Math 210C. Existence of the coroot

Let G be a non-commutative connected compact Lie group, T a maximal torus in G, and $a \in \Phi(G,T)$ a root. Let $T_a = (\ker a)^0$ be the codimension-1 subtorus of T killed by $a : T \twoheadrightarrow S^1$, so $Z_G(T_a)$ is a connected closed subgroup containing T such that $\Phi(Z_G(T_a),T) = \{\pm a\}$.

We have seen in class that $Z_G(T_a)$ is the almost direct product of its maximal central torus T_a and its closed commutator subgroup $G_a := Z_G(T_a)'$ that is either SU(2) or SO(3) and has as a 1-dimensional maximal torus $T'_a := T \cap G_a$. Any element of G_a centralizes T_a and so if it normalizes T'_a then it normalizes $T_aT'_a = T$. Thus, $N_{G_a}(T'_a) \subset N_G(T)$, and since all elements of G_a centralize T_a it follows that

$$N_{G_a}(T'_a) \cap T = N_{G_a}(T'_a) \cap Z_G(T) = N_{G_a}(T'_a) \cap Z_G(T'_a) = Z_{G_a}(T'_a) = T'_a.$$

In other words, we have an inclusion $j : W(G_a, T'_a) \hookrightarrow W(Z_G(T_a), T)$.

The action of $W(Z_G(T_a),T)$ on $\mathrm{X}(T)_{\mathbf{Q}} = \mathrm{X}(T_a)_{\mathbf{Q}} \oplus \mathrm{X}(T'_a)_{\mathbf{Q}}$ respects the direct sum decomposition since all elements of $Z_G(T_a)$ normalizing T certainly centralize T_a and normalize $T \cap Z_G(T_a)' =: T'_a$. Thus, the action of $W(Z_G(T_a),T)$ on $\mathrm{X}(T)$ preserves $\mathrm{X}(T'_a)$ and is determined by its effect on this $\mathbf{Z}$-line, so we have an injection

$$W(Z_G(T_a),T) \hookrightarrow \mathrm{GL}(\mathrm{X}(T'_a)) = \mathrm{GL}(\mathbf{Z}) = \mathbf{Z}^\times = \{\pm 1\}$$

and it is straightforward to check (do it!) that this is compatible via j with the natural action $W(G_a, T'_a) \hookrightarrow \mathrm{GL}(\mathrm{X}(T'_a)) = \mathbf{Z}^\times$. In other words, we have compatibly

$$W(G_a, T'_a) \subset W(Z_G(T_a), T) \subset \{\pm 1\}.$$

But by case-checking for $G_a = \mathrm{SU}(2)$ and $G_a = \mathrm{SO}(3)$ with specific maximal tori, we saw in class that $W(G_a, T'_a)$ has order 2. Hence, by squeezing, $W(Z_G(T_a),T)$ also has order 2. Explicitly, $W(Z_G(T_a),T) = \{1, r_a\}$ for an element r_a of order 2 represented by $n_a \in N_G(T)$ that centralizes the codimension-1 torus T_a and induces inversion on its 1-dimensional isogeny-complement T'_a.

As we explained in class, the effect of the reflection r_a on $\mathrm{X}(T)_{\mathbf{Q}} = \mathrm{X}(T_a)_{\mathbf{Q}} \oplus \mathrm{X}(T'_a)_{\mathbf{Q}}$ is the identity on the hyperplane $\mathrm{X}(T_a)_{\mathbf{Q}}$ and is negation on the line $\mathrm{X}(T'_a)_{\mathbf{Q}}$ that contains a (as a is trivial on $T_a = (\ker a)^0$, so a has vanishing component along the hyperplane factor $\mathrm{X}(T_a)_{\mathbf{Q}}$ of $\mathrm{X}(T)_{\mathbf{Q}}$). Hence, there is a unique linear form $\ell_a : \mathrm{X}(T)_{\mathbf{Q}} \to \mathbf{Q}$ such that $r_a(x) = x - \ell_a(x)a$. The $\mathbf{Z}$-dual of $\mathrm{X}(T)$ is the cocharacter lattice $\mathrm{X}_*(T)$ via the perfect pairing

$$\langle \cdot, \cdot \rangle : \mathrm{X}(T) \times \mathrm{X}_*(T) \to \mathrm{End}(S^1) = \mathbf{Z}$$

defined by $\langle \chi, \lambda \rangle = \chi \circ \lambda$, so we can thereby identify the $\mathbf{Q}$-dual of $\mathrm{X}(T)_{\mathbf{Q}}$ with $\mathrm{X}_*(T)_{\mathbf{Q}}$.

The key point in the story is that $\ell_a : \mathrm{X}(T)_{\mathbf{Q}} \to \mathbf{Q}$ carries $\mathrm{X}(T)$ into $\mathbf{Z}$, which is to say that it lies in the $\mathbf{Z}$-dual $\mathrm{X}_*(T)$ of $\mathrm{X}(T)$ inside the $\mathbf{Q}$-dual of $\mathrm{X}(T)_{\mathbf{Q}}$. In other words:

Proposition 0.1. *There exists a unique $a^\vee : S^1 \to T$ in $\mathrm{X}_*(T)$ such that $r_a(x) = x - \langle x, a^\vee \rangle a$ for all $x \in \mathrm{X}(T)$. Moreover, $a^\vee$ is valued in the 1-dimensional subtorus T'_a.*

We call $a^\vee$ the *coroot* attached to a.

Proof. The uniqueness is clear, since the content of the existence of $a^\vee$ is precisely that $\ell_a \in \mathrm{X}_*(T)_{\mathbf{Q}}$ happens to lie inside $\mathrm{X}_*(T)$ (and as such is then renamed as $a^\vee$). To prove existence, we shall give two proofs: one abstract and one by computing with SO(3) and SU(2) (depending on what $G_a = Z_G(T_a)'$ is).

For the computational proof, let $a' = a|_{T'_a}$. Note that r_a is the identity on the hyperplane $\mathrm{X}(T_a)_{\mathbf{Q}}$ and its restriction to the line $\mathrm{X}(T'_a)$ is exactly through the reflection $r_{a'} \in W(G_a, T'_a)$ (via how we made r_a using the equality $W(G_a, T'_a) = W(Z_G(T_a), T)$!), For any $\lambda \in \mathrm{X}_*(T'_a) \subset \mathrm{X}_*(T)$ certainly $x \mapsto x - \langle x, \lambda \rangle_T a$ is the identity on the hyperplane $\mathrm{X}(T_a)_{\mathbf{Q}}$ and has restriction to $\mathrm{X}(T'_a)_{\mathbf{Q}}$ given by $x \mapsto x - \langle x, \lambda \rangle_{T'_a} a'$, so a solution in $\mathrm{X}_*(T'_a)$ for $r_{a'}$ is also a solution in $\mathrm{X}_*(T)$ for r_a. Hence, we may replace (G, T, a) with (G_a, T'_a, a') to reduce to the case when G is SU(2) or SO(3), so $\dim T = 1$ and $\Phi = \{\pm a\}$.

It is harmless to replace a with $-a$ for our purposes (negate the result to handle the other root). It is also sufficient to treat a single maximal torus (due to the Conjugacy Theorem!). The desired formula $r_a(x) = x - \langle x, a^\vee \rangle a$ on $\mathrm{X}(T)_{\mathbf{Q}}$ for some $a^\vee \in \mathrm{X}_*(T)$ is a comparison of linear endomorphisms of a 1-dimensional $\mathbf{Q}$-vector space. Thus, it suffices to verify such a formula on a single nonzero element. We may take a to be that element, and since r_a is negation on $\mathrm{X}(T)_{\mathbf{Q}}$, our task comes down to finding $a^\vee : S^1 \to T$ such that $\langle a, a^\vee \rangle = 2$. But $T \simeq S^1$, so $a : T \to S^1$ can be identified with a nonzero endomorphism of S^1 (so $\ker a = \mu_n$ for some $n \geq 1$). Our task is to show squaring on S^1 factors through a; equivalently, $\ker a$ has order 1 or 2. By inspection, for SU(2) the roots have kernel of order 2. The quotient map $\mathrm{SU}(2) \to \mathrm{SO}(3)$ induces an isomorphism on Lie algebras and a degree-2 isogeny $T \to \overline{T}$ between maximal tori, so the roots $\overline{T} \to S^1$ for SO(3) are isomorphisms (!).

Now we give a conceptual proof. The key idea is to consider not just the group $Z_G(T_a) = Z_G((\ker a)^0)$ but also the centralizer $Z_G(\ker a)$ of the entire kernel, or rather its identity component $Z_G(\ker a)^0$. This contains T as a maximal torus too, and T_a as a central subtorus (therefore maximal as such), so we have a closed subgroup inclusion

$$Z_G(\ker a)^0/T_a \subset Z_G(T_a)/T_a$$

between rank-1 connected compact subgroups whose maximal torus T/T_a is *not* central (as $\mathrm{Lie}(Z_G(\ker a)^0/T_a) = \mathrm{Lie}(Z_G(\ker a))/\mathrm{Lie}(T_a)$ supports the $\pm a$-weight spaces after complexification, since the possibly disconnected $\ker a$ certainly acts trivially on those weight spaces!). These *connected* groups have dimension 3, so the inclusion between them is an *equality*.

For any $n \in N_{Z_G(T_a)}(T)$, its conjugation action on T is unaffected by changing it by multiplication against an element of T, such as against an element of T_a. Hence, the equality of 3-dimensional groups implies that r_a can be represented by an element $n_a \in N_G(T)$ that centralizes the entirety of $\ker a$! Hence, the endomorphism $f : T \to T$ defined by $t \mapsto t/r_a(t)$ kills the entire $\ker a$, so it factors through the quotient map $a : T \twoheadrightarrow T/(\ker a) = S^1$. In other words, we obtain a Lie group map $a^\vee : S^1 \to T$ such that $f = a^\vee \circ a$. This says that for all $t \in T$, $t/r_a(t) = a^\vee(t^a)$; i.e., $r_a(t) = t/a^\vee(t^a)$. Applying a character $x \in \mathrm{X}(T)$, we get

$$x(r_a(t)) = \frac{x(t)}{x(a^\vee(t^a))} = \frac{x(t)}{(t^a)^{\langle x, a^\vee \rangle}} = \frac{x(t)}{t^{\langle x, a^\vee \rangle a}}.$$

In other words, $x \circ r_a = x - \langle x, a^\vee \rangle a$ in $\mathrm{X}(T)$. By definition of the action of $W(G,T)$ on $\mathrm{X}(T) = \mathrm{Hom}(T, S^1)$ through inner composition, $x \circ r_a$ is the action on x by $r_a^{-1} = r_a \in W(G,T)$, so $r_a(x) = x - \langle x, a^\vee \rangle a$ for all $x \in \mathrm{X}(T)$. ■