as a submanifold of G, we want the composite injective immersive mapping $\ell_{i(h)} \circ i : H \to G$ to factor through $i : H \to G$ set-theoretically (but we'll even get such a factorization smoothly). To make the picture a little clearer, instead of considering the maps $\ell_{i(h)}$ that are C^∞ automorphisms of the manifold G, let us consider a general smooth automorphism φ of a general manifold M and a general integrable subbundle $E \subseteq TM$. The mapping $\mathrm{d}\varphi$ is an automorphism of TM over φ, so $(\mathrm{d}\varphi)(E)$ is a subbundle of TM, and if N is an integral manifold in M for E then the submanifold $\varphi(N)$ is clearly an integral manifold for $(\mathrm{d}\varphi)(E)$ in M. If N is a maximal integral manifold for E then the integral manifold $\varphi(N)$ must be *maximal* for the subbundle $(\mathrm{d}\varphi)(E)$. Indeed, if it is not maximal then (by the global Frobenius theorem!) $\varphi(N) \to M$ factors smoothly through a strictly larger integral submanifold $N' \to M$ for the subbundle $(\mathrm{d}\varphi)(E)$, and so applying φ^{-1} then gives $\varphi^{-1}(N')$ as an integral submanifold for E in M that strictly contains N, contradicting the assumed maximality of N. (Here we have used that $\mathrm{d}\varphi$ and $\mathrm{d}\varphi^{-1}$ are inverse maps on TM with matrices having C^∞ entries over local coordinate domains, as follows from the Chain Rule.)

Now in our special situation, the integrable subbundle $E \stackrel{\mathrm{def}}{=} \widetilde{\mathfrak{h}}$ of TG satisfies $(\mathrm{d}\ell_g)(E) = E$ for all $g \in G$ in the sense that $(\mathrm{d}\ell_g)(g')$ carries $E(g')$ to $E(gg')$ inside of $\mathrm{T}_{gg'}(G)$. This holds because the subspaces $E(g) \subset \mathrm{T}_g(G)$ were *constructed* using the left translation maps on tangent spaces. Hence, the preceding generalities imply that ℓ_g carries maximal integral manifolds for $E = \widetilde{\mathfrak{h}}$ to maximal integral manifolds for E. In particular, for the maximal integral manifold $i : H \to G$, we conclude that $\ell_{i(h)} \circ i : H \to G$ is also a maximal integral manifold for E. But the image of the latter contains the point $he = h \in i(H)$, so these two integral submanifold touch! Hence, by uniqueness of maximal integral manifolds through any single point (such as that point of common touch) they must coincide as submanifolds, which is to say that left multiplication by h on G carries H smoothly isomorphically back to itself (as a smooth submanifold of G).

This not only proves that H is algebraically a subgroup, but also that for all $h \in H$ the left multiplication mapping on G restricts to a bijective self-map (even smooth automorphism) of H. Since the identity e lies in H, it follows that $hh' = e$ for some $h' \in H$, which is to say that the unique inverse $h^{-1} \in G$ lies in H. That is, H is stable under inversion, and so it is algebraically a subgroup of G. If we let $\mathrm{inv} : G \simeq G$ be the smooth inversion mapping, then this says that the composite of inv with the smooth inclusion of H into G lands in the subset $H \subseteq G$ set-theoretically. Hence, by (2) in the global Frobenius theorem (applied to the maximal integral manifold H for E in G) we conclude that $\mathrm{inv}|_H : H \to G$ factors *smoothly* through the inclusion of H into G, which is to say that inversion on the subgroup H of G is a *smooth* self-map of the manifold H.

To conclude that H is a Lie subgroup, it remains to check smoothness for the group law. That is, we want the composite smooth mapping

$$H \times H \stackrel{i \times i}{\to} G \times G \to G$$

(the second step being the smooth group law of G) to factor smoothly through the inclusion i of H into G. But it does factor through this inclusion set-theoretically because H is a subgroup of G, and so again by (2) in the Frobenius theorem we get the desired smooth factorization. Hence, H is a Lie subgroup of G.

Finally, we have to prove the uniqueness aspect: if H' is a connected Lie subgroup of H with Lie algebra equal to the Lie algebra $\mathfrak{h}$ of H inside of $\mathfrak{g}$, then we want $H' = H$ as Lie subgroups of G. The discussion at the beginning of the proof shows that H' must at least smoothly factor through the maximal integral submanifold through the identity for the integrable subbundle $E = \widetilde{\mathfrak{h}} \subseteq TG$, which is to say that H' factors smoothly through H. Hence, we have a smooth injective immersion $H' \hookrightarrow H$ (as submanifolds of G) and we just need this to be an isomorphism. Any Lie group has the same dimension at all points (due to left translation automorphisms that identify all tangent spaces with the one at the identity), so H' and H have the same dimension at all points (as their tangent spaces at the identity coincide inside of $\mathfrak{g}$). Thus, the injective tangent mappings for the immersion $H' \to H$ are isomorphisms for dimension reasons, so the injective map $H' \to H$ is a local C^∞ isomorphism by the inverse function theorem! As such, it has *open* image onto which it is bijective, so H' is an open submanifold of H and thus is an open Lie subgroup of H.

Now comes the magical trick (which is actually a powerful method for proving global properties of a connected group): a *connected* topological group (such as H) has no proper open subgroups. This will certainly force the open immersion $H' \to H$ to be surjective and thus $H' = H$ as Lie subgroups of G. Rather more generally, an open subgroup of a topological group is always *closed* (giving what we need in the connected case). To see closedness, it is equivalent to prove openness of the complement, and by group theory we know that the complement of a subgroup of a group is a disjoint union of left cosets. Since any coset for an open subgroup is open (as it is an image of the open subgroup under a left-translation map that is necessarily a homeomorphism), any union of such cosets is open. ■

In 3.11 of Chapter I of the course text, the exponential map in the theory of Lie groups (that has no logical dependence on the "Lie subgroups to Lie subalgebras" correspondence discussed in this handout) is used to prove that closed subgroups of Lie groups are smooth submanifolds (with the subspace topology). Here is a further application of the exponential map:

Proposition 4.4. *Let $f : G \to G'$ be a Lie group homomorphism. For any closed Lie subgroup $H' \subseteq G'$, the preimage $f^{-1}(H')$ is a closed Lie subgroup of G, with Lie algebra equal to $\mathrm{Lie}(f)^{-1}(\mathfrak{h}') \subseteq \mathfrak{g}$. In particular, taking $H' = \{e'\}$, $\ker f$ is a closed Lie subgroup of G with Lie algebra $\ker(\mathrm{Lie}(f))$.*

If G_1 and G_2 are closed Lie subgroups of G then the closed Lie subgroup $G_1 \cap G_2$ has Lie algebra $\mathfrak{g}_1 \cap \mathfrak{g}_2$ inside $\mathfrak{g}$.

Beware that in the second part of this proposition, G_1 and G_2 do *not* necessarily have transverse intersection inside G at their common points. A counterexample is $G = \mathbf{R}^4$ with G_1 and G_2 planes through the origin that share a common line. Consequently, it is remarkable that $G_1 \cap G_2$ is always a submanifold (with the subspace topology).

Proof. The closed Lie subgroup $f^{-1}(H')$ in G is a submanifold (with the subspace topology), but it is not obvious that its tangent space exhausts $\mathrm{Lie}(f)^{-1}(\mathfrak{h}')$ (rather than being a proper subspace). So we forget the manifold structure on $f^{-1}(H')$ and aim to construct a connected Lie subgroup that *does* have the desired Lie algebra, and eventually inferring that it recovers $f^{-1}(H')^0$.

Since $\mathrm{Lie}(f)^{-1}(\mathfrak{h}')$ is a Lie subalgebra of G, it has the form $\mathrm{Lie}(H)$ for a unique connected Lie subgroup H of G (which we do *not* yet know to have the subspace topology from G). We want to show that the Lie group homomorphism $f|_H : H \to G'$ smoothly factors through the connected Lie subgroup ${H'}^0$. By the mapping property for ${H'}^0$ as the maximal integral submanifold at e' to the subbundle $\widetilde{\mathfrak{h}}'$ of $T(G')$ (see the second part of the global Frobenius theorem), it suffices to show that $f|_H$ factors through H' (and hence ${H'}^0$) set-theoretically.

It is enough to show that an open neighborhood U of e in H is carried by f into H', as H is generated by U algebraically (since H is *connected*, so it has no open subgroup aside from itself).

But consider the commutative diagram of exponential maps

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\mathrm{Lie}(f)} & \mathfrak{g}' \\ {\scriptstyle \exp_H}\downarrow & & \downarrow{\scriptstyle \exp_{G'}} \\ H & \xrightarrow[f|_H]{} & G' \end{array}$$

with $\mathfrak{h} := \mathrm{Lie}(H) = \mathrm{Lie}(f)^{-1}(\mathfrak{h}')$. The right vertical map carries an open neighborhood of 0 in $\mathfrak{h}'$ diffeomorphically onto an open neighborhood of e' in the connected Lie subgroup ${H'}^0$, since $\exp_{G'}|_{H'} = \exp_{H'}$ (functoriality of the exponential map). But $\mathrm{Lie}(f)$ carries $\mathfrak{h}$ into $\mathfrak{h}'$, so since $\exp_H$ carries an open neighborhood of 0 in $\mathfrak{h}$ diffeomorphically onto an open neighborhood of e in H we see from the commutativity of the diagram that $f|_H$ must carry an open neighborhood of e in H into ${H'}^0$ as required. This completes the proof that $f|_H$ factors through ${H'}^0$, so $H \subseteq f^{-1}({H'}^0) \subseteq f^{-1}(H')$. Note that we do *not* yet know if H has the subspace topology from G!

Recall that $f^{-1}(H')$ is a closed submanifold of G (with its subspace topology). The injective immersion $H \hookrightarrow f^{-1}(H')^0$ between connected Lie groups is an isomorphism on Lie algebras since we have the reverse inclusion

$$\mathrm{Lie}(f^{-1}(H')^0) = \mathrm{Lie}(f^{-1}(H')) \subseteq \mathrm{Lie}(f)^{-1}(H') =: \mathfrak{h}$$

(the middle inclusion step due to the fact that $f : G \to G'$ carries $f^{-1}(H')$ into H'), so $H = f^{-1}(H')^0$. It follows that $\mathrm{Lie}(f^{-1}(H')) = \mathfrak{h} = \mathrm{Lie}(f)^{-1}(H')$, as desired.

Now we turn to the behavior of intersections for closed Lie subgroups G_1 and G_2 of G. By the preceding generalities applied to $G_1 \hookrightarrow G$ in the role of f and to the closed Lie subgroup G_2 of G, the preimage $G_1 \cap G_2$ is a closed Lie subgroup of G whose Lie algebra is the preimage of $\mathfrak{g}_2$ under the inclusion $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$. This says exactly that $\mathfrak{g}_1 \cap \mathfrak{g}_2 = \mathrm{Lie}(G_1 \cap G_2)$. ■

Corollary 4.5. *Let $\{G_\alpha\}$ be closed Lie subgroups of a Lie group G. The intersection $H = \bigcap G_\alpha$ is a closed Lie subgroup of G (with the subspace topology) and its Lie algebra is $\bigcap \mathfrak{g}_\alpha$.*

Proof. By finite-dimensionality of $\mathfrak{g}$, we can find a finite collection of indices $\alpha_1, \dots, \alpha_n$ so that

$$\bigcap \mathfrak{g}_\alpha = \mathfrak{g}_{\alpha_1} \cap \cdots \cap \mathfrak{g}_{\alpha_n}.$$

The previous proposition ensures that *finite* intersections among closed Lie subgroups are closed Lie subgroups with the expected Lie algebra, so $G' := G_{\alpha_1} \cap \cdots \cap G_{\alpha_n}$ is a closed Lie subgroup of G with $\mathfrak{g}' = \cap_\alpha \mathfrak{g}_\alpha$. Hence, for any α outside of the α_i's, the intersection $G' \cap G_\alpha$ is a closed Lie subgroup whose Lie algebra is the same as that of G', so its identity component coincides with that of G'. In other words, $G' \cap G_\alpha$ is a union of cosets of ${G'}^0$ inside G'. Thus, intersecting over all such α, we conclude that H is a union of ${G'}^0$-cosets inside G'. Any such union is an open and closed submanifold of G', so it is a closed submanifold of G (with the subspace topology) and its Lie algebra is $\mathrm{Lie}({G'}^0) = \mathrm{Lie}(G') = \cap \mathfrak{g}_\alpha$. ■

Corollary 4.6. *Let H be a closed Lie subgroup of G. The centralizer $Z_G(H)$ is a closed Lie subgroup of G, and $\mathrm{Lie}(Z_G(H)) = \mathfrak{g}^{\mathrm{Ad}_G(H)}$.*

If H is connected then the normalizer $N_G(H)$ is a closed Lie subgroup of G and

$$\mathrm{Lie}(N_G(H)) = \mathfrak{n}_\mathfrak{g}(\mathfrak{h}) := \{v \in \mathfrak{g} \,|\, [v, \mathfrak{h}] \subseteq \mathfrak{h}\}, \;\; \mathrm{Lie}(Z_G(H)) = \mathfrak{z}_\mathfrak{g}(\mathfrak{h}) := \{v \in \mathfrak{g} \,|\, [v, \mathfrak{h}] = 0\}.$$

Before proving this corollary, we make some remarks. When defining $N_G(H)$, the equality $gHg^{-1} = H$ (ensures stability of $N_G(H)$ under inversion!) is equivalent to the inclusion $gHg^{-1} \subseteq H$ (more convenient to use) for dimension reasons due to *connectedness* of H, and so also holds if $\pi_0(H)$

is finite; it is used without comment. The expressions $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ are the *Lie algebra normalizer* and *Lie algebra centralizer*. By the Jacobi identity, each is a Lie subalgebra of $\mathfrak{g}$.

Also, the equality $\mathrm{Lie}(Z_G(H)) = \mathfrak{g}^{\mathrm{Ad}_G(H)}$ can be proved very quickly using the exponential map: the inclusion "$\subseteq$" is obvious by functoriality of the tangent space at the identity since H-conjugation on G has no effect on $Z_G(H)$, and the reserve inclusion holds because any $v \in \mathfrak{g}^{\mathrm{Ad}_G(H)}$ is the velocity at $t = 0$ to the 1-parameter subgroup $\alpha_v : t \mapsto \exp_G(tv)$ that is visibly valued in the closed submanifold $Z_G(H)$ (by functoriality of α_v in (G, v)). We give a rather different proof of this equality below because we prefer arguments that minimize the role of the exponential map (that we have limited to its role in the proof of Proposition 4.4) because such arguments are more robust for adapting to other contexts such as the algebro-geometric study of linear algebraic groups over general fields (where one has a result like Proposition 4.4 in characteristic 0 for purely algebraic reasons unrelated to an "exponential map").

Finally, the equality $\mathrm{Lie}(Z_G(H)) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ generally *fails* when H is not assumed to be connected (e.g., consider discrete H). Likewise, if $\pi_0(H)$ is infinite then the condition "$gHg^{-1} \subseteq H$" on g is generally not equivalent to "$gHg^{-1} = H$" (equivalently, this inclusion condition is not stable under inversion on g); an example is $G \subset \mathrm{GL}_2(\mathbf{R})$ the closed subgroup of upper-triangular matrices, H the subgroup of unipotent matrices $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ with $n \in \mathbf{Z}$, and diagonal $g = \mathrm{diag}(m, 1)$ for $m \in \mathbf{Z} - \{0, 1, -1\}$.

Proof. The closedness is clear since G is Hausdorff and H is closed in G. Clearly $Z_G(H) = \bigcap_{h \in H} Z_G(h)$, so by Corollary 4.5 we have $\mathrm{Lie}(Z_G(H)) = \bigcap_{h \in H} \mathrm{Lie}(Z_G(h))$. Since $\mathfrak{g}^{\mathrm{Ad}_G(H)} = \bigcap_{h \in H} \mathfrak{g}^{\mathrm{Ad}_G(h)=1}$, to prove the equality $\mathrm{Lie}(Z_G(H)) = \mathfrak{g}^{\mathrm{Ad}_G(H)}$ it suffices to show that $\mathrm{Lie}(Z_G(g)) = \mathfrak{g}^{\mathrm{Ad}_G(g)=1}$ for any $g \in G$. The centralizer $Z_G(g)$ is the preimage of the diagonal $\Delta : G \hookrightarrow G \times G$ under the Lie group homomorphism $f : G \to G \times G$ given by $x \mapsto (gxg^{-1}, x)$. Thus, by Proposition 4.4, $\mathrm{Lie}(Z_G(g))$ is the preimage of the diagonal $\mathrm{Lie}(\Delta) : \mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$ under $\mathrm{Lie}(f) = (\mathrm{Ad}_G(g), \mathrm{id})$. This preimage is exactly $\mathfrak{g}^{\mathrm{Ad}_G(g)=1}$.

Now we assume H is connected for the rest of the argument. The conjugation action of $N_G(H)$ on G carries H into itself, so the restriction of Ad_G to $N_G(H)$ carries $\mathfrak{h}$ into itself. That is,

$$\mathrm{Ad}_G : N_G(H) \to \mathrm{GL}(\mathfrak{g})$$

lands inside the closed subgroup $\mathrm{Stab}(\mathfrak{h})$ of linear automorphisms that carry $\mathfrak{h}$ into itself. Thus, the map $\mathrm{ad}_{\mathfrak{g}} = \mathrm{d}(\mathrm{Ad}_G)(e)$ from $\mathfrak{g}$ into $\mathrm{End}(\mathfrak{g})$ carries $\mathrm{Lie}(N_G(H))$ into $\mathrm{Lie}(\mathrm{Stab}(\mathfrak{h}))$.

A computation with block upper triangular matrices inside $\mathrm{End}(\mathfrak{g})$ using a basis of $\mathfrak{g}$ extending a basis of $\mathfrak{h}$ shows that the Lie algebra of $\mathrm{Stab}(\mathfrak{h})$ is equal to the vector space of linear endomorphisms of $\mathfrak{g}$ that carry $\mathfrak{h}$ into itself. This establishes the containment "$\subseteq$" for $N_G(H)$. Arguining similarly with $Z_G(H)$-conjugation on G that restricts to the identity on H, and replacing $\mathrm{Stab}(\mathfrak{h})$ with the closed subgroup $\mathrm{Fix}(\mathfrak{h})$ of elements of $\mathrm{GL}(\mathfrak{g})$ that fix $\mathfrak{h}$ pointwise, we deduce $\mathrm{Lie}(Z_G(H)) \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$.

So far we have not used that H is connected! We explain how to establish the desired equality for $N_G(H)$ by using the connectedness of H, and the same method works (check!) to show that $\mathrm{Lie}(Z_G(H)) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$. Consider the connected Lie subgroup N' of G whose Lie algebra is the Lie subalgebra $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{g}$. Beware that we do not yet know that N' has the subspace topology (equivalently, is closed in G). Since $N_G(H)^0$ and N' are connected Lie subgroups of G and

$$\mathrm{Lie}(N_G(H)^0) = \mathrm{Lie}(N_G(H)) \subseteq \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathrm{Lie}(N'),$$

we have $N_G(H)^0 \subseteq N'$. Hence, to prove the desired equality of Lie algebras we just have to show that $N' \subseteq N_G(H)$ as subsets of G. That is, for $n' \in N'$ we want that $c_{n'} : G \simeq G$ carries H into itself. By the connectedness of H, $c_{n'}(H) \subseteq H$ if and only if there is such a containment on Lie algebras, which is to say $\mathrm{Ad}_G(n')(\mathfrak{h}) = \mathfrak{h}$ inside $\mathfrak{g}$. In other words, we want that $\mathrm{Ad}_{G'}$

carries N' into $\mathrm{Stab}(\mathfrak{h})$. But since N' is connected by design, it is equivalent to show that the map $\mathrm{ad}_{\mathfrak{g}} = \mathrm{d}(\mathrm{Ad}_G)(e)$ carries $\mathrm{Lie}(N')$ into the vector space $\mathrm{Lie}(\mathrm{Stab}(\mathfrak{h}))$ of linear endomorphisms of $\mathfrak{g}$ that carry $\mathfrak{h}$ into itself. But N' was created precisely so that $\mathrm{Lie}(N') = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$. ■

We conclude our tour through the dictionary between Lie groups and Lie algebras by posing a natural question: if G and G' are *connected* Lie groups, does any map of the associated Lie algebras $T : \mathfrak{g}' \to \mathfrak{g}$ ($\mathbf{R}$-linear respecting the brackets) necessarily arise from a map $f : G' \to G$ of Lie groups (smooth map of manifolds and group homomorphism)? If such an f exists then it is unique, but there is a topological obstruction to existence. To analyze this, we use the method of graphs.

To see the uniqueness, first note that $f : G' \to G$ gives rise to a smooth graph mapping $\Gamma_f : G' \to G' \times G$ (via $g' \mapsto (g', f(g'))$) that is a homeomorphism onto its image (using pr_1 as an inverse), and its image is closed (the preimage of the diagonal in $G \times G$ under the map $f \times \mathrm{id} : G' \times G \to G \times G$). Thus, Γ_f defines a connected *closed* Lie subgroup of $G' \times G$ when the latter is made into a group with product operations, and so it is a closed submanifold (as for any injective immersion between manifolds that is a homeomorphism onto a closed image).

Via the method of left-translations, the natural identification $\mathrm{T}_{(e',e)}(G' \times G) \simeq \mathrm{T}_{e'}(G') \oplus \mathrm{T}_e(G)$ carries the Lie bracket on the left over to the direct sum of the Lie brackets on the right. That is, $\mathrm{Lie}(G' \times G) \simeq \mathrm{Lie}(G') \oplus \mathrm{Lie}(G)$. In this way, the mapping

$$\mathrm{Lie}(\Gamma_f) : \mathrm{Lie}(G') \to \mathrm{Lie}(G' \times G) \simeq \mathrm{Lie}(G') \oplus \mathrm{Lie}(G)$$

is identified with the linear-algebra "graph" of the map $\mathrm{Lie}(f) = (\mathrm{d}f)(e') : \mathrm{Lie}(G') \to \mathrm{Lie}(G)$ that is assumed to be T. Hence, Γ_f corresponds to a connected Lie subgroup of $G' \times G$ whose associated Lie subalgebra in $\mathfrak{g}' \oplus \mathfrak{g}$ is the graph of the linear map T. By the uniqueness aspect of the passage from connected Lie subgroups to Lie subalgebras, it follows that the mapping $\Gamma_f : G' \to G' \times G$ is uniquely determined (if f is to exist), and so composing it with the projection $G' \times G \to G$ recovers f. This verifies the uniqueness of f.

How about existence? To this end, we try to reverse the above procedure: we use the injective graph mapping $\Gamma_T : \mathfrak{g}' \to \mathfrak{g}' \oplus \mathfrak{g}$ that is a mapping of Lie algebras precisely because T is a map of Lie algebras (and because the direct sum is given the "componentwise" bracket). By the general existence/uniqueness theorem, there is a unique connected Lie subgroup $H \subseteq G' \times G$ whose associated Lie subalgebra is the image of Γ_T. In particular, the first projection $H \to G'$ induces an isomorphism on Lie algebras, and if this mapping of connected Lie groups were an isomorphism then we could compose its inverse with the other projection $H \to G$ to get the desired mapping. (Conversely, it is clear that if the existence problem is to have an affirmative answer, then the first projection $H \to G'$ *must* be an isomorphism.) Hence, the problem is reduced to this: can a mapping $\pi : H \to G'$ between *connected* Lie groups induce an isomorphism on Lie algebras without being an isomorphism?

Such a mapping must be a local isomorphism near the identities (by the inverse function theorem), and so the image subgroup is open (as it contains an open in G' around the identity, and hence around all of its points via left translation in the image subgroup). But we have seen above that *connected* topological groups have no proper open subgroups, so the mapping π must be surjective. Also, $\ker(\pi)$ is a closed subgroup that meets a neighborhood of the identity in H in exactly the identity point (as π is a local isomorphism near identity elements), so the identity is an open point in $\ker(\pi)$. It follows by translations that the closed topological normal subgroup $\ker(\pi)$ must have the *discrete* topology. But if Γ is a discrete closed normal subgroup of a connected Lie group H then we can make the quotient H/Γ as a C^∞ manifold and in fact a Lie group. The induced C^∞ map $H/\Gamma \to G$ is a bijective Lie group map that is an isomorphism on Lie algebras and so (via translations!) is an isomorphism between tangent spaces at all points, whence by the inverse function theorem it is a C^∞ isomorphism (whence is an isomorphism of Lie groups).

In the general construction of quotients N'/N for Lie groups modulo closed subgroups, there is a covering of N'/N by open sets U_α over which the preimage in N' can be identified N-equivariantly with $U_\alpha \times N$ in such a way that the quotient map back to U_α is pr_1. Thus, the identification $H/\Gamma \simeq G$ and the discreteness of Γ imply that G admits an open covering $\{U_\alpha\}$ whose preimage in H is a disjoint union of copies of U_α (indexed by Γ). In topology, a *covering space* of a topological space X is a surjective continuous map $q : E \to X$ so that for some open covering $\{U_\alpha\}$ of X each preimage $q^{-1}(U_\alpha)$ is a disjoint union of copies of U_α. Thus, $H \to G$ is a *connected covering space.*

To summarize, we have found the precise topological obstruction to our problem: if G admits nontrivial connected covering spaces then there may be problems in promoting Lie algebra homomorphisms $\mathfrak{g} \to \mathfrak{g}'$ to Lie group homomorphisms $G \to G'$.

For a connected manifold, the existence of a nontrivial connected covering space is equivalent to the nontriviality of the fundamental group (this will be immediate from HW5 Exercise 3 and HW9 Exercise 3). Thus, for any connected Lie group G the map $\mathrm{Hom}(G, G') \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}')$ is bijective when $\pi_1(G) = 1$. (This holds for $\mathrm{SL}_n(\mathbf{C})$, $\mathrm{Sp}_{2n}(\mathbf{C})$, $\mathrm{SU}(n)$, and $\mathrm{Sp}(n)$, but fails for $\mathrm{SL}_n(\mathbf{R})$ and $\mathrm{SO}(n)$ for $n \geq 2$.)

5. Vector fields and local coordinates

We now turn to the task of proving Theorem 3.2. First we consider a more general situation. Let M be a smooth manifold and let $\vec{v}_1, \dots, \vec{v}_n$ be pointwise linearly independent smooth vector fields on an open subset $U \subseteq M$ ($n \geq 1$). One simple example of such vector fields is $\partial_{x_1}, \dots, \partial_{x_n}$ on a coordinate domain for local smooth coordinates $\{x_1, \dots, x_N\}$ on an open set U in M. Can all examples be described in this way (locally) for suitable smooth coordinates?

Choose a point $m_0 \in U$. It is very natural (e.g., to simplify local calculations) to ask if there exists a local C^∞ coordinate system $\{x_1, \dots, x_N\}$ on an open subset $U_0 \subseteq U$ around m_0 such that $\vec{v}_i|_{U_0} = \partial_{x_i}$ in $\mathrm{Vec}_M(U_0) = (TM)(U_0)$ for $1 \leq i \leq n$. The crux of the matter is to have such an identity across an entire open neighborhood of m_0. If we only work in the tangent space *at* the point m_0, which is to say we inquire about the identity $\vec{v}_i(m_0) = \partial_{x_i}|_{m_0}$ in $\mathrm{T}_{m_0}(U_0) = \mathrm{T}_{m_0}(M)$ for $1 \leq i \leq n$, then the answer is trivial (and not particularly useful): we choose local C^∞ coordinates $\{y_1, \dots, y_N\}$ near m_0 and write $\vec{v}_j(m_0) = \sum c_{ij} \partial_{y_i}|_{m_0}$, so the $N \times n$ matrix (c_{ij}) has independent columns. We extend this to an invertible $N \times N$ matrix, and then make a *constant* linear change of coordinates on the y_j's via the inverse matrix to get to the case $c_{ij} = \delta_{ij}$ for $i \leq n$ and $c_{ij} = 0$ for $i > n$. Of course, such new coordinates are only adapted to the situation at m_0. If we try to do the same construction by considering the matrix of functions (h_{ij}) with $\vec{v}_j = \sum h_{ij} \partial_{y_i}$ near m_0, the change of coordinates will now typically have to be *non-constant*, thereby leading to a big mess due to the appearance of differentiation in the transformation formulas for ∂_{t_i}'s with respect to change of local coordinates (having "non-constant" coefficients).

There is a very good reason why the problem over an open set (as opposed to at a single point) is complicated: usually no such coordinates exist! Indeed, if $n \geq 2$ then the question generally has a negative answer because there is an obstruction that is often non-trivial: since the commutator vector field $[\partial_{x_i}, \partial_{x_j}]$ vanishes for any i, j, if such coordinates are to exist around m_0 then the commutator vector fields $[\vec{v}_i, \vec{v}_j]$ must vanish near m_0. (Note that the concept of commutator of vector fields is meaningless when working on a single tangent space; it only has meaning when working with vector fields over open sets. This is "why" we had no difficulties when working at a single point m_0.)

For $n \geq 2$, the necessary condition of vanishing of commutators for pointwise independent vector fields usually fails. For example, on an open set $U \subseteq \mathbf{R}^3$ consider a pair of smooth vector fields

$$\vec{v} = \partial_x + f\partial_z, \quad \vec{w} = \partial_y + g\partial_z$$

for smooth functions f and g on U. These are visibly pointwise independent vector fields but

$$[\vec{v}, \vec{w}] = ((\partial_x g + f\partial_z g) - (\partial_y f + g\partial_z f))\partial_z,$$

so a necessary condition to have $\vec{v} = \partial_{x_1}$ and $\vec{w} = \partial_{x_2}$ for local C^∞ coordinates $\{x_1, x_2, x_3\}$ near $m_0 \in U$ is

$$\partial_x g + f\partial_z g = \partial_y f + g\partial_z f$$

near m_0. There is a special case in which the vanishing condition on the commutators $[\vec{v}_i, \vec{v}_j]$ for all i, j is vacuous: $n = 1$. Indeed, since $[\vec{v}, \vec{v}] = 0$ for any smooth vector field, in the case $n = 1$ we see no obvious reason why our question cannot always have an affirmative answer. The technique of vector flow along integral curves will prove such a result.

In the case $n = 1$, pointwise-independence for the singleton $\{\vec{v}_1\}$ amounts to pointwise non-vanishing. Hence, we may restate the goal: if $\vec{v}$ is a smooth vector field on an open set $U \subseteq M$ and $\vec{v}(m_0) \neq 0$ for some $m_0 \in U$ (so $\vec{v}(m) \neq 0$ for m near m_0, by continuity of $\vec{v} : U \to TM$), then there exists a local C^∞ coordinate system $\{x_1, \ldots, x_N\}$ near m_0 in U such that $\vec{v} = \partial_{x_1}$ near m_0.

Example 5.1. Consider the circular vector field $\vec{v} = -y\partial_x + x\partial_y$ on $M = \mathbf{R}^2$ with constant speed $r \geq 0$ on the circle of radius r centered at the origin. This vector field vanishes at the origin, but for $m_0 \neq (0, 0)$ we have $\vec{v}(m_0) \neq 0$. Let $U_0 = \mathbf{R}^2 - L$ for a closed half-line L emanating from the origin and not passing through m_0. For a suitable θ_0, trigonometry provides a C^∞ parameterization $(0, \infty) \times (\theta_0, \theta_0 + 2\pi) \simeq U_0$ given by $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$, and $\partial_\theta = \vec{v}|_{U_0}$. Thus, in this special case we get lucky: we already "know" the right coordinate system to solve the problem. But what if we didn't already know trigonometry? How would we have been able to figure out the answer in this simple special case?

Example 5.2. In order to appreciate the non-trivial nature of the general assertion we are trying to prove, let us try to prove it in general "by hand" (i.e., using just basic definitions, and no substantial theoretical input such as the theory of vector flow along integral curves). We shrink U around m_0 so that there exist local C^∞ coordinates $\{y_1, \ldots, y_N\}$ on U. Hence, $\vec{v} = \sum h_j \partial_{y_j}$, and since $\vec{v}(m_0) = \sum h_j(m_0)\partial_{y_j}|_{m_0}$ is nonzero, we have $h_j(m_0) \neq 0$ for some j. By relabelling, we may assume $h_1(m_0) \neq 0$. By shrinking U around m_0, we may assume h_1 is non-vanishing on U (so $\vec{v}$ is non-vanishing on U). We wish to find a C^∞ coordinate system $\{x_1, \ldots, x_N\}$ near m_0 inside of U such that $\vec{v} = \partial_{x_1}$ near m_0.

What conditions are imposed on the x_i's in terms of the y_j's? For any smooth coordinate system $\{x_i\}$ near m_0, $\partial_{y_j} = \sum (\partial_{y_j} x_i)\partial_{x_i}$ near m_0, so near m_0 we have

$$\vec{v} = \sum_j h_j \sum_i (\partial_{y_j} x_i)\partial_{x_i} = \sum_i (\sum_j h_j \partial_{y_j}(x_i))\partial_{x_i}.$$

Thus, the necessary and sufficient conditions are two-fold: $x_1, \ldots, x_N$ are smooth functions near m_0 such that $\det((\partial_{y_j} x_i)(m_0)) \neq 0$ (this ensures that the x_i's are local smooth coordinates near m_0, by the inverse function theorem) and

$$\sum_j h_j \partial_{y_j}(x_i) = \delta_{ij}$$

for $1 \leq i \leq N$. This is a system of linear first-order PDE's in the N unknown functions $x_i = x_i(y_1, \ldots, y_N)$ near m_0. We have already seen that the theory of first-order linear ODE's is quite substantial, and here were are faced with a PDE problem. Hence, our task now looks to be considerably less straightforward than it may have seemed to be at the outset.

The apparent complications are an illusion: it is because we have written out the explicit PDE's in local coordinates that things look complicated. As will be seen in the proof below, when we restate our problem in *geometric* language the idea for how to solve the problem essentially drops into our lap without any pain at all.

The fundamental theorem is this (a restatement of Theorem 3.2):

Theorem 5.3. *Let M be a smooth manifold and $\vec{v}$ a smooth vector field on an open set $U \subseteq M$. Let $m_0 \in U$ be a point such that $\vec{v}(m_0) \neq 0$. There exists a local C^∞ coordinate system $\{x_1, \dots, x_N\}$ on an open set $U_0 \subseteq U$ containing m_0 such that $\vec{v}|_{U_0} = \partial_{x_1}$.*

Proof. What is the geometric meaning of what we are trying to do? We are trying to find local coordinates $\{x_i\}$ an open open U_0 in U around m_0 so that the integral curves for $\vec{v}|_{U_0}$ are exactly flow along the x_1-direction at unit speed. That is, in this coordinate system for any point ξ near m_0 the integral curve for $\vec{v}$ through ξ is coordinatized as $c_\xi(t) = (t + x_1(\xi), x_2(\xi), \dots, x_N(\xi))$ for t near 0. This suggests that we try to find a local coordinate system around m_0 such that the first coordinate is "time of vector flow". The study of flow along integral curves in manifolds shows that for a sufficiently small open $U_0 \subseteq U$ around m_0 there exists $\varepsilon > 0$ such that for all $\xi \in U_0$ the maximal interval of definition for the integral curve c_ξ contains $(-\varepsilon, \varepsilon)$. More specifically, the vector-flow mapping

$$\Phi : \Omega \to M$$

defined by $(t, \xi) \mapsto c_\xi(t)$ (using t varying through the maximal open interval of definition I_ξ around 0 for each ξ) has *open* domain of definition $\Omega \subset \mathbf{R} \times M$ and is a smooth mapping on Ω. Thus, for small $\varepsilon > 0$ and small $U_0 \subseteq U$ around m_0, we have that $(-\varepsilon, \varepsilon) \times U_0$ is contained in Ω (as $\{0\} \times M \subseteq \Omega$). The mapping Φ, restricted to $(-\varepsilon, \varepsilon) \times U_0$, will be the key to creating a coordinate system on M near m_0 such that the time-of-flow parameter t is the first coordinate.

Here is the construction. We first choose an arbitrary smooth coordinate system $\phi : W \to \mathbf{R}^N$ on an open around m_0 that "solves the problem at m_0". That is, if $\{y_1, \dots, y_N\}$ are the component functions of ϕ, then $\partial_{y_1}|_{m_0} = \vec{v}(m_0)$. This is the trivial pointwise version of the problem that we considered at the beginning of this handout (and it has an affirmative answer precisely because the singleton $\{\vec{v}(m_0)\}$ in $\mathrm{T}_{m_0}(M)$ is an independent set; i.e., $\vec{v}(m_0) \neq 0$). Making a constant translation (for ease of notation), we may assume $y_j(m_0) = 0$ for all j. In general this coordinate system will fail to "work" at any other points, and we use vector flow to fix it. Consider points on the slice $W \cap \{y_1 = 0\}$ in M near m_0. In terms of y-coordinates, these are points $(0, a_2, \dots, a_N)$ with small $|a_j|$'s. By openness of the domain of flow $\Omega \subseteq \mathbf{R} \times M$, there exists $\varepsilon > 0$ such that, after perhaps shrinking W around m_0, $(-\varepsilon, \varepsilon) \times W \subseteq \Omega$.

By the definition of the y_i's in terms of ϕ, $\phi(W \cap \{y_1 = 0\})$ is an open subset in $\{0\} \times \mathbf{R}^{N-1} = \mathbf{R}^{N-1}$, and ϕ restricts to a C^∞ isomorphism from the smooth hypersurface $W \cap \{y_1 = 0\}$ onto $\phi(W \cap \{y_1 = 0\})$. Consider the vector-flow mapping

$$\Psi : (-\varepsilon, \varepsilon) \times \phi(W \cap \{y_1 = 0\}) \to M$$

defined by

$$(t, a_2, \dots, a_N) \mapsto \Phi(t, \phi^{-1}(0, a_2, \dots, a_N)) = c_{\phi^{-1}(0, a_2, \dots, a_N)}(t).$$

By the theory of vector flow, this is a *smooth* mapping. (This is the family of solutions to a first-order initial-value problem with varying initial parameters $a_2, \dots, a_N$ near 0. Thus, the smoothness of the map is an instance of smooth dependence on varying initial conditions for solutions to first-order ODE's.) Geometrically, we are trying to parameterize M near m_0 by starting on the hypersurface

$H = \{y_1 = 0\}$ in W (with coordinates given by the restrictions $y'_2, \dots, y'_N$ of $y_2, \dots, y_N$ to H) and flowing away from H along the vector field $\vec{v}$; the time t of flow provides the first parameter in our attempted parameterization of M near m_0.

Note that $\Psi(0, 0, \dots, 0) = c_{m_0}(0) = m_0$. Is Ψ a parameterization of M near m_0? That is, is Ψ a local C^∞ isomorphism near the origin? If so, then its local inverse near m_0 provides a C^∞ coordinate system $\{x_1, \dots, x_N\}$ with $x_1 = t$ measuring time of flow along integral curves for $\vec{v}$ with their *canonical* parameterization (as integral curves). Thus, it is "physically obvious" that in such a coordinate system we will have $\vec{v} = \partial_{x_1}$ (but we will also derive this by direct calculation below). To check the local isomorphism property for Ψ near the origin, we use the inverse function theorem: we have to check $\mathrm{d}\Psi(0, \dots, 0)$ is invertible. In terms of the local C^∞ coordinates $\{t, y'_2, \dots, y'_N\}$ near the origin on the source of Ψ and $\{y_1, \dots, y_N\}$ near $m_0 = \Psi(0, \dots, 0)$ on the target of Ψ, the $N \times N$ Jacobian matrix for $\mathrm{d}\Psi(0, \dots, 0)$ has lower $(N-1) \times (N-1)$ block given by the identity matrix (i.e., $(\partial_{y'_j} y_i)(0, \dots, 0) = \delta_{ij}$) because $\partial_{y'_j} y_i = \delta_{ij}$ at points on $W \cap \{y_1 = 0\}$ (check! It is *not* true at most other points of W).

What is the left column of the Jacobian matrix at $(0, \dots, 0)$? Rather generally, if ξ is the point with y-coordinates $(t_0, a_2, \dots, a_N)$ then the t-partials $(\partial_t y_i)(t_0, a_2, \dots, a_N)$ are the coefficients of the velocity vector $c'_\xi(t_0)$ to the integral curve c_ξ of $\vec{v}$ at time t_0, and such a velocity vector is equal to $\vec{v}(c_\xi(t_0))$ by the *definition* of the concept of integral curve. Hence, setting $t_0 = 0$, $c'_\xi(0) = \vec{v}(c_\xi(0)) = \vec{v}(\xi)$, so taking $\xi = m_0 = \Psi(0, \dots, 0)$ gives that $(\partial_t y_i)(0, \dots, 0)$ is the coefficient of $\partial_{y_i}|_{m_0}$ in $\vec{v}(m_0)$. Aha, but recall that we *chose* $\{y_1, \dots, y_N\}$ at the outset so that $\vec{v}(m_0) = \partial_{y_1}|_{m_0}$. Hence, the left column of the Jacobian matrix at the origin has $(1, 1)$ entry 1 and all other entries equal to 0. Since the lower right $(N-1) \times (N-1)$ block of the Jacobian matrix is the identity, this finishes the verification of invertibility of $\mathrm{d}\Psi(0, \dots, 0)$, so Ψ gives a local C^∞ isomorphism between opens around $(0, \dots, 0)$ and m_0.

Let $\{x_1, \dots, x_N\}$ be the C^∞ coordinate system near m_0 on M given by the local inverse to Ψ. We claim that $\vec{v} = \partial_{x_1}$ near m_0. By definition of the x-coordinate system, $(a_1, \dots, a_n)$ is the tuple of x-coordinates of the point $\Phi(a_1, \phi^{-1}(0, a_2, \dots, a_n)) \in M$. Thus, ∂_{x_1} is the field of velocity vectors along the parameteric curves $\Phi(t, \phi^{-1}(0, a_2, \dots, a_n)) = c_{\phi^{-1}(0, a_2, \dots, a_n)}(t)$ that are the integral curves for the smooth vector field $\vec{v}$ with initial positions (time 0) at points

$$\phi^{-1}(0, a_2, \dots, a_n) \in W \cap \{y_1 = 0\}$$

near m_0. Thus, the velocity vectors along these parametric curves are exactly the vectors from the smooth vector field $\vec{v}$! This shows that the smooth vector fields ∂_{x_1} and $\vec{v}$ coincide near m_0. ■