## Math 210C. The dual root system and the $\mathbf{Q}$-structure on root systems

By our definition, for a root system $(V, \Phi)$ we assume $V$ is a finite-dimensional vector space over some field $k$ of characteristic 0. In practice, the only cases of interest are $k = \mathbf{Q}$ (for "algebraic" aspects) and $k = \mathbf{R}$ (for geometric arguments with Weyl chambers later, as well as for applications to compact Lie groups). In this handout, we explain how the general case reduces to the case with $k = \mathbf{Q}$. Along the way, we introduce and use the notion of the *dual root system.*

Let $\langle \cdot, \cdot \rangle : V \times V^* \to k$ be the evaluation pairing. For each $a \in \Phi$, the uniquely determined reflection $r_a : V \simeq V$ has the form

$$r_a(v) = v - \langle v, a^\vee \rangle a$$

for a unique $a^\vee \in V^*$ (the *coroot* associated to $a$) that is required to satisfy the integrality condition $a^\vee(\Phi) \subset \mathbf{Z} \subset k$. The condition $r_a(a) = -a$ forces $\langle a, a^\vee \rangle = 2$; in particular, $a^\vee \neq 0$. We saw in class that $a^\vee$ uniquely determines $a$, so the set $\Phi^\vee \subset V^* - \{0\}$ of coroots is in bijection with $\Phi$ via $a \mapsto a^\vee$. We define the reflection $r_{a^\vee} = (r_a)^* : V^* \simeq V^*$ to be dual to $r_a$ (i.e., $r_{a^\vee}(\ell) = \ell \circ r_a$); this *is* a reflection since it is dual to a reflection. More specifically:

$$r_{a^\vee}(v^*) = v^* - \langle a, v^* \rangle a^\vee$$

since evaluating the left side on $v' \in V$ gives

$$v^*(r_a(v')) = v^*(v' - \langle v', a^\vee \rangle a) = v^*(v') - \langle v', a^\vee v^*(a) = v^*(v') - \langle a, v^* \rangle a^\vee(v'),$$

which is the right side evaluated on $v'$.

We aim to show that $(V^*, \Phi^\vee)$ equipped with these dual reflections is a root system (called the *dual root system*). This requires establishing two properties: $\Phi^\vee$ spans $V^*$ over $k$, and $r_{a^\vee}(\Phi^\vee) = \Phi^\vee$ for all $a \in \Phi$. For this latter equality, we will actually prove the more precise result that $r_{a^\vee}(b^\vee) = r_a(b)^\vee$ for all $a, b \in \Phi$. The spanning property turns out to lie a bit deeper for general $k$, and is tied up with proving that root systems have canonical $\mathbf{Q}$-structures.

### 1. Coroot reflections and spanning over $\mathbf{Q}$

Let's first show that $r_{a^\vee}(\Phi^\vee) = \Phi^\vee$ for all $a \in \Phi$, or more precisely:

**Proposition 1.1.** *For all $a, b \in \Phi$, $r_{a^\vee}(b^\vee) = r_a(b)^\vee$.*

*Proof.* By the unique characterization of the coroot associated to a root, we want to show that the linear form $r_{a^\vee}(b^\vee) \in V^*$ satisfies the condition that

$$r_{r_a(b)}(x) = x - \langle x, r_{a^\vee}(b^\vee) \rangle r_a(b)$$

for all $x \in V$. To do this, we seek another expression for $r_{r_a(b)}$.

Let $\Gamma$ be the *finite* subgroup of elements of $\mathrm{GL}(V)$ that preserve the finite spanning set $\Phi$, so all reflections $r_c$ lie in $\Gamma$ ($c \in \Phi$). Inside $\Gamma$ there is at most one reflection negating a given line (since $\Gamma$ is finite and $k$ has characteristic 0), so since $r_b \in \Gamma$ is uniquely determined by the property that it is a reflection negating the line $kb$, it follows that $r_a r_b r_a^{-1} \in \Gamma$ is uniquely determined as being a reflection that negates $r_a(kb) = kr_a(b)$. But $r_{r_a(b)} \in \Gamma$ is also such an element, so we conclude that

$$r_{r_a(b)} = r_a r_b r_a^{-1} = r_a r_b r_a.$$

Evaluating on $v \in V$, we get that

$$v - \langle v, r_a(b)^\vee \rangle r_a(b) = r_a(r_b(r_a(v))).$$

Applying $r_a = r_a^{-1}$ to both sides, we get

$$r_a(v) - \langle v, r_a(b)^\vee \rangle b = r_b(r_a(v)) = r_a(v) - \langle r_a(v), b^\vee \rangle b.$$

Hence, $\langle v, r_a(b)^\vee \rangle = \langle r_a(v), b^\vee \rangle = \langle v, r_{a^\vee}(b^\vee) \rangle$ by definition of the dual reflection $r_{a^\vee} := (r_a)^*$. This holds for all $v \in V$, so $r_{a^\vee}(b^\vee) = r_a(b)^\vee$. ■

To prove that $\Phi^\vee$ spans $V^*$, we will first give an argument that works when $k = \mathbf{Q}$, and then we will bootstrap that to the general case.

**Proposition 1.2.** *If $k = \mathbf{Q}$ then $\Phi^\vee$ spans $V^*$. In particular, $(V^*, \Phi^\vee)$ is a root system when $k = \mathbf{Q}$.*

The proof we give works verbatim over $\mathbf{R}$, or any ordered field at all. (The special roles of $\mathbf{Q}$ and $\mathbf{R}$ is that they admit unique order structures as fields.)

*Proof.* Choose a positive-definite quadratic form $q : V \to \mathbf{Q}$, and by averaging this over the finite Weyl group $W = W(\Phi)$ we arrive at a positive-definite $q$ that is $W$-invariant. Hence, the associated symmetric bilinear form $B = B_q$ is $W$-invariant in the sense that $B_q(w.v, w.v') = B_q(v, v')$ for all $v, v' \in V$ and $w \in W$, and it is *non-degenerate* since $B_q(v, v) = 2q(v) > 0$ for $v \neq 0$. This bilinear form defines a $W$-equivariant isomorphism $V \simeq V^*$ via $v \mapsto B_q(v, \cdot) = B_q(\cdot, v)$.

For each root $a$, the reflection $r_a : V \simeq V$ induces negation on the line $L$ spanned by $a$, so it restricts to an automorphism of the $B_q$-orthogonal hyperplane $H = L^\perp$. But $L \cap L^\perp = 0$ since $q$ is positive-definite, so addition $L \oplus L^\perp \to V$ is an isomorphism. Since $L^\perp$ is characterized in terms of $L$ and $B_q$, and $W$ leaves $B_q$ invariant, so $r_a$ leaves $B_q$ invariant, the $r_a$-stability of $L$ implies the same for $L^\perp$. But the eigenvalue $-1$ for $r_a$ is already accounted for on $L$, so the finite-order automorphism of $L^\perp$ arising from $r_a$ has only 1 as an eigenvalue, and hence $r_a|_{L^\perp}$ must be the identity. Writing $v \in V$ as $v = v' + ca$ for $v' \in L^\perp$ and a scalar $c$,

$$r_a(v) = v' - ca = (v' + ca) - 2ca = v - 2ca$$

and $B_q(v, a) = B_q(v', a) + cB_q(a, a) = cB_q(a, a)$ with $B_q(a, a) \neq 0$.

We conclude that $c = B_q(v, a)/B_q(a, a)$, so

$$r_a(v) = v - 2ca = v - \frac{2B_q(v, a)}{B_q(a, a)} a = v - B_q(v, a')a$$

where $a' := 2a/B_q(a, a)$. In other words, the identification of $V^*$ with $V$ via $B_q$ identifies $a^\vee \in V^*$ with $a' = 2a/B_q(a, a) \in V$. This is traditionally written as:

$$a^\vee = \frac{2a}{(a|a)}$$

with $(\cdot|\cdot)$ denoting a positive-definite symmetric bilinear form on $V$ that is $W$-invariant (and the role of this *choice* of such positive-definite form in the identification of $V^*$ with $V$ has to be remembered when using that formula!).

Now we're ready to show $\Phi^\vee$ spans $V^*$. If not, its span is contained in some hyperplane in $V^*$, and a hyperplane in $V^*$ is nothing more or less than the set of linear forms that kill a specified nonzero $v \in V$. Hence, there would exist some nonzero $v \in V$ such that $\langle v, a^\vee \rangle = 0$ for all $a \in \Phi$. The identification of $V^*$ with $V$ via $B_q$ carries the evaluation pairing between $V^*$ and $V$ over to the symmetric bilinear form $B_q$, so the coroot $a^\vee$ is brought to $a' = 2a/B_q(a,a)$. Thus,

$$0 = \langle v, a^\vee \rangle = B_q(v, a') = \frac{2B_q(v,a)}{B_q(a,a)}$$

for all $a \in \Phi$. In other words, $v$ is $B_q$-orthogonal to all $a \in \Phi$. But $\Phi$ spans $V$ (!), so $v$ is $B_q$-orthogonal to the entirety of $V$, a contradiction since $v \neq 0$ and $B_q$ is non-degenerate. ■

## 2. The spanning property over general $k$

The verification of the root system properties for $(V^*, \Phi^\vee)$ when $k$ is general shall now be deduced from the settled case $k = \mathbf{Q}$. The trick is to introduce an auxiliary $\mathbf{Q}$-structure, apply the result over $\mathbf{Q}$ there, and then return to the situation over $k$. To that end, let $V_0 = \mathbf{Q}\Phi$ denote the $\mathbf{Q}$-span of $\Phi$ inside $V$, and write $a_0$ to denote $a$ viewed inside $V_0$. Also write $\Phi_0 \subset V_0$ to denote $\Phi$ viewed inside $V_0$.

Note that since $r_a(\Phi) = \Phi$ for all $a$, we see that $r_a(V_0) = V_0$ for all $a$. Likewise, by the integrality hypothesis, $a^\vee(\Phi) \subset \mathbf{Z} \subset \mathbf{Q}$ for all $a$, so $a^\vee(V_0) \subset \mathbf{Q}$ for all $a$. Hence, we get $\mathbf{Q}$-linear form $a_0^\vee : V_0 \to \mathbf{Q}$ that is the restriction of $a^\vee$, and for all $v_0 \in V_0$ we have

$$r_a(v_0) = v_0 - \langle v_0, a^\vee \rangle a_0 = v_0 - \langle v_0, a_0^\vee \rangle a_0.$$

Thus, $(V_0, \Phi_0)$ is a root system over $\mathbf{Q}$ with associated reflections $r_{a_0} = r_a|_{V_0}$ for all $a_0 \in \Phi_0 = \Phi$, so the associated coroot is $a_0^\vee$. It follows from the settled case over $\mathbf{Q}$ that we have a dual root system $(V_0^*, \Phi_0^\vee)$ where $V_0^*$ denotes the $\mathbf{Q}$-dual of $V_0$ and $\Phi_0^\vee$ is the set of $\mathbf{Q}$-linear forms $a_0^\vee$. In particular, the elements $a_0^\vee \in V_0^*$ are a spanning set over $\mathbf{Q}$ by the settled case over $\mathbf{Q}$!

Consider the natural $k$-linear map $f : k \otimes_{\mathbf{Q}} V_0 \to V$. This carries $1 \otimes a_0$ to $a$ for all $a \in \Phi$, so it is surjective (since $\Phi$ spans $V$ over $k$). Moreover, the $k$-linear form $a^\vee : V \to k$ is compatible with the scalar extension of $a_0^\vee : V_0 \to \mathbf{Q}$ under this surjection since we can compare against the sets $\Phi_0$ and $\Phi$ that compatibly span $V$ over $k$ and $V_0$ over $\mathbf{Q}$ respectively. Once we show that $f$ is also injective, it follows that $a^\vee$ is identified with the scalar extension of $a_0^\vee$, so in fact the initial root system $(V, \Phi)$ is obtained by scalar extension from the root system $(V_0, \Phi_0)$ over $\mathbf{Q}$. (The notion of *scalar extension* for root systems is defined in an evident manner.) In this sense, *every root system will have a canonical* $\mathbf{Q}$*-structure*. This is why the case $k = \mathbf{Q}$ is essentially the "general" case (though it is very convenient to perform certain later arguments after scalar extension from $\mathbf{Q}$ to $\mathbf{R}$).

Why is $f$ also injective? It is equivalent to show that the dual $k$-linear map $f^* : V^* \to k \otimes_{\mathbf{Q}} V_0^*$ is surjective. In other words, we seek a spanning set in $V^*$ over $k$ that is carried to a spanning set for $k \otimes_{\mathbf{Q}} V_0^*$ over $k$. Well, $f^*(a^\vee) = a^\vee \circ f = 1 \otimes a_0^\vee$ (the compatibility of $a^\vee$ and $a_0^\vee$ that has already been noted), so it remains to recall that the coroots $a_0^\vee$ in $V_0^*$ are a spanning set over $\mathbf{Q}$!