# Math 210C. Clifford algebras and spin groups

Clifford algebras were discovered by Clifford in the late 19th century as part of his search for generalizations of quaternions. He considered an algebra generated by $V = \mathbf{R}^n$ subject to the relation $v^2 = -\|v\|^2$ for all $v \in V$. (For $n = 2$ this gives the quaternions via $i = e_1$, $j = e_2$, and $k = e_1 e_2$.) They were rediscovered by Dirac. In this handout we explain some general features of Clifford algebras beyond the setting of $\mathbf{R}^n$, including its role in the definition of spin groups. This may be regarded as a supplement to the discussion in 6.1–6.19 in Chapter I of the course text, putting those constructions into a broader context. Our discussion is generally self-contained, but we punt to the course text for some arguments.

## 1. Quadratic spaces and associated orthogonal groups

Let $V$ be a finite-dimensional nonzero vector space over a field $k$ and let $q : V \to k$ be a *quadratic form*; i.e., $q(cv) = c^2 q(v)$ for all $v \in V$ and $c \in k$, and the symmetric function $B_q : V \times V \to k$ defined by $B_q(v, w) := q(v + w) - q(v) - q(w)$ is $k$-bilinear. This is less abstract than it may seem to be: if $\{e_i\}$ is a $k$-basis of $V$ then

$$q(\sum x_i e_i) = \sum q(e_i) x_i^2 + \sum_{i<j} B_q(e_i, e_j) x_i x_j.$$

Thus, a quadratic form is just a degree-2 homogeneous polynomial function in linear coordinates. Conversely, any such function on $k^n$ is a quadratic form. You should intrinsically define the *scalar extension* quadratic form $q_{k'} : V_{k'} \to k'$ for any field extension $k'/k$.

As long as $\mathrm{char}(k) \neq 2$ (the cases of most interest to use will be $\mathbf{R}$ and $\mathbf{C}$) we can reconstruct $q$ from $B_q$ since the equality $B_q(v, v) = q(2v) - 2q(v) = 2q(v)$ yields that $q(v) = B_q(v, v)/2$. It is easy to check that if $\mathrm{char}(k) \neq 2$ and $B : V \times V \to k$ is *any* symmetric $k$-bilinear form then $q_B : v \mapsto B(v, v)/2$ is a quadratic form whose associated symmetric bilinear form is $B$. In other words, if $\mathrm{char}(k) \neq 2$ then we have a natural isomorphism of $k$-vector spaces

$$\mathrm{Quad}(V) \simeq \mathrm{SymBil}(V)$$

given by $q \mapsto B_q$ and $B \mapsto q_B$ between the $k$-vector spaces of quadratic forms on $V$ and symmetric bilinear forms on $V$.

By successive "completion of the square" (check!), if $\mathrm{char}(k) \neq 2$ then any quadratic form $q$ on $V$ admits an $B_q$*-orthogonal basis*: a basis $\{e_i\}$ such that $B_q(e_i, e_j) = 0$ for all $i \neq j$, which is to say that

$$q(\sum x_i e_i) = \sum q(e_i) x_i^2;$$

in other words, $q$ can be put in "diagonal form". (The spectral theorem gives a much deeper result over $k = \mathbf{R}$, namely that there exists a $B_q$-orthogonal basis that is also orthogonal for a choice of positive-definite inner product. That has nothing to do with the purely algebraic diagonalization of quadratic forms, for which there is no geometric meaning akin to that in the Spectral Theorem when $k = \mathbf{R}$.)

In what follows we shall *always assume* $\mathrm{char}(k) \neq 2$ for ease of discussion (we only need the cases $k = \mathbf{R}, \mathbf{C}$). If one is attentive to certain technical details and brings in ideas from algebraic geometry then it is possible to appropriately formulate the definitions, results, and proofs so that essentially everything works in characteristic 2 (and most arguments wind up being characteristic-free). This is genuinely useful in number theory for the integral theory of quadratic forms, as it is important to have $p$-adic results for *all* primes $p$, including $p = 2$, for which the integral theory requires input over the residue field $\mathbf{F}_p$.

**Definition 1.1.** A quadratic space $(V, q)$ is *non-degenerate* if the symmetric bilinear form $B_q : V \times V \to k$ is non-degenerate; i.e., the linear map $V \to V^*$ defined by $v \mapsto B_q(v, \cdot) = B_q(\cdot, v)$ is an isomorphism.

*Example* 1.2. When working with a $B_q$-orthogonal basis, non-degeneracy is just the condition that the diagonal form $q = \sum c_i x_i^2$ for $q$ has all coefficients $c_i \neq 0$. Thus, the notion of "non-degenerate quadratic form" on an $n$-dimensional vector space is just a coordinate-free way to think about diagonal quadratic forms $q = \sum c_i x_i^2$ in $n$ variables with all $c_i \neq 0$.

Given a non-degenerate $(V, q)$, the *orthogonal group* and *special orthogonal group* are respectively defined to be the groups

$$\mathrm{O}(q) = \{L \in \mathrm{GL}(V) \mid q \circ L = q\} = \{L \in \mathrm{GL}(V) \mid B_q \circ (L \times L) = B_q\}, \quad \mathrm{SO}(q) = \mathrm{O}(q) \cap \mathrm{SL}(V).$$

In down-to-earth terms, if we choose a basis $\{e_i\}$ of $V$ and let $[B_q]$ be the *symmetric* matrix $(B_q(e_i, e_j))$ then identifying $\mathrm{GL}(V)$ with $\mathrm{GL}_n(k)$ via $\{e_i\}$ identifies $\mathrm{O}(q)$ with the set of $L \in \mathrm{GL}_n(k)$ such that

$$({}^{\top}L)[B_q]L = [B_q]$$

and $\mathrm{SO}(q)$ entails the additional condition $\det(L) = 1$. (The groups $\mathrm{O}(q)$ and $\mathrm{SO}(q)$ can certainly be defined without requiring non-degeneracy, but they have rather poor properties in the degenerate case.)

Applying the determinant to both sides of the matrix equation describing $\mathrm{O}(q)$ gives that $\det(L)^2 \det([B_q]) = \det([B_q])$, and $\det([B_q]) \neq 0$ by the non-degeneracy hypothesis, so $\det(L)^2 = 1$. In other words, $\det(L) = \pm 1$. Thus, $\mathrm{SO}(q)$ has index in $\mathrm{O}(q)$ at most 2.

Define $n := \dim(V)$. If $n$ is odd then $-1 \in \mathrm{O}(q)$ and $\det(-1) = (-1)^n = -1$ in $k$, so $\mathrm{O}(q) = \{\pm 1\} \times \mathrm{SO}(q)$ in such cases. If $n$ is even then $-1 \in \mathrm{SO}(q)$ and so to show that $\mathrm{O}(q) \neq \mathrm{SO}(q)$ in general we choose a basis diagonalizing $q$ and use $\mathrm{diag}(-1, 1, 1, \ldots, 1)$. This is the $B_q$-orthogonal reflection in $e_1$, and suitable reflections always provide elements of $\mathrm{O}(q) - \mathrm{SO}(q)$:

**Proposition 1.3.** *For any $v_0 \in V$ such that $q(v_0) \neq 0$, the subspace $L^{\perp} := \ker B_q(v_0, \cdot)$ that is $B_q$-orthogonal to the line $L = kv_0$ is a hyperplane in $V$ not containing $v_0$ and the endomorphism*

$$r_{v_0} : v \mapsto v - \frac{B_q(v, v_0)}{q(v_0)} v_0$$

*of $V$ lies in $\mathrm{O}(q)$ and fixes $L^{\perp}$ pointwise but acts as negation on $L$. In particular, $r_{v_0} \in \mathrm{O}(q)$ and $\det r_{v_0} = -1$.*

The formula for $r_{v_0}$ is an algebraic analogue of the formula for orthogonal reflection through a hyperplane $H = \mathbf{R}v_0^{\perp}$ in $\mathbf{R}^n$:

$$x \mapsto x - 2\left(x \cdot \frac{v_0}{\|v_0\|}\right)\frac{v_0}{\|v_0\|} = x - \frac{2(x \cdot v_0)v_0}{v_0 \cdot v_0},$$

noting that $B_q(x, v_0) = \|x + v_0\|^2 - \|x\|^2 - \|v_0\|^2 = 2(x \cdot v_0)$.

*Proof.* Since $B_q(v_0, v_0) = 2q(v_0) \neq 0$, certainly $v_0 \notin L^\perp$. Thus, the linear functional $B_q(v_0, \cdot)$ on $V$ is not identically zero, so its kernel $L^\perp$ is a hyperplane in $V$ (not containing $v_0$). From the definition it is clear that $r_{v_0}$ fixes $L^\perp$ pointwise, and

$$r_{v_0}(v_0) = v_0 - \frac{B_q(v_0, v_0)}{q(v_0)} v_0 = v_0 - 2v_0 = -v_0.$$

Finally, to check that $r_{v_0} \in \mathrm{O}(q)$ we want to show that $q(r_{v_0}(v)) = q(v)$ for all $v \in V$. Let's write $v = v' + cv_0$ for $v' \in L^\perp$ and $c \in k$. We have $r_{v_0}(v) = v' - cv_0$, so

$$q(r_{v_0}(v)) = q(v' - cv_0) = q(v') + c^2 q(v_0) - 2cB_q(v', v_0) = q(v') + c^2 q(v_0)$$

since $B_q(v', v_0) = 0$ (as $v' \in L^\perp$). This is clearly equal to $q(v' + cv_0) = q(v)$ by a similar calculation. ■

We conclude that $\mathrm{SO}(q)$ has index 2 in $\mathrm{O}(q)$. Our main aim is to construct an auxiliary "matrix group" $\mathrm{Spin}(q)$ equipped with a surjective homomorphism $\mathrm{Spin}(q) \to \mathrm{SO}(q)$ whose kernel is central with order 2, and to show that in a specific case over $k = \mathbf{R}$ this group is *connected*. (It is also "Zariski-connected" when constructed in a more general algebro-geometric framework, but that lies beyond the level of this course.) Actually, in the end we will only succeed when the nonzero values of $-q$ are squares in $k$, such as when $q$ is negative-definite over $k = \mathbf{R}$ (e.g., $V = \mathbf{R}^n$, $q = -\sum x_i^2$, with $\mathrm{SO}(q) = \mathrm{SO}(-q) = \mathrm{SO}(n)$) or when $k$ is separably closed (recall $\mathrm{char}(k) \neq 2$). Our failure for more general settings can be overcome in an appropriate sense by introducing ideas in the theory of linear algebraic groups that lie beyond the level of this course.

The key to constructing an interesting exact sequence

$$1 \to \mathbf{Z}/2\mathbf{Z} \to \mathrm{Spin}(q) \to \mathrm{SO}(q) \to 1$$

is to introduce a certain associative $k$-algebra $\mathrm{C}(q) = \mathrm{C}(V, q)$ that contains $V$ and has dimension $2^n$ as a $k$-vector space; its group of units will contain $\mathrm{Spin}(q)$. Such algebras are called *Clifford algebras*, and they will be defined and studied in the next section.

## 2. Clifford algebras

For our initial construction of the Clifford algebra associated to $(V, q)$ we make no non-degeneracy hypothesis; the best properties occur only when $(V, q)$ is non-degenerate, but for the purpose of some early examples the case $q = 0$ (with $V \neq 0$!) is worth keeping in mind.

The problem that Clifford algebras universally solve is that of finding a $k$-algebra containing $V$ in which $q$ looks like squaring. More specifically, consider pairs $(A, j)$ where $A$ is an associative $k$-algebra and $j : V \to A$ is a $k$-linear map (not necessarily injective) such that $j(v)^2 = q(v) \cdot 1_A$ in $A$ for all $v \in V$. Since the tensor algebra

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} = k \oplus V \oplus V^{\otimes 2} \oplus \ldots$$

equipped with its evident map $V \to T(V)$ is the *initial* assocative $k$-algebra equipped with a $k$-linear map from $V$, we can construct a pair $(A, j)$ by imposing the relation $v \otimes v = q(v)$ on the tensor algebra. That is:

**Definition 2.1.** The *Clifford algebra* of $(V, q)$ is the associated $k$-algebra

$$\mathrm{C}(V, q) = \mathrm{C}(q) := T(V)/\langle v \otimes v - q(v) \rangle$$

equipped with the $k$-linear map $V \to \mathrm{C}(q)$ induced by the natural inclusion of $V$ into $T(V)$; the quotient is by the 2-sided ideal in $T(V)$ generated by elements of the form $v \otimes v - q(v)$.

In view of the universal property of the tensor algebra, it is easy to see (check!) that $\mathrm{C}(q)$ equipped with its canonical $k$-linear map from $V$ is the initial pair $(A, j)$ as above: for any such pair there is a unique $k$-algebra map $\mathrm{C}(q) \to A$ compatible with the given $k$-linear maps from $V$ into each. Also, if $k'/k$ is an extension field then it is easy to see that the scalar extension $V_{k'} \to \mathrm{C}(q)_{k'}$ uniquely factors through $V_{k'} \to \mathrm{C}(q_{k'})$ via a $k'$-algebra map $\mathrm{C}(q_{k'}) \to \mathrm{C}(q)_{k'}$ that is an isomorphism. Thus, the formation of $\mathrm{C}(q)$ commutes with ground field extension. We will soon show that the canonical map $V \to \mathrm{C}(q)$ is injective.

*Example* 2.2. As an instance of the universal property, if $f : (V, q) \to (V', q')$ is a map of quadratic spaces (i.e., $f : V \to V'$ is $k$-linear and $q' \circ f = q$) then there is a unique map of $k$-algebras $\mathrm{C}(q) \to \mathrm{C}(q')$ compatible with $f : V \to V'$ via the natural maps from $V$ and $V'$ into their respective Clifford algebras. In other words, the formation of the Clifford algebra is functorial in the quadratic space.

Note that upon imposing the relation $v \otimes v = q(v)$ for some $v \in V$, the relation $v' \otimes v' = q(v')$ automatically holds for any $v' \in kv$. Thus, we only need to impose the relation $v \otimes v = q(v)$ for a single $v$ on each line in $V$. In fact, since

$$(v_1 + v_2) \otimes (v_1 + v_2) = v_1 \otimes v_1 + v_2 \otimes v_2 + (v_1 \otimes v_2 + v_2 \otimes v_1)$$

and

$$q(v_1 + v_2) = q(v_1) + q(v_2) + B_q(v_1, v_2),$$

if $\{e_i\}$ is a basis of $V$ then imposing the necessary relations

$$e_i \otimes e_i = q(e_i), \;\; e_i \otimes e_j + e_j \otimes e_i = B_q(e_i, e_j) \tag{2.1}$$

for all $i, j$ implies that $v \otimes v = q(v)$ for all $v \in V$. Thus, the 2-sided ideal used to define $\mathrm{C}(q)$ as a quotient of $T(V)$ is generated (as a 2-sided ideal!) by the relations (2.1) with the basis vectors.

*Example* 2.3. Suppose $V = k$ and $q(x) = cx^2$ for some $c \in k$ (allowing $c = 0$!). In this case the tensor algebra $T(V)$ is the *commutative* 1-variable polynomial ring $k[t]$ (with $t$ corresponding to the element $1 \in V \subset T(V)$) and $\mathrm{C}(q) = k[t]/(t^2 - c)$ since $1 \in k = V$ is a basis (ensuring that the single relation $1 \otimes 1 = q(1) = c$, which is to say $t^2 = c$, is all we need to impose).

For example, if $V = k = \mathbf{R}$ and $q(x) = -x^2$ then $\mathrm{C}(V) = \mathbf{C} = \mathbf{R} \oplus i\mathbf{R}$.

*Example* 2.4. If $q = 0$ then $\mathrm{C}(V, q) = \wedge^\bullet(V) = \bigoplus_{n \geq 0} \wedge^n(V)$ is exactly the exterior algebra of $V$. This has dimension $2^{\dim(V)}$.

By construction, if $\{e_i\}$ is a basis of $V$ then $\mathrm{C}(q)$ is spanned over $k$ by the products $e_{i_1} \cdots e_{i_r}$ for indices $i_1, \dots, i_r$. Systematic use of the relation

$$e_i e_j = -e_j e_i + B_q(e_i, e_j)$$

with $B_q(e_i, e_j) \in k$ allows us to "rearrange terms" in these basis products at the cost of introducing the scaling factor $B_q(e_i, e_j) \in k$ into some $k$-coefficients in the $k$-linear combination, so we may arrange that $i_1 \leq \cdots \leq i_r$. We can then use the relation $e_i^2 = q(e_i) \in k$ to eliminate the appearance of any repeated indices (by absorbing such repetition into a $k$-multiplier coefficient). Thus, exactly as with exterior algebras (the case $q = 0$), a spanning set is given by the products $e_{i_1} \cdots e_{i_r}$ with $1 \leq i_1 < \cdots < i_r \leq n := \dim(V)$. Hence, $\dim \mathrm{C}(q) \leq 2^n$, with equality if and only if these products are linearly independent inside $\mathrm{C}(q)$. When such equality holds, the $e_i$'s (singleton products) are $k$-linearly independent inside $\mathrm{C}(q)$, which is to say that the canonical map $V \to \mathrm{C}(q)$ is *injective*.

*Example* 2.5. Suppose $k = \mathbf{R}$, $V = \mathbf{R}^2$ (with standard basis $e_1, e_2$) and $q(x_1, x_2) = -x_1^2 - x_2^2$, Then $\mathrm{C}(q)$ is generated over $\mathbf{R}$ by elements $e_1$, $e_2$, and $e_1 e_2$ with $e_1^2 = e_2^2 = -1$, $e_1 e_2 = -e_2 e_1$ (since $e_1$ is $B_q$-orthogonal to $e_2$, as $q$ is diagonal with respect to the $\{e_1, e_2\}$ basis) and $(e_1 e_2)^2 = -1$. Thus, there is a well-defined ring map $\mathrm{C}(q) \to \mathbf{H}$ via $e_1 \mapsto j$ and $e_2 \mapsto k$ (so $e_1 e_2 \mapsto i$), and it is surjective, so the inequality $\dim \mathrm{C}(q) \leq 4$ is an equality; i.e., $\mathrm{C}(q) = \mathbf{H}$.

In order to show that the "expected dimension" $2^{\dim V}$ of $\mathrm{C}(q)$ is always the actual dimension, we need to understand how the formation of the Clifford algebra interacts with orthogonal direct sums of quadratic spaces. To that end, and for other purposes, we now digress to discuss a natural grading on $\mathrm{C}(V)$.

Note that $T(V)$ is a $\mathbf{Z}$-graded $k$-algebra: it is a direct sum of terms $\oplus_{n \in \mathbf{Z}} A_n$ as a $k$-vector space ($A_n = V^{\otimes n}$ with $n \geq 0$ and $A_n = 0$ for $n < 0$) such that $A_n A_m \subset A_{n+m}$ and $1 \in A_0$. The elements of a single $A_n$ are called *homogeneous*. The relations that we impose in the construction of the quotient $\mathrm{C}(q)$ are generated by expressions $v \otimes v - q(v)$ that straddle two separate degrees, namely degree 2 (for $v \otimes v$) and degree 0 (for $-q(v)$). Thus, the $\mathbf{Z}$-grading is *not* inherited in a well-defined manner by the quotient $\mathrm{C}(q)$ of $T(V)$, but the *parity* of the $\mathbf{Z}$-grading on $T(V)$ survives passage to the quotient. (You should check this, by considering left and right multiples of $v \otimes v - q(v)$ against homogeneous elements in $T(V)$, and using that every element in $T(V)$ is a finite sum of homogeneous terms.)

We may therefore speak of the "even part" $\mathrm{C}(q)_0$ and the "odd part" $\mathrm{C}(q)_1$ of $\mathrm{C}(q)$, which are the respective images of the even and odd parts of the tensor algebra $T(V)$. It is clear that $\mathrm{C}(q)_0$ is a $k$-subalgebra of $\mathrm{C}(q)$ and that $\mathrm{C}(q)_1$ is a 2-sided $\mathrm{C}(q)_0$-module inside $\mathrm{C}(q)$.

*Example* 2.6. For $V = k$ and $q(x) = cx^2$ we have $\mathrm{C}(q) = k[t]/(t^2 - c)$ with $\mathrm{C}(q)_0 = k$ and $\mathrm{C}(q)_1$ is the $k$-line spanned by the residue class of $t$. As a special case, if $k = \mathbf{R}$ and $V = k$ with $q(x) = -x^2$ then $\mathrm{C}(q) = \mathbf{C}$ with even part $\mathbf{R}$ and odd part $i\mathbf{R}$,

Likewise, if $k = \mathbf{R}$ and $V = \mathbf{R}^2$ with $q(x_1, x_2) = -x_1^2 - x_2^2$ then we have seen that $\mathrm{C}(q) = \mathbf{H}$. The even part is $\mathbf{C} = \mathbf{R} \oplus \mathbf{R}i$ and the odd part $\mathbf{C}j = \mathbf{R}j \oplus \mathbf{R}k$, with $j = e_1$, $k = e_2$, and $i = e_1 e_2$.

The decomposition

$$\mathrm{C}(q) = \mathrm{C}(q)_0 \oplus \mathrm{C}(q)_1$$

thereby makes $\mathrm{C}(q)$ into a $\mathbf{Z}/2\mathbf{Z}$-graded algebra with $\mathrm{C}(q)_0$ as the "degree 0" part and $\mathrm{C}(q)_1$ as the "degree 1" part. To be precise, a *$\mathbf{Z}/2\mathbf{Z}$-graded $k$-algebra* is an associative $k$-algebra $A$ equipped with a direct sum decomposition $A = A_0 \oplus A_1$ as $k$-vector spaces such that

$A_iA_j \subset A_{i+j}$ (for indices $i, j \in \mathbf{Z}/2\mathbf{Z}$) and $1 \in A_0$. An element of $A$ is *homogeneous* when it lies in either $A_0$ or $A_1$.

*Example* 2.7. The $\mathbf{Z}/2\mathbf{Z}$-graded algebra property for $\mathrm{C}(q)$ is easily checked by computing in the tensor algebra $T(V)$ to see that $\mathrm{C}(q)_1 \cdot \mathrm{C}(q)_1 \subset \mathrm{C}(q)_0$, etc.

In the special case $q = 0$ on $V$, the grading on the exterior algebra $\mathrm{C}(V, 0) = \wedge^\bullet(V)$ has even part $\oplus_{j\geq 0} \wedge^{2j}(V)$ and odd part $\oplus_{j\geq 0} \wedge^{2j+1}(V)$.

If $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ are two $\mathbf{Z}/2\mathbf{Z}$-graded $k$-algebras then we define the $\mathbf{Z}/2\mathbf{Z}$-*graded tensor product* $A\widehat{\otimes}B$ to have as its underlying $k$-vector space exactly the usual tensor product

$$A \otimes_k B = ((A_0 \otimes_k B_0) \oplus (A_1 \otimes_k B_1)) \oplus ((A_1 \otimes_k B_0) \oplus (A_0 \otimes B_1))$$

on which we declare $A_i \otimes_k B_j$ to lie in the graded piece for $i + j \bmod 2$ and we impose the skew-commutative $k$-algebra structure (as for exterior algebras)

$$b_j \cdot a_i = (-1)^{ij} a_i \otimes b_j$$

for $a_i \in A_i$ and $b_j \in B_j$. In other words, we define

$$(a_i \otimes b_j)(a'_{i'} \otimes b'_{j'}) = (-1)^{i'j} a_i a'_{i'} \otimes b_j b'_{j'}.$$

This is not generally isomorphic *as a $k$-algebra* to the usual tensor product of algebras $A \otimes_k B$ in which $A$ and $B$ are made to commute with each other; sometimes the $k$-algebra $A\widehat{\otimes}B$ is called the *super tensor product* to distinguish it from the usual one.

For our purposes, the main reason for interest in the $k$-algebra construction $A\widehat{\otimes}B$ is its role in expressing how Clifford algebras interact with orthogonal direct sums of quadratic spaces, as we shall now explain. As a preliminary observation, here is the "universal property" for our $\mathbf{Z}/2\mathbf{Z}$-graded tensor product. If $f : A \to C$ and $g : B \to C$ are maps of $\mathbf{Z}/2\mathbf{Z}$-graded algebras such that $f(a_i)g(b_j) = (-1)^{ij} g(b_j)f(a_i)$ for all $a_i \in A_i$ and $b_j \in B_j$ then the $k$-bilinear map

$$A \otimes_k B \to C$$

defined by $a \otimes b \mapsto f(a)g(b)$ is a map of $\mathbf{Z}/2\mathbf{Z}$-graded $k$-algebras $h : A\widehat{\otimes}B \to C$. Indeed, by $k$-bilinearity it suffices to check that for $a_i \in A_i$, $a'_{i'} \in A_{i'}$, $b_j \in B_j$, and $b'_{j'} \in B_{j'}$ we have

$$h(a_i \otimes b_j)h(a'_{i'} \otimes b'_{j'}) = h((a_i \otimes b_j)(a'_{i'} \otimes b'_{j'})).$$

Since

$$h : (a_i \otimes b_j)(a'_{i'} \otimes b'_{j'}) = (-1)^{i'j}(a_i a'_{i'} \otimes b_j b'_{j'}) \mapsto (-1)^{i'j} f(a_i a'_{i'})g(b_j b'_{j'}),$$

the problem is to show that

$$f(a_i)g(b_j)f(a'_{i'})g(b'_{j'}) = (-1)^{i'j} f(a_i a'_{i'})g(b_j b'_{j'}).$$

This reduces to the identity $g(b_j)f(a'_{i'}) = (-1)^{i'j} f(a'_{i'})g(b_j)$, which is exactly our initial hypothesis on $f$ and $g$.

**Lemma 2.8.** *Consider an orthogonal direct sum* $(V, q) = (V_1, q_1)\widehat{\oplus}(V_2, q_2)$ *of quadratic spaces (i.e.,* $V = V_1 \oplus V_2$ *and* $q(v_1, v_2) = q_1(v_1) + q_2(v_2)$*, so* $V_1$ *is* $B_q$*-orthogonal to* $V_2$*). The pair of* $k$*-algebra maps*

$$\mathrm{C}(q_1), \mathrm{C}(q_2) \rightrightarrows \mathrm{C}(q)$$

*satisfies the required skew-commutativity to combine them to define a map*

$$\mathrm{C}(q_1)\widehat{\otimes}\mathrm{C}(q_2) \to \mathrm{C}(q)$$

*of* $\mathbf{Z}/2\mathbf{Z}$*-graded* $k$*-algebras. This latter map is an isomorphism. In particular,* $\dim \mathrm{C}(q) = \dim \mathrm{C}(q_1) \cdot \dim \mathrm{C}(q_2)$*.*

*Proof.* The desired skew-commutativity says that for homogeneous elements $x_i$ in $\mathrm{C}(q_1)$ of degree $i \in \mathbf{Z}/2\mathbf{Z}$ and $y_j \in \mathrm{C}(q_2)$ of degree $j \in \mathbf{Z}/2\mathbf{Z}$, $x_i y_j = (-1)^{ij} y_j x_i$ inside $\mathrm{C}(q)$. By $k$-bilinearity, to check this property we may assume $x_i$ and $y_j$ are products of vectors in $V_1$ and $V_2$ respectively in number having the same parity as $i$ and $j$. By induction on the number of such vectors in the products, we're reduced to the case $x_i = v \in V_1$ and $y_j = v' \in V_2$. We want to show that $vv' = -v'v$ in $\mathrm{C}(q)$. By the laws of Clifford algebras,

$$vv' + v'v = (v + v')^2 - v^2 - v'^2 = B_q(v, v'),$$

and $B_q(v, v') = 0$ precisely because $(V, q)$ is an orthogonal direct sum of $(V_1, q_1)$ and $(V_2, q_2)$.

Having constructed the desired $k$-algebra map, it remains to prove that it is an isomorphism. We will construct an inverse. Let $j_1 : V_1 \to \mathrm{C}(q_1)$ and $j_2 : V_2 \to \mathrm{C}(q_2)$ be the natural $k$-linear maps. Thus, the $k$-linear map

$$j : V = V_1 \oplus V_2 \to \mathrm{C}(q_1)\widehat{\otimes}\mathrm{C}(q_2)$$

given by $v_1 + v_2 \mapsto j_1(v_1) \otimes 1 + 1 \otimes j_2(v_2)$ satisfies

$$(j_1(v_1) \otimes 1 + 1 \otimes j_2(v_2))^2 = q(v_1 + v_2)$$

since

$$q(v_1 + v_2) = q_1(v_1) + q_2(v_2) = j_1(v_1)^2 + j_2(v_2)^2, \quad (1 \otimes j_2(v_2))(j_1(v_1) \otimes 1) = -j_1(v_1) \otimes j_2(v_2)$$

due to the laws of Clifford multiplication and the $B_q$-orthgonality of $v_1, v_2$ in $V$. It follows from the universal property of Clifford algebras that we obtain a unique $k$-algebra map

$$\mathrm{C}(q) \to \mathrm{C}(q_1)\widehat{\otimes}\mathrm{C}(q_2)$$

extending the map $j$ above. By computing on algebra generators coming from $V_1$ and $V_2$, this is readily checked to be an inverse to the map we have built in the opposite direction. ■

**Proposition 2.9.** *For* $n = \dim(V) > 0$*,* $\dim \mathrm{C}(q) = 2^n$ *and* $\dim \mathrm{C}(q)_0 = \dim \mathrm{C}(q)_1 = 2^{n-1}$*.*

*Proof.* By diagonalizing $q$ we obtain an orthogonal direct sum decomposition of $(V, q)$ into quadratic spaces $(V_i, q_i)$ of dimension 1. Since the preceding lemma ensures that the formation of Clifford algebras turns orthogonal direct sums of quadratic spaces into $\mathbf{Z}/2\mathbf{Z}$-graded tensor products, we have

$$\dim \mathrm{C}(q) = \prod \dim \mathrm{C}(q_i)$$

for $n$ quadratic spaces $(V_i, q_i)$ with $\dim V_i = 1$. Hence, to establish the formula for $\dim \mathrm{C}(q)$ it suffices to show that if $\dim V = 1$ then $\dim \mathrm{C}(V) = 2$. This follows from Example 2.3.

To show that the even and odd parts of $\mathrm{C}(q)$ each have dimension $2^{n-1} = (1/2)2^n$, it suffices to show that their dimensions coincide (as the sum of their dimensions is $2^n$). It is clear from the basis description that the even and odd parts of $\mathrm{C}(q)$ have the same dimension as in the case $q = 0$, which is to say the exterior algebra. The difference between the dimensions of its even and odd parts is given by a difference of sums of binomial coefficients that is readily seen to concide with the binomial exapansion of $(1-1)^n = 0$. ■

*Example* 2.10. The Clifford algebra $\mathrm{C}(\mathbf{R}^n, -\sum x_i^2)$ is generated by elements $e_1, \dots, e_n$ subject to the relations $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ when $i \neq j$. It has as an $\mathbf{R}$-basis the $2^n$ products $e_{i_1} \cdots e_{i_r}$ for $1 \leq i_1 < \cdots < i_r \leq n$ $(r \geq 0)$. These were the Clifford algebras introduced by Clifford, and in an evident sense they generalize the construction of quaternions. The description of this Clifford algebra as an $n$-fold $\mathbf{Z}/2\mathbf{Z}$-graded tensor product (in the $\widehat{\otimes}$-sense) of copies of $\mathrm{C}(\mathbf{R}, -x^2) = \mathbf{C}$ should not be confused with the usual $n$-fold tensor power of $\mathbf{C}$ as a commutative $\mathbf{R}$-algebra; the former is highly non-commutative!

*Remark* 2.11. A deeper study of the structure of Clifford algebras reveals that when $(V, q)$ is *non-degenerate* the Clifford algebra has very nice properties, depending on the parity of $n := \dim(V)$. If $n$ is even then $\mathrm{C}(q)$ has center $k$ and is a central simple $k$-algebra; i.e., a "twisted form" of the matrix algebra $\mathrm{Mat}_{2^{n/2}}(k)$ in the sense that $\mathrm{C}(q)$ and $\mathrm{Mat}_{2^{n/2}}(k)$ become isomorphic after a scalar extension to some finite Galois extension of $k$. Also, in such cases $\mathrm{C}(q)_0$ has center $Z$ that is either a quadratic field extension or $k \times k$ and $\mathrm{C}(q)_0$ is a "twisted form" of $\mathrm{Mat}_{2^{(n-2)/2}}(Z)$ (i.e., they become isomorphic after scalar extension to a finite Galois extension of $k$). Moreover, $\mathrm{C}(q)_0$ is the centralizer of $Z$ inside $\mathrm{C}(q)$.

Explicitly, if $n$ is even and $\{e_i\}$ is a basis of $V$ diagonalizing $q$ then the center $Z$ of $\mathrm{C}(q)_0$ has $k$-basis $\{1, z\}$ where $z = e_1 \cdots e_n$ (central in $\mathrm{C}(q)_0$, *not* in $\mathrm{C}(q)$: $ze_i = -e_i z$ for all $i$) and $z^2 = (-1)^{n(n-1)/2} \prod q(e_i) \in k^\times$. For instance, if $k = \mathbf{R}$ and $V = \mathbf{R}^n$ with $q = -\sum x_i^2$ then $z^2 = (-1)^{n(n+1)/2}$, so $\mathrm{C}(q)_0$ has center $Z = \mathbf{C}$ if $n \equiv 2 \bmod 4$ and $Z = \mathbf{R} \times \mathbf{R}$ if $n \equiv 0 \bmod 4$.

Suppose instead that $n$ is odd. In these cases the behavior is somewhat "opposite" that for even $n$ in the sense that it is $\mathrm{C}(q)_0$ that has center $k$ and $\mathrm{C}(q)$ that has larger center. More precisely, $\mathrm{C}(q)$ has center $Z$ of dimension 2 over $k$ that is either a quadratic field extension or $k \times k$ and it inherits a $\mathbf{Z}/2\mathbf{Z}$-grading from $\mathrm{C}(q)$ with $Z_0 = k$ and $Z_1 \otimes_k Z_1 \simeq Z_0 = k$ via multiplication. (That is, $Z = k[t]/(t^2 - c)$ for some $c \in k^\times$ with $Z_1$ the line spanned by the residue class of $t$.) The subalgebra $\mathrm{C}(q)_0$ is a "twisted form" of $\mathrm{Mat}_{2^{(n-1)/2}}(k)$ and the natural map

$$\mathrm{C}(q)_0 \otimes_k Z \to \mathrm{C}(q)$$

of $\mathbf{Z}/2\mathbf{Z}$-graded algebras is an isomorphism. (In particular, $\mathrm{C}(q)_1$ is free of rank 1 as a left or right $\mathrm{C}(q)_0$-module with basis given by a $k$-basis of $Z_1$.)

Explicitly, if $q$ diagonalizes with respect to a basis $\{e_i\}$ of $V$ (so $e_i e_j = -e_j e_i$ for all $i \neq j$) then $Z = k \oplus Z_1$ where the $k$-line $Z_1$ is spanned by the element $z = e_1 \cdots e_n \in \mathrm{C}(q)_1$ satisfying $z^2 = (-1)^{n(n-1)/2} \prod q(e_i) \in k^\times$. For instance, if $k = \mathbf{R}$ and $V = \mathbf{R}^n$ with $q = -\sum x_i^2$ then $z^2 = (-1)^{n(n+1)/2}$, so $Z = \mathbf{C}$ if $n \equiv 1 \bmod 4$ and $Z = \mathbf{R} \times \mathbf{R}$ if $n \equiv 3 \bmod 4$.

*Example* 2.12. Consider a 3-dimensional quadratic space $(V, q)$. In this case $\mathrm{C}(q)$ has dimension 8 and so its subalgebra $\mathrm{C}(q)_0$ has dimension 4. If $(V, q)$ is non-degenerate, $\mathrm{C}(q)_0$ is either a quaternion division algebra over $k$ or it is isomorphic to $\mathrm{Mat}_2(k)$. For example, if $V = \mathbf{R}^3$ and $q = -x_1^2 - x_2^2 - x_3^2$ then $\mathrm{C}(q)_0 \simeq \mathbf{H}$ via $i \mapsto e_1 e_2$, $j \mapsto e_2 e_3$, and $k \mapsto e_3 e_1$.

**Definition 2.13.** The *classical Clifford algebras* are $\mathrm{C}_n = \mathrm{C}(\mathbf{R}^n, -\sum x_i^2)$ for $n \geq 1$.

The non-degenerate quadratic forms on $\mathbf{R}^n$ up to isomorphism are

$$q_{r,n-r} = \sum_{i=1}^{r} x_i^2 - \sum_{j=r+1}^{n} x_j^2,$$

and the corresponding Clifford algebra $\mathrm{C}(\mathbf{R}^n, q_{r,n-r})$ is sometimes denoted $\mathrm{Cl}_{r,s}$ (so $\mathrm{C}_n = \mathrm{Cl}_{0,n}$). As we have noted above, $\mathrm{C}_1 = \mathbf{C}$ and $\mathrm{C}_2 = \mathbf{H}$. One can show that as $\mathbf{R}$-algebras $\mathrm{C}_3 = \mathbf{H} \times \mathbf{H}$, $\mathrm{C}_4 = \mathrm{Mat}_2(\mathbf{H})$, $\mathrm{C}_5 = \mathrm{Mat}_4(\mathbf{C})$, $\mathrm{C}_6 = \mathrm{Mat}_8(\mathbf{R})$, $\mathrm{C}_7 = \mathrm{Mat}_8(\mathbf{R}) \times \mathrm{Mat}_8(\mathbf{R})$, $\mathrm{C}_8 = \mathrm{Mat}_{16}(\mathbf{R})$, and then remarkably a periodicity property applies: as $\mathbf{R}$-algebras $\mathrm{C}_{n+8} = \mathrm{Mat}_{16}(\mathrm{C}_n) = \mathrm{Mat}_{16}(\mathbf{R}) \otimes_{\mathbf{R}} \mathrm{C}_n$ for $n > 0$. This is closely related to "Bott periodicity" in topology.

## 3. Clifford groups

Now assume that $(V, q)$ is *non-degenerate*. Since $V$ naturally occurs as a subspace of $\mathrm{C}(q)$, it makes sense to consider the units $u \in \mathrm{C}(q)^\times$ such that $u$-conjugation on $V$ inside $\mathrm{C}(q)$ preserves $V$. Let's define the *naive Clifford group*

$$\Gamma(q)' = \{u \in \mathrm{C}(q)^\times \mid uVu^{-1} = V\}.$$

(This really is a group, since if $uVu^{-1} = V$ then multiplying suitably on the left and right gives $V = u^{-1}Vu$.) Clearly there is a natural representation $\Gamma(q)' \to \mathrm{GL}(V)$ via $u \mapsto (v \mapsto uvu^{-1})$, and it lands inside $\mathrm{O}(q)$ because inside $\mathrm{C}(q)$ we may do the computation

$$q(uvu^{-1}) = (uvu^{-1})^2 = uv^2u^{-1} = q(v).$$

This representation has kernel that contains the multiplicative group of the center of $\mathrm{C}(q)$, so we'd like to cut it down to size in order to have a chance at making a degree-2 "cover" of $\mathrm{O}(q)$ (which we can then restrict over $\mathrm{SO}(q)$ and hope to get something connected for the case of $(\mathbf{R}^n, -\sum x_i^2)$).

What we call the "naive" Clifford group is called the "Clifford group" in the classic book by Chevalley on Clifford algebras and spinors. It does give rise to the correct notion of "spin group" inside the unit groups of the classical Clifford algebras over $\mathbf{R}$ (following a procedure explained below), but it is *not* the definition used in the course text (introduced below, to be called the "Clifford group"). The two definitions give *distinct* subgroups of $\mathrm{C}(q)^\times$ when $n$ is odd; the advantage of the definition in the course text is that it avoids a problem with $\Gamma(q)'$ for odd $n$ that we now explain.

The difficulty with the naive Clifford group can be seen by noticing that there is a nuisance lurking in the above natural-looking representation of $\Gamma(q)'$ on $V$ via conjugation inside the Clifford algebra:

*Example* 3.1. A basic source of elements in $\Gamma(q)'$ is the $q$-isotropic vectors $u \in V$; i.e., those $u$ for which $q(u) \neq 0$. These $u$ are certainly units in $\mathrm{C}(q)$ since $u^2 = q(u) \in k^\times$ (so $u^{-1} = u/q(u)$), and since $uv + vu = B_q(u, v)$ for $v \in V$ we have

$$uvu^{-1} = -v + B_q(u,v)u^{-1} = -v + (B_q(u,v)/q(u))u = -(v - (B_q(u,v)/q(u))u) \in V.$$

In other words, $u \in \Gamma(q)'$ and its conjugation action on $V$ is the *negative* of the $q$-orthogonal reflection $r_u$ through $u$. Since $-r_u$ leaves $u$ fixed and negates the hyperplane $B_q$-orthogonal to $u$, its determinant $(-1)^{\dim(V)-1}$ depends on $\dim(V)$ whereas $\det r_u = -1$.

In order to fix the glitch in the preceding example, we'd like to negate the effect of such $u$ under the representation, and extend this correction *homomorphically* to the entire group $\Gamma(q)' \subset \mathrm{C}(q)^\times$. The natural idea is to modify the action of any *homogeneous* element $u \in \Gamma(q)' \cap \mathrm{C}(q)_j$ via the sign $(-1)^j$ since in the case of $q$-isotropic vectors $u \in V$ we have homogeneity with $j = 1$. But then we are faced with a puzzle: are all elements of $\Gamma(q)'$ necessary homogeneous inside the $\mathbf{Z}/2\mathbf{Z}$-graded $\mathrm{C}(q)$? The answer is affirmative when $n := \dim(V)$ is even (via a calculation that we omit) but *negative* when $n$ is odd.

The failure for odd $n$ is obvious when $n = 1$: $\mathrm{C}(q) = k[t]/(t^2 - c)$ for some $c \in k^\times$, so this algebra is commutative and hence $\Gamma(q)' = \mathrm{C}(q)^\times$, which has plenty of non-homogeneous elements ($a + bt$ with $a, b \in k^\times$ such that $a^2 - cb^2 \in k^\times$). The failure for general odd $n$ is also caused by the structure of the center $Z$ of $\mathrm{C}(q)$: as we noted (without proof) in Remark 2.11 that as a $\mathbf{Z}/2\mathbf{Z}$-graded algebra $Z = k[t]/(t^2 - c)$ for some $c \in k^\times$ with $Z_1$ spanned by the class of $t$, so $Z^\times$ contains a lot of non-homogeneous elements (as for the case $n = 1$) and these all lie inside the naive Clifford group $\Gamma(q)'$.

*Remark* 3.2. A closer study reveals that the inhomogeneity in the center $Z$ of the Clifford algebra for odd $n$ is the entire source of the problem, in the sense that for odd $n$ the group $\Gamma(q)'$ is generated by its subgroup of homogeneous elements and the central subgroup $Z^\times$ (whose homogeneous part is generated by the degree-0 part $k^\times$ and an additional element in degree 1 whose square lies in $k^\times$, but which has lots of non-homogeneous elements).

One way to fix this problem is to manually insert a homogeneity condition into the definition of $\Gamma(q)'$: define the *Clifford group* $\Gamma(q)$ to be the group of *homogeneous* units $u$ in the Clifford algebra such that $uVu^{-1} = V$ (any homogeneous unit has inverse that is also homogeneous). For even $n$ this turns out to recover the naive Clifford group $\Gamma(q)'$, and personally I find this alternative procedure to be the most elegant one. We can then define the representation $\Gamma(q) \to \mathrm{O}(q)$ by letting $u \in \Gamma(q)$ act on $V$ via $v \mapsto (-1)^{i_u} uvu^{-1}$ where $i_u \in \mathbf{Z}/2\mathbf{Z}$ is the degree of the homogeneous $u \in \Gamma(q)$. This "works" (e.g., isotropic $u \in V$ acts as $r_u$) but it is not the procedure used in the course text (or in many other references, which generally follow an idea of Atiyah, Bott, and Shapiro that we now explain).

Following the course text, we shall base our development on a different-looking definition that yields the same Clifford group $\Gamma(q)$. It rests on useful involutions of the Clifford algebra:

**Definition 3.3.** Let $\alpha : \mathrm{C}(q) \simeq \mathrm{C}(q)$ be the automorphism induced by the negation automorphism of $(V, q)$. In other words, $\alpha$ acts as the identity on the even part and negation on the odd part. Let $t : \mathrm{C}(q) \simeq \mathrm{C}(q)$ by the anti-automorphism induced by $v_1 \otimes \cdots \otimes v_j \mapsto v_j \otimes \cdots \otimes v_1$ on the tensor algebra (swapping the order of multiplication in the tensor algebra and preserving the relations $v \otimes v - q(v)$, hence passing to the Clifford algebra quotient as an anti-automorphism).

It is clear that $t$ commutes with $\alpha$, and our main interest will be in the anti-automorphism $\alpha \circ t : \mathrm{C}(q) \simeq \mathrm{C}(q)$, which we call *conjugation* and denote as $x \mapsto \overline{x}$. For $v_1, \dots, v_j \in V$, the effect of conjugation is $v_1 \cdots v_j \mapsto (-1)^j v_j \cdots v_1$. Thus, on the quaternions $\mathbf{H} = \mathrm{C}_2$ (see Example 2.12) this is the usual conjugation, and likewise on $\mathbf{C} = \mathrm{C}_1$. In general

$$\overline{xy} = \alpha(t(xy)) = \alpha(t(y)t(x)) = \overline{y}\,\overline{x},\ \ \overline{\overline{x}} = \alpha(t(\alpha(t(x)))) = \alpha(\alpha(t(t(x)))) = x.$$

The effect of both $\alpha$ and conjugation on $V$ is negation, but since $\alpha$ is a homomorphism whereas conjugation is an anti-homomorphism we need to use $\alpha$ as the "generalized negation" in the following definition that fixes the glitch with the naive Clifford group noted above.

**Definition 3.4.** The *Clifford group* $\Gamma(V,q) = \Gamma(q)$ is the group of units $u \in \mathrm{C}(q)^\times$ such that $\alpha(u)Vu^{-1} = V$ inside $\mathrm{C}(q)$. Its *natural representation* on $V$ is the homomorphism $\rho : \Gamma(q) \to \mathrm{GL}(V)$ defined by $\rho(u) : v \mapsto \alpha(u)vu^{-1}$. The *classical Clifford group* is $\Gamma_n = \Gamma(\mathbf{R}^n, -\sum x_i^2) \subset \mathrm{C}_n^\times$.

The map $\alpha$ restricts to $(-1)^j$ on $\mathrm{C}(q)_j$, so $\Gamma(q)$ meets the homogeneous parts of $\mathrm{C}(q)$ exactly where the naive Clifford group $\Gamma(q)'$ does. In particular, $k^\times$ is a central subgroup of $\Gamma(q)$ and any discrepancy between $\Gamma(q)$ and $\Gamma(q)'$ is in the non-homogeneous aspect. (In Corollary 3.8 we will see that all elements of $\Gamma(q)$ are homogeneous.)

Since $\alpha$ restricts to negation on $V$, an element $u \in V$ lies in $\Gamma(q)$ precisely when $u$ is a unit in $\mathrm{C}(q)$ and $uVu^{-1} = V$; i.e., precisely when $u$ lies in the naive Clifford group. But $u^2 = q(u) \in k$, so $u \in \mathrm{C}(q)^\times$ precisely when $u$ is isotropic (i.e., $q(u) \neq 0$). Hence, $\Gamma(q) \cap V$ consists of exactly the isotropic vectors $u \in V$, and for such $u$ the automorphism $\rho(u)$ of $V$ is reflection through $u$ in the quadratic space $(V,q)$ precisely because $\alpha(u) = -u$ (this bypasses the glitch which affects the natural representation on $V$ by the naive Clifford group). The proof of Lemma 6.9 in Chapter I of the course text applies verbatim in our generality to show that the automorphism $\alpha$ and anti-automorphism $t$ of $\mathrm{C}(q)$ carry the Clifford group $\Gamma(q)$ isomorphically back onto itself. Hence, conjugation on the Clifford algebra also restricts to an anti-automorphism of the Clifford group.

On the classical Clifford algebras $\mathrm{C}_1 = \mathbf{C}$ and $\mathrm{C}_2 = \mathbf{H}$ over $k = \mathbf{R}$, since the conjugation anti-automorphism is the usual one, we see that $\overline{c}$ commutes with $c$ and $c\overline{c} \in k$, so $c \mapsto c\overline{c}$ is a multiplicative map from the Clifford algebra into $k$ in such cases. This holds whenever $\dim(V) \leq 2$. To see this we may assume $k$ is algebraically closed, so with a suitable basis $(V,q)$ is $(k,x^2)$ or $(k,xy)$. In the first case $\mathrm{C}(q) = k \times k$ and conjugation is the "flip" automorphism. In the second case $\mathrm{C}(q) \simeq \mathrm{Mat}_2(k)$ via universal property of $\mathrm{C}(k^2,xy)$ applied to the $k$-linear map $k^2 \to \mathrm{End}(k \oplus ke_1) = \mathrm{Mat}_2(k)$ carrying $(a,b)$ to $\left(\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right)$, and conjugation is thereby the map $\left(\begin{smallmatrix} r & s \\ t & z \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} z & -s \\ -t & r \end{smallmatrix}\right)$ that is $M \mapsto w({}^\top M)w^{-1}$ with $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ the standard "Weyl element" of $\mathrm{SL}_2(k)$. Thus, on the Zariski-open subset $\mathrm{C}(q)^\times$ of units in $\mathrm{C}(q)$ we have $\overline{u} = \det(u)u^{-1}$, making it evident that $\overline{u}$ commutes with $u$ and that $u\overline{u}$ is a scalar; we also directly compute that $c\overline{c} = \overline{c}c = \det(c)$ for all $c \in \mathrm{C}(q) = \mathrm{Mat}_2(k)$.

These properties break down in higher dimensions. If $\dim V = 3$ then $c$ commutes with $\overline{c}$ but the map $c \mapsto c\overline{c}$ has image equal to the center $Z$ that is larger than $k$. Explicitly, for algebraically closed $k$ we have $\mathrm{C}(q) \simeq \mathrm{Mat}_2(k) \times \mathrm{Mat}_2(k)$ with conjugation given as above on each $\mathrm{Mat}_2(k)$ factor. If $\dim(V) \geq 4$ then in general $\overline{c}$ does not commute with $c$ and the map $c \mapsto c\overline{c}$ is not even valued in the center. For example, if $k = \overline{k}$ and $\dim V = 4$ then $\mathrm{C}(q) \simeq \mathrm{Mat}_4(k)$ with conjugation given by $\left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} {}^\top X & -{}^\top Z \\ -{}^\top Y & {}^\top W \end{smallmatrix}\right)$ for $X, Y, Z, W \in \mathrm{Mat}_2(k)$.

Restricting conjugation to the Clifford group $\Gamma(q)$ (recall that conjugation preserves the Clifford group), the above problems with conjugation and the map $c \mapsto c\overline{c}$ on the entire Clifford algebra disappear. The proof rests on a lemma of independent interest:

**Lemma 3.5.** *The kernel of $\rho$ is equal to the central subgroup $k^\times$.*

*Proof.* We know that $k^\times$ is a central subgroup of $\mathrm{C}(q)^\times$ on which $\alpha$ is the identity, so certainly it lies inside $\ker\rho$. To show that any $u \in \ker\rho$ lies in $k^\times$, we adapt the proof of 6.11 in Chapter I of the text (which treats the classical Clifford groups over $k = \mathbf{R}$) by computing relative to a diagonalizing basis of $V$ for $q$.

Let $\{e_i\}$ be a basis of $V$ such that $q(\sum x_i e_i) = \sum q(e_i)x_i^2$ (so all $q(e_i) \neq 0$ by non-degeneracy). Thus, $\mathrm{C}(q)$ has a basis given by products of the $e_i$'s with strictly increasing indices, and $e_ie_j = -e_je_i$ for $i \neq j$.

By hypothesis $vu = \alpha(u)v$ for all $v \in V$, so if we write $u = u_0 + u_1$ as a sum of homogeneous parts in the Clifford algebra then we wish to show that $u_1 = 0$ and $u_0 \in k$. Since $\alpha(u) = u_0 - u_1$, comparing homogeneous parts on both sides of the equality $vu = \alpha(u)v$ gives

$$vu_0 = u_0v, \;\; vu_1 = -u_1v$$

for all $v \in V$. We will show that $u_0 \in k$ by proving that its expression in terms of the basis of products of an even number of $e_i$'s involves no term with any $e_i$'s (leaving only the option of the empty product, which is to say $k$). Likewise, to prove that the odd $u_1$ vanishes, it suffices to show that its basis expansion involves no appearance of any $e_i$.

Fix an index $i$, so we may uniquely write

$$u_0 = y_0 + e_iy_1$$

with even $y_0$ and odd $y_1$ involving no appearances of $e_i$ in their basis expansion inside the Clifford algebra. Thus, the equality $vu_0 = u_0v$ with $v = e_i$ gives

$$e_iy_0 + q(e_i)y_1 = y_0e_i + e_iy_1e_i = e_iy_0 - q(e_i)y_1.$$

(Here we have used that $y_0$ and $y_1$ do not involve $e_i$, and that $e_i$ anti-commutes with $e_j$ for any $j \neq i$.) Thus, $q(e_i)y_1 = -q(e_i)y_1$, and since $q(e_i) \neq 0$ it follows that $y_1 = -y_1$, so $y_1 = 0$ since $\mathrm{char}(k) \neq 2$. This shows that $u_0$ doesn't involve $e_i$, and since $i$ was arbitrary we may conclude (as indicated already) that $u_0 \in k$.

Now we analyze $u_1$ using that $vu_1 = -u_1v$ for all $v \in V$. Choose an index $i$, so uniquely

$$u_1 = w_1 + e_iw_0$$

for even $w_0$ and odd $w_1$ that involve no appearance of $e_i$ in their basis expansion inside the Clifford algebra. Hence, the equation $vu_1 = -u_1v$ with $v = e_i$ gives

$$e_iw_1 + q(e_i)w_0 = -w_1e_i - e_iw_0e_i = e_iw_1 - q(e_i)w_0.$$

But $q(e_i) \in k^\times$, so $w_0 = 0$ and $u_1 = w_1$ does not involve $e_i$. Varying $i$, we get $u_1 = 0$. ■

**Proposition 3.6.** *For $u \in \Gamma(q)$, the element $u\overline{u} \in \Gamma(q)$ lies in $k^\times$ and is equal to $\overline{u}u$. In particular, $\mathrm{N} : u \mapsto u\overline{u}$ is a multiplicative map from $\Gamma(q)$ into $k^\times$ that extends squaring on $k^\times$. Moreover, its restriction to the subset of isotropic vectors in $V$ is $-q$.*

We call $\mathrm{N} : \Gamma(q) \to k^\times$ the *Clifford norm*. The multiplicativity properties of N are obvious on elements $u = v_1 \cdots v_r$ for isotropic $v_j$ (with $\mathrm{N}(u) = \prod q(v_j)$ in such cases). We will see in Corollary 3.8 that all elements in $\Gamma(q)$ are of this type, and in particular are homogeneous in $\mathrm{C}(q)$, but the proof rests on the multiplicativity properties of the Clifford norm.

*Proof.* For any $v \in V$, $v\overline{v} = -v^2 = -q(v)$. For $u \in \Gamma(q)$, provided that $u\overline{u} = c \in k^\times$ it follows that $\overline{u} = cu^{-1}$, so $\overline{u}u = c$ as well. To show that $u\overline{u} \in k^\times$, the key idea is to show that $\rho(u\overline{u}) = 1$, so we can apply Lemma 3.5 to conclude. The calculation $\rho(u\overline{u}) = 1$ is carried out in the proof of 6.12 in Chapter I of the course text (the computation given there, which involves working with both $\alpha$ and $t$, carries over without change to the general case). ■

Although it was obvious that the naive Clifford group $\Gamma(q)'$ acts on $V$ via elements of $\mathrm{O}(q)$, this is not quite as obvious for the Clifford group $\Gamma(q)$ due to the intervention of $\alpha$ (since we have not addressed homogeneity properties of elements of $\Gamma(q)$!), but it is true. The proof requires applying $\alpha$, $t$, and conjugation to elements of the Clifford group, and gives more:

**Proposition 3.7.** *The image $\rho(\Gamma(q))$ is equal to $\mathrm{O}(q)$.*

*Proof.* We have seen that every reflection in $V$ through an isotropic vector $u$ lies in $\rho(\Gamma(q))$, as such $u$ lie in $\Gamma(q)$ and the associated reflection literally is $\rho(u)$. Thus, to show that $\mathrm{O}(q)$ is contained in $\rho(\Gamma(q))$ it suffices to prove that every element in $\mathrm{O}(q)$ is a product of finitely many (in fact, at most $\dim(V)$) such reflections. The course text explains this in the classical case by an elementary induction argument in the proof of 6.15 in Chapter I. The method works in general using a diagonalizing basis for $q$ *provided* that for any vectors $v, v' \in V$ satisfying $q(v) = q(v')$ there exists $g \in \mathrm{O}(q)$ such that $g(v) = v'$. (For $q = -\sum x_i^2$ on $V = \mathbf{R}^n$ with $k = \mathbf{R}$ it is proved by using Gramm-Schmidt with orthonormal bases.) This is a special case of Witt's Extension theorem in the theory of quadratic forms: the orthogonal group of a non-degenerate quadratic space acts transitively on the set of embeddings of a fixed quadratic space (such as embeddings of $(k, cx^2)$ with $c = q(v) = q(v')$).

To prove that $\rho(\Gamma(q)) \subset \mathrm{O}(q)$, we have to carry out a more sophisticated calculation than in the case of the naive Clifford group. For $u \in \Gamma(q)$, we want to prove that $q(\alpha(u)vu^{-1}) = q(v)$ for any $v \in V$. Recall that the function $\mathrm{N} : x \mapsto x\overline{x}$ on the Clifford algebra restricts to $-q$ on $V$. Hence, it is equivalent to show that $\mathrm{N}(\alpha(u)vu^{-1}) = \mathrm{N}(v)$ for all $v \in V$. This proposed equality for fixed $u$ and varying $v$ is an equality of polynomial functions on $V$. But in general $\mathrm{N}$ only has good multiplicative properties on the Clifford group $\Gamma(q)$, not on the entire Clifford algebra. Thus, we want to restrict to attention to $v$ that is $q$-isotropic (as then $v \in \Gamma(q)$). Note that the restriction to isotropic $v$ is harmless in the classical cases with $(\mathbf{R}^n, -\sum x_i^2)$, as there all nonzero $v$ are isotropic.

To reduce to treating isotropic $v$ in general, we wish to appeal to a density argument with the Zariski topology. For that purpose, we want to work over an algebraically closed field. The problem of proving $q = q \circ \rho$ is visibly sufficient to check after applying an extension of the ground field, so we may and do now assume that $k$ is algebraically closed. Thus, the locus of isotropic vectors is the Zariski-open subset of the affine space $V$ complementary to the hypersurface $q = 0$. This is a proper hypersurface (i.e., $q$ isn't identically zero), so the open complement is non-empty (and hence is Zariski-dense). An equality of polynomial functions on $V$ holds everywhere if it does so on a Zariski-dense subset. In other words, it suffices to prove $\mathrm{N}(\alpha(u)vu^{-1}) = \mathrm{N}(v)$ for $v \in V$ that are isotropic for $q$, so $v \in \Gamma(q)$.

The multiplicativity of $\mathrm{N} : \Gamma(q) \to k^\times$ gives

$$\mathrm{N}(\alpha(u)vu^{-1}) = \mathrm{N}(\alpha(u))\mathrm{N}(v)\mathrm{N}(u)^{-1}$$

in $k^\times$, so it suffices to show that $\mathrm{N}\circ\alpha = \mathrm{N}$ on $\Gamma(q)$. Since conjugation is the composition of the commuting operations $t$ and $\alpha$, it follows that conjugation commutes with $\alpha$, so for $u \in \Gamma(q)$ we have $\mathrm{N}(\alpha(u)) = \alpha(u)\overline{\alpha(u)} = \alpha(u)\alpha(\overline{u}) = \alpha(u\overline{u}) = u\overline{u}$ since $u\overline{u} \in k$ for $u \in \Gamma(q)$ and $\alpha$ is the identity on $k$. ■

**Corollary 3.8.** *Every element in* $\Gamma(q)$ *is a* $k^\times$*-multiple of a product of at most* $\dim(V)$ *isotropic vectors. In partcular, each* $u \in \Gamma(q)$ *is homogeneous in* $\mathrm{C}(q)$ *and* $\det(\rho(u)) = (-1)^{\deg(u)}$*, where* $\deg(u) \in \mathbf{Z}/2\mathbf{Z}$ *is the degree of* $u$ *in* $\mathrm{C}(q)$*.*

*Proof.* As we noted in the preceding proof, every element in $\mathrm{O}(q)$ is a product of at most $\dim(V)$ reflections in isotropic vectors. Writing

$$\rho(u) = r_{v_1}\cdots r_{v_m} = \rho(v_1)\cdots\rho(v_m) = \rho(v_1\cdots v_m)$$

for isotropic $v_j$'s, we see that $u(v_1\cdots v_m)^{-1} \in \ker\rho$. But our work with the multiplicative Clifford norm on $\Gamma(q)$ showed that $\ker\rho = k^\times$. Since reflections in isotropic vectors have determinant $-1$, we're done. ■

## 4. Pin and Spin

Let $(V,q)$ be a non-degenerate quadratic space over $k$. The homomorphism $\rho : \Gamma(q) \twoheadrightarrow \mathrm{O}(q)$ has kernel $k^\times$, and we'd like to cut this kernel down to the subgroup $\mu_2 = \{\pm 1\}$ of order 2. To do this, we note that the Clifford norm $\mathrm{N} : \Gamma(q) \to k^\times$ restricts to squaring on $k^\times$, so its kernel meets $k^\times$ in $\{\pm 1\}$. This suggests trying the subgroup $\ker\mathrm{N}$ as a candidate for a "natural" double cover of $\mathrm{O}(q)$. But does $\ker\mathrm{N}$ maps onto $\mathrm{O}(q)$?

In view of how we proved the surjectivity of $\rho$ onto $\mathrm{O}(q)$ using reflections, it would suffice to show that if $r \in \mathrm{O}(q)$ is a reflection through an isotropic vector $u$ then we can find a $k^\times$-multiple $u'$ so that $\mathrm{N}(u') = 1$ (as such a $u'$ has the same associated reflection as $u$). The Clifford norm on $\Gamma(q)$ restricts to $-q$ on the set $\Gamma(q)\cap V$ of isotropic vectors, so we seek a $k^\times$-multiple $u'$ of $u$ such that $-q(u') = 1$. Writing $u' = cu$ with $c \in k^\times$ to be determined, this amounts to asking that $-c^2q(u) = 1$, so we need to be able to extract a square root of $-q(u)$ in $k$. This may not be possible (depending on $k$ and $q$)! Of course, it could be that our strategy is simply too naive, and that if we used deeper properties of $\mathrm{O}(q)$ then we might be able to prove surjectivity.

In fact there is a genuine obstruction called the *spinor norm* which is nontrivial even for indefinite non-degenerate $(V,q)$ over $\mathbf{R}$. These matters for general $(V,q)$ are best understood via the theory of linear algebraic groups (exact sequences of algebraic groups, Galois cohomology, etc.). To avoid this, *now* we finally specialize our attention to the classical case $(\mathbf{R}^n, -\sum x_i^2)$ over $k = \mathbf{R}$ and the representation $\rho : \Gamma_n \to \mathrm{O}(n)$. In this case $-q = \sum x_i^2$ is positive-definite, so all nonzero $v \in V$ are isotropic and admit a $k^\times$-multiple on which $-q$ takes the value 1 (i.e., a unit-vector multiple). This brings us to:

**Definition 4.1.** The *Pin group* $\mathrm{Pin}(n)$ is the kernel of the Clifford norm $\mathrm{N} : \Gamma_n \to \mathbf{R}^\times$.

Note that $\mathrm{C}_n^\times$ is naturally a Lie group, being the units of a finite-dimensional associative $\mathbf{R}$-algebra, and $\Gamma_n$ is visibly a closed subgroup (check!), so it inherits Lie group structure from $\mathrm{C}_n^\times$. Also, from the construction inside the Clifford algebra it is straightforward to check (do it!) that the representation $\rho : \Gamma_n \twoheadrightarrow \mathrm{O}(n) \subset \mathrm{GL}_n(\mathbf{R})$ and Clifford norm $\mathrm{N} : \Gamma_n \to \mathbf{R}^\times$

are $C^\infty$ (equivalently: continuous), so $\mathrm{Pin}(n)$ is a closed Lie subgroup and the restriction $\rho_n : \mathrm{Pin}(n) \to \mathrm{O}(n)$ of $\rho$ is $C^\infty$. We have seen that $\rho_n$ is surjective (as $\mathbf{R}^\times_{>0}$ admits square roots in $\mathbf{R}^\times$) and it fits into an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Pin}(n) \to \mathrm{O}(n) \to 1.$$

In particular, $\mathrm{Pin}(n)$ is *compact* since it is a degree-2 covering space of the compact $\mathrm{O}(n)$.

The preimage $\mathrm{Spin}(n) = \rho_n^{-1}(\mathrm{SO}(n)) \subset \mathrm{Pin}(n)$ is an open and closed subgroup of index 2 (exactly as for $\mathrm{SO}(n)$ inside $\mathrm{O}(n)$); it consists of the even elements of $\mathrm{Pin}(n)$, by Corollary 3.8. Thus, $\mathrm{Spin}(n)$ is a compact Lie group fitting into an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$$

that realizes $\mathrm{Spin}(n)$ as a degree-2 cover of $\mathrm{SO}(n)$. We have to make sure that $\mathrm{Spin}(n)$ is *connected* (e.g., to know we haven't wound up with $\mathrm{SO}(n) \times \{\pm 1\}$ by another name).

The surjective Lie group map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is necessarily surjective on Lie algebras (as for any surjective map of Lie groups), so by the usual arguments with the submersion theorem we know via the connectedness of $\mathrm{SO}(n)$ that $\mathrm{Spin}(n)^0 \to \mathrm{SO}(n)$ is surjective. Hence, $\mathrm{Spin}(n)$ is generated by its identity component $\mathrm{Spin}(n)^0$ together with the central kernel $\{\pm 1\}$ of order 2. Consequently, to prove that $\mathrm{Spin}(n)$ is connected, it suffices to show that this kernel lies in the identity component. More specifically, it suffices to find a path in $\mathrm{Spin}(n)$ that links the identity to the nontrivial element of the kernel. Such a path is written down explicitly in the proof of 6.17 in Chapter I by working inside the Clifford algebra $\mathrm{C}_n$.

*Remark* 4.2. In Chevalley's book on Clifford algebras and spinors, specialized to the classical case $(\mathbf{R}^n, -\sum x_i^2)$, he calls the preimage of $\mathrm{SO}(n)$ in $\Gamma_n$ the *special Clifford group* and he calls its intersection $\mathrm{Spin}(n)$ with the kernel of the Clifford norm the *reduced Clifford group*. He gives no name to $\mathrm{Pin}(n)$. The name for $\mathrm{Pin}(n)$ was coined in the early 1960's: much as the notation $\mathrm{O}(n)$ is obtained typographically from $\mathrm{SO}(n)$ by removing the initial letter "S", due to the fortuitous coincidence that the word "Spin" begins with the letter "S" we can apply the same procedure to arrive at the name $\mathrm{Pin}(n)$ from $\mathrm{Spin}(n)$.