Math 210C. Elementary properties of characters

1. Basic definitions and facts

Let G be a Lie group, and consider a nonzero left-invariant top-degree differential form ω. For any $g \in G$, $(r_g)^*(\omega)$ is also left-invariant (as r_g commutes with left translations!), so $(r_g)^*(\omega) = c(g)\omega$ for some constant $c(g) \in \mathbf{R}^\times$ (computed by comparing at the identity point). Clearly $c(g)$ is unaffected by replacing ω with an $\mathbf{R}^\times$-multiple, and since such multiples exhaust all choices for ω it follows that $c(g)$ depends only on g and not ω. It is also easy to check that $g \mapsto c(g)$ is a homomorphism.

In class, $|c|$ was called the *modulus character* of G. (In some references c is called the modulus character, or the "algebraic" modulus character, due to certain considerations with matrix groups over general fields, lying beyond the level of this course.) On HW4 you show that $|c|$ is continuous, and the same method of proof will certainly show that c itself is continuous (hence C^∞). As was noted in class, such continuity forces $|c|$ to be trivial when G is compact. Also, if G is *connected* then c has constant sign and hence $c = |c|$ (as $c(e) = 1$), so in general if G is compact then ω is right-invariant under G^0 but in general the potential sign problems for the effect of right-translation on ω by points of G in other connected components really can occur:

Example 1.1. Let $G = \mathrm{O}(n) = \mathrm{SO}(n) \rtimes \langle \iota \rangle$ for the diagonal matrix $\iota = \mathrm{diag}(-1, 1, 1, \dots, 1)$. The adjoint action of ι on the subspace $\mathfrak{so}(n) \subseteq \mathfrak{gl}_n(\mathbf{R})$ of skew-symmetric matrices is the restriction of the adjoint action of $\mathrm{GL}_n(\mathbf{R})$ on $\mathfrak{gl}_n(\mathbf{R}) = \mathrm{Mat}_n(\mathbf{R})$, which is the ordinary conjugation-action of $\mathrm{GL}_n(\mathbf{R})$ on $\mathrm{Mat}_n(\mathbf{R})$ (why?). Thus, we see that $\mathrm{Ad}_G(\iota)$ acts on $\mathfrak{so}(n)$ by negating the first row and first column (keep in mind that the diagonal entries vanish, due to skew-symmetry). Since $\mathfrak{so}(n)$ projects isomorphically onto the vector space of strictly upper-triangular matrices, we conclude that $\mathrm{Ad}_G(\iota)$ has -1-eigenspace of dimension $n-1$, so its determinant is $(-1)^{n-1}$. But this determinant is the scaling effect of $\mathrm{Ad}_G(\iota)$ on the top exterior power of $\mathfrak{so}(n)$ and hence likewise on the top exterior power of its dual.

In the language of differential forms, it follows that $r_\iota^*(\omega) = (-1)^{n-1}(\omega)$ for any left-invariant top-degree differential form on G due to two facts: $\omega = \ell_\iota^*(\omega)$ and the composition of $r_\iota = r_{\iota^{-1}}$ with ℓ_ι is ι-conjugation (whose effect on the tangent space at the identity is $\mathrm{Ad}_G(\iota)$). Thus, if n is even it follows that left-invariant top-degree differential forms are *not* right-invariant.

For the remainder of this handout, assume G is compact (but not necessarily connected, since we wish to include finite groups as a special case of our discussion). Thus, for any left-invariant ω, the measure $|\omega|$ is bi-invarant (as $|c| = 1$) even though ω might not be right-invariant under the action of some connected components away from G^0. However, this bi-invariant measure is not uniquely determined since if we scale ω by some $a > 0$ then the measure scales by a. But there is a canonical choice! Indeed, the integral $\int_G |\omega|$ converges to some positive number since G is compact, so by scaling ω it can be uniquely determined up to a sign by the condition $\int_G |\omega| = 1$. This defines a *canonical* bi-invariant measure on G, called the "volume 1 measure". We denote this measure with the suggestive notation $\mathrm{d}g$ (though it is *not* a differential form, and has no dependence on orientations).

Example 1.2. If $G = S^1$ inside $\mathbf{R}^2$ then $\mathrm{d}g = |\mathrm{d}\theta|/2\pi$. If G is a finite group then $\mathrm{d}g$ is the measure that assigns each element of the group the mass $1/|G|$, so $\int_G f(g)\,\mathrm{d}g = (1/|G|)\sum_{g\in G} f(g)$. Thus, integration against $\mathrm{d}g$ in the finite case is precisely the averaging process that pervades the representation theory of finite groups (in characteristic 0).

For an irreducible finite-dimensional $\mathbf{C}$-linear representation $\rho : G \to \mathrm{GL}(V)$, we define the *character* χ_V (or χ_ρ) to be the function $g \mapsto \mathrm{Tr}(\rho(g))$. This is a smooth $\mathbf{C}$-valued function on G since ρ is C^∞ (so its matrix entries relative to a $\mathbf{C}$-basis of V are smooth $\mathbf{C}$-valued functions). Obviously $\chi_{V\oplus W} = \chi_V + \chi_W$, and in class some other related identities were explained:

$$\chi_{V^*} = \overline{\chi}_V, \quad \chi_{V\otimes W} = \chi_V \cdot \chi_W, \quad \chi_{\mathrm{Hom}(V,W)} = \overline{\chi}_V \cdot \chi_W$$

where $\mathrm{Hom}(V, W)$ denotes the space of $\mathbf{C}$-linear maps $V \to W$ and is equipped with a *left* G-action via $g.T = \rho_W(g) \circ T \circ \rho_V(g)^{-1}$. (This ensures the crucial fact that the subspace $\mathrm{Hom}(V, W)^G$ is precise the subspace $\mathrm{Hom}_G(V, W)$ of G-equivariant homomorphisms.)

To save notation, we shall now write $g.v$ rather than $\rho(g)(v)$. The following lemma will be used a lot.

Lemma 1.3. *Let $L : W' \to W$ be a $\mathbf{C}$-linear map between finite-dimensional $\mathbf{C}$-vector spaces and $f : G \to W'$ a continuous function. Then $L(\int_G f(g)\,\mathrm{d}g) = \int_G (L \circ f)(g)\,\mathrm{d}g$.*

In this statement $\int_G f(g)\,\mathrm{d}g$ is a "vector-valued" integral.

Proof. By computing relative to $\mathbf{C}$-bases of W and W', we reduce to the case $W = W' = \mathbf{C}$ and $L(z) = cz$ for some $c \in \mathbf{C}$. This case is obvious. ■

2. Applications

Consider the linear operator $T : V \to V$ defined by the vector-valued integral $v \mapsto \int_G g.v\,\mathrm{d}g$. In the case of finite groups, this is the usual averaging projector onto V^G. Let's see that it has the same property in general. Since $\int_G \mathrm{d}g = 1$, it is clear (e.g., by computing relative to a $\mathbf{C}$-basis of V) that $T(v) = v$ if v lies in the subspace V^G of G-invariant vectors. In fact, T lands inside V^G: by Lemma 1.3, we may compute

$$g'.T(v) = \int_G g'.(g.v)\,\mathrm{d}g = \int_G (g'g).v\,\mathrm{d}g = \int_G g.v\,\mathrm{d}g = T(v),$$

where the second to last equality uses the invariance of the measure under left translation by g'^{-1}.

Since T is a linear projector on V^G, its trace as an endomorphism of V is $\dim(V^G)$. In terms of integration of continuous fuctions $G \to \mathrm{End}(V)$, we see that $T = \int_G \rho(g)\,\mathrm{d}g$. Thus, applying Lemma 1.3 to the linear map $\mathrm{Tr} : \mathrm{End}(V) \to \mathbf{C}$, we conclude that

$$\dim(V^G) = \mathrm{Tr}(T) = \int_G \mathrm{Tr}(\rho(g))\,\mathrm{d}g = \int_G \chi_V(g)\,\mathrm{d}g.$$

Applying this identity to the G-representation space $\mathrm{Hom}(V, W)$ mentioned earlier (with V and W any two finite-dimensional G-representations), we obtain:

Proposition 2.1.
$$\dim_{\mathbf{C}} \mathrm{Hom}_G(V, W) = \int_G \overline{\chi}_V(g)\chi_W(g)\,\mathrm{d}g.$$

Rather generally, for any continuous functions $\psi, \phi : G \rightrightarrows \mathbf{C}$ we define
$$\langle \psi, \phi \rangle = \int_G \overline{\psi}(g)\phi(g)\,\mathrm{d}g,$$
so $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W)$. Thus, by Schur's Lemma for irreducible finite-dimensional $\mathbf{C}$-linear representations of G (same proof as for finite groups), we conclude that if V and W are *irreducible* then $\langle \chi_V, \chi_W \rangle$ is equal to 1 if $V \simeq W$ and it vanishes otherwise, exactly as in the special case of finite groups. These are the *orthogonality relations* among the characters.

By using integration in place of averaging, any quotient map $V \twoheadrightarrow W$ between finite-dimensional continuous representations of G admits a G-equivariant section, so a general finite-dimensional continuous representation V of G over $\mathbf{C}$ decomposes up to isomorphism as a direct sum $\oplus V_j^{\oplus n_j}$ where the V_j's are the pairwise non-isomorphic irreducible subrepresentations of G inside V and n_j is the multiplicity with which it occurs (explicitly, $n_j = \dim \mathrm{Hom}_G(V_j, V)$ due to Schur's Lemma, as for finite groups). Thus, in view of the orthogonality relations,
$$\langle \chi_V, \chi_V \rangle = \sum n_j^2.$$
This yields:

Corollary 2.2. *A representation (ρ, V) of G is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$; in the reducible case this pairing is larger than 1. In particular, χ_V determines whether or not V is an irreducible representation of G.*

In fact, we can push this computation a bit further: if W is an irreducible representation of G and $W \not\simeq V_j$ for any j (i.e., W does not occur inside V) then $\langle \chi_V, \chi_W \rangle = 0$, so an irreducible representation W of G occurs inside V if and only if $\langle \chi_V, \chi_W \rangle \neq 0$, in which case this pairing is equal to the multiplicity of W inside V. Thus, writing $n_{V,W} := \langle \chi_V, \chi_W \rangle$, we have
$$V \simeq \bigoplus_{n_{V,W} \neq 0} W^{\oplus n_{V,W}}.$$
This reconstructs V from data depending solely on χ_V (and general information associated to every irreducible representation of G). In particular, it proves:

Corollary 2.3. *Every representation V of G is determined up to isomorphism by χ_V.*