Math 210C. Class Functions and Weyl groups

As an application of the Conjugacy Theorem, we can describe the continuous class functions $f : G \to \mathbf{C}$ on a connected compact Lie group G in terms of a choice of maximal torus $T \subset G$. This will be an important "Step 0" in our later formulation of the Weyl character formula. If we consider G acting on itself through conjugation, the quotient $\mathrm{Conj}(G)$ by that action is the space of conjugacy classes. We give it the quotient topology from G, so then the $\mathbf{C}$-algebra of continuous $\mathbf{C}$-valued class functions on G is the same as the $\mathbf{C}$-algebra $C^0(\mathrm{Conj}(G))$ of continuous $\mathbf{C}$-valued class functions on $\mathrm{Conj}(G)$.

Let $W = N_G(T)/T$ be the (finite) Weyl group, so W naturally acts on T. The W-action on T is induced by the conjugation action of $N_G(T)$ on G, so we get an induced continuous map of quotient spaces $T/W \to \mathrm{Conj}(G)$.

Proposition 0.1. *The natural continuous map $T/W \to \mathrm{Conj}(G)$ is bijective.*

Proof. By the Conjugacy Theorem, every element of G belongs to a maximal torus, and such tori are G-conjugate to T, so surjectivity is clear. For injectivity, consider $t, t' \in T$ that are conjugate in G. We want to show that they below to the same W-orbit in T.

Pick $g \in G$ so that $t' = gtg^{-1}$. The two tori T, gTg^{-1} then contain t', so by connectedness and commutativity of tori we have $T, gTg^{-1} \subset Z_G(t')^0$. But these are *maximal* tori in $Z_G(t')^0$ since they're even maximal in G, and $Z_G(t')^0$ is a connected compact Lie group. Hence, by the Conjugacy Theorem applied to this group we can find $z \in Z_G(t')^0$ such that $z(gTg^{-1})z^{-1} = T$. That is, zg conjugates T onto itself, or in other words $zg \in N_G(T)$. Moreover,

$$(zg)t(zg)^{-1} = z(gtg^{-1})z^{-1} = zt'z^{-1} = t',$$

the final equality because $z \in Z_G(t')$. Thus, the class of zg in $W = N_G(T)/T$ carries t to t', as desired. ∎

To fully exploit the preceding result, we need the continuous bijection $T/W \to \mathrm{Conj}(G)$ to be a homeomorphism. Both source and target are compact spaces, so to get the homeomorphism property we just need to check that each is Hausdorff. The Hausdorff property for these is a special case of:

Lemma 0.2. *Let X be a locally compact Hausdorff topological space equipped with a continuous action by a compact topological group H. The quotient space X/H with the quotient topology is Hausdorff.*

Proof. This is an exercise in definitions and point-set topology. ∎

Combining this lemma with the proposition, it follows that the $\mathbf{C}$-algebra $C^0(\mathrm{Conj}(G))$ of continuous $\mathbf{C}$-valued class functions on G is naturally identified with $C^0(T/W)$, and it is elementary (check!) to identify $C^0(T/W)$ with the the $\mathbf{C}$-algebra $C^0(T)^W$ of W-invariant continuous $\mathbf{C}$-valued functions on T. Unraveling the definitions, the composite identification

$$C^0(\mathrm{Conj}(G)) \simeq C^0(T)^W$$

of the $\mathbf{C}$-algebras of continuous $\mathbf{C}$-valued class functions on G and W-invariant continuous $\mathbf{C}$-valued functions on T is given by $f \mapsto f|_T$.