Math 210C. The "Weyl Jacobian" formula

1. Introduction

Let G be a connected compact Lie group, and T a maximal torus in G. Let $q : (G/T)\times T \to G$ be the map $(\overline{g}, t) \mapsto gtg^{-1}$. Fix choices of nonzero left-invariant top-degree differential forms $\mathrm{d}g$ on G and $\mathrm{d}t$ on T, and let $\mathrm{d}\overline{g}$ be the associated left-invariant top-degree differential form on G/T, determined by the property that

$$\mathrm{d}\overline{g}(e) \otimes \mathrm{d}t(\overline{e}) = \mathrm{d}g(e)$$

via the canonical isomorphism $\det(\mathrm{Tan}^*_{\overline{e}}(G/T)) \otimes \det(\mathrm{Tan}^*_e(T)) \simeq \det(\mathrm{Tan}^*_e(G))$. These differential forms are used to define orientations on G, T, and G/T, hence also on $(G/T)\times T$ via the product orientation.

Remark 1.1. Recall that for oriented smooth manifolds X and X' with respective dimensions d and d', the *product orientation* on $X \times X'$ declares the positive ordered bases of

$$\mathrm{T}_{(x,x')}(X \times X') = \mathrm{T}_x(X) \oplus \mathrm{T}_{x'}(X')$$

to be the ones in the common orientation class of the ordered bases $\{v_1, \dots, v_d, v'_1, \dots, v'_{d'}\}$, where $\{v_i\}$ is an oriented basis of $\mathrm{T}_x(X)$ and $\{v'_{j'}\}$ is an oriented basis of $\mathrm{T}_{x'}(X')$. If we swap the order of the factors (i.e., we make an ordered basis for $\mathrm{T}_{(x,x')}(X \times X')$ by putting the $v'_{j'}$'s ahead of the v_i's) then the orientation on $X \times X'$ changes by a sign of $(-1)^{dd'}$. Consequently, as long as one of d or d' is *even*, there is no confusion about the orientation on $X \times X'$. Fortunately, it will turn out that G/T has even dimension!

Since $\mathrm{d}t \wedge \mathrm{d}\overline{g}$ is a nowhere-vanishing top-degree C^∞ differential form on $(G/T) \times T$, there is a unique C^∞ function F on $(G/T) \times T$ satisfying

$$q^*(\mathrm{d}g) = F \cdot \mathrm{d}t \wedge \mathrm{d}\overline{g}.$$

(In class we used $\mathrm{d}\overline{g} \wedge \mathrm{d}t$, as in the course text; this discrepancy will not matter since we'll eventually show that $\dim(G/T)$ is even.) Let's describe the meaning of $F(\overline{g}_0, t_0) \in \mathbf{R}$ as a Jacobian determinant. For any point $(\overline{g}_0, t_0) \in (G/T)\times T$, we may and do choose an oriented ordered basis of $\mathrm{Tan}_{t_0}(T)$ whose ordered dual basis has wedge product equal to $\mathrm{d}t(t_0)$. We also may and do choose an oriented ordered basis of $\mathrm{Tan}_{\overline{g}_0}(G/T)$ whose ordered dual basis has wedge product equal to $\mathrm{d}\overline{g}(\overline{g}_0)$. Use these to define an ordered basis of $\mathrm{Tan}_{(\overline{g}_0,t_0)}((G/T)\times T)$ by following the convention for product orientation putting T ahead of G/T. Finally, choose an oriented ordered basis of $\mathrm{Tan}_{q(\overline{g}_0,t_0)}(G)$ whose associated ordered dual basis has wedge product equal to $\mathrm{d}g(q(\overline{g}_0, t_0))$.

Consider the matrix of the linear map

$$\mathrm{d}q(\overline{g}_0, t_0) : \mathrm{Tan}_{(\overline{g}_0,t_0)}((G/T) \times T) \to \mathrm{Tan}_{q(\overline{g}_0,t_0)}(G)$$

relative to the specified ordered bases on the source and target. The determinant of this matrix is exactly $F(\overline{g}_0, t_0)$. (Check this!) In particular, $F(\overline{g}_0, t_0) \neq 0$ precisely when q is a local C^∞ isomorphism near $(\overline{g}_0, t_0)$, and $F(\overline{g}_0, t_0) > 0$ precisely when q is an orientation-preserving local C^∞ isomorphism near $(\overline{g}_0, t_0)$. Provided that $\dim(G/T)$ is even, it won't matter which way we impose the orientation on $(G/T)\times T$ (i.e., which factor we put "first").

In view of the preceding discussion, it is reasonable to introduce a suggestive notation for F: we shall write $\det(\mathrm{d}q(\overline{g}, t))$ rather than $F(\overline{g}, t)$. Of course, this depends on more than just the linear map $\mathrm{d}q(\overline{g}, t)$: it also depends on the initial choices of invariant differential forms $\mathrm{d}g$ on G and $\mathrm{d}t$ on T, as well as the convention to orient $(G/T) \times T$ by putting T ahead of G/T (the latter convention becoming irrelevant once we establish that $\dim(G/T)$ is even). Such dependence is suppressed in the notation, but should not be forgotten.

The purpose of this handout is to establish an explicit formula for the Jacobian determinant $\det(\mathrm{d}q)$ associated to the map q. It is expressed in terms of the map

$$\mathrm{Ad}_{G/T} : T \to \mathrm{GL}(\mathrm{Tan}_{\overline{e}}(G/T)),$$

so let's recall the definition of $\mathrm{Ad}_{G/T}$ more generally (given dually in Exercise 2 of HW5).

If H is a closed subgroup of a Lie group G then $\mathrm{Ad}_{G/H} : H \to \mathrm{GL}(\mathrm{T}_{\overline{e}}(G/H))$ is the C^∞ homomorphism that carries $h \in H$ to the linear automorphism $\mathrm{d}(\ell_h)(e)$ of $\mathrm{T}_{\overline{e}}(G/H) = \mathfrak{g}/\mathfrak{h}$ arising from the left translation on G/H by h (which fixes the coset $\overline{e} = \{H\}$ of h). It is equivalent to consider the effect on $\mathfrak{g}/\mathfrak{h}$ of the conjugation map $c_h : x \mapsto hxh^{-1}$ since right-translation on G by h^{-1} has no effect upon passage to G/H. Put in other terms, $\mathrm{Ad}_{G/H}(h)$ is the effect on $\mathfrak{g}/\mathfrak{h}$ of the automorphism $\mathrm{d}c_h(e) = \mathrm{Ad}_G(h)$ of $\mathfrak{g}$ that preserves $\mathfrak{h}$ (since $\mathrm{Ad}_G(h)|_{\mathfrak{h}} = \mathrm{Ad}_H(h)$ due to functoriality in the Lie group for the adjoint representation).

2. Main result

We shall prove the following formula for the "Weyl Jacobian" $\det(\mathrm{d}q)$:

Theorem 2.1. *For any $(\overline{g}, t) \in (G/T) \times T$,*

$$\det(\mathrm{d}q(\overline{g}, t)) = \det(\mathrm{Ad}_{G/T}(t^{-1}) - 1).$$

In class we will show that $\dim(G/T)$ is even, so there is no possible sign ambiguity in the orientation on $(G/T) \times T$ that underlies the definition of $\det(\mathrm{d}q)$. Our computation in the proof of the Theorem will use the orientation which puts T ahead of G/T; this is why we used $\mathrm{d}t \wedge \mathrm{d}\overline{g}$ rather than $\mathrm{d}\overline{g} \wedge \mathrm{d}t$ when initially defining $\det(\mathrm{d}q)$. Only after the evenness of $\dim(G/T)$ is proved in class will the Theorem hold without sign ambiguity. (The proof of such evenness will be insensitive to such sign problems.)

Proof. We first use left translation to pass to the case $\overline{g} = \overline{e}$, as follows. For $g_0 \in G$, we have a commutative diagram

$$\begin{array}{ccc} (G/T) \times T & \xrightarrow{q} & G \\ \lambda \downarrow & & \downarrow c_{g_0} \\ (G/T) \times T & \xrightarrow[q]{} & G \end{array}$$

where $\lambda(\overline{g}, t) = (g_0.\overline{g}, t)$. The left-translation by g_0 on G/T pulls $\mathrm{d}\overline{g}$ back to itself due to left invariance of this differential form on G/T. The conjugation c_{g_0} pulls $\mathrm{d}g$ back to itself because $\mathrm{d}g$ is *bi-invariant* (i.e., also right-invariant) due to the triviality of the algebraic modulus character Δ_G (since G is compact and *connected*). It therefore follows via the definition of $\det(\mathrm{d}q)$ that

$$\det(\mathrm{d}q(g_0.\overline{g}, t)) = \det(\mathrm{d}q(\overline{g}, t)).$$

Hence, by choose g_0 to represent the inverse of a representative of $\overline{g}$, we may and do restrict attention to the case $\overline{g} = \overline{e}$.

Our aim is to show that for any $t \in T$, $\det(\mathrm{d}q(\overline{e}, t_0)) = \det(\mathrm{Ad}_{G/T}(t_0^{-1}) - 1)$ for all $t_0 \in T$. Consider the composite map

$$f : (G/T) \times T \xrightarrow{1 \times \ell_{t_0}} (G/T) \times T \xrightarrow{q} G \xrightarrow{\ell_{t_0^{-1}}} G.$$

This carries $(\overline{g}, t)$ to $(t_0^{-1} g t_0)(t g^{-1})$ (which visibly depends only on gT rather than on g, as it must); this map clearly carries $(\overline{e}, e)$ to e. The first and third steps pull the chosen differential forms on $(G/T) \times T$ and G (namely, $\mathrm{d}t \wedge \mathrm{d}\overline{g}$ and $\mathrm{d}g$) back to themselves due to the arranged left-invariance properties. Consequently, $\det(\mathrm{d}q(\overline{e}, t_0)) = \det(\mathrm{d}f(\overline{e}, e))$, where the "determinant" of

$$\mathrm{d}f(\overline{e}, e) : (\mathfrak{g}/\mathfrak{t}) \oplus \mathfrak{t} \to \mathfrak{g}$$

is defined using oriented ordered bases of $\mathfrak{t}$, $\mathfrak{g}/\mathfrak{t}$, and $\mathfrak{g}$ whose wedge products are respectively dual to $\mathrm{d}t(e)$, $\mathrm{d}\overline{g}(\overline{e})$, and $\mathrm{d}g(e)$ (and the direct sum is oriented by putting $\mathfrak{t}$ ahead of $\mathfrak{g}/\mathfrak{t}$).

The *definition* of $\mathrm{d}\overline{g}(\overline{e})$ uses several pieces of data: $\mathrm{d}g(e)$, $\mathrm{d}t(e)$, and the natural isomorphism $\det(V') \otimes \det(V'') \simeq \det(V)$ associated to a short exact sequence of finite-dimensional vector spaces $0 \to V' \to V \to V'' \to 0$ (applied to $0 \to \mathfrak{t} \to \mathfrak{g} \to \mathrm{Tan}_{\overline{e}}(G/T) \to 0$). Thus, we can *choose* the oriented ordered basis of $\mathrm{Tan}_{\overline{e}}(G/T) = \mathfrak{g}/\mathfrak{t}$ adapted to $\mathrm{d}\overline{g}(\overline{e})$ to be induced by an oriented ordered basis of $\mathfrak{g}$ adapted to $\mathrm{d}g(e)$ that has as its inital part an ordered oriented basis of $\mathfrak{t}$ adapted to $\mathrm{d}t(e)$. To summarize, the matrix for $\mathrm{d}f(\overline{e}, e) : (\mathfrak{g}/\mathfrak{t}) \oplus \mathfrak{t} \to \mathfrak{g}$ rests on: an ordered basis of $\mathfrak{t}$, an extension of this to an ordered basis of $\mathfrak{g}$ by appending additional vectors at the end of the ordered list, and the resulting quotient ordered basis of $\mathfrak{g}/\mathfrak{t}$.

The restriction $\mathrm{d}f(\overline{e}, e)|_{\mathfrak{t}}$ to the direct summand $\mathfrak{t}$ is the differential of $f(\overline{e}, \cdot) : T \to G$ that sends $t \in T$ to $(t_0^{-1} e t_0) t e^{-1} = t \in G$. In other words, this restriction is the natural inclusion of $\mathfrak{t}$ into $\mathfrak{g}$.

Composing the restriction $\mathrm{d}f(\overline{e}, e)|_{\mathfrak{g}/\mathfrak{t}}$ to the direct summand $\mathfrak{g}/\mathfrak{t}$ with the quotient map $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{t}$ is the endomorphism of $\mathfrak{g}/\mathfrak{t}$ that is the differential at $\overline{e}$ of the map $k : G/T \to G/T$ defined by

$$\overline{g} \mapsto (t_0^{-1} g t_0) e g^{-1} \bmod T = c_{t_0^{-1}}(g) g^{-1} \bmod T = m(c_{t_0^{-1}}(g), g^{-1}) \bmod T.$$

Since the group law $m : G \times G \to G$ has differential at (e, e) equal to addition in $\mathfrak{g}$, and inversion $G \to G$ has differential at the identity equal to negation on $\mathfrak{g}$, clearly

$$\mathrm{d}k(\overline{e}) = (\mathrm{Ad}_G(t_0^{-1}) \bmod \mathfrak{t}) - 1 = \mathrm{Ad}_{G/T}(t_0^{-1}) - 1.$$

Our ordered basis of $\mathfrak{g}$ begins with an ordered basis of $\mathfrak{t}$ and the remaining part lifts our chosen ordered basis of $\mathfrak{g}/\mathfrak{t}$, so the matrix used for $\mathrm{d}f(\overline{e}, e)$ has the upper triangular form

$$\begin{pmatrix} 1 & * \\ 0 & M(t_0) \end{pmatrix}$$

where the lower-right square $M(t_0)$ is the matrix of the endomorphism $\mathrm{Ad}_{G/T}(t_0^{-1}) - 1$ of $\mathfrak{g}/\mathfrak{t}$ relative to the *same* ordered basis on its source and target. Consequently, the determinant of this matrix for $\mathrm{d}f(\overline{e}, e)$ is equal to $\det M(t_0) = \det(\mathrm{Ad}_{G/T}(t_0^{-1}) - 1)$ (the intrinsic determinant using *any* fixed choice of ordered basis for the common source and target $\mathfrak{g}/\mathfrak{t}$). ∎